MAT 4574 Vector/Complex Analysis

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Contents

Preface

This class is essentially a survey of two different course – vector analysis and complex analysis – and the notes strive to follow the two main course texts:

- *Calculus: Early Transcendentals* by J. Stewart (9th ed)
- *Fundamentals of Complex Analysis with Applications to Engineering and Science* by E. Saff and A. Snider (3rd ed)

In an attempt to keep things cohesive, section numbers correspond to those found in the respective texts. There are some discrepancies, however, which I will outline below.

- Theorem numbers are internally consistent, but may not align with the numbers found in the course texts. If a theorem in the text is named, I've done my best to retain that naming.
- Some notation may differ from the textbook.

Every effort has been made in these notes to pick examples different from those in the text so that students may have a cornucopia of worked examples to look at. I will reiterate the old adage, however, that "math is not a spectator sport" and that the real learning comes from working through an example, not just reading it over.

The target audience for this course is largely senior undergraduate engineering students, who would likely be perfectly content to never see the word "proof" ever again. However, this is still a mathematics class and proofs can absolutely contribute to understanding the abstract concepts, and so I've tried to strike a balance and include only proofs (or sketches) which I find relatively simple, clever, or in some way illuminating. Many of these proofs can be found in full in the course text.

These notes are ever-evolving, and I welcome any feedback or corrections from my students.

Part I

Vector Analysis (from Stewart)

16 Vector Calculus

Preliminary Notation

In the first part of this class, it will be important to differentiate between points in space and vectors. As such, we use the following notation

- $\mathbb{R}$ – short-hand for the real numbers/scalars
- $\mathbb{X}^n$ – these are n-tuples of real numbers $p = (x_1, \ldots, x_n)$, representing points in space.
  This is nonstandard notation.
- $\mathbb{R}^n$ – these are n-element arrays of real numbers, $\mathbf{v} = \begin{bmatrix} x_1, \ldots, x_n \end{bmatrix}$ or $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, representing vectors. Square brackets replace the angle bracket notation used in the book.
  Square bracket notation is consistent with our treatment of vectors in Math 2114. As well, the math majors will more commonly reserve angle brackets for inner products/quadratic forms and physics majors will reserve them for bra-ket notation.
- $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ – these are the standard basis vectors for $\mathbb{R}^n$. These replace the $\hat{\mathbf{\imath}}$, $\hat{\mathbf{\jmath}}$, $\hat{\mathbf{k}}$ notation found in the book.
  Part of this class involves the imaginary unit i and I'm worried that $\hat{\mathbf{\imath}}$ will be hard to distinguish when handwritten.

16.1 Vector Fields

Definition

Let U be some region in $\mathbb{X}^n$. A **vector field on** U is a function $\mathbf{F} : U \to \mathbb{R}^n$. The **component functions** of $\mathbf{F}$, denoted $F_1, \ldots, F_n$, are just the functions in each of the entries of the output vector. That is, we can write

$$\begin{aligned} \mathbf{F}(x_1, \ldots, x_n) &= F_1(x_1, \ldots, x_n)\mathbf{e}_1 + \cdots + F_n(x_1, \ldots, x_n)\mathbf{e}_n \\ &= \begin{bmatrix} F_1(x_1, \ldots, x_n) \\ F_2(x_1, \ldots, x_n) \\ \vdots \\ F_n(x_1, \ldots, x_n) \end{bmatrix} \end{aligned}$$

Remark. Since this course is taking place exclusively in dimensions $n = 2$ and $n = 3$, in practice we will follow your book's convention and use P, Q, R for the component functions instead ofF_1, F_2, F_3.

One can visualize a vector field by placing a vector in the direction $\mathbf{F}(x_1, \ldots, x_n)$ emanating from the point $(x_1, \ldots, x_n)$

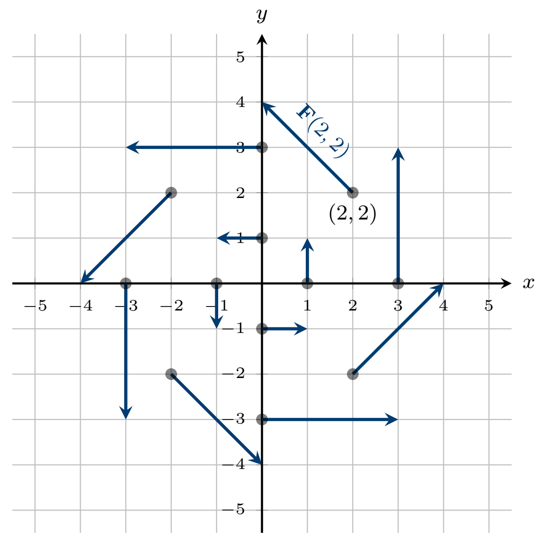
Example 16.1.1

Find the component functions of the vector field below and plot it.

$$\mathbf{F} : \mathbb{X}^2 \to \mathbb{R}^2$$
$$\mathbf{F}(x, y) = -y\mathbf{e}_1 + x\mathbf{e}_2$$

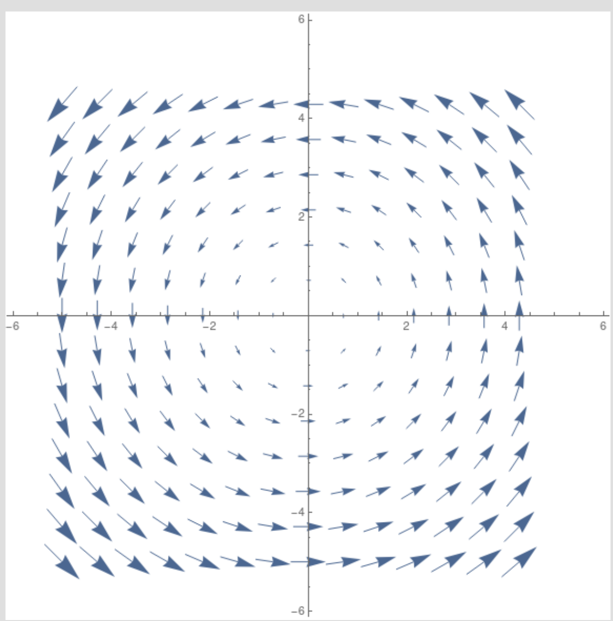
The component functions are $P(x, y) = -y$ and $Q(x, y) = x$. To visualize this vector field, we compute some values of $\mathbf{F}$.

(x, y)	$\mathbf{F}(x, y)$
(1,0)	[0,1]
(0,1)	[-1,0]
(-1,0)	[0,-1]
(0,-1)	[1,0]
(2,2)	[-2 2]
(-2,2)	[-2,-2]
(-2,-2)	[2,-2]
(2,-2)	[2,2]
(3,0)	[0,3]
(0,3)	[-3,0]
(-3,0)	[0,-3]
(0,-3)	[3,0]



Mathematica (as well as MATLAB, Octave, etc.) can plot an entire vector field to give a more complete ideas as to what's happening.

```
F[x_, y_] := {-y, x};
VectorPlot[ F[x,y], {x,-5,5}, {y,-5,5}, Frame->False, Axes->True]
```



The arrows in the vector field plot above kind of look like they follow along circles.

Exercise 16.1.2

If $\mathbf{F} : \mathbb{X}^n \to \mathbb{R}^n$ is a vector field on $\mathbb{X}^n$ and $\sigma(t) = (x_1(t), \ldots, x_n(t))$ is a smooth parameterized curve in $\mathbb{X}^n$, then σ is called an **integral curve** (or a **flow line** or a **streamline**) for $\mathbf{F}$ if, for every parameter value t, we have that

$$\mathbf{F}(\sigma(t)) = [\dot{x_1}(t), \ldots, \dot{x_n}(t)].$$

In other words, σ is an integral curve if the vector field arrows along the curve are precisely the tangent vectors of σ.

(a) Show that the circle $\sigma(t) = (\cos(t), \sin(t))$ is an integral curve for the vector field $\mathbf{F}$ in Example 16.1.1.

(b) Find all possible integral curves for the vector field $\mathbf{F}$ in Example 16.1.1.

Exercise 16.1.3

Let $\mathbf{F} : \mathbb{X}^2 \to \mathbb{R}^2$ be the vector field $\mathbf{F}(x, y) = [-x, y]$. Find all possible parameterized integral curves for this vector field.
Hint: It may be useful to recall that $\frac{d}{dt}e^t = e^t$.

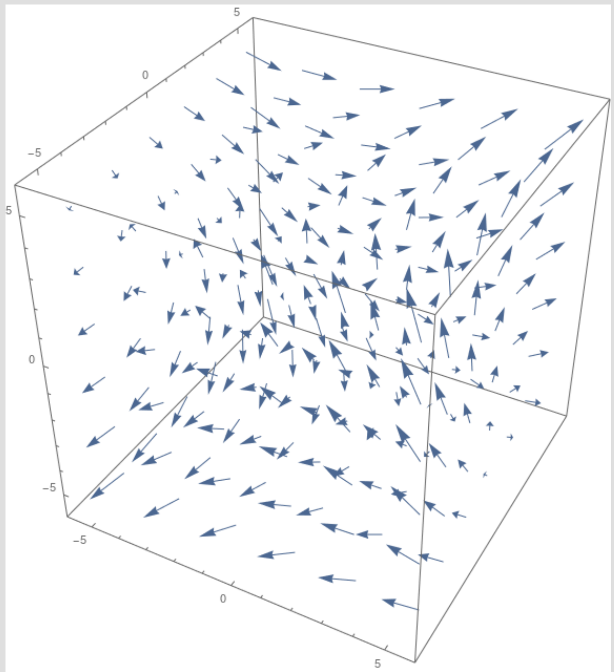
Example 16.1.4

Find the component functions of the vector field below and plot it.

$$\mathbf{F} : \mathbb{X}^3 \to \mathbb{R}^3$$
$$F(x, y, z) = y\mathbf{e}_1 + z\mathbf{e}_2 + x\mathbf{e}_3$$

The component functions are $P(x, y, z) = y$, $Q(x, y, z) = z$, and $R(x, y, z) = x$. Visually (using Mathematica),

```
F[x_, y_, z_] := {y, z, x};
VectorPlot3D[F[x,y,z], {x,-5,5}, {y,-5,5}, {z,-5,5}, Axes->True]
```



16.1.1 Gradient Fields

Recall that a **scalar function** is a function $f : \mathbb{X}^n \to \mathbb{R}$ and the **gradient** of f at the point $(x_1, \ldots, x_n)$ is the vector (denoted either ∇f or $\operatorname{grad} f$) in $\mathbb{R}^n$ given by

$$\begin{aligned}\nabla f(x_1, x_2, \ldots, x_n) &= \frac{\partial}{\partial x_1} f(x_1, \ldots, x_n)\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_n} f(x_1, \ldots, x_n)\mathbf{e}_n \\ &= \begin{bmatrix} \frac{\partial}{\partial x_1} f(x_1, \ldots, x_n) \\ \vdots \\ \frac{\partial}{\partial x_n} f(x_1, \ldots, x_n) \end{bmatrix}\end{aligned}$$

This means that ∇f is a function whose inputs are in $\mathbb{X}^n$ and whose outputs are in $\mathbb{R}^n$, i.e., is a vector field!

Definition

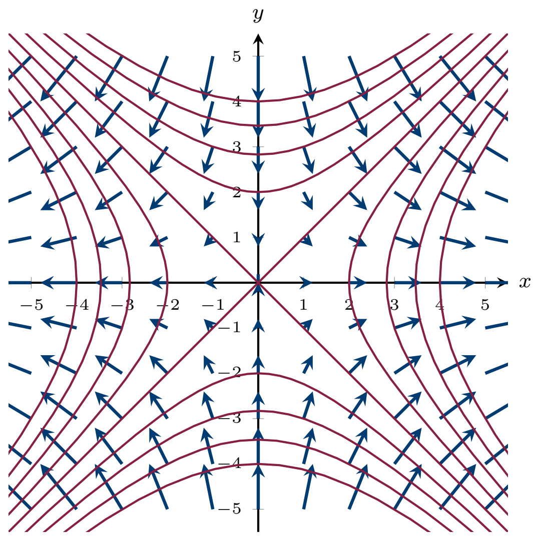
Given a scalar function $f : \mathbb{X}^n \to \mathbb{R}$, the corresponding **gradient field** is $\nabla f : \mathbb{X}^n \to \mathbb{R}^n$. The component functions are the partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_k}$.

Example 16.1.5

Find the corresponding gradient field for the function $f(x, y) = x^2 - y^2$.

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{e}_1 + \frac{\partial f}{\partial y}\mathbf{e}_2 = 2x\mathbf{e}_1 - 2y\mathbf{e}_2$$

When plotting the gradient field and the contour lines, one sees that the vectors are perpendicular to the contour lines, and the longer arrows occur when contour lines are close together. This makes sense as close contour lines imply that the corresponding surface is steeper at that point.



Definition

A vector field $\mathbf{F}$ is **conservative** if we can write $\mathbf{F} = \nabla f$ for some scalar function f. In this case, f is called the **potential function** for $\mathbf{V}$.

Remark. Not every vector field is conservative.

If a vector field is conservative, we can find the potential function.

Example 16.1.6

The vector field $\mathbf{F}$ given by $\mathbf{F}(x,y) = [2x, 2y]$ is conservative. Find a potential function f.

By definition of a conservative vector field, we should have that

$$\big[2x, 2y\big] = \mathbf{F}(x,y) = \nabla f(x,y) = \left[\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}\right]$$

So

$$\frac{\partial f}{\partial x} = 2x \qquad \Longrightarrow \qquad f(x,y) = x^2 + K(y)$$

where K is a single-variable function solely in terms of y. From the other partial derivative, we get

$$2y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\big(x^2 + K(y)\big) = K'(y) \qquad \Longrightarrow \qquad K(y) = y^2 + Const.$$

and therefore any function $f(x,y) = x^2 + y^2 + Const.$ is a potential function for $\mathbf{F}$.

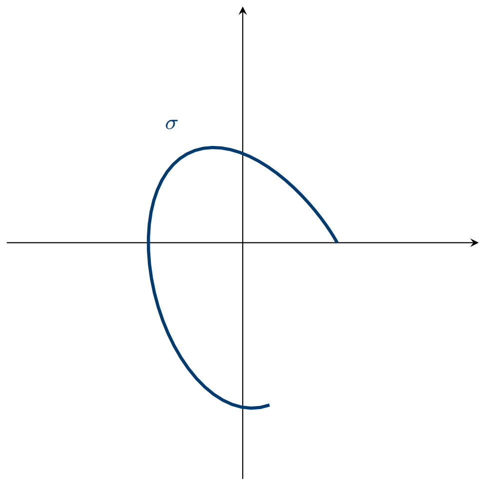
Exercise 16.1.7

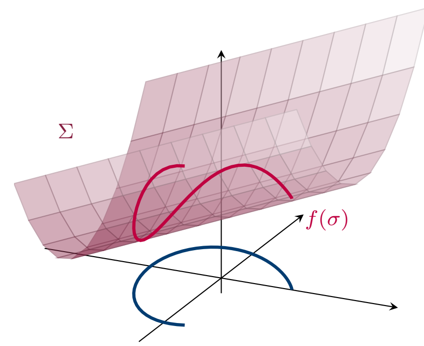
Using the technique outlined in Example 16.1.6, show that the vector field in Example 16.1.1 is not conservative.

We'll come back to find that there are easier ways to test whether a vector field is conservative.

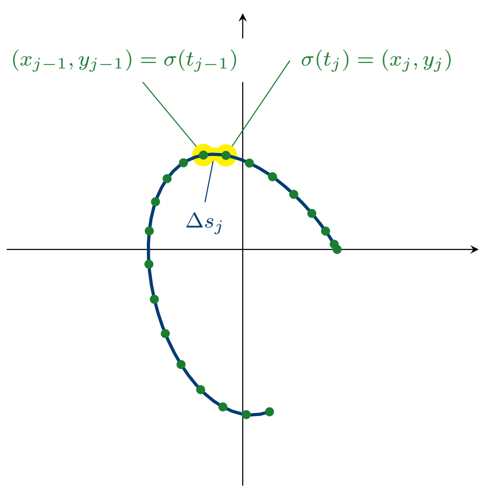
16.2 Line Integrals

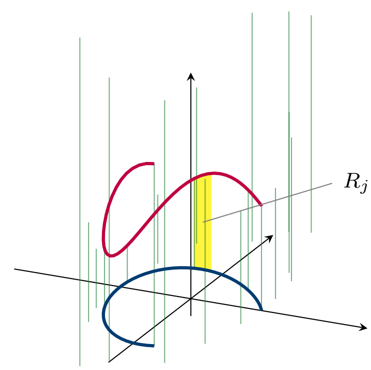
For simplicity, suppose Σ is a surface in $\mathbb{X}^3$ given by $z = f(x, y)$ and σ is a smooth curve in the xy-plane. Then $z = f(\sigma)$ is a parameterized curve in the surface lying above σ.





The compute the area under the curve $f(\sigma(t))$ for $a \le t \le b$, we'll begin by partitioning $[a, b]$ into k subintervals, $[t_0, t_1], \ldots, [t_{k-1}, t_k]$ which divides σ into k arcs of varying lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_k$. This also divides the area under the curve $f(\sigma)$ into k regions $R_1, \ldots, R_k$.





Since region R_j is approximately a rectangle of height $f(x_j, y_j)$ (where $(x_j, y_j) = \sigma(t_j)$) and width Δs_j, the area of the j^{th} region is approximately $f(x_j, y_j)\,\Delta s_j$, hence the region under the curve $f(\sigma)$ has area approximately

$$\sum_{j=1}^{k} f(x_j, y_j)\Delta s_j.$$

This is a Riemann sum, and so we can turn it into an integral by taking a limit as $n \to \infty$. This gives us the following general definition.

Definition: Line integral with respect to arc length

Suppose σ is a smooth curve in $\mathbb{X}^n$ and f is a scalar function defined on σ. Then the **line**

integral (or **path integral**) of f along σ is given by

$$\int_\sigma f(x_1, \ldots, x_n)\, ds = \lim_{k\to\infty} \sum_{j=1}^{k} f(x_{1,j}, x_{2,j}, \ldots, x_{n,j})\Delta s_j,$$

provided this limit exists.

Recall that, for a parameterized smooth planar curve $\sigma(t) = (x(t), y(t))$ where $a \le t \le b$, the arc length of σ is given by

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Intuitively the integrand here is the infinitesimal change in arc length of σ, and that's exactly what ds is in the definition of a line integral. One reasonably expects, then, that we can rewrite the line integral as

$$\int_\sigma f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

or more generally,

Proposition 16.2.1

Suppose $\sigma : [a, b] \to \mathbb{X}^n$ is a parameterized smooth curve and $f : \mathbb{X}^n \to \mathbb{R}$ is a scalar function that is continuous on σ. Then

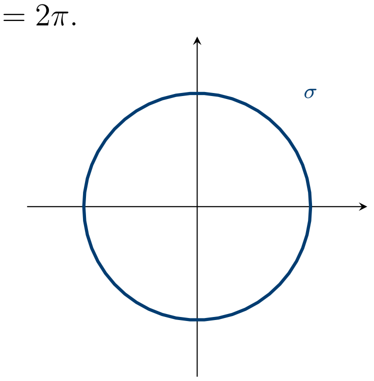
$$\int_\sigma f(x_1, \ldots, x_n)\, ds = \int_a^b f(x_1(t), \ldots, x_n(t))\sqrt{\left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2}\, dt.$$

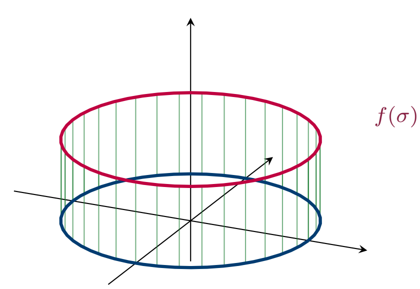
Let's check that all of this agrees with our intuition.

Example 16.2.2

Let $f(x, y) = 1$ and let σ be the unit circle. Evaluate $\int_\sigma f\, ds$.

In this case, $f(\sigma)$ is also the unit circle, but sitting at height $z = 1$, so the area under this curve is just the surface area of a cylinder of height 1. Thus it should be that $\int_\sigma f(x, y)\, ds = 2\pi r h = 2\pi(1)(1) = 2\pi$.





Writing $\sigma(t) = (\cos(t), \sin(t))$, this parameterization traverses the unit circle one time counter-clockwise when $0 \leq t \leq 2\pi$ and applying Proposition 16.2.1, we have

$$\begin{aligned}\int_\sigma f(x,y)\,ds &= \int_0^{2\pi} f(\cos(t), \sin(t))\sqrt{\left(\frac{d}{dt}\cos(t)\right)^2 + \left(\frac{d}{dt}\sin(t)\right)^2}\,dt \\ &= \int_0^{2\pi} (1)\sqrt{(-\sin(t))^2 + (\cos(t))^2}\,dt \\ &= \int_0^{2\pi} (1)\sqrt{1}\,dt = 2\pi.\end{aligned}$$

Remark. The value of a line integral is independent of the particular parameterization of σ.

Exercise 16.2.3

The planar curve $\sigma(t) = (\sin(7t), \cos(7t))$ traverses the unit circle one time *clockwise* when $0 \leq t \leq \frac{2\pi}{7}$. Show that $\int_\sigma f(x,y)\,ds = 2\pi$ using this different parameterization of σ.

Definition

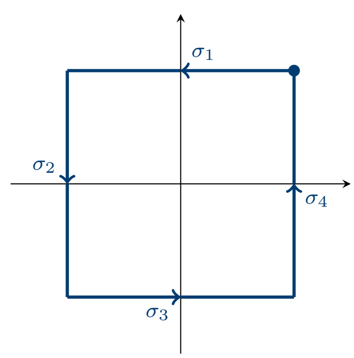
A curve $\sigma : [a,b] \to \mathbb{X}^n$ is called **piecewise smooth** if it is the union of a finite number of smooth curves whose endpoints agree. Explicitly, σ is piecewise smooth if we can find smooth curves $\sigma_1, \ldots, \sigma_k$ so that

$$\sigma(t) = \begin{cases} \sigma_1(t) & \text{when } a \leq t < t_1 \\ \sigma_1(t) & \text{when } t_1 \leq t < t_2 \\ & \vdots \\ \sigma_k(t) & \text{when } t_{k-1} \leq t \leq b \end{cases}$$

Example 16.2.4

The planar curve $\sigma : [0,4] \to \mathbb{X}^2$ is piecewise smooth and traverses the square in the figure below counter-clockwise, starting at the point $(1,1)$.

$$\sigma(t) = \begin{cases} \sigma_t(1) = (1-2t, 1) & \text{when } 0 \leq t < 1, \\ \sigma_2(t) = (-1, -2t+3) & \text{when } 1 \leq t < 2, \\ \sigma_3(t) = (2t-5, -1) & \text{when } 2 \leq t < 3, \\ \sigma_3(t) = (1, 2t-7) & \text{when } 3 \leq t \leq 4. \end{cases}$$



We can also evaluate line integrals along piecewise smooth curves by evaluating along each smooth piece separately and then summing them.

Proposition 16.2.5

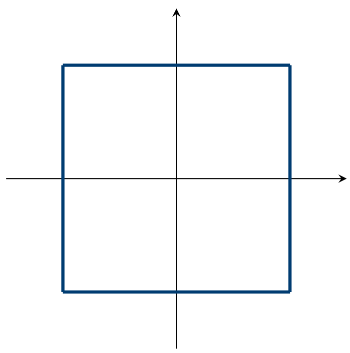
Suppose $\sigma : [a, b] \to \mathbb{X}^n$ is a piecewise smooth curve comprised of the smooth curves $\sigma_1, \dots, \sigma_k$. Suppose also that $f : \mathbb{X}^n \to \mathbb{R}$ is a scalar function that is continuous on σ. Then

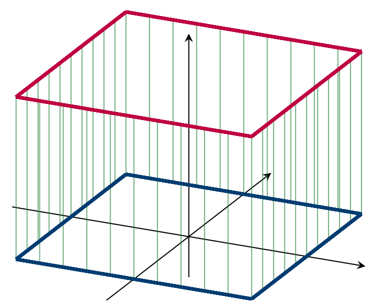
$$\int_\sigma f(x_1, \dots, x_n)\, ds = \int_{\sigma_1} f(x_1, \dots, x_n)\, ds + \cdots + \int_{\sigma_k} f(x_1, \dots, x_n)\, ds$$

Note that, in practice, we'll have to apply Proposition 16.2.1 to each smooth piece. Let's check to see that this proposition agrees with our intuition.

Example 16.2.6

Let $f(x, y) = 2$ and let σ be the square in the plane with endpoints $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$ (that is, the curve from Example 16.2.4). Compute $\int_\sigma f(x, y)\, ds$.





Intuitively, this line integral is computing the surface area of four sides of a cube (with side length 2), so the value should be $4(2)(2) = 16$.
Using the same parameterization as in Exercise 16.2.4, we compute

$$\begin{aligned}
\int_\sigma f(x, y)\, ds &= \int_{\sigma_1} f(x, y)\, ds + \int_{\sigma_2} f(x, y)\, ds + \int_{\sigma_3} f(x, y)\, ds + \int_{\sigma_4} f(x, y)\, ds \\
&= \int_0^1 f(1 - 2t, 1)\sqrt{\left(\frac{d}{dt}(1 - 2t)\right)^2 + \left(\frac{d}{dt}(1)\right)^2}\, dt \\
&\quad + \int_1^2 f(-1, -2t + 3)\sqrt{\left(\frac{d}{dt}(-1)\right)^2 + \left(\frac{d}{dt}(-2t + 3)\right)^2}\, dt \\
&\quad + \int_2^3 f(2t - 5, -1)\sqrt{\left(\frac{d}{dt}(2t - 5)\right)^2 + \left(\frac{d}{dt}(-1)\right)^2}\, dt \\
&\quad + \int_3^4 f(1, 2t - 7)\sqrt{\left(\frac{d}{dt}(1)\right)^2 + \left(\frac{d}{dt}(2t - 7)\right)^2}\, dt \\
&= \int_0^1 2\sqrt{(-2)^2 + 0}\, dt \\
&\quad + \int_1^2 2\sqrt{0 + (-2)^2}\, dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_2^3 2\sqrt{(2)^2+0}\,dt \\
&+\int_3^4 2\sqrt{0+(2)^2}\,dt \\
&= 2(2)+2(2)+2(2)+2(2) = 16.
\end{aligned}
$$

16.2.1 A Word About Parameterizations

For us, probably the two most important types of planar curve pieces you'll want to parameterize are line segments and arcs of circles.

The line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is given by

$$\sigma(t) = ((1-t)x_1 + tx_2, (1-t)y_1 + ty_2)\,.$$

(The obvious extension works to parameterize line segments between points in $\mathbb{X}^n$.)

The circle of radius r centered at (x_0, y_0), traversed counter-clockwise, is parameterized by

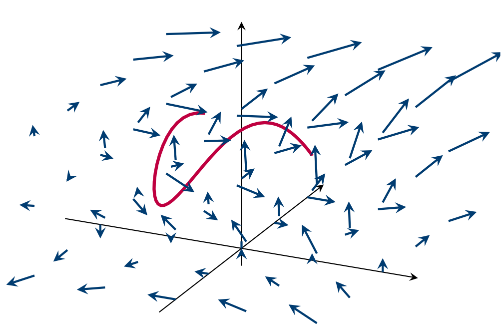
$$\sigma(t) = (x_0 + r\cos(t), y_0 + r\sin(t))\,.$$

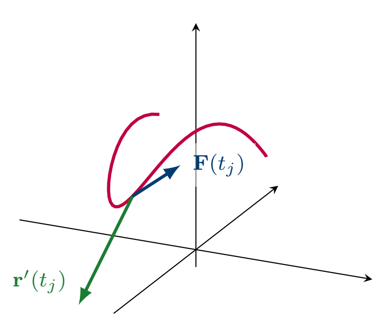
16.2.2 Line Integrals of Vector Fields; Work

A slight abuse of notation: Suppose σ is a smooth curve in $\mathbb{X}^n$ given by $\sigma(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ and let $\mathbf{r}$ be the vector $\mathbf{r}(t) = [x_1(t), \ldots, x_n(t)]$. It is common to write $\mathbf{r}(t)$ refer to the curve itself. This will give us some intuitive notational benefit and allow us to write things like $\mathbf{r}'(t_0)$ for the tangent vector of the curve $\mathbf{r}(t)$ at the point $\mathbf{r}(t_0)$.

Recall that if $\mathbf{F}$ is a constant force vector and $\mathbf{D}$ is a displacement vector, then the work in moving an object along $\mathbf{D}$ is given by $W = \mathbf{F}\cdot\mathbf{D}$.

Suppose $\mathbf{F}$ is a continuous vector field on $\mathbb{X}^n$ and $\sigma(t)$ is a smooth parameterized curve. Consider the arc along σ from (x_j, y_j, z_j) to $(x_{j+1}, y_{j+1}, z_{j+1})$ with arc length Δs_j. Let $\mathbf{T}(x_j, y_j, z_j)$ be the unit tangent vector of σ at (x_j, y_j, z_j).





Then, the work done in moving an object along this arc is approximately

$$W_j = \mathbf{F}(x_j, y_j, z_j)\cdot(\Delta s_j \mathbf{T}(x_j, y_j, z_j)) = (\mathbf{F}(x_j, y_j, z_j)\cdot\mathbf{T}(x_j, y_j, z_j))\,\Delta s_j$$

hence the work done in moving along the whole curve is approximately

$$W = \sum_j \left(\mathbf{F}(x_j, y_j, z_j) \cdot \mathbf{T}(x_j, y_j, z_j)\right) \Delta s_j$$

This looks like a Riemann sum, so the following should be of no surprise

Definition

Let σ be a smooth curve in $\mathbb{X}^n$, let $\mathbf{T}(x_1, \dots, x_n)$ be the unit tangent vector at each point $(x_1, \dots, x_n)$ on σ, and let $\mathbf{F}$ be a vector field on $\mathbb{X}^n$. Then the **work W done by $\mathbf{F}$ along σ** is

$$W = \int_\sigma \mathbf{F} \cdot \mathbf{T}\, ds$$

We give the curve via a parameterized vector $\mathbf{r}(t)$ for $a \leq t \leq b$, then $ds = |\mathbf{r}'(t)|\, dt$ and $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, hence the work equation above becomes

$$\int_\sigma \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

and this is often shortened to $\displaystyle\int_\sigma \mathbf{F} \cdot d\mathbf{r}$.

Definition

Let σ be a smooth curve in $\mathbb{X}^n$ defined by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$ and let $\mathbf{F}$ be a vector field that is continuous on σ. Then the **line integral of $\mathbf{F}$ along σ** is

$$\int_\sigma \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_\sigma \mathbf{F} \cdot \mathbf{T}\, ds.$$

Example 16.2.7

Evaluate the line integral of $\mathbf{F}$ along σ where $\mathbf{F}(x, y) = \begin{bmatrix} xy \\ -4x \end{bmatrix}$ and σ is the line segment from $(1, 0)$ to $(3, 1)$.

Notice that we can parameterize σ via the vector equation

$$\mathbf{r}(t) = \begin{bmatrix} 1 + 2t \\ t \end{bmatrix} \qquad \text{and thus } \mathbf{r}'(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

where $0 \leq t \leq 1$. We then have that

$$\begin{aligned} \int_\sigma \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^1 \begin{bmatrix} (1+2t)t \\ -4(1+2t) \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} dt \\ &= \int_0^1 (4t^2 - 6t - 4)\, dt = -\frac{17}{3} \end{aligned}$$

Example 16.2.8

Evaluate $\int_\sigma \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = 8x^2yz\mathbf{e}_1 + 5z\mathbf{e}_2 - 4xy\mathbf{e}_3$ and σ is the curve given by $\mathbf{r}(t) = t\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3$ for $0 \leq t \leq 1$.

We have that

$$\begin{aligned} \mathbf{r}'(t) &= \mathbf{e}_1 + 2t\mathbf{e}_2 + 3t^2\mathbf{e}_3, \\ \mathbf{F}(\mathbf{r}(t)) &= 8t^7\mathbf{e}_1 + 5t^3\mathbf{e}_2 - 4t^3\mathbf{e}_3, \\ \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= 8t^7 + 2t(5t^3) - 3t^2(4t^3) \\ &= 8t^7 + 10t^4 - 12t^5. \end{aligned}$$

It follows that

$$\begin{aligned} \int_\sigma \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^1 8t^7 + 10t^4 - 12t^5\, dt = 1. \end{aligned}$$

16.2.3 Line Integrals with respect to x, y, z

When defining the line integral, we integrated with respect to arc length. One may encounter integrals that are not with respect to arc length. Recalling the basic theory of differentials, that if $x = x(t)$, then $dx = x'(t)\, dt$, the following definition occurs.

Definition

Let $\sigma : [a, b] \to \mathbb{X}^n$ be a (piecewise) smooth curve in space given by $\sigma(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ and let $f : \mathbb{X}^n \to \mathbb{R}$ be a scalar function that is continuous on σ. Then, for $j = 1, \ldots, n$, the **line integral of f along σ with respect to x_j** is

$$\int_\sigma f(x_1, \ldots, x_n)\, dx_j = \int_a^b f(x_1(t), \ldots, x_n(t))\, x_j'(t)\, dt$$

Example 16.2.9

Evaluate $\displaystyle\int_\sigma \left(x^2y + \sin(x)\right)\, dy$ where σ is the arc of the parabola $y = x^2$ from $(0, 0)$ to (π, π^2)

First we note that we can parameterize σ as follows:

$$\sigma(t) = (x(t), y(t)) = \left(x(t), (x(t))^2\right) = (t, t^2) \ \text{ where } 0 \leq t \leq \pi$$

Now,

$$\int_\sigma \left(x^2y + \sin(x)\right)\, dy = \int_0^\pi \left(t^2(t^2) + \sin(t)\right)(2t)\, dt$$

$$= \int_0^{\pi} 2t^5 + 2t\sin(t)\,dt$$
$$= \frac{t^6}{3} - 2t\cos(t) + 2\sin(t)\Big|_0^{\pi} = \frac{\pi^6}{3} + 2\pi.$$

If two line integrals occur along the same path, it is common to simplify notation a bit. Explicitly, one might write

$$\int_\sigma P(x,y)\,dx + \int_\sigma Q(x,y)\,dy = \int_\sigma P(x,y)\,dx + Q(x,y)\,dy, \qquad \text{and}$$
$$\int_\sigma P(x,y,z)\,dx + \int_\sigma Q(x,y,z)\,dy + \int_\sigma R(x,y,z)\,dz = \int_\sigma P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

In fact, the above occur quite naturally

Example 16.2.10

Here we see the relationship between line integrals with respect to arc length and x, y, z.

Suppose you are given a vector field

$$\mathbf{F}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

and a (piecewise) smooth curve σ determined by the vector equation

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

with $a \le t \le b$. Writing

$$\mathbf{r}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$$

it follows that

$$\begin{aligned}
\int_\sigma \mathbf{F}\cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\
&= \int_a^b \begin{bmatrix} P(x(t),y(t)) \\ Q(x(t),y(t)) \end{bmatrix} \cdot \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} dt \\
&= \int_a^b P(x(t),y(t))\dot{x}(t) \;+\; Q(x(t),y(t))y'(t)\,dt \\
&= \int_\sigma P(x,y)\,dx + Q(x,y)\,dy
\end{aligned}$$

A similar computation holds in the 3-dimensional case.

Example 16.2.11

Evaluate $\displaystyle\int_\sigma y\,dx + z\,dy + x\,dz$ where $x = \sqrt{t}$, $y = t$, $z = t^2$ and $1 \le t \le 4$.

$$\begin{aligned}
\int_\sigma y\,dx + z\,dy + x\,dz &= \int_1^4 y(t)x'(t)\,dt \ + \ z(t)y'(t)\,dt \ + \ x(t)z'(t)\,dt \\
&= \int_1^4 \left[(t)\left(\frac{1}{2\sqrt{t}}\right) + \left(t^2\right)(1) + \left(\sqrt{t}\right)(2t)\right]\,dt \\
&= \int_1^4 \frac{1}{2}t^{1/2} + t^2 + 2t^{3/2}\,dt \\
&= \left[\frac{1}{3}t^{3/2} + \frac{1}{3}t^3 + \frac{4}{5}t^{5/2}\right]_1^4 = \frac{722}{15}
\end{aligned}$$

16.3 The Fundamental Theorem for Line Integrals

Recall the following

Theorem 16.3.1: Fundamental Theorem of Calculus, pt II

If F is differentiable and F' is continuous on $[a, b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

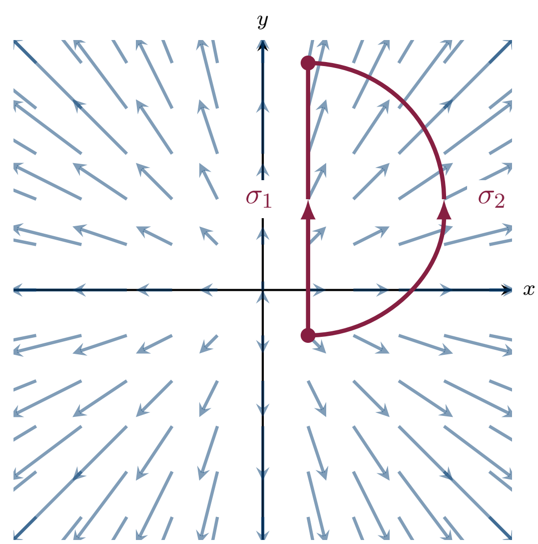
In short, the value of the integral only depended on the endpoints of the interval. It would be nice if there was some equivalent version for line integrals, where the value of the integral was determined only by the path's endpoints and not the particular path taken. Indeed, this is sometimes the case.

Example 16.3.2

Let $\mathbf{F}$ be the vector field on $\mathbb{X}^2$ from Example 16.1.6:

$$\mathbf{F}(x, y) = [2x, 2y]$$

Evaluate $\displaystyle\int_\sigma \mathbf{F} \cdot d\mathbf{r}$ for two different paths, σ_1 and σ_2, from $(1, -1)$ to $(1, 5)$.



Take σ_1 to be the straight line defined by the vector equation $\mathbf{r}_1(t) = [1, t]$ for $-1 \leq t \leq 5$. We then have that

$$\begin{aligned}
\int_{\sigma_1} \mathbf{F} \cdot d\mathbf{r}_1 &= \int_{-1}^{5} \mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t)\,dt \\
&= \int_{-1}^{5} [2, 2t] \cdot [0, 1]\,dt \\
&= \int_{-1}^{5} 2t\,dt
\end{aligned}$$

$$= t^2 \Big|_{-1}^{5} = 24$$

Now take σ_2 to be the circular arc defined by the vector equation $\mathbf{r}_2(t) = [1+3\cos(t), 2+3\sin(t)]$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

$$\begin{aligned}
\int_{\sigma_1} \mathbf{F} \cdot d\mathbf{r}_2 &= \int_{-\pi/2}^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_{-\pi/2}^{\pi/2} [2+6\cos(t), 4+6\sin(t)] \cdot [-3\sin(t), 3\cos(t)]\, dt \\
&= \int_{-\pi/2}^{\pi/2} 12\cos(t) - 6\sin(t)\, dt \\
&= 12\sin(t) + 6\cos(t) \Big|_{-\pi/2}^{\pi/2} = 24
\end{aligned}$$

Recall that the vector field in this last example is special – *it's a conservative vector field*. This leads one to conjecture the following:

Theorem 16.3.3: Fundamental Theorem for Line Integrals

Let σ be a smooth curve in $\mathbb{X}^n$ given by the vector equation $\mathbf{r}(t)$, where $a \le t \le b$. Lert f be a scalar function that is differentiable and whose gradient vector field ∇f is continuous on σ. Then

$$\int_\sigma \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(b)).$$

Proof. Let $\mathbf{r}(t) = [x_1(t), \cdots, x_n(t)]$ and write $\mathbf{r}'(t) = [\frac{dx_1}{dt}, \cdots, \frac{dx_n}{dt}]$. Then

$$\begin{aligned}
\int_\sigma \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_a^b \begin{bmatrix} \frac{\partial f}{\partial x_1}\Big|_{\mathbf{r}(t)} \\ \vdots \\ \frac{\partial f}{\partial x_n}\Big|_{\mathbf{r}(t)} \end{bmatrix} \cdot \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \vdots \\ \frac{dx_n(t)}{dt} \end{bmatrix} dt \\
&= \int_a^b \frac{\partial f}{\partial x_1}\Big|_{\mathbf{r}(t)} \frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\Big|_{\mathbf{r}(t)} \frac{dx_n}{dt}\, dt \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt && \text{(multi-variate chain rule)} \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a))
\end{aligned}$$

□

Dot products generally describe "how much" of one vector lies in the direction of another vector, so the quantity $\mathbf{F} \cdot d\mathbf{r}$ describes how much of the tangent vector to $\mathbf{r}(t)$ lies in the direction of $\mathbf{F}(\mathbf{r}(t))$.

The integral is then a total of how much this occurs along the curve. What this theorem says is that any curve measures the same total amount.

Definition

Let D be some region in $\mathbb{X}^n$ and let $\mathbf{F} : D \to \mathbb{R}^n$ be a continuous vector field on D. We say that the line integral $\displaystyle\int_\sigma \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if $\displaystyle\int_{\sigma_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\sigma_2} \mathbf{F} \cdot d\mathbf{r}$ for any two smooth curves σ_1, σ_2 in D with the same initial and terminal points.

Corollary 16.3.4

If $\mathbf{F}$ is a conservative vector field and σ is a smooth curve, then $\displaystyle\int_\sigma \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

Definition

A curve $\sigma : [a, b] \to \mathbb{X}^n$ is said to be

- **closed** if $\sigma(a) = \sigma(b)$ (that is, if the initial and terminal points are the same).
- **simple** if $\sigma(t_1) \neq \sigma(t_2)$ whenever $a < t_1 < t_2 < b$ (that is, the curve doesn't intersect itself except possibly at the endpoints).

Example 16.3.5

The following figures show examples of curves in the various combinations of closed/simple.

Figure 16.3.1: Not closed, not simple

Figure 16.3.2: Not closed, simple

Figure 16.3.3: Closed, not simple

Figure 16.3.4: Closed, simple

Suppose A, B are two fixed points in D (some region in $\mathbb{X}^n$), and suppose that $\int_\sigma \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D. Then we can come up with two piecewise-smooth curves, σ_1 and σ_2 from A to B and from B to A (respectively) that form a closed path. Let γ be the name of this closed path from A to A.

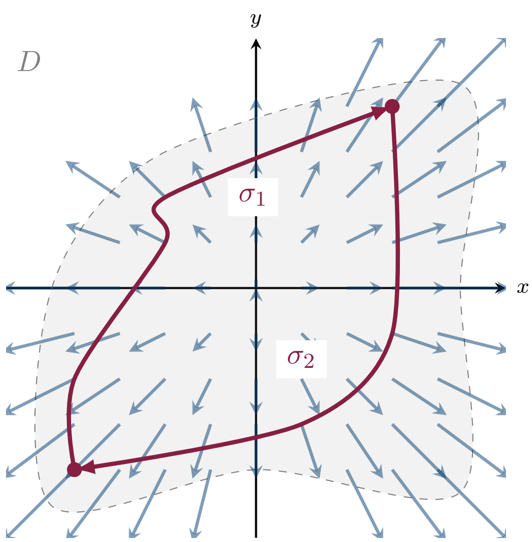


Figure 16.3.5: Two paths

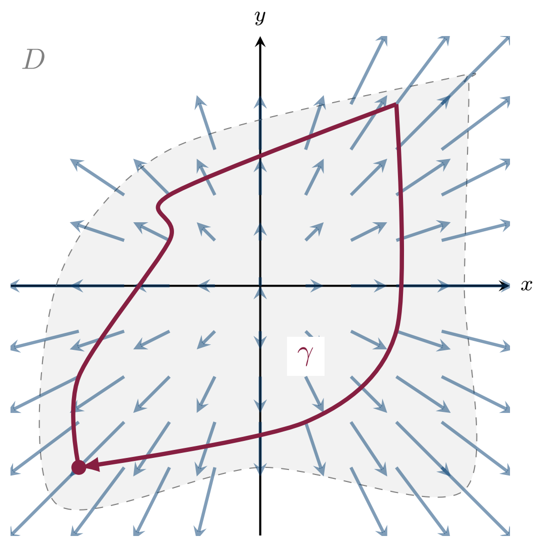


Figure 16.3.6: One closed path

By independence of path, we know that

$$\int_{\sigma_1} \mathbf{F} \cdot d\mathbf{r} = -\int_{\sigma_2} \mathbf{F} \cdot d\mathbf{r}$$

which rearranges to

$$0 = \int_{\sigma_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\sigma_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\gamma} \mathbf{F} \cdot d\mathbf{r}.$$

This proves the following

Theorem 16.3.6

$\int_\gamma \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_\gamma \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path γ in D.

What we generally have is that line integrals evaluate to zero on loops in conservative vector fields. The following tells us that this actually uniquely classifies whether or not a vector field is conservative.

Definition

A region D in $\mathbb{X}^n$ is said to be **open** if, for every point $x = (x_1, \ldots, x_n)$ in $\mathbb{X}^n$, we can draw a small ball/disk around x that is entirely contained in D. A region D is said to be **connected** if any two points in D can be joined by a path in D. A region D is said to be **simply-connected** if it is connected and has no “holes” in it.

Remark. Simple connectedness is actually quite tedious to define rigorously (because “hole” is hard to define rigorously), although fairly intuitive as a concept.

Example 16.3.7: Open Region, non-Open Region

$D_1 =$ is an open region in $\mathbb{X}^2$. $D_2 =$ is not an open region in $\mathbb{X}^2$.

Figure 16.3.7: Open Region

Figure 16.3.8: Not an Open Region

Example 16.3.8: Connected Region, Disconnected Region

$D_1 =$ is a connected region in $\mathbb{X}^2$. $D_2 =$ is a disconnected region in $\mathbb{X}^2$.

Figure 16.3.9: Connected Region

Figure 16.3.10: Disconnected Region

Example 16.3.9: Simply-Connected Region, non-Simply-Connected Region

$D_1 =$ is a simply-connected region in $\mathbb{X}^2$. $D_2 =$ is not a simply-connected region in $\mathbb{X}^2$.

Figure 16.3.11: Connected Region

Figure 16.3.12: Disconnected Region

Theorem 16.3.10

Suppose $\mathbf{F}$ is a vector field that is conservative on some open connected region D in $\mathbb{X}^n$. If $\int_\sigma \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D.

(or equivalently)

Suppose $\mathbf{F}$ is a vector field that is conservative on some open connected region D in $\mathbb{X}^n$. If $\int_\gamma \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path γ in D, then $\mathbf{F}$ is a conservative vector field on D.

The proof works by explicitly cooking up the scalar function f for which $\mathbf{F} = \nabla f$. We'll only give the proof in the 2-dimensional case, but it should be clear that the process can extend to the n-dimensional case.

Proof. INSERT PROOF SKETCH HERE

□

It should be noted that Theorem 16.3.10 is a good tool from a theoretical perspective, but isn't really useful in practice to check if a vector field is conservative (because you would have to check EVERY possible path or closed curve). In practice, given a vector field $\mathbf{F}(x_1, \ldots, x_n) = \sum_{j=1}^{n} F_j(x_1, \ldots, x_n)\mathbf{e}_n$, we'd like a simple condition on the component functions.

We restrict to the 2-dimensional case, $\mathbb{X}^2$, for the time being.

Suppose that $\mathbf{F} = P\mathbf{e}_1 + Q\mathbf{e}_2$ is a conservative vetor field. Then there's a potential function f for which $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$. But then

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

So conservative vector fields satisfy this relatively simple relationship on their component functions. And it turns out (as a consequence of the next section) that this is actually sufficient to determine if a vector field is conservative.

Theorem 16.3.11

Let $\mathbf{F} = P\mathbf{e}_1 + Q\mathbf{e}_2$ be a vector field on an open simply-connected region D. Suppose P and Q have continuous first-order partial derivatives. Then $\mathbf{F}$ is conservative if and only if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ throughout D.

Example 16.3.12

Let $\mathbf{F}$ be the vector field from Example 16.1.6:

$$\mathbf{F}(x, y) = 2x\mathbf{e}_1 + 2y\mathbf{e}_2$$

Verify that it is conservative using Theorem 16.3.11.

Since

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[2x] = 0$$
$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[2y] = 0$$

then **F** is conservative.

Example 16.3.13

Let **F** be the vector field from Example 16.1.1:

$$\mathbf{F}(x,y) = -y\mathbf{e}_1 + x\mathbf{e}_2$$

Verify that it is not conservative using Theorem 16.3.11.

Since

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[-y] = -1$$
$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[x] = 1$$

then **F** is not conservative.

16.4 Green's Theorem

?? is the crucial element in proving that conservative vector fields on $\mathbb{X}^2$ are characterized by some first partial derivatives.

Theorem 16.4.1: Green's Theorem

Let σ be a parameterized piecewise-smooth, simple closed curve in $\mathbb{X}^2$ and let D be the open simply-connected region bounded by σ. If $P(x, y), Q(x, y)$ are scalar functions with continuoous partial derivatives on (an open region containing) D, then

$$\int_\sigma P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Remark. When σ is a parameterized simple closed curve, it's common to use the symbol $\displaystyle\oint_\sigma$ instead of just $\displaystyle\int_\sigma$. Furthermore, if σ is the boundary of a region D (as above), one may write ∂D instead of σ. In this way, the crux of Green's Theorem is often written

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dA$$

Proof. Sketch? □

Example 16.4.2

Evaluate $\int_\sigma x^4\,dx + xy\,dy$ where σ traverses counter-clockwise the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$.

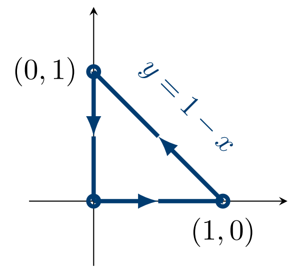


Figure 16.4.1: Triangular path

Path Integral

Homework Exercise

Surface Integral

Notice that the region D bounded by the triangle is open and simply-connected, hence $\sigma = \partial D$. Letting $P(x, y) = x^4$ and $Q(x, y) = xy$ we have

$$\begin{aligned}
\oint_\sigma x^4\,dx - xy\,dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA \\
&= \int_0^1 \int_0^{1-x} (y - 0)\,dy\,dx \\
&= \int_0^1 \left[\frac{1}{2}y^2\right]_0^{1-x}\,dx \\
&= \int_0^1 \frac{1}{2}(1-x)^2\,dx \\
&= \frac{1}{2}\int_0^1 1 - 2x + x^2\,dx \\
&= \frac{1}{2}\left[x - x^2\frac{1}{3}x^3\right]_0^1 = \frac{1}{6}.
\end{aligned}$$

16.4.1 Green's Theorem to Find Area

Recall that for an open, simply-connected region, D, the area of D is given by

$$\text{Area}(D) = \iint_D dA$$

Notice that if we $P(x, y)$ and $Q(x, y)$ so that $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$, then the above becomes

$$\text{Area}(D) = \iint_D dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = \oint_{\partial D} P\,dx + Q\,dy$$

which reduces potentially complicated area computations to a single-variable integral. To make life even easier, there are several usefully simple choices of P and Q, namely

$P(x,y)$	$Q(x,y)$	Area(D)
$-y$	0	$-\oint_{\partial D} y\,dx$
0	x	$\oint_{\partial D} x\,dy$
$-\frac{1}{2}y$	$\frac{1}{2}x$	$\frac{1}{2}\oint_{\partial D} x\,dy - y\,dx$

Example 16.4.3

Find the area of the ellipse given by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

We can parameterize the boundary of the ellipse via

$$\sigma(t) = (x(t), y(t)) = (a\cos(t), b\sin(t)), \quad 0 \le t \le 2\pi.$$

Using the third equation from the table above, we get

$$\begin{aligned}
\text{Area(ellipse)} &= \frac{1}{2}\oint_{\sigma} x\,dy - y\,dx \\
&= \frac{1}{2}\int_0^{2\pi} x(t)y'(t)\,dt - y(t)x'(t)\,dt \\
&= \frac{1}{2}\int_0^{2\pi} ab\cos^2(t) + ab\sin^2(t)\,dt \\
&= \frac{ab}{2}\int_0^{2\pi} dt = ab\pi.
\end{aligned}$$

16.4.2 Extended Version of Green's Theorem

It turns out that we can also use Green's Theorem to find areas of regions that are connected, but not simply-connected (or, more specifically, a connected union of finitely-many simply-connected regions). For simplicity, we introduce the following (non-standard) notation to use with our integrals: let $\sigma_1, \sigma_2 : [0,1] \to \mathbb{X}^n$ be parameterized (piecewise) smooth curves. Then

- $\sigma_1 \oplus \sigma_2$ denotes the piecewise curve obtained from traversing σ_1 and then traversing σ_2.

$$\sigma_1 \oplus \sigma_2(t) = \begin{cases} \sigma_1(2t) & \text{when } 0 \le t \le \frac{1}{2} \\ \sigma_2(2t-1) & \text{when } \frac{1}{2} < t \le 1 \end{cases}$$

- $-\sigma_1$ denotes the curve obtained from traversing σ_1 backwards.

$$-\sigma_1(t) = \sigma_1(-t).$$

In this way we have the following

$$\int_{\sigma_1 \oplus \sigma_2} f = \int_{\sigma_1} f + \int_{\sigma_2} f$$
$$\text{and} \qquad \int_{(-\sigma_1)} f = -\int_{\sigma_1} f.$$

Remark. When it comes to regions with multiple boundary curves, one has to take care that their orientations agree. Our standard has been to traverse the outer boundary counter-clockwise, which means that the bounded region is to the left (if traveling along the curve). As a result, any inner boundary curves have to be traversed clockwise to keep the bounded region to the left.

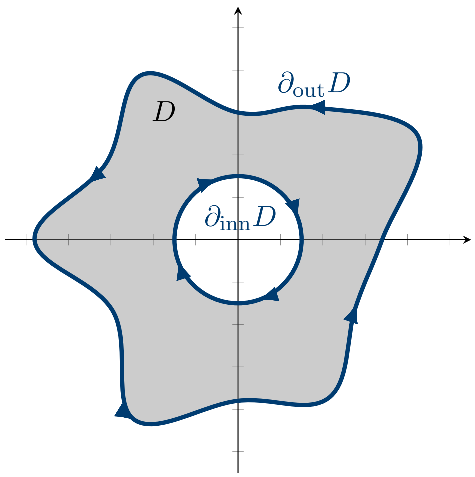


Figure 16.4.2: Non-simply connected region

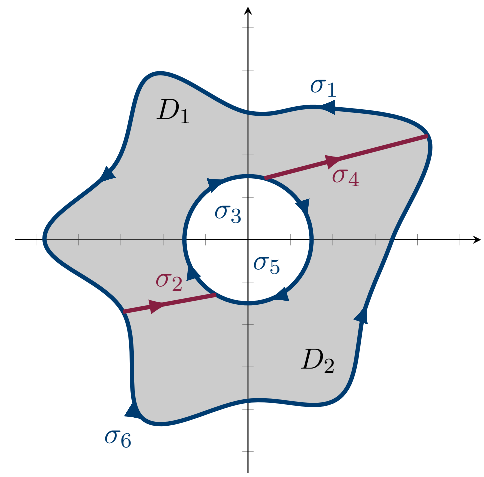


Figure 16.4.3: Decomposition into simply-connected regions

We will apply Green's Theorem to these regions separately.

$$\begin{aligned}
\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= \left(\oint_{\sigma_1 \oplus \sigma_2 \oplus \sigma_3 \oplus \sigma_4} P\,dx + Q\,dy\right) + \left(\oint_{\sigma_6 \oplus (-\sigma_4) \oplus \sigma_5 \oplus (-\sigma_2)} P\,dx + Q\,dy\right) \\
&= \int_{\sigma_1} P\,dx + Q\,dy + \int_{\sigma_2} P\,dx + Q\,dy + \int_{\sigma_3} P\,dx + Q\,dy + \int_{\sigma_4} P\,dx + Q\,dy \\
&\quad + \int_{\sigma_6} P\,dx + Q\,dy - \int_{\sigma_4} P\,dx + Q\,dy + \int_{\sigma_5} P\,dx + Q\,dy - \int_{\sigma_2} P\,dx + Q\,dy \\
&= \int_{\sigma_1} P\,dx + Q\,dy + \int_{\sigma_6} P\,dx + Q\,dy + \int_3 P\,dx + Q\,dy + \int_4 P\,dx + Q\,dy \\
&= \oint_{\sigma_1 \oplus \sigma_6} P\,dx + Q\,dy + \oint_{\sigma_3 \oplus \sigma_4} P\,dx + Q\,dy \\
&= \oint_{\partial_{\text{out}} D} P\,dx + Q\,dy + \oint_{\partial_{\text{in}} D} P\,dx + Q\,dy.
\end{aligned}$$

And this last term is sometimes just written as $\oint_{\partial D} P\,dx + Q\,dy$ with the implication that it potentially has multiple boundary components and they are summed in this way.

Example 16.4.4

Compute the area of the annulus A_{r_1,r_2}, the open region bounded between the circles $x^2+y^2=r_1^2$ and $x^2+y^2=r_2^2$ with $r_2>r_1$.

16.4.3 A Deformation Theorem

The same diagram as before gives us something else here. Suppose $\mathbf{F}$ is a continuous vector field on D and is conservative on D_1 and D_2 (it may be conservative on all of D too, but that's not strictly necessary).

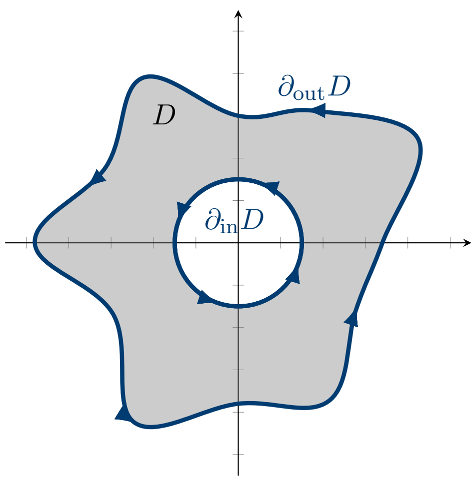


Figure 16.4.4: Non-simply connected region

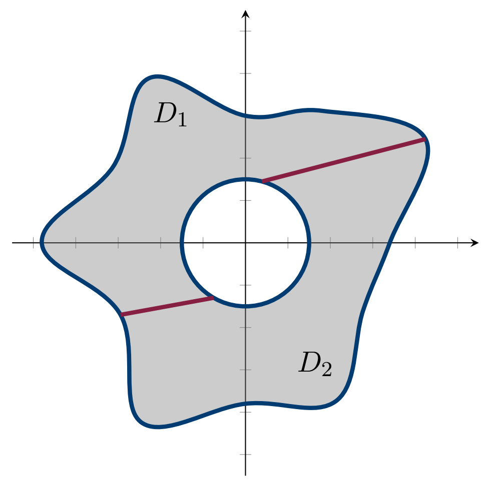


Figure 16.4.5: Decomposition into simply-connected regions

Noting that we have reversed the natural orientation of $\partial_{\text{in}} D$ above, the previous computations yielded

$$\oint_{\partial_{\text{out}} D} \mathbf{F}\cdot d\mathbf{r} - \oint_{\partial_{\text{in}} D} \mathbf{F}\cdot d\mathbf{r} = \oint_{\partial D_1} \mathbf{F}\cdot d\mathbf{r} + \oint_{\partial D_2} \mathbf{F}\cdot d\mathbf{r} \tag{16.4.1}$$

Since $\mathbf{F}$ is assumed to be conservative on both D_1 and D_2, then $\oint_{\partial D_1} \mathbf{F}\cdot d\mathbf{r} = 0$ and $\oint_{\partial D_2} \mathbf{F}\cdot d\mathbf{r} = 0$. Hence Equation **??** becomes

$$\oint_{\partial_{\text{out}} D} \mathbf{F}\cdot d\mathbf{r} = \oint_{\partial_{\text{in}} D} \mathbf{F}\cdot d\mathbf{r}$$

So evaluating the line integral around the outer loop is the same as integrating the line integral around the inner loop; we can pick whichever one is easier! This gives us the following

Proposition 16.4.5: A One-Holed Deformation Theorem

With D and $\mathbf{F}$ as above, if σ is any closed curve enclosing the hole and is traversed in the same direction as ∂D, then $\int_{\partial_{\text{in}} D} \mathbf{F}\cdot d\mathbf{r} = \int_\sigma \mathbf{F}\cdot d\mathbf{r}$.

The name comes from this visualization

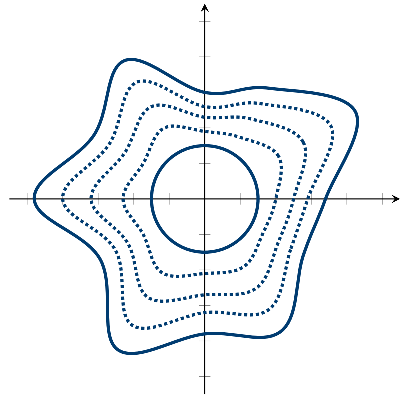


Figure 16.4.6: Step-by-step, the outer path loop is being "deformed" and shrinking into the inner loop

In practice, what this means is that you can always pick the most convenient curve possible!

Example 16.4.6

Let $\mathbf{F}(x,y) = \dfrac{1}{x^2+y^2}\begin{bmatrix}-y\\ x\end{bmatrix}$ and let σ be *any* closed curve that encloses the origin. Evaluate $\int_\sigma \mathbf{F}\, d\mathbf{r}$.

$\mathbf{F}(x,y)$ is not defined at the origin, but it is quick to check that $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, hence $\mathbf{F}$ is conservative on any simply-connected region *NOT* containing the origin.
Since σ is any closed curve enclosing the origin, we can find a small circle of radius $r > 0$ around the origin that is entirely enclosed by σ; call this $\tilde{\sigma}$ and parameterize it in the usual fashion: $\mathbf{r}(t) = [r\cos(t), r\sin(t)]$. Letting γ_1, γ_2 be lines along the x-axis adjoining σ and $\tilde{\sigma}$, we create two simply-connected regions on which $\mathbf{F}$ is conservative. By the Deformation theorem, we have that

$$\begin{aligned}\int_\sigma P(x,y)\,dx + Q(x,y)\,dy &= \int_{\tilde{\sigma}} P(x,y)\,dx + Q(x,y)\,dy\\ &= \int_0^{2\pi} \frac{-r\sin(t)}{r^2}\left(-r\sin(t)\right)\,dt + \frac{r\cos(t)}{r^2}\left(r\cos(t)\right)\,dt\\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

Remark. It's worth noting that, even though $\mathbf{F}$ above is conservative on any simply-connected region not containing the origin, that's not enough for it to be conservative on all of $\mathbb{X}^2$ minus the origin. As an exercise, show that $f(x,y) = \arctan(x/y)$ is a potential function for F which defined on the whole plane except along the x-axis). Moreover, for every nonzero x-value, $\lim_{y\to 0^+} f(x,y) \neq \lim_{y\to 0^-} f(x,y)$, so this potential function cannot be extended to a differentiable function on all of $\mathbb{X}^2$ minus the origin.

16.5 Curl and Divergence

Definition 16.5.1: Del Operator

The **del operator**, ∇, is the differential operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{e}_1 + \frac{\partial}{\partial y}\mathbf{e}_2 + \frac{\partial}{\partial z}\mathbf{e}_3$$

At this point we've only thought of this operator as a function from 3-variable scalar functions to vectors in $\mathbb{R}^3$ (the gradient), but given its vector form, we can also think about its behavior with vector fields on $\mathbb{X}^3$.

Definition 16.5.2

Let $\mathbf{F} = P\mathbf{e}_1 + Q\mathbf{e}_2 + R\mathbf{e}_3$ be a vector field on (a region in) $\mathbb{X}^3$. If the partial derivatives of P, Q, R exist, then we define the **curl of F** to be

$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{e}_1 + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{e}_2 + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{e}_3$$

and the **divergence of F** to be

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Remark. Since the cross product is special to 3 dimensions (and 7, weirdly), curl is only defined for vectors on $\mathbb{X}^3$. The definitions of ∇ and div extend naturally to all dimensions.

Example 16.5.3: I

$\mathbf{F}(x, y, z) = [\sin(y), \cos(x)]$, find $\operatorname{curl}(\mathbf{F})$.

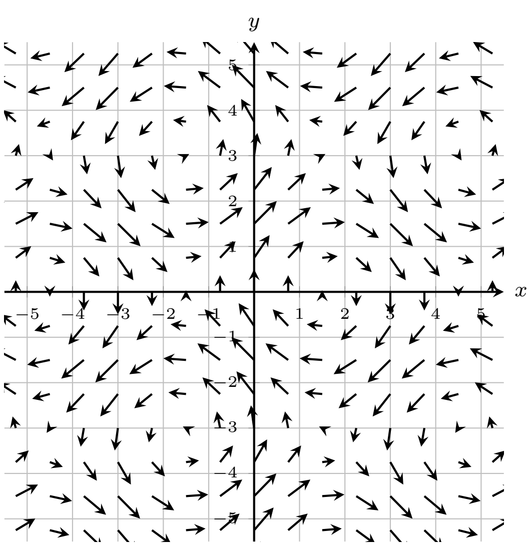


Figure 16.5.1: Vector Field

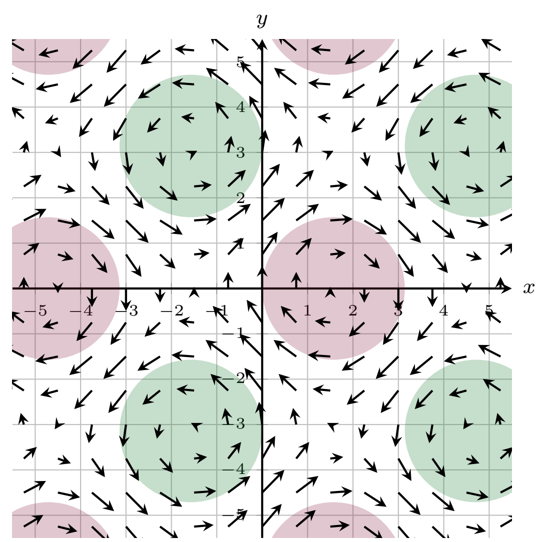


Figure 16.5.2: Vector Field w/ 3rd Component of curl(F) positive/negative/zero

The curl is a new vector that records how much "swirly" is in your vector field at a point, as well as the direction of "swirly"

Definition

If $\text{curl}(\mathbf{F}) = \mathbf{0}$, then $\mathbf{F}$ is called **irrotational**.

Example 16.5.4

f $\mathbf{F}(x, y, z) = [\cos(x + y), \sin(x - y)]$, find $\text{div}(\mathbf{F})$.

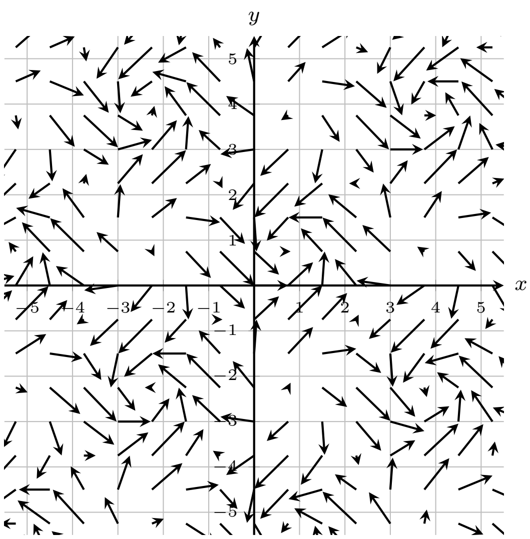


Figure 16.5.3: Vector Field

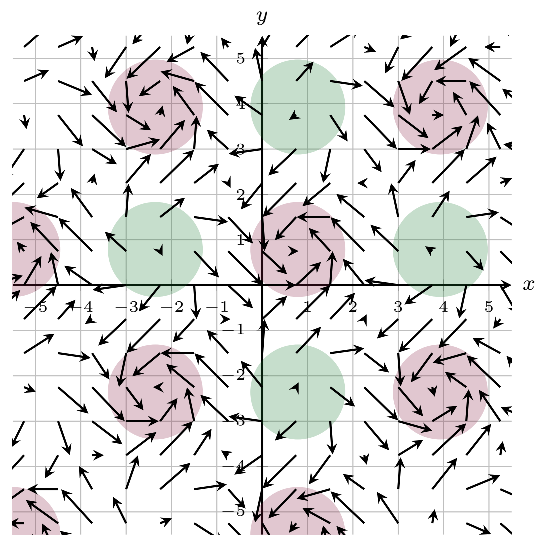


Figure 16.5.4: Vector Field w/ $\text{div}(F)$ positive/negative/zero

The divergence measures how much a point is a "source" (vectors flow away from it; $\text{div}(\mathbf{F}) > 0$) or a "sink" (vectors flow toward it; $\text{div}(\mathbf{F}) < 0$).

Definition

If $\operatorname{div}(\mathbf{F}) = 0$, then $\mathbf{F}$ is called **incompressible**.

Theorem 16.5.5: Checking for Conservative Vector Fields on $\mathbb{X}^3$

Let $\mathbf{F}$ be a vector field on a region D in $\mathbb{X}^3$ with continuous first partial derivatives.

1. If $\mathbf{F}$ is conservative on D, then $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$.
2. If D is simply connected and $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$, then $\mathbf{F}$ is conservative.

Proof. Write $\mathbf{F} = P\mathbf{e}_1 + Q\mathbf{e}_2 + R\mathbf{e}_3$.

1. Suppose $\mathbf{F}$ is conserative. Then there is a function f so that $\mathbf{F} = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$. Computing $\operatorname{curl}(\nabla f)$, we have

$$\begin{aligned}
\operatorname{curl}(\nabla f) &= \nabla \times \nabla f \\
&= \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{e}_1 - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\mathbf{e}_2 + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{e}_3.
\end{aligned}$$

By Clairaut's Theorem, the mixed partial derivatives $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, hence all of the coefficients in $\operatorname{curl}(\nabla f)$ are 0.

2. This is a consequence of Stokes' Theorem (which we'll encounter soon).

$\square$

Exercise 16.5.6

Let $\mathbf{F}$ be the vector field from Example 16.4.6:

$$\mathbf{F}(x, y) = \frac{1}{x^2 + y^2}\begin{bmatrix} -y \\ x \end{bmatrix}$$

Let $D = \mathbb{X}^3 - \{z\text{-axis}\}$. Define a new vector field $\mathbf{G}$ on D via

$$\mathbf{G}(x, y, z) = \frac{1}{x^2 + y^2}\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$$

1. Let σ be some curve in D from the point $A(-1, 0, 0)$ to the point $B(1, 0, 0)$. Show that $\mathbf{G}$ is not conservative on D by showing that $\int_\sigma \mathbf{G}\, ds$ is not independent of path.
2. Compute $\operatorname{curl}(\mathbf{G})$.

This shows that the simply-connected assumption in the previous theorem is actually necessary.

What's the interplay with div and curl?

Theorem 16.5.1. *Let $\mathbf{F}$ be a vector field on a region D in $\mathbb{X}^3$ with continuous first partial* *derivatives.*

$$\mathrm{div}(\mathrm{curl}(\mathbf{F})) = 0.$$

Proof. This is a straightforward computation. $\square$

This result actually tells us that we can quickly check if a given vector field can ever arise as the curl of another vector field.

Example 16.5.7

Let $\mathbf{F}(x, y, z) = [xz\,, \; xyz\,, \; -y^2]$. Find some vector field $\mathbf{G}$ such that $\mathbf{F} = \mathrm{curl}(\mathbf{G})$, or show that one cannot exist.

If such a $\mathbf{G}$ did exist, then we have that

$$\begin{aligned} \mathbf{F} &= \mathrm{curl}(\mathbf{G}) \\ \mathrm{div}(\mathbf{F}) &= \mathrm{div}(\mathrm{curl}(\mathbf{G})) = 0 \end{aligned}$$

but

$$\mathrm{div}(\mathbf{F}) = z + xz + 0 \neq 0$$

hence no such $\mathbf{G}$ can exist.

16.5.1 Vector Form of Green's Theorem

Let D be a simply-connected region with boundary ∂D parameterized by the vector equation $\mathbf{r}(t) = [x(t), y(t)]$ for $a \leq t \leq b$. Notice that the unit tangent vector $\mathbf{T}(t)$ and unit normal vector $\mathbf{n}(t)$ to the curve at time t are given by

$$\begin{aligned} \mathbf{T}(t) &= \frac{1}{|\mathbf{r}'(t)|}(x'(t) + y'(t)) \\ \mathbf{n}(t) &= \frac{1}{|\mathbf{r}'(t)|}(y'(t) - x'(t)) \end{aligned}$$

Green's Theorem originally came from exploring the tangent vectors, but the tangent space to an n-dimensional object is n-dimensional, so with the aim of generalizing to higher dimensions, that contour integral seems like it might get complicated. However, the tangent space is always uniquely determined by the outward pointing normal vector, which is just 1-dimensional. So what if we explored the normal vectors instead? Let $\mathbf{F}(x, y) = [P(x, y)\,, \; Q(x, y)]$ be a vector field with continuous partial derivatives on D.

$$\begin{aligned} \oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds &= \int_a^b (\mathbf{F} \cdot \mathbf{n})(t) |\mathbf{r}'(t)|\, dt \\ &= \int_a^b \frac{1}{|\mathbf{r}'(t)} \left(P(x(t), y(t))\; y'(t)\, dt - Q(x(t), y(t))\; x'(t)\, dt \right) |\mathbf{r}'(t)|\, dt \\ &= \oint_{\partial D} -Q\, dx + P\, dy \end{aligned}$$

$$= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$
$$= \iint_D \operatorname{div}(\mathbf{F}) \, dA$$

Theorem 16.5.8: Green's Theorem - Vectorized

Let D be a simply-connected region in $\mathbb{X}^2$ and $\mathbf{F}$ a vector field with continuous first partial derivatives on D. Then

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div}(\mathbf{F}) \, dA.$$

Remark. It might not be obvious in its present form, but since $\mathbf{T}$ and $\mathbf{n}$ are an orthonormal basis for $\mathbb{R}^2$, then $(\mathbf{F} \cdot \mathbf{T}) + (\mathbf{F} \cdot \mathbf{N}) = \mathbf{F} \cdot [1, 1]$, so with this relationship, the lef

As a bit of foreshadowing, div is not special to 2- or 3-dimensional space and can work for all dimensions. This particular form of Green's Theorem gives us a clue as to how we might go about extending it to work in all dimensions...

16.5.2 The Laplacian

If $\mathbf{F} = \nabla f$ for some scalar function f, then

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Definition 1. Let f be a scalar function whose second partial derivatives exist), then the **Laplacian of f** is

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and we call ∇^2 the **Laplace operator** (which is also sometimes denoted as Δ instead).

16.6 Parametric Surfaces and Their Axes

Recall that the vector equation

$$\mathbf{r}(t) = \begin{bmatrix} x(t) & , & y(t) & , & z(t) \end{bmatrix}$$

parameterizes a curve (a 1-dimensional object) in $\mathbb{X}^3$. Similarly,

$$\mathbf{r}(u, v) = \begin{bmatrix} x(u, v) & , & y(u, v) & , & z(u, v) \end{bmatrix}$$

parameterizes a surface (a 2-dimensional object) in $\mathbb{X}^3$.

Definition: Parameteric Surface, Parametric Equations

Example 16.6.1: Sphere

Consider the parametric surface given by

$$\mathbf{r}(u, v) = \begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) \end{bmatrix}$$

where (u, v) are points in the domain $D = \{0 \leq u \leq 2\pi,\ 0 \leq v \leq \pi\}$.

This is the unit sphere.

```
r[u_, v_] := {Cos[u] Sin[v], Sin[u] Sin[v], Cos[v]};
ParametricPlot3D[r[u, v], {u, 0, 2 Pi}, {v, 0, Pi}]
```

PICTURE

Example 16.6.2: Cylinder

Consider the parametric surface given by

$$\mathbf{r}(u, v) = \begin{bmatrix} 2\cos(u) \\ v \\ 2\sin(v) \end{bmatrix}$$

where (u, v) are points in the domain $D = \{0 \leq u \leq 2\pi,\ -\infty < v < \infty\}$.

This is a cylinder extending infinitely in the y-direction.

```
r[u_, v_] := {2 Cos[u], v, 2 Sin[u]};
ParametricPlot3D[r[u, v], {u, 0, 2 Pi}, {v, -2, 2}]
```

PICTURE

Example 16.6.3: Cavatappi Noodle

Consider the parametric surface given by

$$\mathbf{r}(u,v) = \begin{bmatrix} (2+\sin(v))\cos(u) \\ (2+\sin(v))\sin(u) \\ u+\cos(v) \end{bmatrix}$$

where (u,v) are points in the domain $D = \{-\infty < u < \infty,\, 0 < v < 2\pi\}$.

This is an infinite cavatappi noodle.

```
r[u_, v_] := {(2 + Sin[v]) Cos[u], (2 + Sin[v]) Sin[u], u + Cos[v]};
ParametricPlot3D[r[u, v], {u, -5, 5}, {v, 0, 2 Pi}]
```

PICTURE

With algebraic manipulation, can recover some familiar equations for these surfaces

Example 16.6.4

Find a familiar equation/description for the surface in Example 16.6.1.

Example 16.6.5

Parameterize the surface $z = f(x,y)$

Example 16.6.6

Parameterize the infinite cylinder $x^2 + y^2 = 36$.

Example 16.6.7: Surfaces of Revolution

Suppose f is a continuous function of 1-variable and $f(x) > 0$ for $a < x < b$. Parameterize the surface obtained by rotating $y = f(x)$ about the x-axis.

16.6.1 Tangent Planes

Definition: Tangent Plane

Let S be a surface parameterized by the vector $\mathbf{r}(u, v)$ and let $\mathbf{r}_u$ and $\mathbf{r}_v$ be as above. The **tangent plane to S at the point** $p = \mathbf{r}(u,v_0)$, denoted T_pS, is

$$T_pS = \text{span}\left\{\mathbf{r}_u\Big|_{(u_0,v_0)}, \mathbf{r}_v\Big|_{(u_0,v_0)}\right\}$$

Tangent plane degenerates if $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{0}$,

Definition: Smooth Surface

A surface S is called smooth if the tangent plane never degenerates, i.e., that $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ for all points p in S.

Example 16.6.8

Find equation of tangent plane to the surface in Example 16.6.2 at $p = r(0, 0)$.

$$x + z = -2\sqrt{2}$$

16.6.2 Surface Area

Idea is the same as arc length: Approximate surface near each point p with a piece of the tangent plane. The sum of these areas is a Riemann sum. Take limits to refine approximations and get

Definition: Surface Area

Suppose S is a smooth parametric surface given by the equation

$$\mathbf{r}(u,v) = [x(u,v), y(u,v), z(u,v)] \text{ where } (u,v) \in D$$

where S is covered just once as (u,v) ranges throughout D. Furthermore, let

$$\mathbf{r}_u = \left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right]$$

$$\mathbf{r}_v = \left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right]$$

be the basis vectors for the tangent plane at the point (u,v). Then the surface are of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA.$$

Example 16.6.9

Find the surface area of the sphere of radius R.

Recall that the sphere of radius R is parameterized by

$$\mathbf{r}(u,v) = [R\cos(u)\sin(v), R\sin(u)\sin(v), R\cos(v)]$$

and the parameter domain D is $D = \left\{ \begin{array}{c} 0 \le u \le 2\pi \\ 0 \le v \le \pi \end{array} \right\}$.

$$A(S) = 4\pi R^2.$$

16.7 Surface Integrals

In this section, when referring to a parameterized smooth surface $S = \mathbf{r}(u, v)$ over some domain D, we will say that S is **parameterized once** to mean that $\mathbf{r}$ is a one-to-one function on D (except possibly at the boundary); i.e. that for all $(u_1, v_1) \neq (u_2, v_2)$ in D (except possibly at the boundary), $\mathbf{r}(u_1, v_1) \neq \mathbf{r}(u_2, v_2)$

From the last section, given a surface S parameterized by $\mathbf{r}(u, v)$ over a domain D, we got that the surface area was approximated as follows:

$$\Delta S \approx |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v$$

which, in the language of differentials, became

$$dS = |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

This leads us to the following

Definition: Surface Integrals

Suppose $S = \mathbf{r}(u, v)$ is a surface in $\mathbb{X}^3$ parameterized once over a domain D and let $f : \mathbb{X}^3 \to \mathbb{R}$ be a scalar function whose domain contains S. Then the **integral of f over the surface S** is given by

$$\iint_S f(x, y, z) \, dS = \iint_D f(\mathbf{r}(u, v)) \, |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

Remark. When $f(x, y, z) = 1$, we have that $\iint_S f(x, y, z) \, dS = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \text{Area}(S)$.

Example 16.7.1: S is the Graph of a Function

Evaluate $\displaystyle\iint_S y \, dS$ where S is the surface $z = x + y^2$, $0 \leq x \leq 1$, $0 \leq y \leq 2$.

$$= \frac{13\sqrt{2}}{3}$$

16.7.1 Oriented Surfaces

One key ingredient in Green's Theorem was that the boundary curve was traced counter-clockwise. This is a notion of orientation, and the description of it is straightforward in the 1-dimensional case, but extending it to the 2-dimensional case is a bit more tricky. In the case of a curve, you have two choices: clockwise and counter-clockwise. But the same is not true in higher dimensions. However, in the case of a curve, you have the choice of normal vectors that point inward or outward, and this same binary choice exists for surfaces. So

Definition: Orientation

An **orientation** on a surface is a choice of unit normal vector at every point p on the surface so that the family of normal vectors varies smoothly along the surface. If S is a closed surface (i.e. it bounds a solid region), then it is **positively oriented** if the unit vectors all point outward.

Remark. When $S = \mathbf{r}(u, v)$ is an oriented surface, one often takes $\mathbf{N} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$.

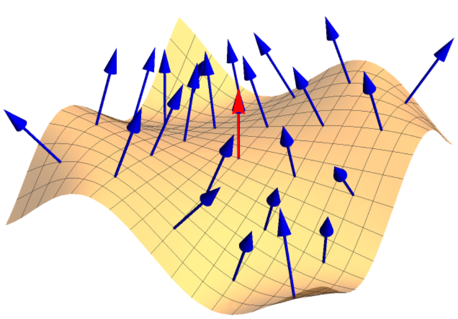


Figure 16.7.1: Orientable Surface

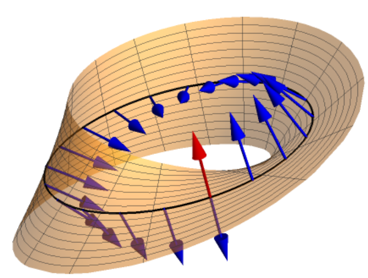


Figure 16.7.2: Nonorientable Surface

16.7.2 Surface Integrals of Vector Fields

Definition

If $\mathbf{F} : \mathbb{X}^3 \to \mathbb{R}^3$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{N}$, then the **surface integral of F over** S (or **flux of F across** S) is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{N}\, dS.$$

When $S = \mathbf{r}(u, v)$ is parameterized exactly once on a region D, then we get

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{N}\, dS \\ &= \iint_D \mathbf{F} \cdot \left(\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right) |\mathbf{r}_u \times \mathbf{r}_v|\, dA \\ &= \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA\end{aligned}$$

Example 16.7.2

Let S be the unit sphere of and let $\mathbf{F}(x, y, z) = [z, y, x]$. Evaluate the surface integral of $\mathbf{F}$ over S.

Note that $S = \mathbf{r}(u, v)$ where $\mathbf{r}(u, v) = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)]$ with parameter domain $D = \{0 \le u < 2\pi, 0 \le v \le \frac{\pi}{2}\}$. Since S is a closed surface, we want the normal vectors to point *outward*, so we take $\mathbf{N} = \mathbf{r}_v \times \mathbf{r}_u$ (one can check that these indeed point outward).

$$\begin{aligned}\mathbf{r}_v \times \mathbf{r}_u &= [\cos(u)\sin^2(v), \sin(u)\sin^2(v), \cos(v)\sin(v)] \\ \mathbf{F}(\mathbf{r}(u, v)) &= [\cos(u), \sin(u)\sin^2(v), \cos(u)\sin(v)] \\ \mathbf{F}(\mathbf{r}(u, v)) \cdot \mathbf{r}_v \times \mathbf{r}_u &= 2\cos(u)\cos(v)\sin^2(v) + \sin^2(u)\sin^3(v)\end{aligned}$$

And thus

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi} 2\cos(u)\cos(v)\sin^2(v) + \sin^2(u)\sin^3(v)\, du\, dv \\ &= \int_0^{\pi}\int_0^{2\pi} 2\cos(u)\cos(v)\sin^2(v) + \frac{1}{2}(1 - \cos(2u))\sin^3(v)\, du\, dv \\ &= \int_0^{\pi} \left[2\sin(u)\sin^2(v)\cos(v) + \frac{1}{2}u\sin^3(v) - \frac{1}{4}\sin(2u)\sin^3(v)\right]_0^{2\pi} dv \\ &= \int_0^{\pi} \pi\sin^3(v)\, dv \\ &= \int_0^{\pi} \pi(1 - \cos^2(v))\sin(v)\, dv\end{aligned}$$

Taking the susbtitution $w = \cos(v)$, we get

$$\int_0^{\pi} \pi(1 - \cos^2(v))\sin(v)\, dv = \pi\int_1^{-1} w^2 - 1\, dw = \frac{4\pi}{3}$$

In the case that our surface is defined by $z = f(x, y)$ (so that the parameterization is $\mathbf{r}(u, v) = [u, v, f(u, v)]$, we notice that

$$\begin{aligned}
\mathbf{r}_u &= [1, 0, \frac{\partial f}{\partial u}] \\
\mathbf{r}_v &= [0, 1, \frac{\partial f}{\partial v}] \\
\mathbf{r}_u \times \mathbf{r}_v &= \left[-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right]
\end{aligned}$$

Hence

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = [P, Q, R] \cdot \left[-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right] = -P\frac{\partial f}{\partial u} - Q\frac{\partial f}{\partial v} + R.$$

Example 16.7.3

Let S be the boundary of the solid region enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane ($z = 0$). let $\mathbf{F}(x, y, z) = [y, x, z]$. Evaluate the surface integral of $\mathbf{F}$ over S.

Note that S is comprised of two regions, S_1 and S_2, where $S_1 = \mathbf{r}_1(u, v) = [u\cos(v), u\sin(v), 1 - u^2]$ with parameter domain $D_1 = \{0 \leq u1, 0 \leq v \leq 2\pi\}$ and $S_2 = \mathbf{r}_2(u, v) = [u\cos(v), u\sin(v), 0]$ with parameter domain $D_2 = \{0 \leq u1, 0 \leq v \leq 2\pi\}$.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

16.8 Stokes' Theorem

Definition: Induced Orientation

Suppose S is an oriented surface with boundary curve ∂S and normal vectors $\mathbf{n}$. The **induced orientation from** S **on** ∂S is described as follows: travel along σ so that the surface with normal vectors pointing upward is always to the left.

Theorem 16.8.1: Stokes' Theorem

Suppose S is an oriented piecewise smooth surface with boundary curve ∂S (with the induced orientation). Let $\mathbf{F} : \mathbb{X}^3 \to \mathbb{R}^3$ be a continuous vector field whose partial derivatives exist in a neighborhood of S. Then

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

Figure 16.8.1: SMBC Comics, 24 February 2014

Proof. In the special case that S is the surface defined by $z = f(x, y)$... $\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} =$

$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} =$

$\square$

Example 16.8.2

Evaluate $\oint_\sigma \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = [-y^2, x, z^2]$ and σ is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Note: orient σ to traverse around the z-axis counter-clockwise when viewed from above).

We could parameterize σ directly, or apply Stokes' theorem instead.

$$\operatorname{curl} \mathbf{F} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{bmatrix} = [0, 0, 1 + 2y]$$

Example 16.8.3

Let S be the part of the sphere of radius 4 lying inside the cylinder $x^2 + y^2 = 1$ above the xy-plane. Let $\mathbf{F}(x, y, z) = [xz, yz, xy]$. Compute $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

Approach by Stokes' Theorem and computing $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$.

Remark. Notice that Stokes' Theorem tells us that any surfaces (with the same niceness assumptions as in the theorem) with the same boundary curve will always have the same integral.

$$\iint_{S_1} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial S_2} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

This allows us to pick the simplest surface.

Example 16.8.4: (Attempt 2 at Example 16.8.3)

Let S be the part of the sphere of radius 4 lying inside the cylinder $x^2 + y^2 = 1$ above the xy-plane. Let $\mathbf{F}(x, y, z) = [xz, yz, xy]$. Compute $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

Approach by computing $\iint_{S_1} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$ where S_1 is the unit disk in the $z = \sqrt{3}$ plane.

16.9 The Divergence Theorem

Recall the vectorized version of Green's Theorem 16.5.8

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div}(\mathbf{F})\, dA.$$

which held when D was a simply-connected region in $\mathbb{X}^2$ and $\mathbf{F}$ had continuous first partial derivatives on D. If we wanted to extend this to 3-dimensions, the natural choice would be to replace the left-hand integral with a surface integral over the boundary of a solid region E, and replace the right-hand side with a volume integral on the whole region E. As it turns out, this is precisely the right notion.

Theorem 16.9.1: Divergence Theorem

Suppose E is a solid region in $\mathbb{X}^3$ and its boundary ∂E is a (piecewise) smooth closed surface oriented with outward-pointing normal vectors. Let $\mathbf{F} :\subset \mathbb{X}^3 \to \mathbb{R}^3$ be a vector field whose partial derivatives are continuous on some open region containing E. Then

$$\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\mathbf{F})\, dV$$

Remark. The Divergence Theorem is actually true in all dimensions, but it takes some real work to describe orientation in higher dimensions and one has to come up with an appropriate notion of $d\mathbf{S}$ that avoids the cross product.

One can prove this for certain "simple" regions in $\mathbb{X}^3$ with no complicated machinery, but the proof is quite tedious. The generic proof is actually quite simple, but relies on some complicated new machinery from the realm of differential topology/Riemannian manifolds. For these reasons, we omit the proof entirely in these notes.

Let's verify that the result is true in a very simple case:

Example 16.9.2

Verify the divergence theorem in the case that E is the unit ball, ∂E is the unit sphere, and $\mathbf{F}(x, y, z) = [z, y, x]$.

Recall the following about the unit sphere:

$$\begin{aligned} \mathbf{r}(u, v) &= [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)] \\ \mathbf{r}_v \times \mathbf{r}_u &= [\cos(u)\sin^2(v), \sin(u)\sin(v)^2, \cos(v)\sin(v)] \qquad \text{(outward-pointing normal)} \end{aligned}$$

with parameter domain $D = \{0 \le u \le 2\pi, 0 \le v \le \pi\}$.
We then have that

$$\begin{aligned} \iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_v \times \mathbf{r}_u)\, dA \\ &= \int_0^{\pi} \int_0^{2\pi} \begin{bmatrix} \cos(v) \\ \sin(u)\sin(v) \\ \cos(u)\sin(v) \end{bmatrix} \cdot \begin{bmatrix} \cos(u)\sin^2(v) \\ \sin(u)\sin^2(v) \\ \cos(v)\sin(v) \end{bmatrix} du\, dv \end{aligned}$$

$$= \int_0^{\pi} \int_0^{2\pi} 2\cos(u)\cos(v)\sin^2(v) + \sin^2(u)\sin^3(v)\, du\, dv$$
$$= \int_0^{\pi} \pi \sin^3(v)\, dv = \frac{4\pi}{3}$$

Alternatively, with the volume integral approach:

$$\iiint_E \operatorname{div}(\mathbf{F})\, dV = \iiint_E 0 + 1 + 0\, dV = \iint_E dV = \frac{4}{3}\pi(1)^2 = \frac{4}{3}\pi.$$

Example 16.9.3

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = xy\mathbf{e}_1 + (y^2 + e^{xz^2})\mathbf{e}_2 + \sin(xy)\mathbf{e}_3$ and S is the surface of the region bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.

The surface of this region is comprised of 4 smooth pieces, so we would need to parameterize four separate pieces and compute four separate integrals - no fun! We also note that the divergence is super simple:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(y^2 + e^{xz^2}) + \frac{\partial}{\partial z}(\sin(xy)) = 3y.$$

So, letting E be the solid region bounded by S and applying the Divergence Theorem, we have

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_E \operatorname{div}(\mathbf{F})\, dV \\
&= \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y\, dy\, dz\, dx \\
&= \int_{-1}^{1} \int_0^{1-x^2} \frac{3}{2}(2-z)^2\, dy\, dz\, dx \\
&= \frac{3}{2} \int_{-1}^{1} \int_0^{1-x^2} 4 - 4z + z^2\, dz\, dx \\
&= \frac{3}{2} \int_{-1}^{1} 4(1-x^2) - 2(1-x^2)^2 + \frac{1}{3}(1-x^2)^3\, dx \\
&= \frac{3}{2} \int_{-1}^{1} 41 - 4x^2 - 2 + 4x^2 - x^4) + \frac{1}{3} - x^2 + x^4 - \frac{1}{3}x^6\, dx \\
&= \frac{184}{35}.
\end{aligned}$$

Part II

Complex Analysis (from Saff–Snider)

1 Complex Numbers

1.1 The Algebra of Complex Numbers

Definition

A **complex number** is a symbol $x + iy$ or $x + yi$, where x, y are real numbers and i satisfies $i^2 = -1$. The collection of complex numbers is denoted $\mathbb{C}$. Writing $z = x + yi$, we say that x is the **real part** of z, denoted $\mathrm{Re}(z)$, and we say that y is the **imaginary part** of z, denoted $\mathrm{Im}(z)$.

Remark. Some authors use $\mathfrak{R}(z)$ and $\mathfrak{I}(z)$ to denote the real and imaginary parts of z, respectively.

The complex numbers satisfy the following rules of arithmetic:

- *Equality*: $a + bi = c + di$ if and only if $a = c$ and $b = d$
- *Addition*: $(a + bi) + (c + di) = (a + c) + (b + d)i$
- *Multiplication*: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Remark. If you think of complex numbers as polynomials with indeterminate i, then the arithmetic operations are the same as those for polynomials, with the added simplification of $i^2 = -1$.

Proposition 1.1.1: Properties of Complex Arithmetic

Complex arithmetic has the following familiar properties from arithmetic of the real numbers. For all $u, v, w \in \mathbb{C}$, we have

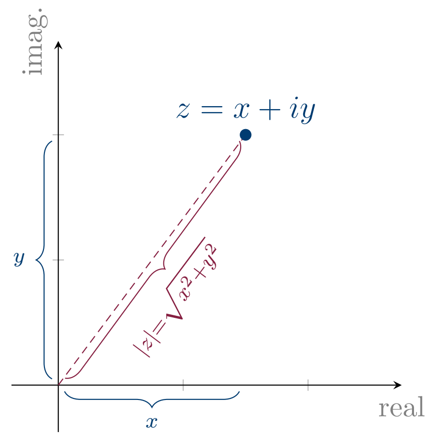
- Associative addition: $u + (v + w) = (u + v) + w$
- Commutative addition: $u + v = v + u$
- Associative multiplication: $u(vw) = (uv)w$
- Commutative multiplication: $uv = vu$
- Distributive law: $u(v + w) = uv + uw$
- $w + 0 = w$
- $1w = w$

Exercise 1.1.2

Prove Proposition 1.1.1.

1.2 Point Representation of Complex numbers

We can identify the complex number $z = x + yi$ with the point (x, y) in $\mathbb{X}^2$ (the plane). Because of this identification, the horizontal axis is known as the **real axis** and the vertical axis is known as the **imaginary axis**.



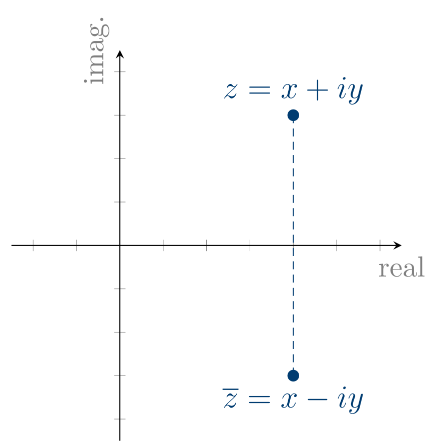
The following definition is thus natural:

Definition

For a complex number $z = x + yi$, the **magnitude** (or **modulus** or **absolute value**) of z is $|z| = |x + yi| = \sqrt{x^2 + y^2}$.

Definition

The **(complex) conjugate** of $z = x + iy$ is $\overline{z} = x - iy$.



Proposition 1.2.1: Properties of the Conjugate and Magnitude

For complex numbers z, w, we have the following properties of the conjugate and magnitude:

1. $\mathrm{Re}(\overline{z}) = \mathrm{Re}(z)$
2. $\mathrm{Im}(\overline{z}) = -\mathrm{Im}(z)$
3. $\overline{\overline{z}} = z$
4. $\overline{z+w} = \overline{z} + \overline{w}$
5. $\overline{(zw)} = (\overline{z})(\overline{w})$
6. If $w \neq 0$, then $\overline{z}/\overline{w} = \overline{(z/w)}$
7. $|\overline{z}| = |z|$
8. $|zw| = |z||w|$
9. $\mathrm{Re}(z) = \dfrac{z+\overline{z}}{2}$
10. $\mathrm{Im}(z) = \dfrac{z-\overline{z}}{2}$
11. $|z| \geq 0$
12. $|z| = 0$ if and only if $z = 0$
13. $z\overline{z} = |z|^2$

Proof. We prove only #4 above and acknowledge that all of the others are similarly boring. Let $z = a_1 + ib_1$ and $w = a_2 + ib_2$. Then

$$\overline{z+w} = \overline{(a_1+a_2) + i(b_1+b_2)} = (a_1+a_2) - i(b_1+b_2) = (a_1 - ib_1) + (a_2 - ib_2) = \overline{z} + \overline{w}.$$

□

Exercise 1.2.2

Prove the remaining parts of Proposition 1.2.1.

Complex conjugation is usually used when computing quotients. In particular

$$\frac{z}{w} = \frac{z}{w}\frac{\overline{w}}{\overline{w}} = \frac{1}{|w|^2}(z\overline{w})$$

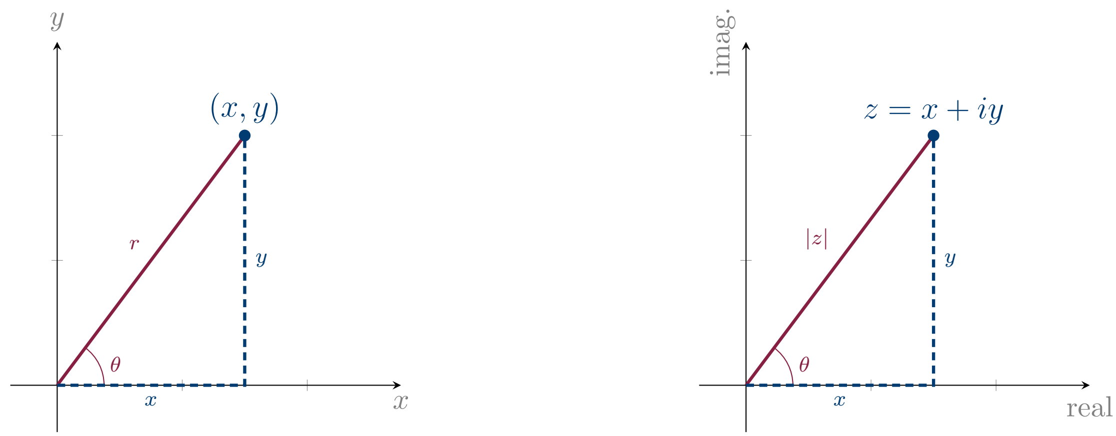
Example 1.2.3

Compute the quotient $\dfrac{7-4i}{3+9i}$.

$$\frac{7-4i}{3+9i} = \frac{7-4i}{3+9i}\left(\frac{\overline{3+9i}}{\overline{3+9i}}\right) = \frac{(7-4i)(3-9i)}{9+81} = -\frac{15}{90} - \frac{75}{90}i = -\frac{1}{6} - \frac{5}{6}i$$

1.3 Vectors and Polar Forms

Recall that any point $(x, y) \in \mathbb{X}^2$ can be written in terms of polar coordinates as $(r\cos\theta, r\sin\theta)$ where r, θ are determined as in the diagram below.



Given a complex number $z = x + iy$, it follows from the above that can then rewrite it as $z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$ where $r = |z|$.

Definition

A nonzero complex number z is in **polar form** if it is written as $z = r(\cos\theta + i\sin\theta)$. The value of θ is called an **argument** for z and is denoted $\arg z$.

Remark. Because $\sin(\theta) = \sin(\theta + 2k\pi)$ and $\cos(\theta) = \cos(\theta + 2k\pi)$ for any integer k, given a complex number, there is no unique argument. Some other texts will even let $\arg(z)$ be a set of infinitely-many angles:

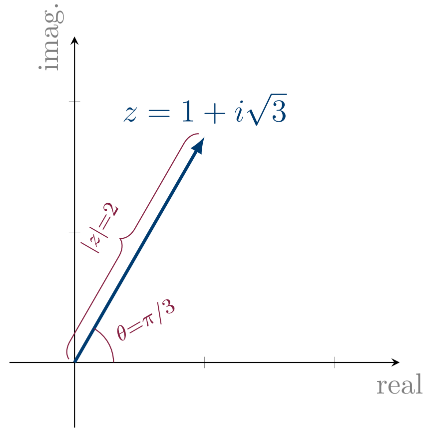
$$\arg(z) = \{\theta + 2k\pi \ : \ \text{for all integers } k\}$$

In practice, one often ends up choosing the range of allowable θ values, say $(-\pi, \pi]$, and working with it. For this reason, we let $\text{Arg}(z)$ denote the θ-value in the range $(-\pi, \pi]$ and we call it the **principal value of the argument**.

Example 1.3.1

Find the polar form for $z = 1 + i\sqrt{3}$.

Since $|z| = \sqrt{1+3} = 2$ and $\arctan(\sqrt{3}) = \frac{\pi}{3}$, then we can write $z = 2e^{i\pi/3}$. Note, of course, that we could also write $z = 2e^{i(\pi/3 + 2k\pi)}$ for any integer k since there are infinitely many possible arguments for z.



Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_1(\cos\theta_2 + i\sin\theta_2$. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\right)) \\ &= r_1 r_2 \left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right) && \text{(angle sum identities)} \end{aligned}$$

so multiplication of complex numbers can be described as encoding both a scaling and a rotation. Specifically

$$\begin{aligned} |z_1 z_2| &= |z_1||z_2| \\ \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \end{aligned}$$

If $z = r(\cos\theta + i\sin\theta)$, then

$$\overline{z} = r\left(\cos\theta - i\sin\theta\right) = r\left(\cos(-\theta) + i\sin(-\theta)\right)$$

so it also follows that

$$\arg(\overline{z}) = -\arg(z)$$

1.4 The Complex Exponential

Recall from integral calculus that, for any real number x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

If we replace x with a purely imaginary complex number iy (and brazenly assume that the same formula holds), we get

$$\begin{aligned}
e^{iy} &= \sum_{n=0}^{\infty} \frac{(i * y)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{i^n y^n}{n!} \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} \\
&= \cos(y) + i\sin(y)
\end{aligned}$$

If we also want the usual multiplicative properties of the complex exponential to hold, namely that $e^{x+iy} = e^x e^{iy}$, then the following definition is the obvious one.

Definition

The **complex exponential** e^z is defined for all $z = x + iy$ as

$$e^z := e^x \left(\cos(y) + i\sin(y)\right)$$

In this way, we have an easier version of our polar form

$$z = |z| \left(\cos\theta + i\sin\theta\right) = |z|e^{i\theta}$$

and it also follows that

$$\overline{z} = |z|e^{-i\theta}$$

From the complex exponential, we can derive some familiar formulae:

Proposition 1.4.1

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Proof. Notice that

$$z + \overline{z} = 2\,\mathrm{Re}(z) = 2|z|\cos\theta$$

and that

$$z + \overline{z} = |z|e^{i\theta} + |z|e^{-i\theta}.$$

We achieve the desired results by solving for $\cos\theta$, and the same argument *mutatis mutandis* works to derive the formula for $\sin\theta$. $\square$

1.5 Powers and Roots

The complex exponential and polar form also give us a convenient way of computing powers of a complex number. For any integer (positive or negative) we have that

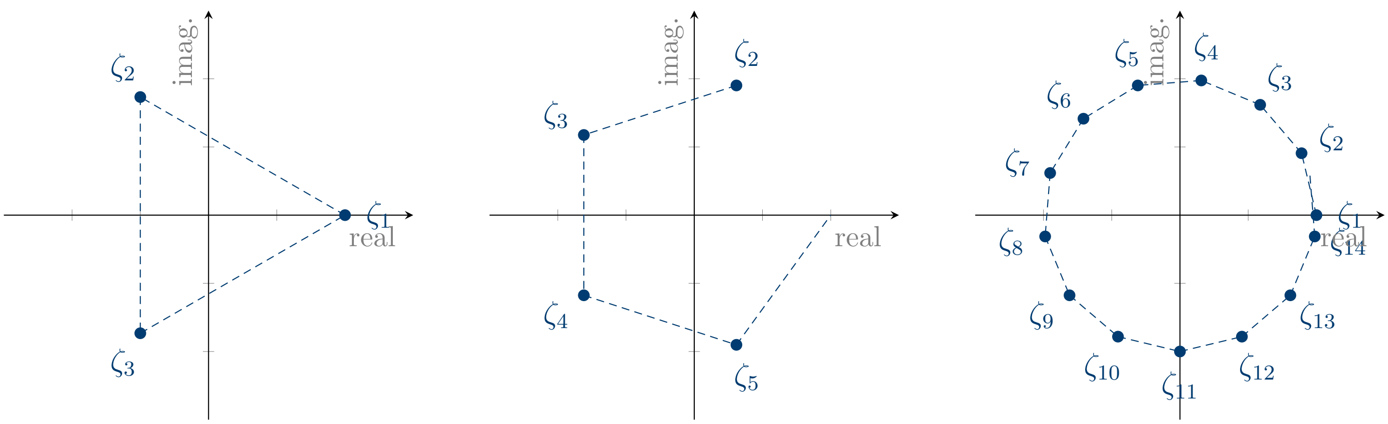
$$z^n = r^n e^{in\theta}.$$

The question is, what about m^{th} roots? We already run into issues with square roots of real numbers: since $(-1)^2 = (1)^2 = 1$, then $\sqrt{1}$ has two possible values (and we just take the convention that it's positive. For complex numbers, it's arguably much worse:

$$(1)^3 = \left(e^{i2\pi/3}\right)^3 = \left(e^{i4\pi/3}\right)^3 = 1$$

so 1 has *three* cube roots, so what possible convention could one even take that is consistent for m^{th} roots? The answer is that we can't, so we take all possible values for the m^{th} root.

Notice that, for a positive number m, there are only m-many m^{th} roots of 1, and they all form the vertices of a regular m-gone in the complex plane.



Definition: Roots of Unity

For any positive integer m, the m^{th} **roots of unity** are

$$1^{1/m} = \zeta_m = e^{i2k\pi/m} \qquad \text{where} \qquad k = 0, 1, \ldots, m-1$$

It follows that

Definition

For any positive integer m and complex number $z = re^{i\theta}$ the m^{th} **roots of** z are

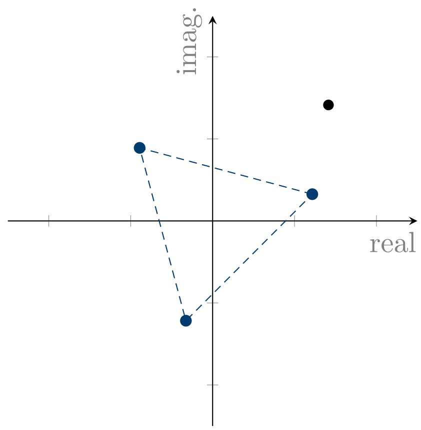
$$z^{1/m} = r^{1/m} e^{i(\theta + 2k\pi)/m} \qquad \text{where} \qquad k = 0, 1, \ldots, m-1$$

Example 1.5.1

Find the cube roots of $\sqrt{2} + i\sqrt{2}$

The polar form of $\sqrt{2} + i\sqrt{2} = 2e^{i\pi/4}$. So

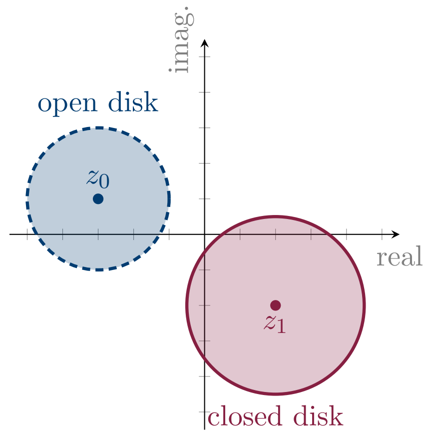
$$\left(\sqrt{2} + i\sqrt{2}\right)^{1/2} = \begin{cases} \sqrt[3]{2}e^{i\pi/12} \\ \sqrt[3]{2}e^{i(\pi/12 + 2\pi/3)} \\ \sqrt[3]{2}e^{i(\pi/12 + 4\pi/3)} \end{cases}$$



1.6 Planar Sets

Definition

Let z_0 be a fixed complex number and let r be a positive real number. The **open disk** of radius r centered at z_0 is the set of all z satisfying $|z - z_0| < r$ and the **closed disk** of radius r centered at z_0 is the set of all z satisfying $|z - z_0| \leq r$.

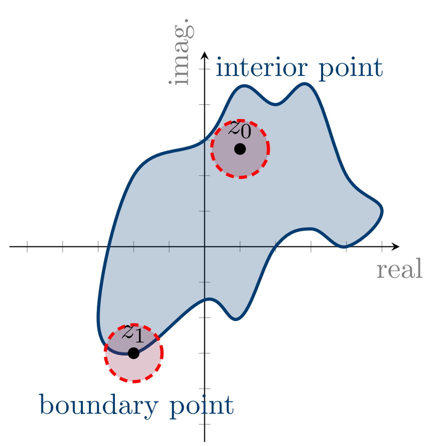


Exercise 1.6.1. Writing $z = x + iy$ and $z_0 = x_0 + iy_0$, convince yourself that these equations look like the familiar equations for open and closed disks in the plane.

Open disks play the same role for complex analysis as open intervals (a, b) do for calculus. Similarly for closed disks and closed intervals $[a, b]$.

Definition

Let S be a set of complex numbers. A point z is an **interior point** if there is a small positive real number for which the open disk of radius r around z is entirely contained within S. A point z is a **boundary point** if every open disk around z contains both a point in S and a point not in S.

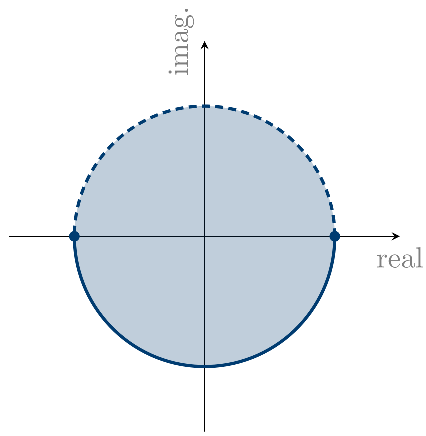


Definition

A set of complex numbers S is **open** if every point in S is an interior point. The set S is **closed** if S contains all boundary points.

Example 1.6.1

Show that the open disk $|z| < 1$ together with the points satisfying $|z| = 1$ and $\text{Im}(z) \leq 0$ is neither open nor closed.



Remark. To head off the question, yes, a set can actually be both open and closed (all of $\mathbb{C}$ is one such set, for example), but understanding when this happens is considerably more subtle and probably best left for office hours or MATH 4324; that is, you don't need to know it for this course.

Remark. The definition of a closed set we gave is actually not entirely accurate, but it's perfectly sufficient for this class. We're really only concerned with closed sets that can contain an open disk within them. There are plenty of other closed sets around, like a finite collection of points for instance, but this is maybe best explored in MATH 4324 or my office hours.

1.7 The Riemann Sphere and Stereographic Projection

1.7.1 The Point at Infinity

If f has a pole at z_0, it has no doubt occured that we might consider taking the value of $f(z_0) = \infty$. Let's look at what happens if we try to define $1/0 = \infty$.

If $z \to 0$ (from the right) along the real axis, we might say that $\frac{1}{z}$ approaches "$+\infty$." If $z \to 0$ (from the left) along the real axis, we might say that $\frac{1}{z}$ approaches "$-\infty$. If $z \to 0$ along the imaginary axis, what would we say $\frac{1}{z}$ approaches? "$\pm i\infty$"?

If we want to try to define this limit, it has to agree from all directions, so writing "$\frac{1}{0} = \infty$" implies that we are identifying all of these limits with the same point, which we name ∞. The other implication, of course, is that if we think of a complex number z as growing without bound, then it is necessarily tending to this singular point ∞.

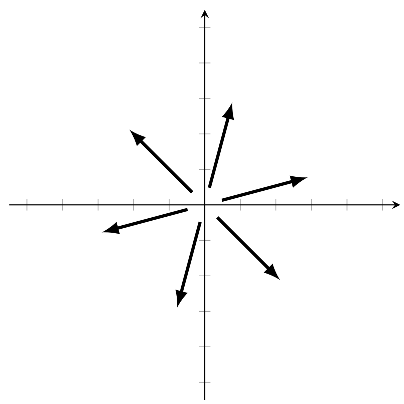


Figure 1.7.1: Growing without bound in any direction, the limit point is always the same: ∞

Definition

The **extended complex plane** $\hat{\mathbb{C}}$ is the set $\mathbb{C} \cup \{\infty\}$. This is also referred to as the **Riemann sphere**.

Example 1.7.1

If $f(z) = \dfrac{2z+1}{z-1}$, then $f(1) = \infty$ and $f(\infty) = 2$.

We might consider certain properties of functions defined on all of $\hat{C}$. Notice that the mapping $z \mapsto \frac{1}{z}$ has the effect of interchanging 0 and ∞, so studying the behavior of a function $f(z)$ at ∞ is equivalent to studying the behavior of the function $g(w) := f\left(\frac{1}{w}\right)$ at 0.

As such, we say that

1. $f(z)$ is differentiable at ∞ if $f\left(\frac{1}{w}\right)$ is differentiable (or has a removable singularity) at $w = 0$.
2. $f(z)$ has a pole of order m at ∞ if $f\left(\frac{1}{w}\right)$ has a pole of order m at $w = 0$.
3. $f(z)$ has an essential singularity at ∞ if $f\left(\frac{1}{w}\right)$ has an essential singularity at $w = 0$.

Example 1.7.2

Let $f(z) = \dfrac{2z+1}{z-1}$. Show that f is differentiable at ∞.

f is certainly analytic for all $z \in \mathbb{C}$ except at 1, so we'll look at a Laurent series expansion of f about 0 on the annulus $1 < |z| < \infty$:

$$\begin{aligned} f(z) = \frac{2z+1}{z-1} &= \frac{2z+1}{z}\left(\frac{1}{z-\frac{1}{z}}\right) \\ &= \frac{2z+1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \\ &= \cdots + \frac{2z+1}{z^4} + \frac{2z+1}{z^3} + \frac{2z+1}{z^2} + \frac{2z+1}{z} \end{aligned}$$

whence

$$\begin{aligned} f\left(\frac{1}{w}\right) &= \cdots + \frac{\frac{2}{w}+1}{\left(\frac{1}{w}\right)^4} + \frac{\frac{2}{w}+1}{\left(\frac{1}{w}\right)^3} + \frac{\frac{2}{w}+1}{\left(\frac{1}{w}\right)^2} + \frac{\frac{2}{w}+1}{\frac{1}{w}} \\ &= (2+w) + (2+w)w + (2+w)w^2 + (2+w)w^3 + \cdots \\ &= 2 + 3w + 3w^2 + 3w^3 + \cdots \end{aligned}$$

and thus $f\left(\frac{1}{w}\right)$ has a removable singularity at $w = 0$.

Example 1.7.3

Let $f(z) = z^3 + 2$. Show that f has a pole of order 3 at ∞.

Notice that

$$f\left(\frac{1}{w}\right) = \frac{1}{w^3} + 2$$

is a Laurent expansion of $f\left(\frac{1}{w}\right)$ about $w = 0$, which is clearly a pole of order 3.

Example 1.7.4

Show that $f(z) = \sin(z)$ has an essential singularity at ∞.

f is analytic for all $z \in \mathbb{C}$, so we'll look at the Laurent (Taylor) series expansion of f about 0.

$$f(z) = \sin(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$$

whence

$$f\left(\frac{1}{w}\right) = \sin\left(\frac{1}{w}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{w}\right)^{2n+1} = \sum_{n=-\infty}^{0}\frac{(-1)^n}{(2n+1)!}w^{2n+1}$$

and thus $f\left(\frac{1}{w}\right)$ has an essential singularity at $w = 0$.

Example 1.7.5

Find all functions that are analytic on all of $\hat{\mathbb{C}}$.

If f has a pole at any z_0, then $\lim_{z\to z_0} f(z) = \infty$. Since f is analytic on all of $\hat{\mathbb{C}}$, then it doesn't have any poles, and f it must be bounded. Since f is bounded on the open set $|z| < 1$, it is constant (by Liouville's theorem), and since f is bounded on the open set $|z| > 1$, it is constant (by Liouville's theorem). By continuity on $\hat{\mathbb{C}}$ (and in particular on $|z| = 1$), these constants must be the same, so f is constant.

Example 1.7.6

Find all functions that have a single pole and are analytic on the rest of $\hat{\mathbb{C}}$.

Suppose f has a pole of order m at some finite point z_0 (so $z_0 \neq \infty$). Then the Laurent expansion of f about z_0 is

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

and it converges for all $z \neq z_0$ (since the only pole is at z_0, f must be bounded on the rest of $\hat{\mathbb{C}}$, including at ∞). For any positive integer n, $c_n(\infty - z_0)^n = \infty$, so it must be that $c_n = 0$ for $n > 0$. As such, f has the form

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{c_{-1}}{(z-z_0)} + c_0.$$

If the pole occurs at $z_0 = \infty$, then $f\left(\frac{1}{w}\right)$ has a pole at 0, so the Laurent series expansion of $f\left(\frac{1}{w}\right)$ about 0 is

$$f\left(\frac{1}{w}\right) = \frac{c_{-m}}{w^m} + \frac{c_{-m+1}}{c^{m-1}} + \cdots + \frac{c_{-1}}{w} + c_0 + c_1 w + c_2 w^2 + \cdots$$

Since $f(z)$ is bounded near 0, then $f\left(\frac{1}{w}\right)$ is bounded near ∞ (i.e. for all sufficiently large $|w|$), and just as last we must have that $c_m = 0$ for $m > 0$. It follows that

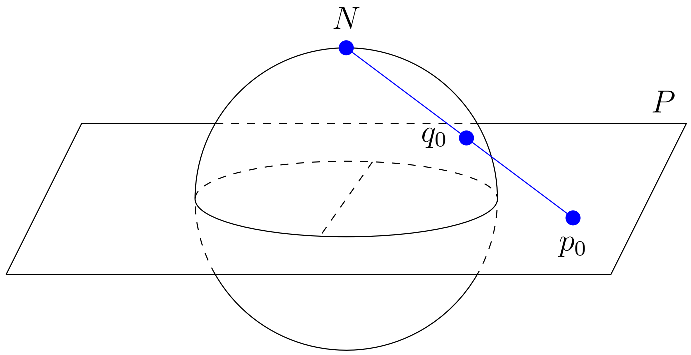
$$f\left(\frac{1}{w}\right) = \frac{c_{-m}}{w^m} + \frac{c_{-m+1}}{c^{m-1}} + \cdots + \frac{c_{-1}}{w} + c_0 \implies \qquad f(z) \qquad = c_0 + c_{-1}z + \cdots + c_{-m}z^m$$

is a polynomial.

1.7.2 Stereographic projection

In $\mathbb{X}^3$, the unit sphere (denoted S^2) is the set $\{(x_1, x_2, x_3) \in \mathbb{X}^3 \; | x_1^2 + x_2^2 + x_3^2 = 1\}$. Letting P be any plane through the origin, P divides S^2 into two hemispheres, call one the northern hemisphere and the other the southern hemisphere.

The northern hemisphere contains a point N, the *north pole*, and so we look at the line passing through N and any other point q_0 on the sphere. This line intersects P in a unique point p_0, and this gives us a way of uniquely identifying points on the sphere (minus the north pole) and points in the plane!



> **Definition**
>
> The function that makes this identification is called the **stereographic projection** from N.

Remark. This same procedure gives a stereographic projection from the n-sphere, S^n (as a subset of $\mathbb{X}^{n+1}$) down to an n-dimensional subspace of $\mathbb{X}^{n+1}$. We will only care about it in dimension 2, however.

If we take P to be the x_1x_2-plane (so the set of points $(x_1, x_2, 0)$), then $N = (0, 0, 1)$. And if we identify P with $\mathbb{C}$ (so $(x_1, x_2, 0)$ is identified with $z = x_1 + ix_2$), then we can describe stereographic projection in coordinates:

$$\begin{aligned} \rho : S^2\text{-}\{N\} &\longrightarrow \mathbb{C} \\ \rho(x_1, x_2, x_3) &= \frac{x_1 + ix_2}{1 - x_3} \end{aligned}$$

$$\begin{aligned} \rho^{-1} : \mathbb{C} &\longrightarrow S^2\text{-}\{N\} \\ \rho^{-1}(z) &= \left(\frac{2\operatorname{Re}(z)}{1+|z|^2}, \frac{2\operatorname{Im}(z)}{1+|z|^2}, \frac{-1+|z|^2}{1+|z|^2} \right) \end{aligned}$$

We notice a couple of things:

1. ρ can be extended to a map $\tilde{\rho} : S^2 \to \hat{\mathbb{C}}$ by defining $\tilde{\rho}(N) = \infty$.
2. ρ sends the equator of S^2 to the unit circle in $\mathbb{C}$.
3. ρ sends the northern hemisphere of S^2-$\{N\}$ to the exterior of the unit disk: $|z| > 1$.
4. ρ sends the southern hemisphere of S^2 to the interior of the unit disk: $|z| < 1$.
5. The map $z \mapsto 1/z$ corresponds to exchanging the northern and southern hemispheres of S^2.

Suppose

$$Ax_1 + Bx_2 + Cx_3 + D = 0 \tag{1.7.1}$$

is some plane passing through S^2. The distance from the origin to this plane is

$$\sqrt{\frac{D^2}{A^2 + B^2 + C^2}}$$

so to pass through S^2 we must have that $A^2 + B^2 + C^2 > D^2$. The corresponding point in $\mathbb{C}$ thus satisfies

$$\begin{aligned} A\left(\frac{2\operatorname{Re}(z)}{1+|z|^2}\right) + B\left(\frac{2\operatorname{Im}(z)}{1+|z|^2}\right) + C\left(\frac{-1+|z|^2}{1+|z|^2}\right) + D = 0 \\ \implies \qquad 2A\operatorname{Re}(z) + 2B\operatorname{Im}(z) + (C+D)|z|^2 = C - D \end{aligned}$$

If the plane contains $N(0,0,1)$, then from Equation 1.7.1 we deduce that $C + D = 0$. Writing $z = x + iy$, then we have

$$2Ax + 2By = C - D$$

which is the equation of a line. If the plane does not contain $N(0,0,1)$, then $C + D \neq 0$. So

$$\begin{aligned} 2A\operatorname{Re}(z) + 2B\operatorname{Im}(z) + (C+D)|z|^2 = C - D \\ \frac{2A\operatorname{Re}(z)}{C+D} + \frac{2B\operatorname{Im}(z)}{C+D} + |z|^2 = \frac{C-D}{C+D}. \end{aligned}$$

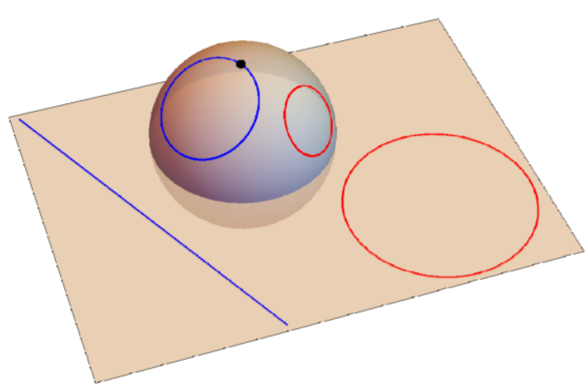
Noting that $|z|^2 = (\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2$ and completing the square, we get that this rearranges to

$$\begin{aligned} \left|z + \frac{A+Bi}{C+D}\right|^2 &= \frac{A^2+B^2+C^2-D^2}{(C+D)^2} \\ \left|z + \frac{A+Bi}{C+D}\right| &= \sqrt{\frac{A^2+B^2+C^2-D^2}{(C+D)^2}} \end{aligned}$$

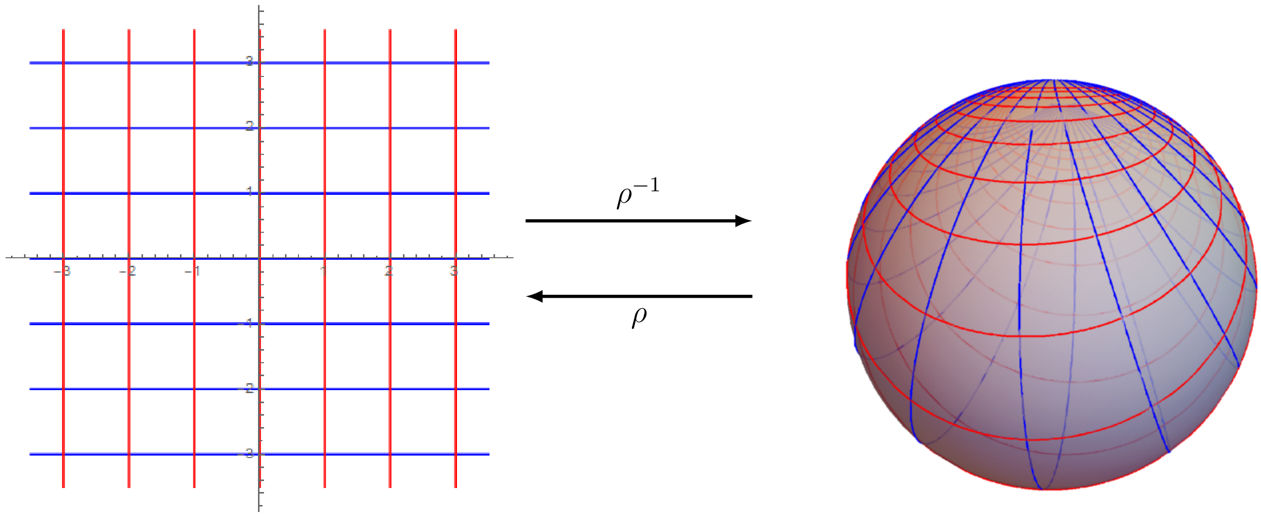
which is the equation of a circle.

Theorem 1.7.7

Stereographic sends circles on S^2 not passing through N to circles in $\mathbb{C}$. It sends circles on S^2 passing through N to lines in $\mathbb{C}$.



One more thing that we may notice is that the (inverse) stereographic projection of the grid on $\mathbb{C}$ extends to a "grid" on the sphere minus N, where the circles all still meet at right angles.



Even though it distorts distances, the stereographic projection map preserves angles. One might wonder about other types of complex functions that preserve angles.

2 Analytic Functions

2.1 Functions of a Complex Variable

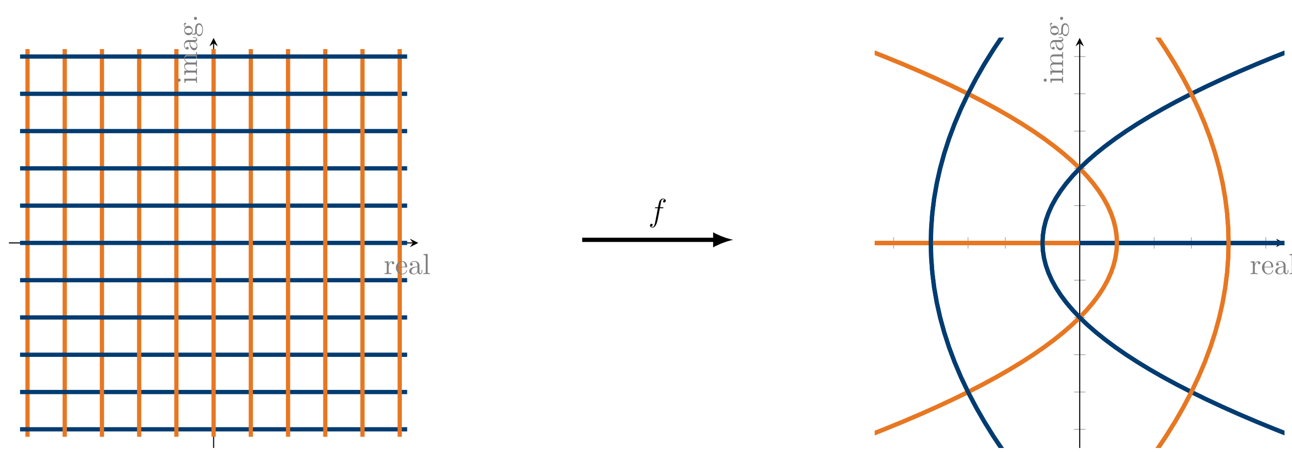
Definition

A **complex function** is a function whose input is a complex number and whose output is a complex number. We say that the function is **defined on a set** S if we restrict our attention to only input values coming from S.

Remark. Complex functions cannot be visualized quite as easily as the real functions you may be familiar with. A function $f : \mathbb{R} \to \mathbb{R}$ has a one-dimensional input and a one-dimensional output, so we can plot a graph of f on a two-dimensional plane. A function $f : \mathbb{C} \to \mathbb{C}$ has a two-dimensional input and a two-dimensional output, so plotting a graph in the same way would require four dimensions.

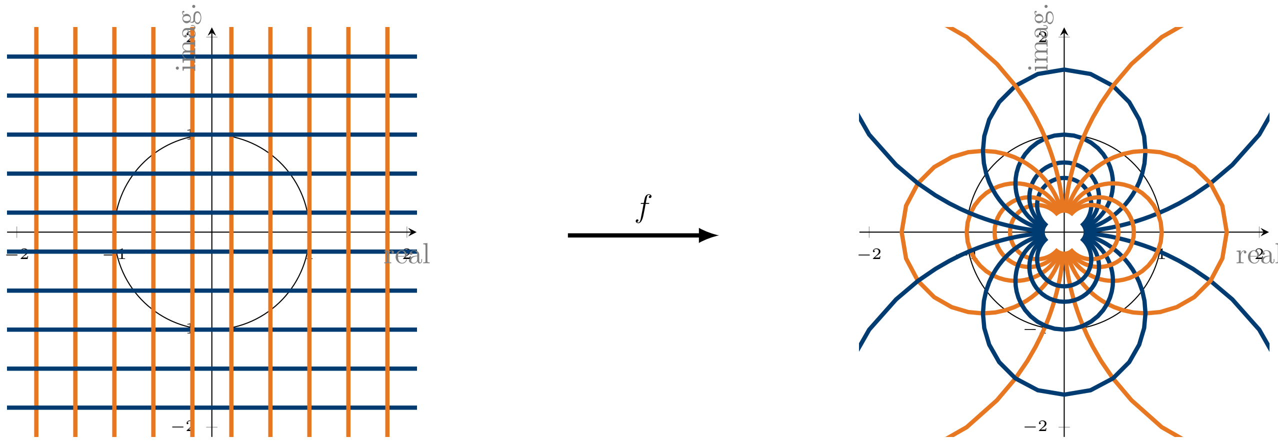
Example 2.1.1

The function $f(z) = z^2$ is a function defined on all of $\mathbb{C}$. We can get a feel for the behavior of this function by examining a grid in $\mathbb{C}$ before and after applying f.



Example 2.1.2

The function $f(z) = \frac{1}{z}$ is a function defined on all of $\mathbb{C} - \{0\}$. We can get a feel for the behavior of this function by examining a grid in $\mathbb{C}$ before and after applying f.



Notice that, if $|z| = 1$ (that is z is on the unit circle), then we have that

$$f(z) = \frac{1}{z} = \frac{1}{z}\left(\frac{\overline{z}}{\overline{z}}\right) = \frac{\overline{z}}{|z|^2} = \overline{z}$$

and $|\overline{z}| = 1$, so $f(z)$ is still on the unit circle. Hence the unit circle is sent back to itself. This map is called the "inversion in the unit circle" (if you remember anything about inversions from Euclidean geometry, it is exactly the same inversion in a circle).

2.2 Limits and Continuity

Definition: Limits

If f is a complex function, then $f(z)$ has a **limit** L as z approaches z_0 if, for every real number $\varepsilon > 0$, there is a real number $\delta > 0$ such that

$$|f(z) - L| < \varepsilon$$

for every z in S satisfying $0 < |z - z_0| < \delta$.

In words, L is a limit if we can arbitrarily approximate it by $f(z)$ when restricting our focus to z-values in a small disk around z_0.

Unlike single-variable calculus where we can check limits from the left and right, in the complex setting (just like in multivariable calculus), the limit has to exist given *any path from z to z_0*.

Proposition 2.2.1: Limit Arithmetic

Limits satisfy all of the familiar properties. If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then

1. $\lim_{z \to z_0} (f(z) \pm g(z)) = A \pm B$
2. $\lim_{z \to z_0} f(z)g(z) = AB$
3. $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{A}{B}$ (provided $B \neq 0$)

Definition: Continuity

Suppose f is a complex function defined on a set S. For z_0 in S, if $\lim_{z \to z_0} f(z) = f(z_0)$, then f is **continuous at** z_0. If f is continuous at every z_0 in S, then we say that f is **continuous on** S.

Proposition 2.2.2: Combinations of Continuous Functions are Continuous

Continuity has all of the familiar properties. If $f(z)$ and $g(z)$ are continuous at z_0, then so are

1. $f(z) + g(z)$
2. $f(z)g(z)$
3. $\dfrac{f(z)}{g(z)}$ (provided $g(z_0) \neq 0$)

As well, if $f(z)$ is continuous at z_0 and $g(z)$ is continuous at $f(z_0)$, then $g(f(z))$ is continuous at z_0.

Proof. The proof of each of these facts is the same as in the real analysis case. □

Proposition

Types of functions that are continuous on their domains:

- Polynomials

- Rational functions
- $f(z) = e^z$
- $f(z) = \overline{z}$
- $f(z) = |z|^2$

(This list is non-exhaustive)

Proof. Each of these proofs is essentially the standarda ε-δ argument from real analysis. The second and fifth bullet points follow directly from the first and fourth, respectively, by applying Proposition 2.2.2. For proving the third and fourth bullet points, it may be worthwhile to point out:

- $|e^z - e^{z_0}| \leq |e^x - e^{x_0}||\cos(y) + i\sin(y)| + e^{x_0}\,|(\cos(y) + i\sin(y)) - (\cos(y_0) + i\sin(y_0))|$ and then you appeal to the fact that $|\cos(y) + i\sin(y)| \leq 1$ and each of these real functions (e, cos, sin are all continuous).
- $|\overline{z} - \overline{z_0}| = |(\overline{z - z_0})| = |z - z_0|$

$\square$

Definition

A set S is **bounded** if there is some positive real number M for which every z in S satisfies $|z| \leq M$. A function f is **bounded** if there is some positive real number K such that $|f(z)| \leq K$ for every z in S.

2.3 Analyticity

Definition: Complex Differentiable

The function f is **(complex) differentiable** at z_0 in S if the following limit exists

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad \left(\text{or equivalently} \qquad \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \right).$$

If this limit exists, we denote it limit $f'(z_0)$ or $\left.\dfrac{df}{dz}\right|_{z=z_0}$ and call it the **(complex) derivative** of f at z_0.

Definition

If f is complex differentiable at every z in S, then we say that f is **(complex) differentiable on** S. In particular, when S is an open set, then we may sometimes say that f is **(complex) analytic**. When f is analytic on the entire complex plane, we say that f is **entire**.

Remark. If you're familiar with the notion of real analytic functions, this definition may seem odd. It turns out that complex differentiation is a *much* stronger notion than real differentiation, so this definition of analyticity is actually equivalent.

Proposition 2.3.1: Complex Differentiation Rules

If $f(z)$ and $g(z)$ are differentiable at z_0, c is any constant, and $n > 0$ is a positive integer, then...

- [**Sum/Difference Rule**] ...$f(z) \pm g(z)$ is differentiable at z_0 and $\left.\frac{d}{dz}\right|_{z_0} (f(z) \pm g(z)) = f'(z_0) \pm g'(z_0)$.
- [**Constant Multiple Rule**] ...$cf(z)$ is differentiable at z_0 and $\left.\frac{d}{dz}\right|_{z_0} (cf(z)) = cf'(z_0)$
- [**Product Rule**] ...$f(z)g(z)$ is differentiable at z_0 and $\left.\frac{d}{dz}\right|_{z_0} (f(z)g(z)) = f'(z_0)g(z_0) + g'(z_0)f(z_0)$
- [**Quotient Rule**] ...$\dfrac{f(z)}{g(z)}$ is differentiable at z_0 and $\left.\frac{d}{dz}\right|_{z_0} \left(\dfrac{f(z)}{g(z)}\right) = \dfrac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{[g(z_0)]^2}$

Also...

- [**Chain Rule**] ...if $f(z)$ is differentiable at z_0 and $g(z)$ is differentiable at $f(z_0)$ then $\left.\dfrac{d}{dz}\right|_{z_0} g(f(z)) = f'(z_0)g'(f(z_0))$.
- [**Power Rule**] ... z^n is differentiable at z_0 and $\left.\frac{d}{dz}\right|_{z_0} z^n = nz_0^{n-1}$.

Proposition 2.3.2

The following types of functions are analytic on their domains:

- Polynomials
- Rational Functions
- $f(z) = e^z$

(The list above is non-exhaustive)

Proof. The proof of the first two follow from Proposition 2.3.1. $f(z) = e^z$ follows from the limit definition of the derivative (with the same observations as in the proof of continuity). $\square$

Theorem 2.3.3

If f is differentiable at z_0, then f is continuous at z_0.

Not every continuous complex function is differentiable, however.

Example 2.3.4

$f(z) = |z|^2$ is only differentiable at $z = 0$.

Notice that the difference quotient for f is

$$\frac{f(z+h) - f(z)}{h} = \frac{|z+h|^2 - |z|^2}{h} = \frac{(z+h)\overline{(z+h)} - z\overline{z}}{h} = \frac{z\overline{z} + z\overline{h} + h\overline{z} + h\overline{h} - z\overline{z}}{h} = \frac{z\overline{h}}{h} + \overline{z}$$

This limit exists precisely when it agrees for every possible path, so we consider two possible pahs that h can take to 0 - where h is real (say $h = t$ for some real number t), and when h is purely imaginary (say $h = it$ for some real number t).

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{t \to 0} \frac{z\overline{t}}{t} + \overline{z} \qquad = \lim_{t \to 0} \frac{zt}{t} + \overline{z} = \lim_{t \to 0} z + \overline{z}$$

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{t \to 0} \frac{z\overline{it}}{it} + \overline{z} \qquad = \lim_{t \to 0} \frac{-zit}{it} + \overline{z} = \lim_{t \to 0} -z + \overline{z}$$

Notice that these values agree precisely when $z = 0$, and nowhere else. In fact, the difference quotient is exactly 0 when $z = 0$, so the limit does indeed exist there.

2.4 The Cauchy–Riemann Equations

Here we're going to see ways to compute complex derivatives (when they exist) without having to use limits.

Let $z = x + iy$. Then $\text{Re}(f(z))$ and $\text{Im}(f(z))$ are real-valued functions of x and y, let's call them $u(x, y)$ and $v(x, y)$, respectively, so we can write:

$$f(z) = u(x, y) + iv(x, y)$$

If f is complex differentiable at $z = x + iy$, then we can compule the limit

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

along two different paths as in Example **??**.

Path 1 (real axis): For a real number h

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h+iy) - f(x+iy)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \\
&= \left(\lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h}\right) + i\left(\lim_{h\to 0} \frac{v(x+h,y) - v(x,y)}{h}\right) \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}
\end{aligned}$$

provided the partial derivatives exist.

Path 2 (imaginary axis): In what follows, h is a real number.

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(z+ih) - f(z)}{ih} \\
&= \lim_{h\to 0} \frac{f(x+ih+iy) - f(x+iy)}{ih} \\
&= \lim_{h\to 0} \frac{u(x,y+h) + iv(x,y+h) - u(x,y) - iv(x,y)}{ih} \\
&= \frac{1}{i}\left(\lim_{h\to 0} \frac{u(x,y+h) - u(x,y)}{h}\right) + \left(\lim_{h\to 0} \frac{v(x,y+h) - v(x,y+h)}{h}\right) \\
&= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}
\end{aligned}$$

provided the partial derivatives exist.

Since the two limits above must agree in both their real and imaginary parts, we must have that

$$\left.\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.
\end{aligned}\right\} \textbf{Cauchy–Riemann equations} \tag{2.4.1}$$

Theorem 2.4.1

Suppose $z = x+iy$ and that $u, v : \mathbb{R}^2 \to \mathbb{R}$ are differentiable at (x, y). If $f(z) = u(x, y)+iv(x, y)$ is complex differentiable at z, then u and v satisfy the Cauchy–Riemann equations 2.4.1. Moreover

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Note that the above implication does not guarantee that f is complex differentiable. We actually require a further assumption about u and v for that to be true.

Theorem 2.4.2

Suppose $z = x + iy$ and that $u, v : \mathbb{R}^2 \to \mathbb{R}$ are differentiable at (x_0, y_0). If the first partial derivatives of u and v are all continuous at (x_0, y_0) and satisfy the Cauchy–Riemann equations at (x_0, y_0), then $f(z) = u(x, y) + iv(x, y)$ is complex differentiable at $z_0 = x_0 + iy_0$.

Example 2.4.3

Cosider the function $f(z) = z^2$ defined on all of $\mathbb{C}$. Find the derivative of f.

We have that $f(x + iy) = u(x, y) + iv(x, y)$ where

$$u(x, y) = x^2 - y^2 \qquad \text{and} \qquad v(x, y) = 2xy.$$

Computing the partial derivatives, we have that

$$\frac{\partial u}{\partial x} = 2x \qquad\qquad \frac{\partial u}{\partial y} = -2y$$
$$\frac{\partial v}{\partial x} = 2y \qquad\qquad \frac{\partial v}{\partial y} = 2x$$

These satisfy the Cauchy–Riemann equations and at every point (x, y) the first partial derivatives are all continuous, so f is complex differentiable (in fact, entire) and

$$\begin{aligned} f'(z) = f'(x + iy) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= 2x + 2iy \\ &= 2z \end{aligned}$$

Example 2.4.4: C

nsider the function $f(z) = z\,\mathrm{Im}(z)$, which is defined on all of $\mathbb{C}$. Determine the points at which f is differentiable.

We have that $f(x + iy) = u(x, y) + iv(x, y)$ where

$$u(x, y) = xy \qquad \text{and} \qquad v(x, y) = y^2.$$

Computing the partial derivatives, we have that

$$\frac{\partial u}{\partial x} = y \qquad \frac{\partial u}{\partial y} = x$$
$$\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = 2y$$

At every point (x, y) the first partial derivatives are all continuous, but the Cauchy–Riemann equations are only satisfied when $x = y = 0$:

$$y = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2y$$
$$x = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

so f is only complex differentiable at $z = 0$.

The Cauchy–Riemann equations also allow us to prove a fact (which is almost obvious in the real case).

Proposition 2.4.5

If $f'(z) = 0$ for all z in a connected open set S, then f is constant on S.

Proof. If $f'(z) = 0$, then it follows that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$$

whence $u = const$ and $v = const$. Therefore f is constant. □

2.5 Harmonic Functions

We now start making the connection between the real analysis and the complex analysis.

Definition: Harmonic Functions

A real-valued function $\varphi(x, y)$ is said to be **harmonic** in a connected open set S if all of its second-order partial derivatives are continuous in S and if

$$\nabla^2 = \nabla \cdot (\nabla \varphi) = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

Theorem 2.5.1

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a connected open set S, then each of the functions $u(x, y)$ and $v(x, y)$ is harmonic in S.

Definition: Harmonic Conjugate

Given a harmonic function $u(x, y)$, the **harmonic conjugate** is another harmonic function $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is an analytic function.

Example 2.5.2

Find an analytic function whose real part is the function $u(x, y) = x^3 - 3xy^2 + y$ (which is harmonic on the entire plane).

By the Cauchy–Riemann equations, we have

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \\ \implies \qquad v(x, y) &= \int \frac{\partial v}{\partial y}\, dy = 3x^2 y - y^3 + K(x) \end{aligned}$$

where K is a function solely in terms of x. Using the other of the Cauchy–Riemann equations, we have

$$6xy - K'(x) = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} - (-6xy + 1)$$

from which it follows that $K(x) = x + \mathit{const}$. Therefore

$$v(x, y) = 3x^2 y - y^3 + x + \mathit{const}.$$

is the harmonic conjugate to $u(x, y)$. Moreover, we have that

$$f(z) = x^3 - 3xy^2 + y + i(3x^2 y - y^3 - x + \mathit{const}) = z^3 - i(z - \mathit{const}).$$

Notice that we have

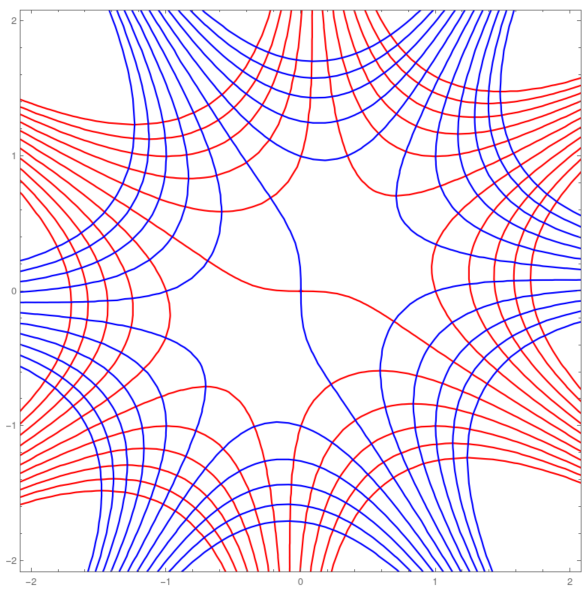
$$\begin{aligned}(\nabla u)\cdot(\nabla v) &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \\ &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \left(-\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial x} = 0\end{aligned}$$

so if the gradients of u and v are nonzero at a point, then they are perpendicular. Hence

Fact. The level curves of harmonic functions and their harmonic conjugates intersect at right angles.

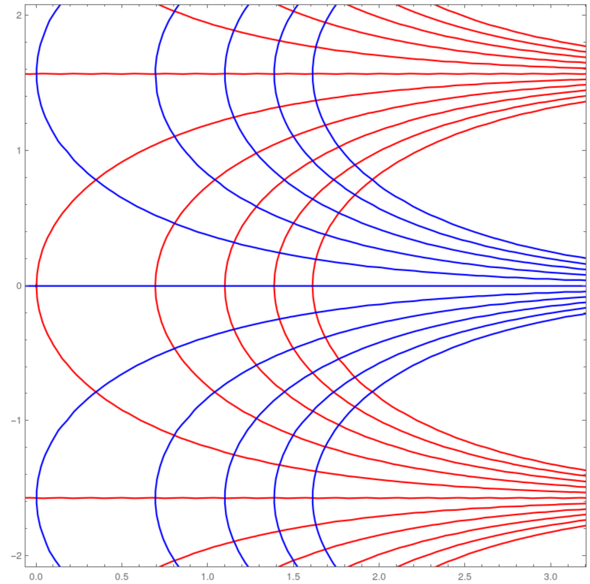
Example 2.5.3

The level curves for u, v as in Example 2.5.2.



Example 2.5.4

The level curves for the curves $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$.


2
1
0
−1
−2
0.0
0.5
1.0
1.5
2.0
2.5
3.0

3 Elementary Functions

3.2 The Exponential, Trigonometric, and Hyperbolic Functions

3.2.1 The Exponential Function - Revisited

Recall that

Definition

The **complex exponential function** e^z is defined for all $z = x + iy$ as

$$e^z := e^x \cos(y) + ie^x \sin(y)$$

Using the Cauchy–Riemann equations, one can show the following fact (whose real equivalent is well known).

Proposition 3.2.1

$\frac{d}{dz}[e^z] = e^z.$

Exercise 3.2.2

Use the Cauchy–Riemann equations to prove Proposition 3.2.1.

The complex exponential does differ from the real exponential in certain behaviors, however.

Example 3.2.3

Find all complex number $z = x + iy$ so that $e^z = 1$.

Certainly this is true when $z = 0$, but using the definition of the complex exponential

$$e^z = e^x(\cos(y) + i\sin(y)) = 1$$

we deduce that $\sin(y) = 0$, and thus $\mathrm{Im}(z) = y = k\pi i$ for some integer k. From that observation it follows that $\cos(y) = \pm 1$, so since we require that $e^x \cos(y) = 1$ and $e^x > 0$, then we must have that k is an even integer and $x = 0$.

Proposition 3.2.4

e^z is periodic with periodicity $2\pi i$.

Proposition 3.2.4 also shows us why the real exponential isn't periodic – all possible periods are purely imaginary!

Example 3.2.5

Find all complex numbers $z = x + iy$ so that $e^z = 2 - 2i$.

Note that $|e^z| = e^x = |2 - 2i| = 2\sqrt{2}$, so we get that $x = \ln(2\sqrt{2})$. Thus, the equation

$$e^z = e^x \cos(y) + ie^x \sin(y) = 2 - 2i$$

implies that

$$2\sqrt{2}\cos(y) = 2 \qquad \text{and} \qquad 2\sqrt{2}\sin(y) = -2.$$

Thus

$$\frac{2\sqrt{2}\sin(y)}{2\sqrt{2}\cos(y)} = \frac{-2}{2} \Longrightarrow \tan(y) = -1$$

and thus $y = \arctan(-1) + 2k\pi = -\frac{\pi}{4} + 2k\pi$ for any integer k. As such, all possible solutions are

$$z = \ln(2\sqrt{2}) - i\left(\frac{\pi}{4} + 2k\pi\right).$$

3.2.2 Trigonometric Functions

Recall from a previous section that, for any real number θ, we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Since the complex exponential is defined for all complex numbers, we can thus define the complex cosine and sine functions.

Definition: F

or any complex number z, we define the **complex cosine function** as

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2},$$

and we define the **complex sine function** as

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Theorem 3.2.6

Both $\cos(z)$ and $\sin(z)$ are complex differentiable on all of $\mathbb{C}$ (i.e. are analytic on all of $\mathbb{C}$).

Moreover,

$$\frac{d}{dz}\sin(z) = \cos(z), \qquad \frac{d}{dz}\cos(z) = -\sin(z).$$

Proof. We will only prove that $\cos(z)$ is complex differentiable, as the proof for $\sin(z)$ is the same *mutatis mutandus*. Letting $z = x + iy$, then we have that $iz = -y + ix$ and $-iz = y - ix$. So

$$\begin{aligned}
\cos(z) &= \frac{1}{2}e^{iz} + \frac{1}{2}e^{-iz} \\
&= \frac{1}{2}e^{-y+ix} + \frac{1}{2}e^{y-ix} \\
&= \frac{1}{2}e^{-y}\left(\cos x + i\sin x\right) + \frac{1}{2}e^{y}\left(\cos x - i\sin x\right) \\
&= \frac{1}{2}\cos(x)\left(e^{y} + e^{-y}\right) + \frac{i}{2}\sin(x)\left(e^{y} - e^{-y}\right).
\end{aligned}$$

Let

$$u(x,y) = \frac{1}{2}\cos(x)\left(e^{y} + e^{-y}\right) \qquad v(x,y) = \frac{i}{2}\sin(x)\left(e^{y} - e^{-y}\right).$$

One can check the u and v satisfy the Cauchy–Riemann equations. Moreover, all of the first partial derivatives of u and v are continuous, so by Theorem 2.4.2, $\cos(z)$ is differentiable. □

Exercise 3.2.7

With u, v as given in the above, check that they do indeed satify the Cauchy–Riemann equations and show that $\frac{d}{dz}\cos(z) = -\sin(z)$.

It's worth noting that the complex trig functions $\cos(z)$ and $\sin(z)$ are *unbounded functions* (to see this, take the imaginary part of z to be arbitrarily large). As it turns out, all of the familiar behavior of sine and cosine happens only in the real numbers. Precisely,

Theorem 3.2.8

1. $\sin(z) = 0$ precisely when $z = k\pi$ for any integer k.
2. $\cos(z) = 0$ precisely when $z = \frac{1}{2}(2k+1)\pi$ for any integer k.
3. The complex sine and cosine functions are period with period 2π.

The remaining trigonometric functions can also be made complex in the obvious ways:

Definition

The complex secant, cosecant, tangent, and cotangent are defined as

$$\sec(z) = \frac{1}{\cos(z)}, \qquad \csc(z) = \frac{1}{\sin(z)},$$
$$\tan(z) = \frac{\sin(z)}{\cos(z)}, \qquad \cot(z) = \frac{\cos(z)}{\sin(z)},$$

provided $\sin(z) \neq 0$ or $\cos(z) \neq 0$ when appropriate.

Theorem 3.2.9

The remaining complex trigonometric functions are complex differentiable on their domains (i.e. analytic) and have the following complex derivatives:

$$\frac{d}{dz}\sec(z) = \sec(z)\tan(z), \qquad \frac{d}{dz}\csc(z) = -\csc(z)\cot(z),$$
$$\frac{d}{dz}\tan(z) = \sec^2(z), \qquad \frac{d}{dz}\cot(z) = -\csc^2(z)$$

Proof. That these functions are complex differentiable is an immediate consequence of the differentiability of $\sin(z)$ and $\cos(z)$ in conjunction with Proposition **??**. Since we now know the complex derivatives of sine and cosine, verification of derivatives is a sraightforward computation and is left as an exercise for the reader. □

3.3 The Logarithmic Function

Before we begin, we mention that functions (by definition) are always single-valued: every input has one single output. However, we've encountered situations where things that would reasonably be called functions, say $f(z) = \arg(z)$ or $f(z) = z^{1/m}$, actually have multiple (even infinitely-many) outputs. Such things are called **multivalued functions** and they usually arise when trying to define the inverse of a function that is not one-to-one.

Remark. Notationally, your book uses "Log" to mean the normal natural logarithm for real numbers and "log" for the complex logarithm. However, this idea is somewhat inconsistent with our usage of arg and Arg, so instead we'll use "ln" for the real natural logarithm.

Suppose z is fixed and we are trying to find all w for which $e^w = z$. We can make things a bit easier if we put z into polar form ($z = re^{i\theta}$) and rewrite $w = u + iv$. Now we have

$$e^u e^{iv} = e^{u+iv} = e^w = z = re^{i\theta} \tag{3.3.1}$$

Since $|e^{iv}| = |e^{i\theta}| = 1$, and since e^u and r are both positive, taking magnitudes of both sides gives us

$$e^u = |e^u||e^{iv}| = |r||e^{i\theta}| = r$$

hence $u = \ln(r) = \ln(|z|)$.

But now Equation 3.3.1 simplifies to

$$e^{iv} = e^{i\theta}.$$

Since the complex exponential has periods of the form $2k\pi i$ for integers k, it must be that $v = \theta + 2k\pi$ for some integer k. As such, the answer to our original equation is

$$w = \ln|z| + i(\theta + 2k\pi)$$

where θ is an argument for z and k is any integer. From this we define

Definition: Complex Logarithm

If z is a <u>nonzero</u> complex number, then the **(complex) logarithm**, $\log(z)$, is the set of complex numbers

$$\begin{aligned} \log(z) &:= \ln|z| + i\arg(z) \\ &:= \ln|z| i\,\mathrm{Arg}(z) + 2k\pi i \end{aligned}$$

where k ranges over all integers. The **principal value of** $\log(z)$ is

$$\mathrm{Log}(z) := \ln|z| + i\,\mathrm{Arg}(z).$$

Note that $\log(z)$ is a multi-valued function, but $\mathrm{Log}(z)$ is a single-valued function.

Definition: A

function $F(z)$ is said to be a **branch** of a multi-valued function $f(z)$ in a domain D if $F(z)$ is a single-valued, continuous function on D and $F(z)$ is one of the values of $f(z)$.

Remark. $\text{Log}(z)$ is a branch of $\log(z)$, $\text{Arg}(z)$ is a branch of $\arg(z)$.

Example 3.3.1

Compute $\log(z)$ where $z = 1 + i\sqrt{3}$.

Since $|z| = 2$ and z has an argument $\theta = \frac{\pi}{3}$, we have that

$$\log(z) = \ln 2 + i\left(\frac{\pi}{3} + 2k\pi\right)$$

Proposition 3.3.1. $\text{Log}(z)$ *is analytic on* $\mathbb{C} - \{$*the non-positive real axis*$\}$. *Moreover,* $\dfrac{d}{dz}[\text{Log}(z)] = \frac{1}{z}$

Proof. We first mention that the strange domain is reasonable - when $z = 0$, $\text{Log}(z)$ is undefined. Also, $\text{Arg}(z)$ isn't continuous on the negative real axis.
To prove that $\text{Log}(z)$ is analytic, we'll approach with the Cauchy–Riemann equations, which means we need

$$\text{Log}(x+iy) = \ln|x+iy| + i\,\text{Arg}(x+iy) = u(x,y) + iv(x,y)$$

We have that

$$u(x,y) = \ln|x+iy| = \ln(\sqrt{x^2+y^2})$$
$$v(x,y) = \text{Arg}(x+iy) = \arctan 2(y,x) = 2\arctan\left(\frac{y}{\sqrt{x^2+y^2}+x}\right)$$

($\arctan(y/x)$ only returns a value between $-\pi/2$ and $\pi/2$, so using the half-angle formula for tangent, this new version returns a value between $-\pi$ and π, as desired.) It is straightforward to compute

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x}$$

so since the first partials of u, v are continuous on the indicated domain, then it follows that $\text{Log}(z)$ is analytic on this domain. Moreover

$$\begin{aligned}\frac{d}{dz}[\text{Log}(z)] &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} \\ &= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}.\end{aligned}$$

□

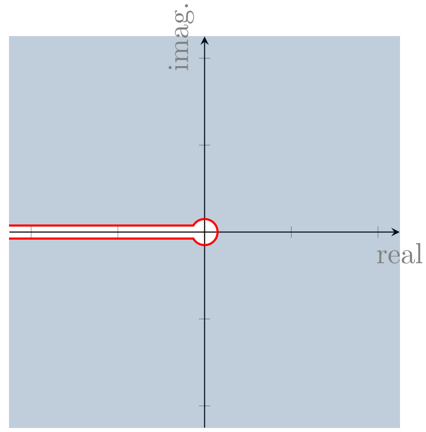


Figure 3.3.1: The domain of Log(z)

Corollary 3.3.2

$\ln|z|$ and $\operatorname{Arg}(z)$ are harmonic on the domain described in the previous proposition.

3.5 Complex Powers and Inverse Trigonometric Functions

3.5.1 Complex Powers

Let z be a nonzero complex number. Then we have that

$$e^{\log(z)} = e^{\ln|z| + \operatorname{Arg}(z) + 2\pi i k} = e^{\ln|z|} e^{\operatorname{Arg}(z)} e^{2\pi i k} = |z| e^{i \operatorname{Arg}(z)} = z$$

It follows that...

Definition

Let $z \neq 0$. For any complex w, we define

$$z^w := e^{w \log(z)}.$$

This definition is consistent with the discussion of roots earlier, and it's easy to check that we get the following:

- If w is a real integer, then z^w has one value.
- If $w = \frac{p}{q}$ is a real rational number, then z^w has q values.
- In any other case, z^w takes infinitely-many values, one for each value of $\log(z)$.

From the chain rule it follows that

Proposition 3.5.1

$f(z) = z^a$ is analytic in the same domain as $\operatorname{Log}(z)$, that is $\mathbb{C} - (-\infty, 0]$. Moreover, $\dfrac{d}{dz}[z^a] = az^{a-1}$.

Example 3.5.2

Find i^i.

$i = e^{i\pi/2}$, so $\log(i) = i(\pi/2 + 2k\pi)$ for any integer k. Hence

$$i^i = e^{i \log(i)} = e^{i^2(\pi/2 + 2k\pi)} = e^{\pi/2 + 2k\pi}$$

for any integer k.

Remark. Yes. i^i produces real numbers.

Example 3.5.3

Another Example Here

3.5.2 Inverse Trig Functions

Recall that for a real function $y = f(x)$, one often finds the inverse by replacing y and x (i.e. writing $y = f(x)$ and trying to solve for solve for y in terms of x.

Let $w = \tan(z)$. We aim to find the inverse, so we write $z = \tan(w)$ and solve for w (which will produce $\arctan(z)$).

$$\begin{aligned}
z = \tan(w) &= \frac{e^{iw} - e^{-iw}}{i\left(e^{iw} + e^{-iw}\right)} \\
\implies 0 &= i\left(e^{iw} + e^{-iw}\right)z - \left(e^{iw} - e^{-iw}\right) \\
0 &= e^{2iw}z - iz + ie^{2iw} - i \qquad\qquad \text{(multiply everything by } -ie^{iz}) \\
0 &= (z+i)e^{2iw} - (iz+i) \\
\implies e^{2iw} &= \frac{iz+i}{z+i} \\
e^{iw} &= \left(\frac{iz+i}{z+i}\right)^{1/2} \\
iw &= \frac{1}{2}\log\left[\frac{iz+i}{z+i}\right] \\
w &= \frac{-i}{2}\log\left[\frac{iz+i}{z+i}\right]
\end{aligned}$$

Fiddling with the algebra just a bit, and applying similar arguments, one obtains the following:

Definition: Inverse Trig Functions

- $\arctan(z) = \dfrac{i}{2}\log\left[\dfrac{i+z}{i-z}\right]$
- $\arcsin(z) = -i\log\left[iz + (1-z^2)^{1/2}\right]$
- $\arccos(z) = -i\log\left[z + (z^2 - 1^2)^{1/2}\right]$

It turns out that the derivatives of the inverse trig functions have familiar derivatives

Proposition 3.5.4

- $\dfrac{d}{dz}\arcsin(z) = \dfrac{1}{(1-z^2)^{1/2}}$
- $\dfrac{d}{dz}\arccos(z) = \dfrac{-1}{(1-z^2)^{1/2}}$
- $\dfrac{d}{dz}\arctan(z) = \dfrac{1}{1+z^2}$

Exercise 3.5.1. Prove the formulae in Proposition **??**.

4 Complex Integration

4.1 Contours

Given our identification of the real plane $\mathbb{X}^2$ and the complex plane $\mathbb{C}$, we can consider curves in the complex plane.

Definition: Contours

Let γ be the function

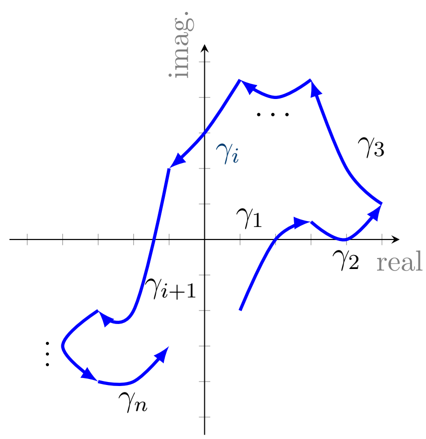
$$\begin{aligned} \gamma &: [a,b] \to \mathbb{C} \\ \gamma(t) &= x(t) + iy(t) \end{aligned}$$

where $x(t)$, and $y(t)$ are real-valued. γ is called a **contour** if $(x(t), y(t))$ is a piecewise smooth curve in the plane, $\mathbb{X}^2$. Just as before, if γ is comprised of several smooth curves $\gamma_1, \ldots, \gamma_n$, we write

$$\gamma(t) = \gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_n(t) = \begin{cases} \gamma_1(t) & \text{when } a \le t \le t_1 \\ \gamma_2(t) & \text{when } t_1 < t \le t_2 \\ & \vdots \\ \gamma_n(t) & \text{when } t_{n-1} < t \le b \end{cases}$$

A contour is **closed** if $\gamma(a) = \gamma(b)$, and is **simple** if $\gamma(t_1) \neq \gamma(t_2)$ for any $a < t_1 < t_2 < b$.

Piecewise smooth curves can have finitely-many corners or cusps.



Example 4.1.1

Parameterize a contour traversing the circle of radius 2, counter-clockwise.

In $\mathbb{X}^2$, the usual parameterization would be

$$\sigma(t) = (2\cos(t), 2\sin(t)), \qquad \text{for } 0 \le t \le \pi$$

Thus, in the complex plane we have

$$\gamma(t) = 2\cos(t) + i2\sin(t) = 2e^{it}$$

Example 4.1.2

Parameterize a contour traversing the circle of radius r, centered at z_0, counter-clockwise.

Letting $z_0 = x_0 + iy_0$, in $\mathbb{X}^2$, the usual parameterization would be

$$\sigma(t) = (x_0 + r\cos(t), y_0 + 2\sin(t)), \qquad \text{for } 0 \leq t \leq \pi$$

Thus, in the complex plane we have

$$\gamma(t) = r\cos(t) + x_0 + ir\sin(t) + iy_0 = re^{it} + z_0$$

Example 4.1.3

Parameterize a contour traversing the line segment from z_0 to z_1.

Letting $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$, the usual parameterization in $\mathbb{X}^2$ is givne by

$$\sigma(t) = (x_0(1-t) + x_1 t, y_0(1-t) + y_1 t), \qquad \text{for } 0 \leq t \leq 1.$$

Thus, in the complex plane, we have

$$\gamma(t) = x_0(1-t) + x_1 t + iy_0(1-t) + iy_1 t = z_0(1-t) + z_1 t$$

4.1.1 Arc Length

Letting $z(t) = x(t) + iy(t)$ for some real parameter t, we have that $\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$, and thus

$$\left|\frac{dz}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

The right-hand-side of this equation is the infinitesimal change in arc length, hence we define arc length of a contour as follows:

Definition

If $\gamma : [a,b] \to \mathbb{C}$ is a contour, then the **length of** γ is given by

$$\text{length}(\gamma) = \int_a^b \left|\frac{dz}{dt}\right| \, dt.$$

4.2 Contour Integrals

Definition

Suppose $\gamma = \gamma_1 \oplus \oplus \gamma_2 \oplus \cdots \oplus \gamma_n$ is a piecewise smooth curve defined for $a \leq t \leq b$ (where $\gamma_k(t)$ is defined for $a_k \leq t \leq b_k$, $a = a_1$, and $b = b_k$) and assume f is continuous at all points along the curve. Then the **contour integral of f along** $\gamma$is

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_{a_k}^{b_k} f(\gamma_k(t))\gamma_k'(t)\,dt.$$

Example 4.2.1

Compute $\int_\gamma \overline{z}\,dz$ for $\gamma = \gamma_1 \oplus \gamma_2$, where

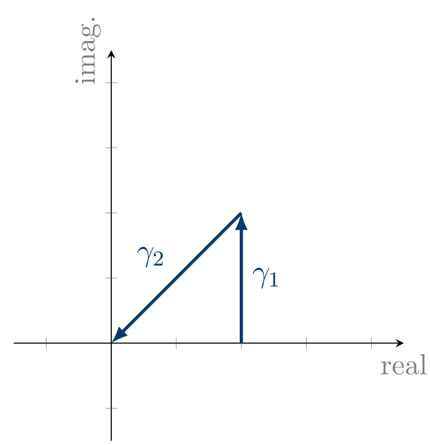
$$\begin{aligned} \gamma_1(t) &= 2 + 2ti && (0 \leq t \leq 1), \\ \gamma_2(t) &= 4 - 2t + i(4 - 2t) && (1 < t \leq 2). \end{aligned}$$

We have that

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = \int_0^1 f(\gamma_1(t))\gamma_1'(t)\,dt + \int_1^2 f(\gamma_2(t))\gamma_2'(t)\,dt.$$

Computing these integrals separately,

$$\begin{aligned} \int_0^1 f(\gamma_1(t))\gamma_1'(t)\,dt &= \int_0^1 (2 - 2ti)(2i)\,dt \\ &= \int_0^1 4t + 4i\,dt = 2 + 4i, \end{aligned}$$



and

$$\begin{aligned} &\int_1^2 f(\gamma_2(t))\gamma_2'(t)\,dt \\ &= \int_1^2 (4 - 2t - 4i + 2ti)(-2 - 2i)\,dt \\ &= \int_1^2 8t - 16\,dt = -4. \end{aligned}$$

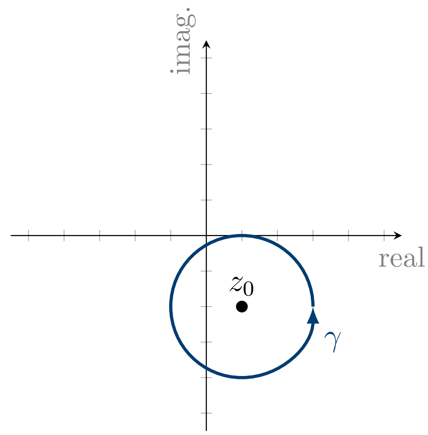
Hence $\int_\gamma f(z)\,dz = (2 + 4i) + (-4) = -2 + 4i$.

Example 4.2.2

For some fixed z_0 and positive real number r, let γ be the circle $|z - z_0| = r$ traversed once in the counterclockwise direction. For each integer $n \neq 1$, compute $\int_\gamma (z - z_0)^n\,dz$.

We know that the circle of radius r centered at the origin is parameterized by re^{it} for $0 \leq t \leq 2\pi$, so we can write $\gamma(t) = z_0 + re^{it}$ for $0 \leq t \leq 2\pi$.

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_\gamma (z-z_0)^n\,dz \\
&= \int_0^{2\pi} (\gamma(t)-z_0)^n \gamma'(t)\,dt \\
&= \int_0^{2\pi} (r^n e^{int})(ire^{it})\,dt \\
&= \int_0^{2\pi} ir^{n+1} e^{i(n+1)t}\,dt \\
&= ir^{n+1}\left(\frac{-i}{n+1}\right) e^{i(n+1)t}\Big|_{t=0}^{t=2\pi} = 0.
\end{aligned}$$

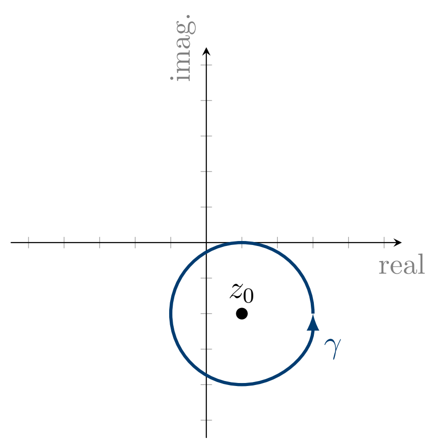


Example 4.2.3

For some fixed z_0 and positive real number r, let γ be the circle $|z-z_0| = r$ traversed once in the counterclockwise direction. Compute $\int_\gamma \frac{1}{z-z_0}\,dz$.

We know that the circle of radius r centered at the origin is parameterized by re^{it} for $0 \leq t \leq 2\pi$, so we can write $\gamma(t) = z_0 + re^{it}$ for $0 \leq t \leq 2\pi$.

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_\gamma \frac{1}{z-z_0}\,dz \\
&= \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)-z_0}\,dt \\
&= \int_0^{2\pi} \frac{ire^{it}}{re^{it}}\,dt \\
&= \int_0^{2\pi} i\,dt \\
&= 2\pi i.
\end{aligned}$$



So now we conclude that, if γ is any circle centered at z_0, then

$$\oint_\gamma (z-z_0)^n\,dz = \begin{cases} 0 & \text{when } n \neq -1, \\ 2\pi i & \text{when } n = -1. \end{cases}$$

Theorem 4.2.4: Properties of Complex Integrals

Let f and g be integrable over some (piecewise) smooth curve γ.

1. $\displaystyle\int_\gamma (f(z)+g(z))\,dz = \int_\gamma f(z)\,dz + \int_\gamma g(z)\,dz.$

2. For any complex number c, $\displaystyle\int_\gamma cf(z)\,dz = c\int_\gamma f(z)\,dz$.
3. Reversing orientation of the curve changes the integral sign. If $\tilde{\gamma}$ is the same curve as γ, traversed in the opposite direction, then

$$\int_{\tilde{\gamma}} f(z)\,dz = -\int_\gamma f(z)\,dz.$$

4. $\displaystyle\int_\gamma f(z)\,dz$ can be written as a sum of two real line integrals: Suppose γ is defined on $[a,b]$. Then we have that

$$f(z) = u(x,y) + iv(x,y) \qquad \text{and} \qquad dz = (x'(t) + iy'(t))\,dt.$$

Then

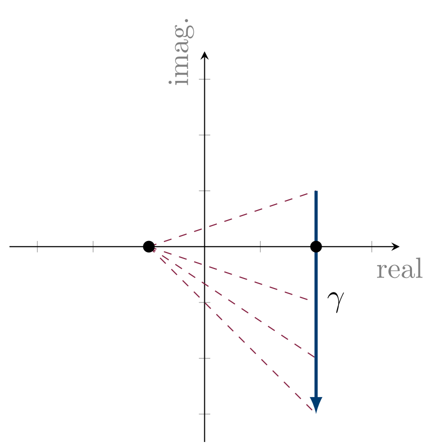
$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b (u(x,y) + iv(x,y))\,(x'(t) + iy'(t))\;dt \\ &= \int_a^b u(x,y)x'(t)\,dt - v(x,y)y'(t),\,dt + iu(x,y)y'(t)\,dt + iv(x,y)x'(t)\,dt \\ &= \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx)\end{aligned}$$

5. Let γ be a smooth curve defined on $[a,b]$, let L be its length, and let f be continuous on γ. If $f(z)$ is bounded by M (i.e. $|f(z)| \le M$ for all z on γ), then

$$\left|\int_\gamma f(z)\,dz\right| \le ML.$$

Example 4.2.5

Using Part 5 of the above Theorem (4.2.4), find a bound for $\displaystyle\left|\int_\gamma \frac{1}{1+z}\,dz\right|$ where γ is the straight line segment from $2+i$ to $2-3i$.



We are looking for a number M so that $\left|\dfrac{1}{z+1}\right| \leq M$ for z-values on γ. Notice that $|z+1| = |z-(-1)|$ is just the distance from a point z to -1, so the smallest value of $|z+1|$ (and thus, the largest value of $\left|\dfrac{1}{z+1}\right|$) occurs when γ is closest to -1. Looking at the picture, it's not hard to see that this happens when γ passes through 2. So we have that, for all z on the curve γ

$$|z+1| \geq |2+1| = 3 \qquad \Longrightarrow \qquad \left|\frac{1}{z+1}\right| \leq \frac{1}{3} = M.$$

Since γ is a line segment with length $L = 4$, then by Theorem 4.2.4, it follows that

$$\left|\int_\gamma \frac{1}{1+z}\,dz\right| \leq ML = \left(\frac{1}{3}\right)(4) = \frac{4}{3}.$$

Computing the integral exactly, we write $\gamma(t) = 2 - ti$ for $-1 \leq t \leq 3$. Then

$$\begin{aligned}
\int_\gamma \frac{1}{1+z}\,dz = \int_{-1}^{3} \frac{\gamma'(t)}{1+\gamma(t)}\,dt \\
&= \int_{-1}^{3} \frac{-i}{3-ti}\,dt \\
&= \int_{-1}^{3} \frac{-i(3+ti)}{9+t^2}\,dt \\
&= \int_{-1}^{3} \frac{t}{9+t^2} - i\frac{3}{9+t^2}\,dt \\
&= \left[\frac{1}{2}\ln\!\left(9+t^2\right) - i\arctan\!\left(\frac{t}{3}\right)\right]_{-1}^{3} \approx 0.293893 - 1.10715i
\end{aligned}$$

and the modulus of this complex number is approximately 1.14549.

4.3 Independence of Path

Suppose $F(z)$ is analytic on some domain D, and write $f(z) = F'(z)$. Let $\gamma : [a, b] \to D$ be a smooth path in D and write $z = \gamma(t)$. Then we have

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt$$

Notice that, by the chain rule,

$$\frac{dF}{dt} = \frac{dF}{dz}\frac{dz}{dt} = f(\gamma(t))\gamma'(t)\, dt$$

whence

$$\int_a^b f(\gamma(t))\gamma'(t)\, dt = \int_a^b \frac{dF(\gamma(t))}{dt}\, dt = F(\gamma(b)) - F(\gamma(a))$$

If γ is a contour (i.e. $\gamma = \gamma_1 \oplus \cdot \oplus \gamma_n$ for some smooth paths γ_j), then this formula extends to sums

$$\int_a^b f(\gamma(t))\gamma'(t)\, dt = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \frac{dF(\gamma_j(t))}{dt}\, dt = \sum_{j=1}^n F(\gamma_j(t_{j-1})) - F(\gamma(t_j))$$

This sum is telescoping, hence reduces to

$$F(\gamma(b)) - F(\gamma(a))$$

With this observation, we conclude the following (which can be thought of as an extension of the Fundamental Theorem of Calculus

Theorem 4.3.1: Independence of Path

If $f(z)$ is continuous in a domain D and has an antiderivative $F(z)$, then for any contour $\gamma : [a, b] \to D$ in D we have that

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a))$$

The following is immediate:

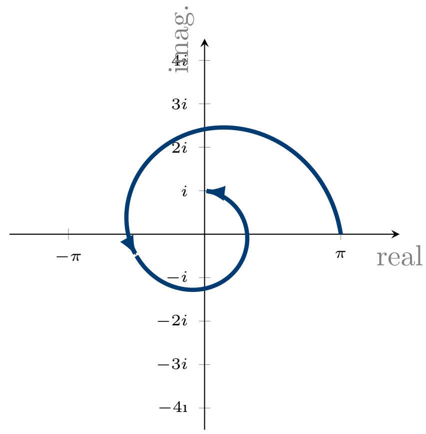
Corollary 4.3.2

If $f(z)$ is continuous in a domain D and has an antiderivative $F(z)$, then for any closed contour $\gamma : [a, b] \to D$ in D we have that

$$\int_\gamma f(z)\, dz = 0$$

Example 4.3.3

Evaluate $\displaystyle\int_\gamma \sin^2(z)\cos(z)\,dz$ where $\gamma(t) = \dfrac{e^{it}(2\pi^2 - \pi t + it)}{2\pi}$ and $0 \le t \le 2\pi$.



We recognize that $F(z) = \frac{1}{3}\sin^3(z)$ is an antiderivative for $f(z) = \sin^2(z)\cos(z)$, so since $\gamma(0) = \pi$ and $\gamma(2\pi) = i$, we have that

$$\int_\gamma \sin^2(z)\cos(z)\,dz = \frac{1}{3}\sin^3(i) - \frac{1}{3}\sin^3(\pi) = \frac{1}{3}\sin^3(i) \approx -0.541023i$$

In Example 4.2.3 we computed $\int_\gamma (z - z_0)^n\,dz$ where z_0 was some fixed number and γ was a circle of radius r centered at 0. Let's look at it again. Specifically, after parameterizing $\gamma(t) = z_0 + re^{it}$ with $a \le t \le b$, we obtained

$$\int_\gamma f(z)\,dz = \int_a^b (\gamma(t) - z_0)^n \gamma'(t)\,dt \frac{r^{n+1}e^{i(n+1)b}}{n+1} - \frac{r^{n+1}e^{i(n+1)a}}{n+1}.$$

However, knowing about the power rule/chain rule, we see that $F(z) = \dfrac{1}{n+1}(z - z_0)^{n+1}$ is an antiderivative for $f(z) = (z - z_0)^n$, and it's fairly clear that the above can be rewritten

$$\frac{r^{n+1}e^{i(n+1)b}}{n+1} - \frac{r^{n+1}e^{i(n+1)a}}{n+1} = \frac{(\gamma(b) - z_0)^{n+1}}{n+1}\frac{(\gamma(a) - z_0)^{n+1}}{n+1} = F(\gamma(b)) - F(\gamma(a))$$

Exercise 4.3.4

Show that $\oint_\gamma \frac{1}{(z - z_0)}\,dz$ by using the antiderivative/independence of path technique on a circle γ centered at z_0. Note that you will require two different branches of $\log(z - z_0)$.

The same cannot be said about Example 4.2.1 – what exactly is the antiderivative of $f(z) = \overline{z}$? Let's compute that integral for a different path from 2 to 0 and see if it's even independent of path.

Example 4.3.5

Compute $\int_\gamma \overline{z}\, dz$ for $\gamma(t) = 2 - t$, where $0 \leq t \leq 2$.

We have that $\gamma(t) = -1$, thus

$$\begin{aligned} \int_\gamma \overline{z}\, dz &= \int_0^2 \overline{\gamma(t)}\gamma'(t)\, dt \\ &= \int_0^2 t - 2\, dt = -2. \end{aligned}$$

This is *not* the value we obtained in Example 4.2.1, so $f(z) = \overline{z}$ is *not* independent of path.

It turns out that the existence of an antiderivative is actually quite tied to the independence of path.

Theorem 4.3.6

Suppose f is continuous in some domain D. The following are equivalent.

1. f has an antiderivative.
2. $\int_\gamma f(z)\, dz$ is independent of path for every contour γ in D.
3. $\int_\gamma f(z)\, dz = 0$ for every closed contour in D.

4.4 Cauchy's Integral Theorem

Notice that, if $f(z)$ has an antiderivative $F(z)$ on some domain D, then $F(z)$ is analytic on the domain D. So Corollary 4.3.2, where

$$\oint_{\gamma} f(z)\,dz = 0$$

kind of feels like it's maybe a statement about analyticity.

As it turns out, Gauss noticed this same thing in 1811 and Cauchy published it in his book in 1890 (which is why it gets Cauchy's name).

Theorem 4.4.1: Cauchy's Theorem

Suppose f is analytic on a simply connected domain S. Then

$$\oint_{\gamma} f(z)\,dz = 0$$

for every closed path γ in S.

There are at least a couple of distinct ways to prove this, and we'll present two strategies – one way using a more topological/complex analytic viewpoint, and another way appealing to Green's Theorem.

4.4.1 A Deformation of Contours Approach

Definition

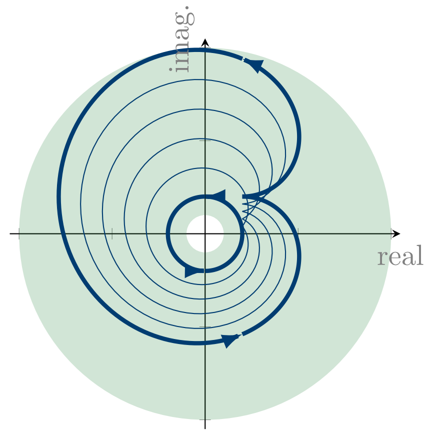
Let γ_0 and γ_1 be contours in a region D. We say that we can **deform** γ_0 into γ_1 if we can find a continuous family of contours $\Gamma(s,t)$ in D with $0 \leq s \leq 1$ so that $\gamma_0(t) = \Gamma(0,t)$ and $\gamma_1(t) = \Gamma(1,t)$.

This definition is a little bit subtle, but the idea is that you treat $\Gamma(s,t)$ as a function of two variables, s and t, and you want it to be a continuous function of two variables.

Example 4.4.2

Show that the curve $\gamma_0(t) = (2+2i) - t(t-2\pi)e^{it}$ (with $0 \leq t \leq 2\pi$) can be deformed into the curve $\gamma_1(t) = 2e^{it}$ entirely within the domain $2 < |z < 10$.

Let $\Gamma(s,t) = (1-s)\gamma_0(t) + s\gamma_1(t)$.



Independence of path suggests that, with minor assumptions, when integrating over a closed path, we can always try to integrate over a more convenient closed path.

Theorem 4.4.3: Deformation Theorem

et Γ and γ be closed contours with γ in the interior of Γ. Suppose f is differentiable on a set S containing both paths <u>and</u> all points in between them. Then

$$\oint_\Gamma f(z)\,dz = \oint_\gamma f(z)\,dz.$$

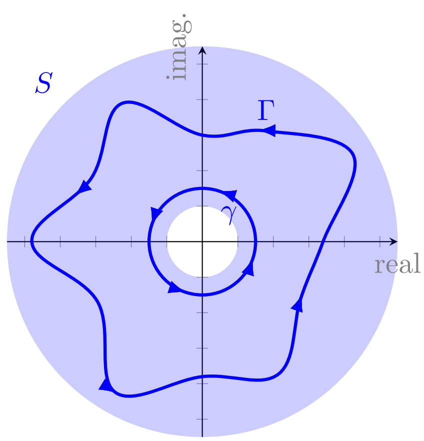


Figure 4.4.1: Assumptions of deformation theorem.

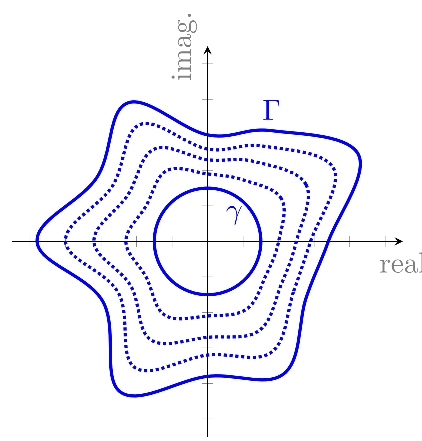
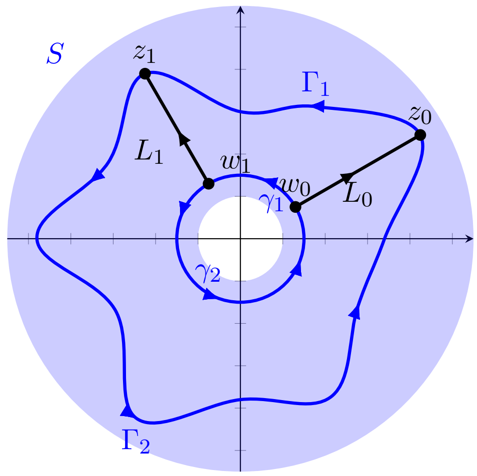


Figure 4.4.2: Step-by-step, the outer path Γ is being "deformed" and shrinking to the inner path γ.

Proof. Let z_0, z_1 be any two distinct points on Γ and let w_0, w_1 be any two distinct points on γ. Let Γ_1 be the path from z_0 to z_1 and Γ_2 the path from z_1 to z_0 so that $\Gamma = \Gamma_1 \oplus \Gamma_2$. Similarly let γ_1 be the path from w_0 to w_1 and γ_2 is the path from w_1 to w_0 so that $\gamma = \gamma_1 \oplus \gamma_2$. Finally, let L_0 be a path from w_0 to z_0, and L_1 a path w_1 to z_1 (without loss of generality, we can choose L_0 and L_1 so that they do not cross). We have the picture below.



By Cauchy's Theorem we have that

$$\oint_{\Gamma_1 \oplus (-L_1) \oplus (-\gamma_1) \oplus L_0} f(z)\, dz = 0,$$
$$\oint_{\Gamma_2 \oplus (-L_0) \oplus (-\gamma_2) \oplus L_1} f(z)\, dz = 0.$$

Recall that

$$\oint_{\Gamma_1 \oplus (-L_1) \oplus (-\gamma_1) \oplus L_0} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz - \int_{L_1} f(z)\, dz - \int_{\gamma_1} f(z)\, dz + \int_{L_0} f(z)\, dz$$
$$\oint_{\Gamma_2 \oplus (-L_0) \oplus (-\gamma_2) \oplus L_1} f(z)\, dz = \int_{\Gamma_2} f(z)\, dz - \int_{L_0} f(z)\, dz - \int_{\gamma_2} f(z)\, dz + \int_{L_1} f(z)\, dz$$

So,

$$\begin{aligned}
0 &= \oint_{\Gamma_1 \oplus (-L_1) \oplus (-\gamma_1) \oplus L_0} f(z)\, dz + \oint_{\Gamma_2 \oplus (-L_0) \oplus (-\gamma_2) \oplus L_1} f(z)\, dz \\
&= \int_{\Gamma_1} f(z)\, dz + \int_{\Gamma_2} f(z)\, dz - \int_{\gamma_1} f(z)\, dz - \int_{\gamma_2} f(z)\, dz \\
&= \int_{\Gamma_1 \oplus \Gamma_2} f(z)\, dz - \int_{\gamma_1 \oplus \gamma_2} f(z)\, dz \\
&= \oint_{\Gamma} f(z)\, dz - \oint_{\gamma} f(z)\, dz
\end{aligned}$$

hence

$$\oint_{\Gamma} f(z)\, dz = \oint_{\gamma} f(z)\, dz.$$

$\square$

4.4.2 Rational Functions

Following from Cauchy's Theorem/The Deformation Theorem, We know that

$$\oint_\gamma (z-z_0)^n\,dz = \begin{cases} 0 & \text{when } n \neq -1, \\ 2\pi i & \text{when } n = -1, \end{cases}$$

for any contour γ that encloses z_0 and

$$\oint_\gamma (z-z_0)^n\,dz = 0$$

for any contour γ that does not enclose z_0.

More generally, if $p(z)$ is any polynomial, then we can do polynomial long division to write

$$\frac{p(z)}{z-z_0} = r(z) + \frac{A}{z-z_0}$$

where r is some other polynomial and A is a constant, which yields

$$\oint_\gamma \frac{p(z)}{z-z_0}\,dz = \oint_\gamma q(z) + \frac{A}{z-z_0}\,dz = 0 + A(2\pi i).$$

Even more generally, if $p(z)$ and $q(z) = (z-z_0)(z-z_1)\cdots(z-z_n)$ are any polynomials, then we can applying polynomial long division and partial fraction decompositions to write

$$\frac{p(z)}{q(z)} = r(z) + \frac{A_0}{z-z_0} + \frac{A_1}{z-z_1} + \cdots + \frac{A_n}{z-z_n}$$

for some other polynomial r and constants $A_0, A_1, \ldots, A_n$. From this it follows that, if γ is a contour enclosing $z_0, \ldots, z_k$ (and not enclosing $z_{k+1}, \ldots, z_n$), then

$$\begin{aligned} \oint_\gamma \frac{p(z)}{q(z)}\,dz &= \oint_\gamma r(z) + \frac{A_0}{z-z_0} + \cdots + \frac{A_k}{z-z_k} + \frac{A_{k+1}}{z-z_{k+1}} + \cdots + \frac{A_n}{z-z_n}\,dz \\ &= 0 + A_0(2\pi i) + A_1(2\pi i) + \cdots + A_k(2\pi i) + 0 \cdots + 0. \end{aligned}$$

None of the above considers multiplicity, so assume that $q(z) = (z-z_0)^n$ for $n > 1$. We know from the rules of partial fractions decompositions that we can write

$$\frac{p(z)}{q(z)} = r(z) + \frac{B_1}{z-z_0} + \frac{B_2}{(z-z_0)^2} + \cdots + \frac{B_2}{(z-z_0)^n}$$

for some polynomial $r(z)$ and constants $B_1, \ldots, B_n$. The boxed facts above implies that, if γ encloses z_0, then

$$\begin{aligned} \oint_\gamma \frac{p(z)}{q(z)}\,dz &= \oint_\gamma r(z) + \frac{B_1}{z-z_0} + \frac{B_2}{(z-z_0)^2} + \cdots + \frac{B_n}{(z-z_0)^n}\,dz \\ &= 0 + B_1(2\pi i) + 0 + \cdots + 0 \end{aligned}$$

In this sense, the *only* things we need to evaluate contour integrals of rational functions are (1) the denominator's roots z_j which are enclosed by γ and (2) the constant coefficients in the numerator for each $(z - z_j)$ term!

But is there an easier way to figure out what that numerator coefficient is? Let's try some examples.

$$\frac{z+2}{z-z_0} = \frac{(z-z_0)+(z_0+2)}{z-z_0} = 1 + \frac{z_0+2}{z-z_0}$$

$$\frac{z^2+3}{z-z_0} = \frac{(z-z_0)^2 + 2z_0 z - 2z_0^2 + z_0^2 + 3}{z-z_0} = (z-z_0) + 2 + \frac{z_0^2+3}{z-z_0}$$

$$\frac{3z^2+5z+7}{z-z_0} = \frac{3(z-z_0)^2 + 6z_0(z-z_0) + 3z_0^2 + 5z_0 + 7}{z-z_0} = 3(z-z_0) + 6z_0 + \frac{3z_0^2+5z+7}{z-z_0}$$

What we're seeing is the classical algebraic fact:

For any polynomial $p(z)$ and any number z_0,

$$\frac{p(z)}{z-z_0} = q(z) + \frac{p(z_0)}{z-z_0}$$

where $q(z)$ is some other polynomial.

Example 4.4.4

Evaluate $\displaystyle\oint_\gamma \frac{z^{800}+1}{z-i}\,dz$ where γ is any loop enclosing i.

$$\oint_\gamma \frac{z^{800}+1}{z-i}\,dz = \oint_\gamma q(z) + \frac{i^{800}+1}{z-i}\,dz = \oint_\gamma q(z) + \frac{2}{z-i}\,dz = 0 + 2(2\pi i).$$

4.5 Cauchy's Integral Formula and Its Consequences

In Example **??**, we saw that, for $z_0 = 2i$

$$\oint_C \frac{1}{z - z_0}\, dz = 2\pi i,$$

where C was a circle of radius 1 centered at z_0. By the Deformation Theorem, we can conclude that in fact that should be true for *any* closed path γ surrounding z_0 (and in fact, the particular choice of z_0 is also unimportant). So it may be reasonable to ask, what happens if we consider a slightly more general integral like the one below?

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz$$

For simplicity, we'll assume γ lives in a domain, that f is differentiable on that domain, and that γ encloses a simply connected domain containing z_0. Then

$$\begin{aligned}
\oint_\gamma \frac{f(z)}{z - z_0}\, dz &= \oint_\gamma \frac{f(z) - f(z_0) + f(z_0)}{z - z_0}\, dz \\
&= \oint_\gamma \frac{f(z_0)}{z - z_0}\, dz + \oint_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz \\
&= 2\pi i f(z_0) + \oint_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz
\end{aligned}$$

What we want to know is that happens to that second integral. By the deformation theorem, it suffices to consider the case when γ is a small circle around z_0, so for some small radius r, we can parameterize γ as

$$\gamma(t) = z_0 + re^{it} \qquad 0 \le t \le 2\pi.$$

Then

$$\begin{aligned}
\oint_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz &= \int_0^{2\pi} \frac{f(z_0 + re^{it}) - f(z_0)}{re^{it}}\, ire^{it}\, dt \\
&= i \int_0^{2\pi} f(z_0 + re^{it}) - f(z_0)\, dt.
\end{aligned}$$

Since the integrand can be negative, it follows that

$$\left| i \int_0^{2\pi} f(z_0 + re^{it}) - f(z_0)\, dt \right| \le \int_0^{2\pi} |f(z_0 + re^{it}) - f(z_0)|\, dt.$$

Now, by continuity of f at z_0,

$$\lim_{r \to 0} |f(z_0 + re^{it}) - f(z_0)| = 0$$

so this implies that

$$\left| i \int_0^{2\pi} f(z_0 + re^{it}) - f(z_0)\, dt \right| = 0$$

and thus

$$\oint_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0.$$

Theorem 4.5.1: Cauchy's Integral Formula

Let f be differentiable on an open set S. Let γ be a closed path in S enclosing only points of S. Then, for any z_0 enclosed by γ,

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,dz$$

Example 4.5.2

Let γ be any closed path that does not pass through i and let $f(z) = 85e^{iz^{100}\pi}$. Evaluate $\displaystyle\oint \frac{f(z)}{z-i}\,dz$.

Note: $f(z)$ is differentiable for all $z \in \mathbb{C}$.

Case 1 (γ does not enclose i): Then by Cauchy's Theorem

$$\oint_\gamma \frac{f(z)}{z-i}\,dz = 0$$

Case 2 (γ encloses i): Then by Cauchy's Integral Formula

$$\oint_\gamma \frac{f(z)}{z-i}\,dz = 2\pi i(85e^{i\pi}) = -170\pi i.$$

Example 4.5.3

Let γ be any closed path that encloses both i or $4i$ and let $f(z) = 85e^{iz^{100}\pi}$. Evaluate $\displaystyle\oint_\gamma \frac{f(z)}{(z-i)(z-4i)}\,dz$.

Let γ_i be a small loop around i and γ_{4i} be a small loop around $4i$. From Cauchy's Theorem, we can write

$$\oint_\gamma \frac{f(z)}{(z-i)(z-4i)}\,dz = \oint_{\gamma_i} \frac{f(z)}{(z-i)(z-4i)}\,dz + \oint_{\gamma_{4i}} \frac{f(z)}{(z-i)(z-4i)}\,dz$$

Cauchy's Integral Formula only works if the denominator is a linear term, so the initial guess is that we'll have to do something like partial fractions. However, notice that $\dfrac{f(z)}{z-i}$ is analytic inside of γ_{4i} and $\dfrac{f(z)}{z-4i}$ is analytic inside γ_i. So wr can further rewrite

$$\oint_{\gamma_i} \frac{f(z)}{(z-i)(z-4i)}\,dz + \oint_{\gamma_{4i}} \frac{f(z)}{(z-i)(z-4i)}\,dz = \oint_{\gamma_i} \frac{f(z)/(z-4i)}{z-i}\,dz + \oint_{\gamma_{4i}} \frac{f(z)/(z-i)}{z-4i}\,dz$$

and now apply Cauchy's Integral Formula

$$= 2\pi i\left(\frac{f(i)}{i-4i}\right) + 2\pi i\left(\frac{f(4i)}{4i-i}\right)$$

$$= 2\pi i \left(\frac{85e^{i \cdot i^{100}\pi}}{-3i} + \frac{85e^{i\cdot(4i)^{100}\pi}}{3i} \right)$$

We can even cleverly use Cauchy's Integral Formula to evaluate some real integrals that would have made us cry in MATH 1226. (It might still make you cry now, but at least it can be solved without resorting to numerical techniques.)

Example 4.5.4

Evaluate $\displaystyle\int_0^{2\pi} e^{\cos(\theta)} \cos(\sin(\theta))\, d\theta$.

Before tackling this head-on, we first examine another contour integral. By Cauchy's Theorem, for any closed path γ that encloses 0,

$$\oint_\gamma \frac{e^z}{z}\, dz = 2\pi i e^0 = 2\pi i.$$

Letting $\gamma(\theta) = e^{i\theta}$ be the unit circle, we have

$$\begin{aligned}
\oint_\gamma \frac{e^z}{z}\, dz &= \int_0^{2\pi} \frac{e^{e^{i\theta}}}{e^{i\theta}} ie^{i\theta}\, d\theta = i\int_0^{2\pi} e^{e^{i\theta}}\, d\theta \\
&= i\int_0^{2\pi} e^{\cos\theta} e^{i\sin\theta}\, d\theta \\
&= i\int_0^{2\pi} e^{\cos\theta} \left(\cos(\sin\theta) + i\sin(\sin\theta)\right)\, d\theta \\
&= -\int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\, d\theta + i\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\, d\theta
\end{aligned}$$

Notice that the imaginary part of this integral is exactly what we set out to solve! So its value must be the same as the imaginary part of $2\pi i$!

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta = 2\pi.$$

Remark. You are not expected to just have the brilliant insight and cleverness to use contour integrals in that way; even WolframAlpha resorts to numerical techniques when given that integral. It's just really interesting to see that, with enough ingenuity, even some complicated real integrals can have deceptively simple values.

Theorem 4.5.5: Cauchy's Integral Formula for Derivatives

Let f, S, γ, and z_0 be as in Cauchy's Integral Formula. Then for any integer $n \geq 0$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

Proof. The proof is a bit tedious, but more-or-less comes down to applying Cauchy's integral formula to the definition of the derivative and simplifying the terms. □

Example 4.5.6

Evaluate $\displaystyle\oint_\gamma \frac{e^{z^3}}{(z-i)^3}\,dz$ where γ is a closed path that encloses i.

Let $f(z) = e^{z^3}$. Since γ encloses i and f is differentiable on all of $\mathbb{C}$, we can compute $f''(z)$ and then apply Cauchy's Integral Formula for Derivatives.

$$f'(z) = 3z^2 e^{z^3}, \qquad \text{and} \qquad f''(z) = (6z + 9z^4)e^{z^3}$$

hence

$$\oint_\gamma \frac{e^{z^3}}{(z-i)^3}\,dz = \frac{2\pi i}{2!} f''(i) = \pi i(6i+9)e^{-i}.$$

Not only does Cauchy's integral formula kind of feel like cheating, it actually has the following completely amazing consequence.

Corollary 4.5.7

Suppose f is complex analytic on an open set S. Then all derivatives f exist on S.

This is *very* different from the behavior in real analysis. For example. $f(x) = x^{2/3}$ is differentiable on all of $\mathbb{R}$, but the second derivative does not exist at $x = 0$.

5 Series Representations for Analytic Functions

5.1 Sequences and Series

We assume familiarity with real sequences and series.

Definition: Sequence

A **sequence** of complex numbers is an infinite collection $\{z_1, z_2, \ldots, \}$ where z_n is a complex number for every nonnegative integer n. We sometimes denote sequnces $\{z_n\}_{n=0}^{\infty}$ or just $\{z_n\}$.

Definition

Let $\{z_n\}$ be a sequence of complex numbers. For every z_n, there are real numbers x_n, y_n such that $z_n = x_n + iy_n$. The **limit** of $\{z_n\}$ is a complex number $L = a + ib$ where

$$\lim_{n\to\infty} x_n = a \qquad \text{and} \qquad \lim_{n\to\infty} y_n = b.$$

In this case we write $\lim\limits_{n\to\infty} z_n = L$ or possibly just $z_n \to L$.

Definition

Given a sequence of complex numbers $\{c_n\}$, a **series** is a sequence of partial sums

$$\sum_{n=0}^{\infty} c_n = \lim_{k\to\infty} \sum_{n=0}^{k} c_n.$$

If this limit exists, we say that the series **converges**.

Proposition 5.1.1

For each complex number c_n in the sequence $\{c_n\}$, write $c_n = a_n + ib_n$ for real a_n, b_n. Then $\sum_{n=0}^{\infty} c_n$ converges to a complex number $C = A + iB$ if and only if $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$.

So convergence of complex series is equivalent to asking about convergence of real series. Although the reader is assumed to be familiar with these convergence tests, we will state them again for complex series.

Theorem 5.1.2: Divergence Test

If $lim_{n\to\infty} z_n \neq 0$, then the series $\displaystyle\sum_{n=0}^{\infty} z_n$ diverges.

Theorem 5.1.3: Comparison Test

Suppose $\displaystyle\sum_{n=0}^{\infty} z_n$ is a series of complex numbers and $\displaystyle\sum_{n=0}^{\infty} M_n$ is a series of real numbers with $|z_n| \leq M_n$ for all n.

1. If $\displaystyle\sum_{n=0}^{\infty} M_n$ converges, then $\displaystyle\sum_{n=0}^{\infty} z_n$ converges.
2. If $\displaystyle\sum_{n=0}^{\infty} z_n$ diverges, then $\displaystyle\sum_{n=0}^{\infty} M_n$ diverges.

Remark. It may be worth noting that the Comparison Test for complex series is slightly different than expected. You may have initially wanted to compare two terms of a series, but given two complex numbers z and w, the inequality $z \leq w$ does not have a meaning (the fancy phrase is that "there is no partial ordering on $\mathbb{C}$ which respects the field structure".) The next best thing, which is how the theorem is stated, is to compare the magnitude of the terms of a real series.

Theorem 5.1.4: Ratio Test

Consider the series $\displaystyle\sum_{n=0}^{\infty} z_n$ and let $\displaystyle\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$.

1. If $L > 1$, $\displaystyle\sum_{n=0}^{\infty} z_n$ diverges.
2. If $L < 1$, $\displaystyle\sum_{n=0}^{\infty} z_n$ converges.

Theorem 5.1.5: Geometric Series

The series $\displaystyle\sum_{n=0}^{\infty} az^n$ converges if and only if $|z| < 1$.

Moreover, if $\displaystyle\sum_{n=0}^{\infty} az^n$ converges, then it converges to $\dfrac{a}{1-z}$.

Definition

A series of complex numbers $\sum_{n=0}^{\infty} c_n$ **converges absolutely** if the real series $\sum_{n=0}^{\infty} |c_n|$ converges.

5.2 Taylor Series

Definition

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots .$$

The complex numbers c_n are called the **coefficients** of the power series, and z_0 is called the **center** of the power series.

Just as for real power series, one fundamental question is about finding z-values for which the power series converges.

Remark. Just as in the real power series case, our goal is to think about the function $f(z) = \displaystyle\sum_{n=0}^{\infty} c_n(z-z_0)^n$. This function is only defined when the output is some finite number, i.e., for z-values where the series converges.

Theorem 5.2.1

Suppose $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges at $z_1 \neq z_0$ (that is, suppose $\sum_{n=0}^{\infty} c_n(z_1-z_0)^n$ is a convergent series). Then this series converges absolutely for all z-values satisfying

$$|z-z_0| < |z_1-z_0|.$$

Proof. Because $\displaystyle\sum_{n=0}^{\infty} c_n(z_1-z_0)^n$ converges,

$$\lim_{n\to\infty} c_n(z_1-z_0)^n = 0.$$

This means that we can find a sufficiently large N so that, for all $n \geq N$,

$$|c_n(z_1-z_0)^n| < 1.$$

As such, for all $n \geq N$,

$$|c_n(z-z_0)^n| = \frac{|(z_1-z_0)^n|}{|(z_1-z_0)^n|}\,|c_n(z-z_0)^n|$$

which rearranges to

$$|c_n(z-z_0)^n| = \frac{|(z-z_0)^n|}{|(z_1-z_0)^n|}|c_n(z_1-z_0)^n| \leq \left|\frac{(z-z_0)^n}{(z_1-z_0)^n}\right|(1) = \left|\frac{z-z_0}{z_1-z_0}\right|^n.$$

When $|z-z_0| < |z_1-z_0|$, then we have that $\left|\dfrac{z-z_0}{z_1-z_0}\right| < 1$, hence the geometric series

$$\sum_{n=1}^{\infty} \left|\frac{z-z_0}{z_1-z_0}\right|^n$$

converges by the Geometric Series Test (5.1.5). By the comparison test (5.1.3), it follows that the series

$$\sum_{n=0}^{\infty} |c_n(z - z_0)^n|$$

converges. As such, the following series converges absolutely:

$$\sum_{n=0}^{\infty} c_n(z - z_0)^n.$$

□

Letting $r = |z_1 - z_0|$, then the equation in the theorem has the form $|z - z_0| < r$, so geometrically, if the series converges on the boundary of a disk of radius r centered at z_0, then it converges absolutely on the interior of that disk.

Definition

The **radius of convergence** R, is the radius of the largest disk around z_0 on which the series $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ converges. The disk $|z - z_0| < R$ is called the **disk of convergence**.

Proposition 5.2.1. *For a given power series, the radius of convergence is unique, and the series diverges outside of this disk (i.e. for z-values satisfying $|z - z_0| > R$.*

Proof. The radius is unique by definition. The series must diverge outside of this disk, for if it didn't, then by Theorem 5.2.1, there would be a disk of larger radius on which the series converged. □

Fact. A power series $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ always converges at the center z_0.

If a power series converges *only* at the center, then we may write $R = 0$, and if it converges for all complex numbers, we write $R = \infty$.

As with real power series, we can sometimes compute the radius of convergence via the ratio test.

Example 5.2.2

Determine the radius of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{(-5)^n}{n+1}(z - i)^n$.

According to the ratio test, this series converges when

$$\begin{aligned} 1 &> \lim_{n\to\infty} \left| \frac{\dfrac{(-5)^{n+1}}{n+2}(z-i)^{n+1}}{\dfrac{(-5)^n}{n+1}(z-i)^n} \right| \\ &= \lim_{n\to\infty} \left| (-5)\left(\frac{n+1}{n+2}\right)(z-i) \right| \\ &= 5|z - i| \end{aligned}$$

$$\implies \qquad \frac{1}{5} > |z - i|.$$

So the radius of convergence for this power series is $R = \frac{1}{5}$.

Example 5.2.3

Determine the radius of convergence for $\displaystyle\sum_{n=0}^{\infty} n!\,(z-2+3i)^n$.

According to the ratio test, this series converges when

$$\begin{aligned} 1 &> \lim_{n\to\infty} \left| \frac{(n+1)!(z-2+3i)^{n+1}}{n!(z-2+3i)^n} \right| \\ &= \lim_{n\to\infty} n\,|(z-2+3i)|\,. \end{aligned}$$

When $z \neq 2-3i$ this series diverges, so it has radius of convergence $R = 0$.

Example 5.2.4

Determine the radius of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}\,(z+2)^n$.

According to the ratio test, this series converges when

$$\begin{aligned} 1 &> \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}(z+2)^{n+1}}{\frac{1}{n}(z+2)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(z+2)}{n+1} \right| \\ &= 0. \end{aligned}$$

The series converges for all $z \in \mathbb{C}$, So the radius of convergence for this power series is $R = \infty$.

The following theorem is analogous to the familiar version from real analysis.

Theorem 5.2.5: Differentiation and Integration of Power Series

Let f be the function given by

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

for z in D, the open disk of convergence.

1. f is complex differentiable with derivative given by

$$f'(z) = \sum_{n=0}^{\infty} \frac{d}{dz} c_n (z - z_0)^n = \sum_{n=1}^{\infty} n c_n (z - z_0)^{n-1} \qquad \text{for } z \in D.$$

   Moreover, the power series for $f'(z)$ has the same radius of convergence as f.

2. If γ is a path within D, then

$$\int_{\gamma} f(z)\, dz = \sum_{n=0}^{\infty} c_n \int_{\gamma} (z - z_0)^n\, dz.$$

Proof. The proof of this is actually longer and less straightforward than one might hope; we can't just quickly apply Cauchy–Riemann. That the derivative and integral are defined the way they are is obvious, but that the sequence of partial sums still converges to the appropriate limit (and with the same disk of convergence) is technical and involves the notion of uniform convergence, which we wont be covering. □

Theorem 5.2.6: Taylor Expansion

Suppose f is differentiable on an open disk D of radius R centered at z_0. Then, for $z \in D$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where

$$c_n = \frac{f^{(n)}(z_0)}{n!}$$

Proof. See the text. The strategy is effectively

- Apply Cauchy's Integral Formula to rewrite $f(z)$ as a contour integral.
- With clever algebraic manipulations, recognize the integrand as the limit of a convergent geometric series
- Integrate this series using 5.2.5.
- Use Cauchy's Integral Formula for Derivatives to rewrite the coefficients of this series.

□

Definition

The series in Theorem 5.2.6 is called the **Taylor series** of f about z_0 (or **Maclaurin series** in the case that $z_0 = 0$). The coefficients are called the **Taylor coefficients** of f at z_0.

Example 5.2.7

Since $\frac{d}{dz}[e^z] = e^z$, just as in the real case, the Maclaurin expansion of e^z should also look like the Maclaurin series for e^x.

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

Example 5.2.8

Since $\frac{d}{dz}[\sin(z)] = \cos(z)$ and $\frac{d}{dz}[\cos(z)] = -\sin(z)$, just as in the real case, the Maclaurin expansions of $\sin(z)$ and $\cos(z)$ should also look like the Maclaurin series for $\sin(x)$ and $\cos(x)$ (respectively).

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1},$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.$$

5.3 Power Series

5.5 Laurent Series

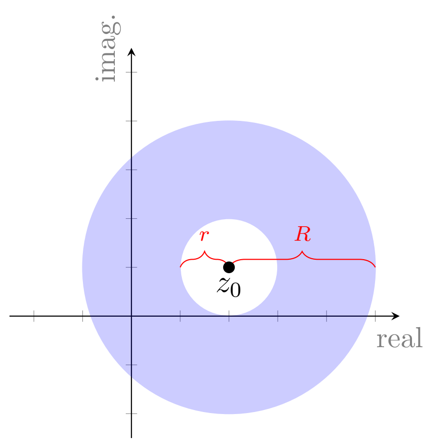
If f is differentiable in a small disk around z_0, then it has a Taylor series expansion at z_0. If f isn't differentiable at z_0 (but is maybe differentiable *near* z_0) it would still be nice to get a series expansion of f centered at z_0.

Definition

An **annulus** is the open set between two concentric circles, and can be written as the set of all z satisfying

$$r < |z - z_0| < R.$$

with $r \geq 0$ and $R \leq \infty$. If $r = 0$ and $R < \infty$, we call this a **punctured disk**. If $r = 0$ and $R = \infty$ we call this a **punctured plane**.



Suppose f is analytic at z_0. Then we can find a disk (the disk of convergence) on which f has a Taylor series expansion $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ where

$$c_n = \frac{f^{(n)}(z_0)}{n!}$$

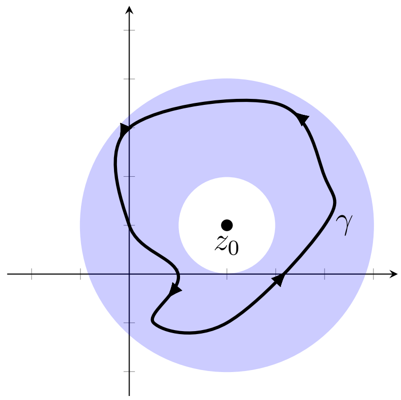
If we draw a closed contour γ within this disk that encloses z_0, then applying Cauchy's Integral Formula for Derivatives, we can rewrite c_n as

$$c_n = \frac{\frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz}{n!} = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

There are a couple of observations one can make about these coefficients:

- Unlike the normal Taylor coefficients, f is not actually required to be defined at z_0.
- The lack of factorial means that n can be negative.

As such, we could reasonably define a series like a Taylor series when f is differentiable in an annulus centered at z_0 so long as γ is entirely within that annulus.



Theorem 5.5.1: Laurent Expansion

Suppose f is differentiable in the annulus $r < |z - z_0| < R$ where $0 \le r < R \le \infty$. Then for each z in this annulus, we have

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where, for each integer n,

$$c_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

for any closed path γ in the annulus enclosing z_0.

Definition

A series as in Theorem 5.5.1 is called a **Laurent series**.

Example 5.5.2

Find a Laurent series expansion for $f(z) = e^{1/z}$ around $z_0 = 0$.

Firing up our trusty computer algebra system (since nobody wants to compute these by hand), one can confirm the following:

$$c_n = \begin{cases} \frac{1}{(-n)!} & \text{if } n \le 0 \\ 0 & \text{if } n > 0 \end{cases}$$

so we have that

$$\begin{aligned} e^{1/z} &= \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n \\ &= \cdots \frac{1}{n!} \cdot \frac{1}{z^n} + \cdots + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{1!} \cdot \frac{1}{z} + 1 \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n. \end{aligned}$$

We notice that the Laurent series obtained for $e^{1/z}$ looks a lot like the Taylor series for e^z with the modification that $z \mapsto \frac{1}{z}$.

Proposition 5.5.3

Given a function f that is differentiable in an annulus centered at z_0, the Laurent expansion about z_0 is unique.

This is great because it means that, no matter how we obtain a Laurent series for our function, it must be the correct one. So often times we'll just manipulate known Taylor series to obtain the correct Laurent expansion (because nobody wants to compute those coefficients by hand if they can help it).

Example 5.5.4

Compute the Laurent series expansion for $f(z) = \dfrac{1}{z-2}$ on the annulus $|z-1| > 1$ (i.e. the punctured plane).

Notice that this annulus is centered at 1, so our Laurent series will be as well. Notice also that this annulus excludes $z = 2$, so in fact $f(z)$ is differentiable on it. Recall that the geometric series $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r| < 1$. Since $|z-1| > 1$, we must have that $\left|\dfrac{1}{z-1}\right| < 1$. This suggests that we want our geometric series' ratio to be $\frac{1}{z-1}$, hence

$$\begin{aligned}
f(z) = \frac{1}{z-2} &= \frac{1}{(z-1)-1} \\
&= \left(\frac{1}{z-1}\right)\frac{1}{1-\frac{1}{z-1}} \\
&= \left(\frac{1}{z-1}\right)\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n \\
&= \frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \cdots \\
&= \sum_{n=1}^{\infty}\frac{1}{(z-1)^n} \\
&= \sum_{n=-\infty}^{-1}(z-1)^n.
\end{aligned}$$

Example 5.5.5

Compute the Laurent series expansion for $f(z) = \dfrac{2z-1-i}{(z-1)(z-i)}$ about $z = 1$.

Clearly f is not differentiable at 1 or i, so whatever our annulus is must avoid these two points. Using partial fractions

$$\frac{2z-1-i}{(z-1)(z-i)} = \frac{1}{z-1} + \frac{1}{z-i}. \tag{5.5.1}$$

The first term is already a Laurent series about $z = 1$ (it only has one term) and its defined on the annulus $|z-1| > 0$, so we focus only on the second term and aim to use the geometric series again. We can rewrite

$$\frac{1}{z-i} = \frac{1}{(z-1)+(1-i)}. \tag{5.5.2}$$

At this points, we have a couple of options for approach: we can factor out $\dfrac{1}{z-1}$ or we can factor out $\dfrac{1}{1-i}$. In both cases we'll work with the geometric series, but the resulting annuli will be different.

(Case 1) We rewrite Equation 5.5.2 as

$$\frac{1}{z-i} = \frac{1}{(z-1)+(1-i)} = \left(\frac{1}{z-1}\right)\frac{1}{1-\frac{i-1}{z-1}}$$

and assuming $\left|\frac{1-i}{z-1}\right| < 1$, from the geometric series this becomes

$$\begin{aligned}
\left(\frac{1}{z-1}\right)\frac{1}{1-\frac{i-1}{z-1}} &= \left(\frac{1}{z-1}\right)\sum_{n=0}^{\infty}\left(\frac{i-1}{z-1}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(i-1)^n}{(z-1)^{n+1}} && \text{(setting } k = -n-1\text{)} \\
&= \sum_{k=-\infty}^{-1}\frac{1}{(1-1)^{k+1}}(z-1)^k.
\end{aligned}$$

Substituting this into the Equation 5.5.1 we get the Laurent expansion for f

$$f(z) = \frac{1}{z-1} + \sum_{k=-\infty}^{-1}\frac{1}{(1-1)^{k+1}}(z-1)^k$$

for z-values satisfying $|z-1| > 0$ and $\left|\dfrac{1-i}{z-1}\right| < 1$, i.e. on the annulus $|z-1| > \sqrt{2}$.

(Case 2) We rewrite Equation 5.5.2 as

$$\frac{1}{z-i} = \frac{1}{(z-1)+(1-i)} = \left(\frac{1}{1-i}\right)\frac{1}{1-\frac{z-1}{i-1}}$$

and assuming $\left|\dfrac{z-1}{1-i}\right| < 1$, for the geometric series this becomes

$$\begin{aligned}\left(\frac{1}{1-i}\right)\frac{1}{1-\frac{z-1}{i-1}} &= \left(\frac{1}{1-i}\right)\sum_{n=0}^{\infty}\left(\frac{z-1}{i-1}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{-1}{(i-1)^{n+1}}(z-1)^n.\end{aligned}$$

Substituting this into the Equation 5.5.1 we get the Lauren expansion for f

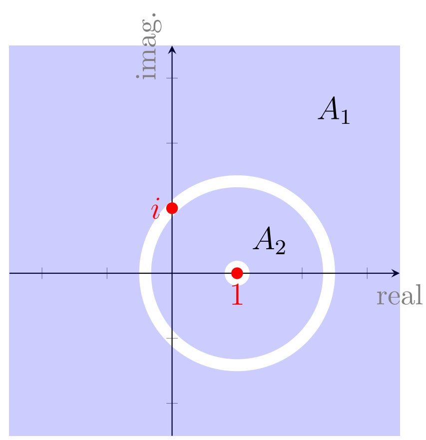
$$f(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty}\frac{-1}{(i-1)^{n+1}}(z-1)^n$$

for z-values satisfying $|z-1| > 0$ and $\left|\dfrac{z-1}{1-i}\right| < 1$, i.e. on the annulus $0 < |z-1| < \sqrt{2}$.

Remark. There's not really a cohesive way to write these series succinctly in the form $\displaystyle\sum_{n=-\infty}^{\infty} c_n(z-1)^n$ because the coefficients don't all follow a nice pattern. That's fine. The same is true of Taylor series of real functions. For example, the Taylor expansion of $1+\cos(x)$ about $x_0 = 0$ is

$$1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}.$$

Remark. The $\sqrt{2}$ in the annulus isn't too surprising when you think about it geometrically - that's the distance between 1 and i. The fact that we had two possible annuli is then not surprising at all - having an annulus centered at $z = 1$, there were only two possible options for which f could be analytic within the whole annulus (either it avoided all singularities, or it avoided one and not the other). Visually, letting A_1 and A_2 be the annuli in cases 1 and 2, respectively,



5.6 Zeros and Singularities

5.6.1 Isolated Zeros

Definition

For a function f, a number z_0 for which $f(z_0) = 0$ is called an **isolated zero** if there is an open disk around z_0 which contains no other zero for f.

Example 5.6.1

The function $f(z) = \sin(z)$ has an isolated zero at $z = 0$.

Given what we know about the real sine function, the fact that $z = 0$ is an isolated zero for $f(z) = \sin(z)$ certainly *seems* reasonable, but how do we know it's actually the case for the complex sine function? It turns out Taylor series can provide the answer.

Let z_0 be a zero for f and consider the Taylor expansion of f in a small disk D around around z_0

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

If every $c_n = 0$, then $f(z) = 0$ for all $z \in D$, so suppose this isn't the case. Let m be the first value for which $c_m \neq 0$ (that is, $c_0 = c_1 = \ldots = c_{m-1} = 0$). Then we have that

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} c_n (z - z_0)^n \\
&= \sum_{n=m}^{\infty} c_n (z - z_0)^n && \text{(first terms all 0)} \\
&= \sum_{k=0}^{\infty} c_{k+m} (z - z_0)^{k+m} && \text{(where } k = n - m\text{)} \\
&= (z - z_0)^m \sum_{k=0}^{\infty} c_{k+m} (z - z_0)^k
\end{aligned}$$

Now let

$$g(z) = \sum_{k=0}^{\infty} c_{k+m} (z - z_0)^k = c_m + \sum_{k=1}^{\infty} c_{k+m} (z - z_0)^k.$$

By construction we have that $g(z_0) = c_m \neq 0$ and

$$f(z) = (z - z_0)^m g(z).$$

Since f is differentiable on D, then so is g; in particular, g is continuous at z_0, so since $g(z_0) \neq 0$, then there is a small disk D_g around z_0 on which $g(z) \neq 0$ (otherwise we would break the intermediate value theorem). It follows that $f(z) \neq 0$ on this same disk as well, making z_0 an isolated zero. So what this says is that

Theorem 5.6.2

Suppose f is differentiable on a domain S, and let $z_0 \in S$ be a zero of f. Then either

$$f(z) = 0 \text{ on all of } S,$$

or

$$\zeta \text{ is an isolated zero.}$$

From the theorem and the proof preceding it, what we have is that, if $f(z)$ has a nonzero Taylor coefficient in the series centered at z_0, then z_0 is an isolated zero.

Example 5.6.3

Show that $f(z) = \sin(z)$ has an isolated zero at $z = 0$.

By the above commentary, we just need to examine the Taylor coefficients of f centered at 0.

$$c_0 = \frac{f(0)}{0!} = \frac{\sin(0)}{1} = 0$$
$$c_1 = \frac{f'(0)}{1!} = \frac{\cos(0)}{1} = 1$$

Since $c_1 \neq 0$, then $z = 0$ must be isolated.

Definition

A point z_0 is said to be a **zero of order** m if f is differentiable at z_0 and the first nonzero coefficient in the Taylor expansion around z_0 is c_m. If z_0 is a zero of order 1, it is sometimes called a **simple zero**.

Example 5.6.4

For $f(z) = \sin(z)$, determine the order of the zero $z = 0$.

$z = 0$ has order 1 per our work in Example 5.6.3.

While proving Theorem 5.6.2, we actually proved the following, but we'll state explicitly.

Proposition 5.6.5

If f is differentiable at z_0, then z_0 is a zero of order m if and only if we can write

$$f(z) = (z - z_0)^m g(z).$$

where $g(z_0) \neq 0$ and g is differentiable at z_0.

Example 5.6.6

Find the order of the zero $z_0 = 0$ of the function $\varphi(z) = \sin^3(z)$.

Looking at the Taylor expansion of $\sin^3(z)$ about $z_0 = 0$, we have

$$\begin{aligned}\sin^3(z) = (\sin(z))^3 &= \left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)^3 \\ &= z^3 - \frac{1}{2}z^5 + \frac{13}{120}z^7 - \frac{41}{3024}z^9 + \cdots \\ &= z^3\left(1 - \frac{1}{2}z^2 + \frac{13}{120}z^4 - \frac{41}{3024}z^6 + \cdots\right)\end{aligned}$$

so taking $g(z) = 1 - \frac{1}{2}z^2 + \frac{13}{120}z^4 - \frac{41}{3024}z^6 + \cdots$, we have that

$$\sin^3(z) = z^3 g(z)$$

with $g(0) \neq 0$, hence $\varphi(z) = \sin^3(z)$ has a zero of order 3 at $z_0 = 0$.

Since we can write $f(z) = (z - z_0)^n g(z)$ with $g(z_0) \neq 0$, we get the following

Corollary 5.6.7

Suppose z_0 is a zero of order m of $h(z)$, and that z_0 is a zero of order n of $k(z)$. Then

1. At z_0, $h(z)k(z)$ has a zero of order $m + n$.
2. If $m > n$, then at z_0, $h(z)/k(z)$ has a zero of order $m - n$.

Proof sketch. Write

$$\begin{aligned}h(z) &= (z - z_0)^m \alpha(z) \\ k(z) &= (z - z_0)^n \beta(z)\end{aligned}$$

Then

$$h(z)k(z) = (z - z_0)^{m+n}\alpha(z)\beta(z)$$

and

$$\frac{h(z)}{k(z)} = \frac{(z - z_0)^m \alpha(z)}{(z - z_0)^n \beta(z)} = (z - z_0)^{m-n}\frac{\alpha(z)}{\beta(z)}$$

□

Remark. The term "zero" in the second item of Corollary 5.6.7 is maybe a bit misleading, because $h(z)/k(z)$ is not even defined at z_0 (and as such, is certainly not 0). The requirement that $m > n$ implies that the limit $L = \lim_{z \to z_0} h(z)/k(z)$ exists. So what we're actually thinking of is a zero of a *continuous extension* (or an *analytic continuation* of $h(z)/k(z)$ at z_0. Explicitly, for z in a small disk around z_0 where $k(z) \neq 0$, we are looking at a zero of the function

$$\widetilde{(h/k)}(z) := \begin{cases} h(z)/k(z) & \text{when } z \neq z_0, \\ L & \text{when } z = z_0. \end{cases}$$

Remark. Some authors may write $h(z)/k(z)$ to refer to the maximal analytic continuation of the quotient of $h(z)$ and $k(z)$. I will not be adopting this convention, but it is out there.

Example 5.6.8

Find the order of the zero $z_0 = 0$ of the function $f(z) = z^2 \sin^2(z)$.

By the previous theorem, it suffices to find the orders of z^2 and $\sin^2(z)$ independently and add them together.

$h(z) = z^2$		$k(z) = \sin^2(z)$	
n	$h^{(n)}(z_0)$	n	$k^{(n)}(z_0)$
1	$2(0) = 0$	1	$2\sin(0)\cos(0) = 0$
2	2	2	$2\cos^2(0) - 2\sin^2(0) = 2$

Since z_0 is a zero of order 2 for z^2 and order 2 for $\sin^2(z)$, then z_0 is a zero of order $2+2=4$ for f.

Example 5.6.9

Find the order of the zero $z_0 = \frac{3\pi}{2}$ of the function $f(z) = \dfrac{\cos^3(z)}{z - \frac{3\pi}{2}}$.

By the previous theorem, it suffices to find the orders of $\cos^3(z)$ and $z - \frac{3\pi}{2}$ independently and add them together.

$h(z) = z - \frac{3\pi}{2}$		$k(z) = \cos^3(z)$	
n	$h^{(n)}(z_0)$	n	$k^{(n)}(z_0)$
1	1	1	$-3\sin\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{2}\right) = 0$
		2	$6\sin^2\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{2}\right) - 3\cos^3\left(\frac{3\pi}{2}\right) = 0$
		3	$21\sin\left(\frac{3\pi}{2}\right)\cos^2\left(\frac{3\pi}{2}\right) - 6\sin^3\left(\frac{3\pi}{2}\right) = 6$

Since z_0 is a zero of order 1 for $z - \frac{3\pi}{2}$ and order 3 for $\cos^3(z)$, then z_0 is a zero of order $3-1=2$ for f.

5.6.2 Poles and Singularities

In this section we'll use the Laurent expansion to find and classify points at which complex functions are not differentiable.

Definition: Isolated Singularity

We say that a function f has an **isolated singularity** at z_0 if f is differentiable in an annulus $0 < |z - z_0| < R$, but not at z_0.

Example 5.6.10

$f(z) = \dfrac{1}{z}$ has an isolated singularity at $z = 0$.

5.6.3 Classification of Singularities

Definition: Poles of Order m

Suppose f has an isolated singularity at z_0. Let the Laurent expasion of $f(z)$ in a punctured disk around z_0 be

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

If the smallest n-value for which $c_n \neq 0$ is...

1. ... $n \geq 0$, then z_0 is a **removable singularity**.
2. ... $-\infty < n < 0$, then z_0 is a **pole of order** $|n|$ (and in particular, when $n = -1$, z_0 is a **simple pole**).
3. ... $n = \infty$, then z_0 is an **essential singularity**.

In other words,

1. z_0 is removable if the Laurent expansion is actually a power series.
2. z_0 is a pole of order m if $\dfrac{1}{(z-z_0)^m}$ is the largest power of $\dfrac{1}{z-z_0}$ appearing in the Laurent expansion.
3. z_0 is essential if the Laurent expansion contains infinitely many powers of $\dfrac{1}{z-z_0}$ with nonzero coefficients.

Example 5.6.11

Find and classify all poles of $f(z) = \dfrac{\sin(z)}{z}$.

f is analytic on all of $\mathbb{C}$ except at $z = 0$ (where it isn't defined). The Laurent expansion of f around $z_0 = 0$ is

$$\begin{aligned}
\frac{1}{z}\sin(z) &= \frac{1}{z}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n} \\
&= 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \frac{1}{7!}z^6 + \cdots
\end{aligned}$$

Since this is a power series about $z_0 = 0$, $z_0 = 0$ is a removable singularity. As such we can extend f to a function φ which is differentiable at $z_0 = 0$:

$$\varphi(z) = \begin{cases} f(z) & \text{when } z \neq 0 \\ 1 & \text{when } z = 0 \end{cases}$$

Definition: Analytic Continuation

The function φ above is called the **analytic continuation** of f at z_0.

Example 5.6.12

Find and classify all poles of $f(z) = \dfrac{1}{(z-i)^5}$.

f has an isolated singularity at $z_0 = i$, which is a pole of order 5. To see this, note that the Laurent expansion of f around $z_0 = i$ is

$$\frac{1}{(z-i)^5} \quad (f \text{ is its own Laurent expansion}).$$

There is no way that f can be extended to be differentiable at $z_0 = i$.

Example 5.6.13

Consider the function $f(z) = \dfrac{\cos(z)}{z^4}$, which is analytic on all of $\mathbb{C}$ except at $z_0 = 0$ where it isn't defined. Classify the singularity $z_0 = 0$.

The Laurent expansion of f about $z_0 = 0$ is

$$\begin{aligned} \frac{1}{z^4}\cos(z) &= \frac{1}{z^4}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n-4} \\ &= \frac{1}{z^4} - \frac{1}{2!z^2} + \frac{1}{4!} - \cdots \end{aligned}$$

so $f(z)$ has a pole of order 4, and thus it cannot be extended to be differentiable at $z_0 = 0$.

Example 5.6.14

Let $f(z) = e^{1/z}$, which is analytic on all of $\mathbb{C}$ except at $z_0 = 0$ where it isn't defined. Classify the singularity $z_0 = 0$.

Since the Laurent expansion of $f(z) = e^{1/z}$ about $z_0 = 0$ is

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n} = \cdots + \frac{1}{3!z^3} + \frac{1}{2!z^2} + \frac{1}{z} + 1.$$

f has an essential singularity at $z_0 = 0$ because infinitely many powers of $\frac{1}{z}$ appear in this Laurent expansion.

Let us consider the Laurent expansion of some function f in an annulus $0 < |z - z_0| < R$:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n.$$

If f has a pole of order m at z_0, then $c_{-m} \neq 0$ but $c_{-m-1} = c_{-m-2} = \cdots = 0$, so the Laurent expansion is

$$\begin{aligned} f(z) &= \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m-1}}{(z-z_0)^{m+1}} + \frac{c_{-m-2}}{(z-z_0)^{m+2}} + \cdots \\ (z-z_0)^m f(z) &= c_{-m} + c_{-m-1}(z-z_0) + c_{-m-2}(z - z_0^2 + \cdots \end{aligned}$$

and so

$$\lim_{z \to z_0} (z - z_0)^m f(z) = c_{-m} \neq 0.$$

As it turns out, the existence of this limit is enough to deduce that a function has a pole of order m at z_0. Explicitly,

Theorem 5.6.15

Suppose f is differentiable in $0 < |z - z_0| < R$. Then f has a pole of order m at z_0 if and only if

$$\lim_{z \to z_0} (z - z_0)^m f(z)$$

exists and is nonzero.

5.6.4 Zeroes and Poles Together

When looking for poles, it seems natural to look for places where the denominator is zero, especially if $f(z) = g(z)/h(z)$ is a quotient of functions.

Lemma 5.6.16

A function f has a pole of order m at z_0 if and only if, in some annulus $0 < |z - z_0| < R$,

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

where g is differentiable at z_0 and $g(z_0) \neq 0$.

Proof. If f has a pole of order m, then its Laurent expansion about z_0 is

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \cdots$$
$$(z-z_0)^m f(z) = c_{-m} + c_{-m+1}(z-z_0) + c_{-m+2}(z-z_0)^2 + \cdots$$

so writing

$$g(z) = c_{-m} + c_{-m+1}(z-z_0) + c_{-m+2}(z-z_0)^2 + \cdots$$

we have that $g(z_0) \neq 0$ and g is defined by its Taylor expansion about z_0 (whence it is differentiable). □

Theorem 5.6.17

Let $f(z) = g(z)/h(z)$ where g, h are analytic in some open disk about z_0. Suppose that z_0 is a zero of order m for g and a zero of order n for h with $n > m$. Then f has a pole of of order $n - m$ at z_0.

Proof. From Proposition **??**,we can write

$$g(z) = (z-z_0)^m \tilde{g}(z)$$
$$h(z) = (z-z_0)^n \tilde{h}(z)$$

where $\tilde{g}$ and $\tilde{h}$ are differentiable and nonzero at z_0. It follows that $\tilde{g}/\tilde{h}$ is differentiable and nonzero at z_0 and

$$\frac{g(z)}{h(z)} = \frac{\tilde{g}(z)/\tilde{h}(z)}{(z-z_0)^{n-m}}$$

thus, by the above lemma, $f(z) = g(z)/h(z)$ has a pole of order $n - m$ at z_0. □

Remark. One could adopt the convention that $f(z)$ has a zero of order 0 (and likewise, a pole of order 0) if f is defined at z and if $f(z) \neq 0$. This is not standard to my knowledge, but the calculations do align with the theorem.

Example 5.6.18

Find the order of the pole of $f(z) = \dfrac{e^z - 1}{\sin^7(z)}$ at $z_0 = 0$.

The motivated student could compute the Laurent expansion of f about 0 directly to get

$$f(z) = \frac{1}{z^6} + \frac{1}{2}\frac{1}{z^5} + \frac{4}{3}\frac{1}{z^4} + \cdots$$

in which case it is quickly seen that f has a pole of order 6 at $z_0 = 0$. Rather than do this, by looking at the Maclaurin series, it is straightforward to see that $e^z - 1$ has a zero of order 1

$$e^z - 1 = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots$$

and that $\sin^7(z)$ has a zero of order 7

$$\begin{aligned}\sin(z) &= z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots \\ \sin^7(z) &= \left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)^7 \\ &= z^7 - \frac{7}{6}z^9 + \frac{77}{120}z^{11} + \cdots\end{aligned}$$

so by the theorem, $f(z)$ has a pole of order $7 - 1 = 6$ at $z_0 = 0$.

Example 5.6.19

Find all poles of $f(z) = \frac{1}{\left(z-\frac{\pi}{2}\right)^3 \cos^4(z)}$ and their orders.

$\cos^4(z)$ has a zero of order 4 at all odd multiples of $\frac{\pi}{2}$, and $\left(z - \frac{\pi}{2}\right)^3$ has a zero of order 3 at $\frac{\pi}{2}$. So f has a pole of order 7 at $\frac{\pi}{2}$ and a pole of order 4 at all other odd multiples of $\frac{\pi}{2}$.

6 Residue Theory

6.1 The Residue Theorem

The aim of this section is to explore the relationship with singularities, the Laurent expansion, and integrals.

Suppose f is differentiable in the annulus $0 < |z - z_0| < R$ and has an isolated singularity at z_0. Let γ be a closed path in this annulus that encloses z_0. The Laurent expansion for f in this annulus is

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where the coefficients c_n are given by

$$c_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

for all integers n. Notice that when $n = -1$, the $\frac{1}{z - z_0}$ term in the series has coefficient

$$c_{-1} = \frac{1}{2\pi i} \oint_\gamma f(z) \, dz$$

which rearranges to

$$\oint_\gamma f(z) \, dz = 2\pi i c_{-1}.$$

This means that finding that one single coefficient is all that we need to evaluate the integral! Magic!

Definition: residue

The coefficient of $\frac{1}{z - z_0}$ in the Laurent expansion of f about z_0 is called the **residue** of f at z_0 and is denoted $\operatorname{Res}(f, z_0)$.

What we have is that

$$\oint_\gamma f(z) \, dz = 2\pi i \operatorname{Res}(f, z_0),$$

but what if γ enclosed multiple isolated singularities $z_1, \ldots, z_n$?

Around each singularity z_k, we can find a small loop γ_k so that none of the γ_k's intersect and none of the γ_k's enclose any other singularity. By the extended deformation theorem,

$$\oint_\gamma f(z) \, dz = \sum_{k=1}^{n} \oint_{\gamma_k} f(z) \, dz$$

and since each of the integrals on the right can be written in terms of the corresponding residues, the following result is an immediate consequence:

Theorem 6.1.1: Residue Theorem

Let γ be a closed path. Suppose f is differentiable on γ and all points enclosed by γ, exceot for $z_1, \ldots, z_n$ which are all of the isolated singularities of f enclosed by γ. Then

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$

In this way, we see that computing an integral is as efficient as our ability to evaluate these residues. Obviously computing Laurent series by hand is a little time consuming, so we want to find a faster way to obtain the residue.

Proposition 6.1.2

If f has a simple pole at z_0, then

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Proof. Since f has a simple pole at z_0, then the Laurent expansion of f in some annulus about z_0 is

$$f(z) = \frac{c_{-1}}{z - z_0} + \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

As such

$$(z - z_0) f(z) = c_{-1} + \sum_{n=0}^{\infty} c_n (z - z_0)^{n+1}$$

and therefore

$$\lim_{z \to z_0} (z - z_0) f(z) = c_{-1}.$$

□

Example 6.1.3

Evaluate $\displaystyle\oint_\gamma \frac{\sin(z)}{z^2}\, dz$ where γ is any closed path enclosing $z_0 = 0$.

Since $z_0 = 0$ is a zero of order 1 for $\sin(z)$ and a zero of order 2 for z^2, then z_0 is a simple pole for $f(z)$ by Theorem 5.6.17. By the preceding theorem, we have that

$$\operatorname{Res}(f, 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{\sin(z)}{z} = 1.$$

Since $z_0 = 0$ is the only singularity of $\dfrac{\sin(z)}{z^2}$, we have

$$\oint_\gamma \frac{\sin(z)}{z^2}\,dz = 2\pi i \operatorname{Res}(f, 0) = 2\pi i.$$

Corollary 6.1.4

Let $f(z) = \dfrac{h(z)}{g(z)}$ where h is continuous at z_0 and $h(z_0) \neq 0$. Suppose g is differentiable at z_0 and has a simple zero there. Then f has a simple pole at z_0, and

$$\operatorname{Res}(f, z_0) = \frac{h(z_0)}{g'(z_0)}.$$

Proof. Since $g(z_0) = 0$ and z_0 is a simple pole for f, then

$$\begin{aligned}
\operatorname{Res}(f, z_0) &= \lim_{z \to z_0} (z - z_0) f(z) \\
&= \lim_{z \to z_0} (z - z_0) \frac{h(z)}{g(z)} \\
&= \lim_{z \to z_0} (z - z_0) \frac{h(z)}{g(z) - g(z_0)} \\
&= \lim_{z \to z_0} \frac{h(z)}{\frac{g(z) - g(z_0)}{z - z_0}} \\
&= \lim_{z \to z_0} \frac{h(z_0)}{g'(z_0)}.
\end{aligned}$$

□

Example 6.1.5

Evaluate $\displaystyle\oint_\gamma f(z)\,dz$ where $f(z) = \dfrac{10 - 2iz}{\cos(z)}$ and γ is the circle $\left|z - \frac{\pi}{2}\right| = 1$.

Let $h(z) = 10 - 2iz$ and $g(z) = \cos(z)$. Then $f(z) = \dfrac{h(z)}{g(z)}$ has a simple pole at $z_0 = \frac{\pi}{2}$ and $g\left(\frac{\pi}{2}\right) = 0$. By the corollary,

$$\operatorname{Res}\left(f, \frac{\pi}{2}\right) = \lim_{z \to \pi/2} \frac{10 - 2iz}{-\sin(z)} = \frac{10 - i\pi}{-1} = i\pi - 10.$$

Hence

$$\oint_\gamma \frac{10 - 2iz}{\cos(z)}\,dz = 2\pi i \operatorname{Res}\left(f, \frac{\pi}{2}\right) = 2\pi i(i\pi - 10).$$

What if we want to compute $\operatorname{Res}(f, z_0)$ when z_0 is a pole of order m for f? In an annular neighborhood about z_0, f has the Laurent expansion

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \frac{c_{-m+2}}{(z-z_0)^{m-2}} + \frac{c_{-m+3}}{(z-z_0)^{m-3}} + \cdots$$

Thus

$$(z-z_0)^m f(z) = c_{-m} + c_{-m+1}(z-z_0) + \cdots + c_{-1}(z-z_0)^{m-1} + \cdots .$$

By differentiating $m-1$ times, we obtain

$$\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] = (m-1)!\, c_{-1} + m!\, c_0(z-z_0) + \cdots .$$

and taking the limit as $z \to z_0$, the right-hand side reduces to

$$(m-1)!\, c_{-1}.$$

This yields the following

Theorem 6.1.6: Residue at a Pole of Order m

Let f have a pole of order m at z_0. Then

$$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right].$$

Remark. Note that Proposition 6.1.2 is the special case of this theorem when $m = 1$.

Example 6.1.7

Evaluate $\oint_\gamma f(z)\, dz$ where $f(z) = \dfrac{2iz - \cos(z)}{z(z-i)^3}$ and γ is any closed path that encloses 0 and i.

0 is a simple pole of f, so

$$\operatorname{Res}(f, 0) = \lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{2iz - \cos(z)}{(z-i)^3} = \frac{-1}{i} = i.$$

i is a pole of order 3, so

$$\begin{aligned}
\operatorname{Res}(f, i) &= \frac{1}{(3-1)!} \lim_{z\to i} \frac{d^{3-1}}{dz^{3-1}}\left[(z-i)^3 f(z)\right] \\
&= \frac{1}{2} \lim_{z\to i} \frac{d^2}{dz^2}\left[\frac{2iz - \cos(z)}{z}\right] \\
&= \frac{1}{2} \lim_{z\to i} \frac{(z^2-2)\cos(z) - 2z\sin(z)}{z^3} \\
&= \frac{1}{2} \frac{-3\cos(i) - 2i\sin(i)}{-i} \\
&= \sin(i) - \frac{3}{2} i \cos(i).
\end{aligned}$$

So, by the Residue Theorem 6.1.1,

$$\begin{aligned}\oint_\gamma \frac{2iz-\cos(z)}{z(z-i)^3}\,dz &= 2\pi i(\operatorname{Res}(f,0)+\operatorname{Res}(f,i))\\ &= 2\pi i\left(i+\sin(i)-\frac{3}{2}i\cos(i)\right) = -2\pi+3\cos(i)+2\pi i\sin(i).\end{aligned}$$

6.2 Trigonometric Integrals over $[0, 2\pi]$

6.2.1 Rational Functions of Cosine and Sine

Let $K(x,y)$ be a rational function of x and y (i.e. a quotient of multivariate polynomials in x and y). For example,

$$K(x,y) = \frac{x^3y + 6y^2 - 7xy + 2x}{y^3 - x^2 + y - 9x}.$$

We are interested in evaluating integrals of the form

$$\int_0^{2\pi} K(\cos\theta, \sin\theta)\, d\theta.$$

Let γ be the unit circle about the origin, which we parameterize as $\gamma(\theta) = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$. On this curve, $z = e^{i\theta}$ and $\overline{z} = e^{-i\theta} = \frac{1}{z}$, so we can write

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{and} \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

What's more, we have

$$dz = ie^{i\theta}\, d\theta = iz\, d\theta \qquad \Longrightarrow \qquad d\theta = \frac{1}{iz}\, dz$$

and so

$$\oint_\Gamma K\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)\frac{1}{iz}\, dz = \int_0^{2\pi} K(\cos(\theta), \sin(\theta))\, d\theta. \tag{6.2.1}$$

As such, we can interpret the integral on the right as the contour integral on the left, and use the Residue Theorem to solve it.

Example 6.2.1

Evaluate $\int_0^{2\pi} \frac{1}{\alpha + \beta\cos(\theta)}\, d\theta$ with $0 < \beta < \alpha$.

With $x = \cos\theta$ and $y = \sin\theta$, the integrand can be thought of as the rational function

$$K(x,y) = \frac{1}{\alpha + \beta x}.$$

Converting into z-coordinates as in Equation 6.2.1, our integrand becomes

$$\begin{aligned} f(z) &= K\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)\frac{1}{iz} \\ &= \frac{1}{\alpha + \frac{\beta}{2}\left(z + \frac{1}{z}\right)}\frac{1}{iz} \\ &= \frac{-2i}{2\alpha z + \beta z^2 + \beta}. \end{aligned}$$

From the quadratic formula, we get that the poles are

$$z = \frac{-2\alpha \pm \sqrt{\alpha^2 - \beta^2}}{\beta}$$

both of which are real because of our assumption that $\alpha > \beta$, however, only

$$z_0 = \frac{-2\alpha + \sqrt{\alpha^2 - \beta^2}}{\beta}$$

is contained within the unit circle. Therefore

$$\begin{aligned}\int_0^{2\pi} \frac{1}{\alpha + \beta\cos(\theta)}\,d\theta &= 2\pi i \operatorname{Res}(f, z_0)\\ &= 2\pi i \frac{-2i}{2\alpha + 2\beta z_0} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2}}.\end{aligned}$$

6.3 Improper Integrals of Rational Functions over $(-\infty, \infty)$

6.3.1 Preliminary

Recall from your first integral calculus course that

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx + \lim_{c\to\infty}\int_b^c f(x)\,dx$$

where c is any real number. That is, evaluating an integral over $(-\infty, \infty)$ required you to compute two limits, and the integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ **converged** if and only if both of those limits exist.

Naively, one could have also tried to evaluate the single limit

$$\lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\,dx$$

and it turns out that if $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ converges in the classical sense, then the value one obtains with this new limit is the same (this is a classical, though nontrivial fact). However, if the integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ does not converge, then it's still possible that this single limit exists.

Example 6.3.1

Since

$$\int_0^{\infty} x\,dx = \infty$$

then the integral $\displaystyle\int_{-\infty}^{\infty} x\,dx$ diverges. However,

$$\lim_{\rho\to\infty}\int_{-\rho}^{\rho} x\,dx = \lim_{\rho\to\infty}\frac{\rho^2}{2} - \frac{(-\rho)^2}{2} = \lim_{\rho\to\infty} 0 = 0.$$

Definition

The **(Cauchy) principal value** of the integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ is

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\,dx$$

Evaluating a single limit is often easier, so provided we have some insight into whether or not our integral converges, this technique is as good as any.

6.3.2 Evaluating Real Integrals

Suppose we're trying to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx$$

where p, q are polynomials. To ensure that this integral converges, we'll assume that $\deg(q) \geq \deg(p) + 2$ and that q has no real roots. For simplicity, we'll also assume that p and q have no common roots (so the fraction is fully reduced).

Since q has no real roots and is a polynomial with real coefficients, all of its roots are complex and come in conjugate pairs, $z_1, \overline{z_1}, \ldots, z_n, \overline{z_n}$. Without loss of generality, assume that all of the z_k's live in the upper half plane (i.e. satisfy $\operatorname{Im}(z_k) > 0$)), and thus all of the $\overline{z}_k$'s live in the lower half plane.

For a positive real number R, let γ_R be the upper semi-circle from R to $-R$ and let S_R be the line segment from $-R$ to R. Let Γ_R be loop formed from these two segments, and take R taken large enough that Γ_R encloses all of $z_1, \ldots, z_n$.

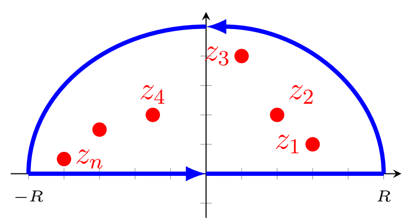


Figure 6.3.1: Γ is the loop formed from the line segment from $-R$ to R and the upper half circle from R to $-R$.

The z_i's are all of the poles of $f(z) = \dfrac{p(z)}{q(z)}$ in the upper half plane, so by the Residue Theorem

$$\int_{S_R} \frac{p(z)}{q(z)}\, dz + \int_{\gamma_R} \frac{p(z)}{q(z)}\, dz = \oint_{\Gamma_R} \frac{p(z)}{q(z)}\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_n).$$

Since $\operatorname{Im}(z) = 0$ for all z on the segment S_R, the equation above can be rewritten

$$\int_{-R}^{R} \frac{p(x)}{q(x)}\, dx + \int_{\gamma_R} \frac{p(z)}{q(z)}\, dz = \oint_{\Gamma_R} \frac{p(z)}{q(z)}\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_n). \tag{6.3.1}$$

We'll state as a fact the following result (which is the complex analog of a familiar result from calculus):

Fact. If $p(z), q(z)$ are polynomials with $\deg(q) \geq \deg(p)$, then the limit as $|z| \to \infty$ of $\frac{p(z)}{q(z)}$ exists.

Since $\deg(q) \geq \deg(p) + 2$ in our case, then we must have that $z^2 p(z)/q(z)$ is bounded for $|z| \geq R$, say

$$\left|\frac{z^2 p(z)}{q(z)}\right| = |z^2| \left|\frac{p(z)}{q(z)}\right| \leq M \qquad \Longrightarrow \qquad \left|\frac{p(z)}{q(z)}\right| \leq \frac{M}{|z|^2}.$$

Since γ_R has length πR, then it follows from Theorem 4.2.4 that

$$\left|\int_{\gamma_R} \frac{p(z)}{q(z)}\, dz\right| \leq \frac{M}{|z|^2}(\pi R) \leq \frac{M}{R^2}(\pi R)$$

and so when we take the limit as $R \to \infty$, Equation 6.3.1 becomes

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx + 0 = 2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(\frac{p(z)}{q(z)}, z_n\right)$$

Example 6.3.2

Compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx$.

Approaching the old-fashioned way,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx &= \lim_{a\to\infty} \int_{-a}^{a} \frac{1}{x^2+1}\, dx \\ &= \lim_{a\to\infty} \arctan(a) - \arctan(-a) \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi.\end{aligned}$$

Approaching the new way, we see that $f(z) = \dfrac{p(z)}{q(z)} = \dfrac{1}{z^2+1}$ has a simple pole at $z = -i$ and a simple pole at $z = i$. Only i lies in the upper half plane, so we compute the residue

$$\operatorname{Res}(f, i) = \lim_{z\to i} (z-i)\left(\frac{1}{(z-i)(z+i)}\right) = \frac{1}{2i}.$$

and from our work above,

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx = 2\pi i \operatorname{Res}(f, i) = \frac{2\pi i}{2i} = \pi.$$

Example 6.3.3

Compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+4}\, dx$.

Writing

$$f(z) = \frac{p(z)}{q(z)} = \frac{1}{z^4+4}$$

We see that $f(z)$ has simple poles at $4^{1/4}e^{\pi i/4}$, $4^{1/4}e^{3\pi i/4}$, $4^{1/4}e^{5\pi i/4}$, and $4^{1/4}e^{7\pi i/4}$. Only two of these have positive imaginary parts, and they are

$$\begin{aligned} z_1 &= 4^{1/4}e^{\pi i/4} = \sqrt{2}\left(\frac{1+i}{\sqrt{2}}\right) = 1+i \\ z_2 &= 4^{1/4}e^{3\pi i/4} = \sqrt{2}\left(\frac{-1+i}{\sqrt{2}}\right) = -1+i\end{aligned}$$

Computing residues, we have

$$\begin{aligned}
\operatorname{Res}(f, z_1) &= \lim_{z \to 1+i} (z-(1+i)) \left(\frac{1}{(z-(1+i))(z-(-1+i))(z-(1-i))(z-(-1-i))} \right) \\
&= \frac{1}{-8+8i} = -\frac{1+i}{16} \\
\operatorname{Res}(f, z_2) &= \lim_{z \to -1+i} (z-(-1+i)) \left(\frac{1}{(z-(1+i))(z-(-1+i))(z-(1-i))(z-(-1-i))} \right) \\
&= \frac{1}{8+8i} = \frac{1-i}{16}
\end{aligned}$$

whence

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{x^4+4}\, dx &= 2\pi i \left(\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2) \right) \\
&= 2\pi i \left(-\frac{1+i}{16} + \frac{1-i}{16} \right) = \frac{\pi}{4}
\end{aligned}$$

Example 6.3.4

Compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx$.

Writing

$$f(z) = \frac{p(z)}{q(z)} = \frac{1}{(z^2+1)^2}$$

We see that $f(z)$ has double poles at $\pm i$, but only i lies in the upper half plane, so we compute the residue

$$\begin{aligned}
\operatorname{Res}(f, i) &= \lim_{z \to i} \frac{d}{dz} \left[(z-i)^2 \frac{1}{(z-i)^2(z+i)^2} \right] \\
&= \lim_{z \to i} \frac{d}{dz} \left[\frac{1}{(z+i)^2} \right] \\
&= \lim_{z \to i} \frac{-2}{(z+i)^3} = \frac{-2}{(2i)^3} = \frac{1}{4i} = -\frac{i}{4}
\end{aligned}$$

whence

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx &= 2\pi i \operatorname{Res}(f, i) \\
&= 2\pi i \left(-\frac{i}{4} \right) = \frac{\pi}{2}
\end{aligned}$$

6.4 Improper Integrals Involving Trig Functions

With p,q,Γ_R,γ_R, and S_R as in the previous subsection, we consider, for a positive real number c, the integral

$$\oint_{\Gamma_R} \frac{p(z)}{q(z)} e^{icz}\, dz$$

By the Residue Theorem,

$$\begin{aligned}
2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(\frac{p(z)}{q(z)} e^{icz}, z_k\right) &= \oint_{\Gamma_R} \frac{p(z)}{q(z)} e^{icz}\, dz \\
&= \int_{\gamma_R} \frac{p(z)}{q(z)} e^{icz}\, dz + \int_{S_R} \frac{p(z)}{q(z)} e^{icz}\, dz \\
&= \int_{\gamma_R} \frac{p(z)}{q(z)} e^{icz}\, dz + \int_{-R}^{R} \frac{p(x)}{q(x)} \cos(x)\, dx + i \int_{-R}^{R} \frac{p(x)}{q(x)} \sin(x)\, dx
\end{aligned}$$

Since $|e^{icz}| = 1$, then the same bound argument holds as in the previous subsection. And so, setting $\lim_{R\to\infty}$, the integral over γ_R tends to 0 and we are left with

$$\int_{-R}^{R} \frac{p(x)}{q(x)} \cos(x)\, dx + i \int_{-R}^{R} \frac{p(x)}{q(x)} \sin(x)\, dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(\frac{p(z)}{q(z)} e^{icz}, z_k\right).$$

As such, but comparing real and imaginary parts, we can solve either of the real integrals on the right via residues.

Example 6.4.1

Let α, β, c be positive real numbers. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(cx)}{(x^2+\alpha^2)(x^2+\beta^2)}\, dx.$

Let $f(z) = \dfrac{e^{icz}}{(z^2+\alpha^2)(z^2+\beta^2)}$, which has simple poles at $x = \pm\alpha i, \pm\beta i$. Computing the residues for the poles in the upper half plane

$$\begin{aligned}
\operatorname{Res}(f, \alpha i) &= \frac{e^{-c\alpha}}{2\alpha i(-\alpha^2+\beta^2)} \\
\operatorname{Res}(f, \beta i) &= \frac{e^{-c\beta}}{2\beta i(-\alpha^2+\beta^2)}
\end{aligned}$$

We thus have that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(cx)}{(x^2+\alpha^2)(x^2+\beta^2)}\, dx &= \operatorname{Im}\left(2\pi i \left[\frac{e^{-c\alpha}}{2\alpha i(-\alpha^2+\beta^2)} + \frac{e^{-c\beta}}{2\beta i(-\alpha^2+\beta^2)}\right]\right) \\
&= \operatorname{Im}\left(\pi \frac{e^{-c\alpha}}{\alpha(-\alpha^2+\beta^2)} + \pi \frac{e^{-c\beta}}{\beta(\alpha^2-\beta^2)}\right) = 0.
\end{aligned}$$

That it's zero isn't too surprising - the integrand is an odd function.

Example 6.4.2

Let α, β, c be positive real numbers. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(cx)}{(x^2+\alpha^2)(x^2+\beta^2)}\,dx$.

From the work we did in the previous example,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(cx)}{(x^2+\alpha^2)(x^2+\beta^2)}\,dx &= \mathrm{Re}\left(2\pi i \left[\frac{e^{-c\alpha}}{2\alpha i(-\alpha^2+\beta^2)} + \frac{e^{-c\beta}}{2\beta i(-\alpha^2+\beta^2)}\right]\right) \\
&= \mathrm{Re}\left(\pi \frac{e^{-c\alpha}}{\alpha(-\alpha^2+\beta^2)} + \pi \frac{e^{-c\beta}}{\beta(\alpha^2-\beta^2)}\right) \\
&= \pi \frac{e^{-c\alpha}}{\alpha(-\alpha^2+\beta^2)} + \pi \frac{e^{-c\beta}}{\beta(\alpha^2-\beta^2)}.
\end{aligned}$$

7 Conformal Mapping

7.2 Geometric Considerations

7.2.1 Construction of Conformal Mappings

One strategy for solving problems is to find the solution on a simple domain (a disk, half-plane, etc) and to use a conformal mapping to pass between them. The following result tells us that this is always possible.

Theorem 7.2.1: Riemann Mapping Theorem

Let D_0 be the unit disk and D_1 any domain in $\mathbb{C}$ (that is not all of $\mathbb{C}$). Then there exists a conformal mapping $f : D_0 \to D_1$ that is both one-to-one and onto.

This major theorem tells us that we can pass between any two domains. Let D_0 be the unit disk and let D_1, D_2 be any two domains (that aren't all of $\mathbb{C}$). Then there exist one-to-one and onto conformal mappings $f_1 : D_0 \to D_1$ and $f_2 : D_0 \to D_2$. Since each of these maps is invertible, we get that $f_2 \circ f_1^{-1}$ is a one-to-one and onto conformal mapping.

Of course, FINDING these conformal maps in practice is generally very hard. Since a conformal mapping must send the boundary of D_1 to the boundary of D_2, one strategy is to try finding a map between the boundaries, and then test to see if the interior points are mapped to interior points. We'll explore this idea in the context of fractional linear transformations.

Example 7.2.2

Find a conformal mapping from the open unit disk to the disk $|z| < 3$.

Clearly the map

$$M(z) = 3z$$

sends the unit circle to the circle of radius 3. If $|z| < 1$, then

$$|M(z)| = |3z| = 3|z| < 3,$$

as desired.

Example 7.2.3

Find a conformal mapping from the open unit disk to the exterior of the disk $|z| > 3$

We know that inversion preserves the circle $|z| = 1$ and that dilation maps this circle conformally onto the circle $|z| = 3$, so we try composing the two

$$M(z) = 3\left(\frac{1}{z}\right).$$

Suppose that $|z| < 1$, then $|1/z| > 1$ and

$$M(z) = \left|\frac{3}{z}\right| = 3\left|\frac{1}{z}\right| > 3,$$

as desired.

Example 7.2.4

Find a conformal mapping from the open unit disk to the open disk $|z-1| < 3$

We know $M(z) = 3z$ takes the unit disk to the disk of radius 3, so we try composing it with a translation by 1

$$M(z) = (3z) + 1.$$

Indeed, if $|z| < 1$, then

$$|M(z) - 1| = |3z + 1 - 1| = |3z| = 3|z| < 3,$$

as desired.

Example 7.2.5

Find a conformal mapping from the right half-plane to the unit disk.

Let's try the conformal mapping f for which

$$\begin{aligned} f(i) &= 1 \\ f(0) &= i \\ f(-i) &= -1 \end{aligned}$$

As in the proof of Theorem 7.3.9, we can look for two fractional linear transformations on $\hat{\mathbb{C}}$ that send our points to 0, 1, and ∞ and then compose them appropriately. Let

$$M_1(z) = \frac{(z-i)(0+i)}{(z+i)(0-i)} = \frac{iz+1}{-iz+1}, \text{ and } \qquad M_2(z) = \frac{(z-1)(i+1)}{(z+1)(i-1)} = \frac{(1+i)z-(1+i)}{(-1+i)z+(-1+i)}.$$

Now we have that

$$\begin{Bmatrix} i \\ 0 \\ -i \end{Bmatrix} \xrightarrow{M_1} \begin{Bmatrix} 0 \\ 1 \\ \infty \end{Bmatrix} \xrightarrow{M_2^{-1}} \begin{Bmatrix} 1 \\ i \\ -1 \end{Bmatrix}$$

and thus can pick

$$f(z) = (M_2^{-1} \circ M_1)(z) = \frac{iz-i}{-z-1}$$

Now we have mapped the boundary of the right half-plane to the boundary of the disk, so it remains to check that the interiors map accordingly. Checking

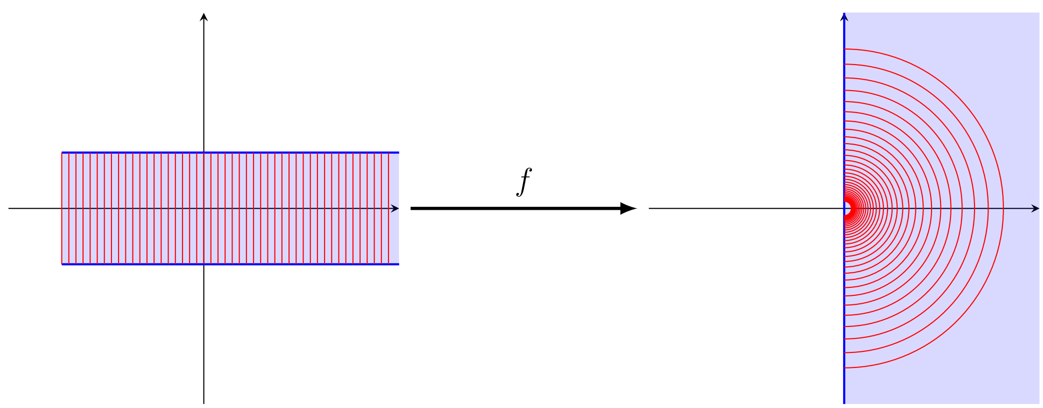
$$f(1) = \frac{i-i}{-1-1} = 0$$

which is in the interior of the disk.

Example 7.2.6

Find a conformal map sending the infinite strip $-\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2}$ to the right half plane

Clearly this cannot be a fractional linear transformation, because there are two boundary lines in the infinite strip and fractional linear transformations send lines to lines (or circles). So instead we look back at the first motivational map we used. Recall that $f(z) = e^z$ sends horizontal lines to rays from the origin (although it technically excludes the origin, this doesn't matter because we're not including the boundary in our map). If $z = x + i\frac{\pi}{2}$ then $f(z) = ie^x$ and if $z = x - i\frac{\pi}{2}$ then $f(z) = -ie^x$, so the boundary of the strip is sent to the imaginary axis (minus the origin). It's quick to check that $f(0) = e^0 = 1$ which is in the right half of the plane.



The complex exponential "opens up" the infinite strip $-\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2}$ like a book into right half plane $\mathrm{Re}(z) > 0$

7.3 Móbius Transformations

Given a set S and a function $f : \mathbb{C} \to \mathbb{C}$, we will write $f(S)$ to be the set of all points $f(z)$ where $z \in S$.

Consider the map $f(z) = e^z$. For any real number s, let V_a be the vertical line consisting of points $z \in \mathbb{C}$ for which $\mathrm{Re}(z) = s$. Similarly, for any real number t let H_t be the horizontal line consisting of points $z \in \mathbb{C}$ for which $\mathrm{Im}(z) = t$.

Let $w \in f(V_a)$. Then $|w| = |e^z| = e^{\mathrm{Re}(z)} = e^a$, which is a circle of radius e^a. So $f(V_a)$ is a circle about the origin, and if $b > a$, then $f(V_b)$ is a larger circle centered at the origin.

Since $e^z = e^{\mathrm{Re}(z)} e^{i\,\mathrm{Im}(z)}$, it follows that every $w \in f(H_c)$ has fixed argument c, hence $f(H_c)$ is a ray from the origin (minus the origin) at an angle of c from the positive real axis.

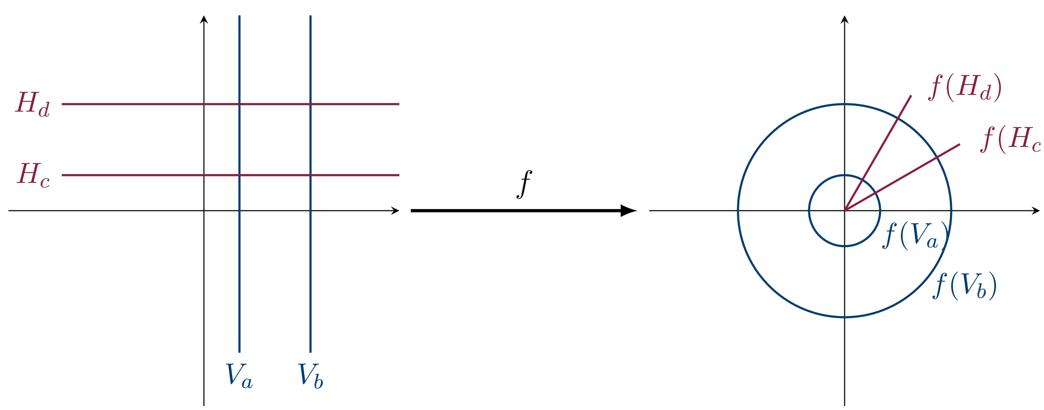
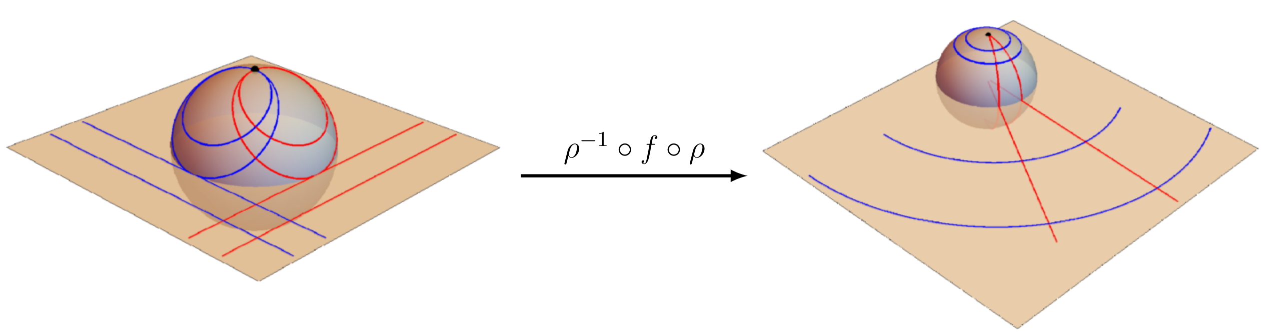
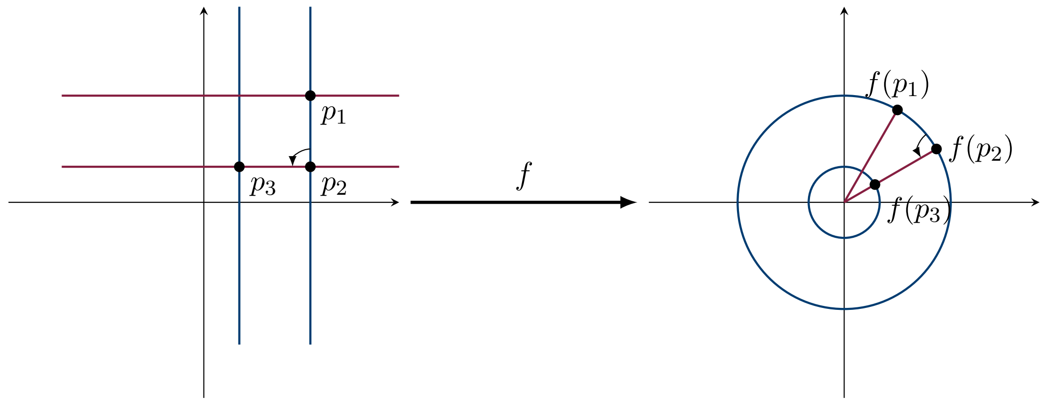


Figure 7.3.1: Not-to-scale image of the grid after applying the map $z \mapsto e^z$

For funsies, here's what it all looks like on the sphere.



Notice that $f(z) = e^z$ preserves angles (the right angles in the square grid get sent to right angles on the circular/curvy grid), but it also preserves *orientation*.



Definition

A function that preserves both angles and orientations on a domain is said to be **conformal** on this domain. Such a function is usually called a **conformal mapping**.

Theorem 7.3.1: Conformal Mappings

Let D_1, D_2 be domains and $f : D_1 \to D_2$. Suppose f is differentiable on D_1 and that $f'(z) \neq 0$ for all $z \in D_1$. Then f is a conformal mapping.

Proof. Let $p \in D_1$ and let $\gamma(t)$ be a smooth curve in D_1 for which $\gamma(0) = p$. Let $\Gamma(t) = f(\gamma(t))$ be the image of γ under f. The tangent vectors of γ and Γ at p and $f(p)$ (respectively) are related by the chain rule

$$\Gamma'(0) = f'(p)\gamma'(0)$$

so applying a differentiable function to any curve through the point p has the effect of multiplying the tangent vector to that curve by the complex number $f'(p) = re^{i\theta}$. Since all curves through p get multiplied by the same complex number, their arguments are all also changed by the same angle θ. $\square$

Proposition 7.3.2

If $f : D_1 \to D_2$ is conformal and one-to-one on D_1 (i.e. f^{-1} is well-defined), then $f^{-1} : D_2 \to D_1$ is conformal.

Proposition 7.3.3

If $f : D_1 \to D_2$ and $g : D_2 \to D_3$ are conformal, then so is the composition $(g \circ f) : D_1 \to D_3$.

Given two domains D_1 and D_2, we may want to construct a conformal mapping $f : D_1 \to D_2$, but this can be very difficult. However, there is s relatively simple class of conformal mappings that we can use for convenient domains (disks, half-planes, etc.)

7.3.1 Fractional Linear Transformations

Definition

A **fractional linear transformation** (or **Möbius transformation**) is a map of the form

$$M(z) = \frac{az+b}{cz+d}$$

where a, b, c, d are constants and $ad - bc \neq 0$. The latter condition ensures the transformation is invertible, and its inverse is given by the fractional linear transformation

$$M^{-1}(z) = \frac{dz-b}{-cz+a}.$$

Remark. Although your book calls these *bilinear mappings*, I don't think this term is quite as common anymore outside of maybe algebraic geometry (when considering special cases of birational maps). Furthermore, "bilinear" is used in other areas of math as well and the meanings do not overlap, so we'll avoid this terminology.

Let A be the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We can use this matrix to encode the fractional linear transformation with the following notation

$$A \bullet z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet z := \frac{az+b}{cz+d} = M(z).$$

Moreover, the invertibility condition on the fractional linear transformation is exactly the requirement that $\det(A) \neq 0$, i.e., that A is invertible!

Exercise 7.3.4

Let M_1, M_2 be fractional linear transformations and let A_1, A_2 be invertible 2×2 matrices so that $A_1 \bullet z = M_1(z)$ and $A_2 \bullet z = M_2(z)$. Show that

1. $M_1^{-1}(z) = A_1^{-1} \bullet z$ and
2. $(M_2 \circ M_1)(z) = (A_2 A_1) \bullet z$.

Because of the parallels, it becomes very convenient to encode fractional linear transformations into a matrix form. What's more, this connection allows us to freely pass between studying an algebraic object (the set of all invertible 2×2 matrices) and studying a geometric object (functions on the complex plane). This is the subject for another class, but is generally the motivation behind the area of "geometric group theory."

Let's look at the properties of some specific types of fractional linear trasnformations.

Definition

A transformation of the form $M(z) = z + b$ is called a **translation**. It translates z by $\mathrm{Re}(b)$-units horizontally and Im(b)-units vertically. The associated matrix A is the unipotent matrix

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

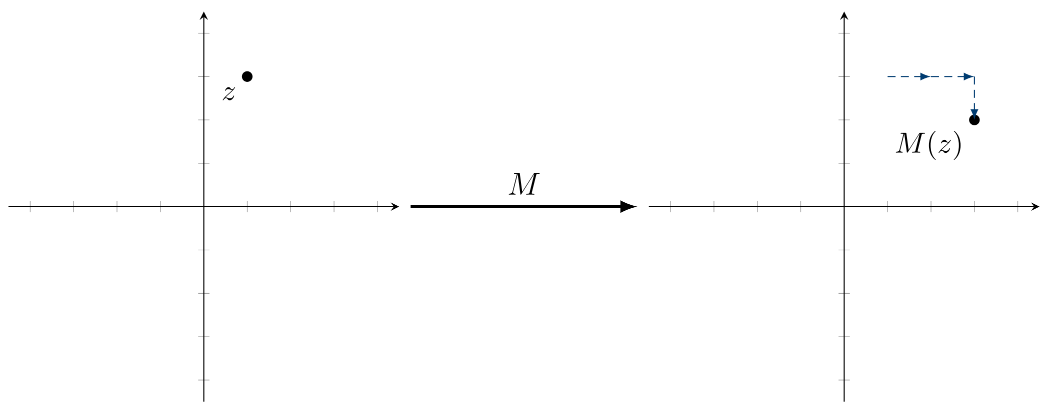
Remark. Technically the matrix could be any invertible matrix of the form

$$\begin{pmatrix} \alpha & \alpha b \\ 0 & \alpha \end{pmatrix},$$

but these are all conjugate, so we take the natural choice having determinant 1.

Example 7.3.5

Consider $M(z) = z + 2 - i$.



Definition

A transformation of the form $M(z) = az$ is called a **(pure) rotation** when $|a| = 1$, and a **(pure) dilation** when a is real. When a is neither of those, then writing it in polar for as $a = re^{i\theta}$ makes it clear that it's a composition of a rotation of angle θ and a dilation with factor r. In any case, the corresponding matrix is the diagonal matrix

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

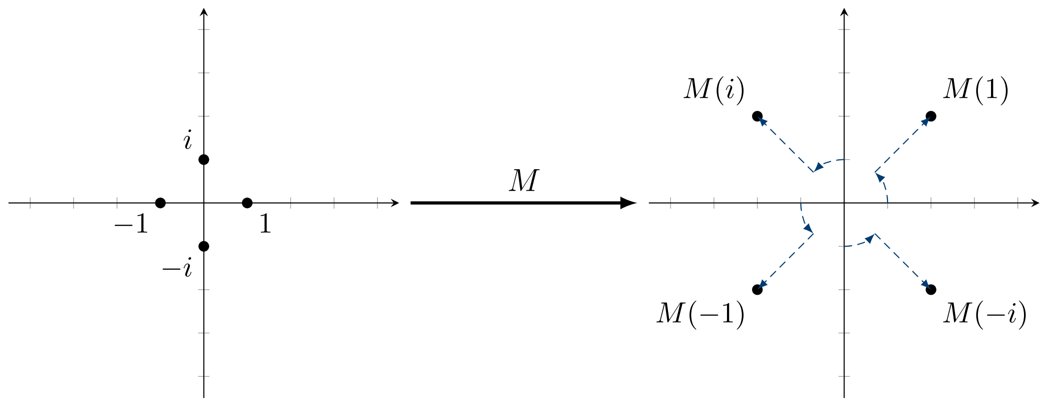
Remark. As before, it's natural to pick the determinant-1 matrix

$$\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}$$

where $\sqrt{a}$ is fixed to be one of $a^{1/2}$.

Example 7.3.6

Consider $M(z) = (2 + 2i)z$. Since $2 + 2i = \sqrt{8}e^{i\pi/4}$, we can picture this as a rotation throuh angle $\pi/4$ and then a dilation with a factor of $\sqrt{8}$.



Definition

A transformation of the form $M(z) = \dfrac{1}{z}$ is called an **inversion**. The associated matrix is the order-2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

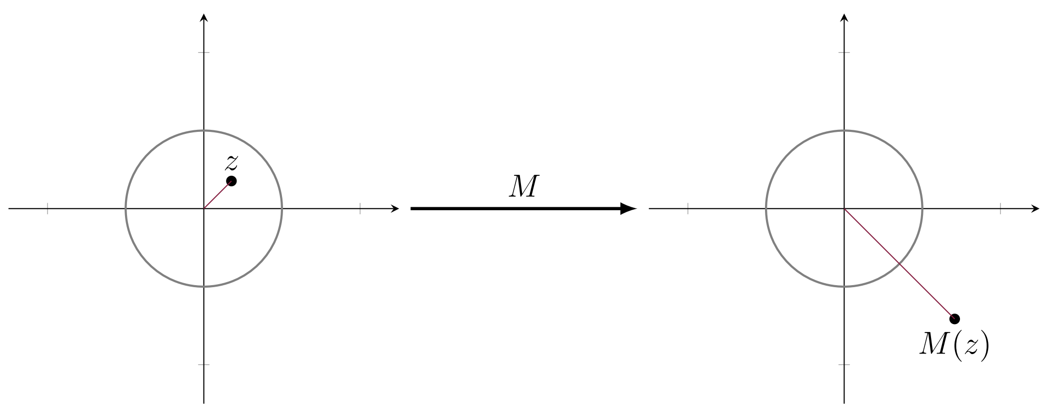
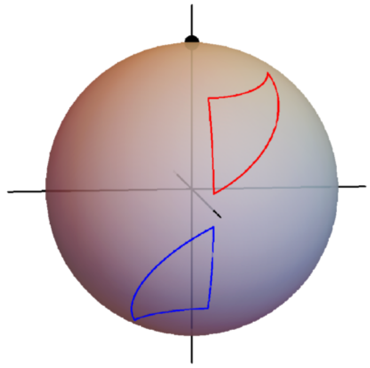


Figure 7.3.2: $\arg(M(z)) = -\arg(z)$ and $|M(z)| = |1/z|$

Remark. The associated determinant-1 matrix is

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Inversion is possibly the least obvious of these transformations, but if we use stereographic projeciton to see what's happening on the sphere, it's more intuitive. Let Δ be a triangle in the plane and let Δ' the inversion of Δ. Using ρ^{-1} to visualize on the sphere, we see that we can pass between $\rho^{-1}(\Delta)$ and $\rho^{-1}(\Delta')$ by a rotation of the sphere around the x-axis by an angle π (which makes sense, the inversion map fixes ± 1 and sends every point $e^{i\theta}$ to $e^{-i\theta}$).



Letting R be the rotation of the sphere around the x-axis by an angle of π, we can thus think of invertion as the composition

$$\rho \circ R \circ \rho^{-1}.$$

Theorem 7.3.7

All fractional linear transformations $M(z) = \dfrac{az+b}{cz+d}$ are a composition of these three types of transformations.

Proof. When $c = 0$, $M(z)$ is an actual linear transformation, which is clearly a rotation/dilation followed by a translation. Since compositions of fractional linear transformations can be represened by products of the corresponding matrices, when $c \neq 0$, we have

$$\underbrace{\begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix}}_{\text{translate}} \underbrace{\begin{pmatrix} \frac{bc-ad}{c} & 0 \\ 0 & 1 \end{pmatrix}}_{\text{rotate/dilate}} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{invert}} \underbrace{\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}}_{\text{translate}} \underbrace{\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}}_{\text{rotate/dilate}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

□

Theorem 7.3.8

Fractional linear transformations send lines to lines or circles, and send circles to lines or circles.

Proof. It's completely obvious that rotations, dilations, and transformations take lines to lines and circles to circles, so it only remains to check inversion. Given that inversion is a composition of stereographic projection (which has the desired property) and a rotation of the sphere (which sends circles on the sphere to circles on the sphere), the inversion has the desired property. □

Theorem 7.3.9: Three point theorem

Given any three points z_1, z_2, z_3 in $\mathbb{C}$, any other three points w_1, w_2, w_3 in $\mathbb{C}$, there is a unique fractional linear transformation M for which $M(z_1) = w_1$, $M(z_2) = w_2$, and $M(z_3) = w_3$.

Proof. Let M_1 and M_2 be the following fractional linear transformations:

$$M_1(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \qquad \text{and} \qquad M_2(z) = \frac{(z - w_1)(w_2 - w_3)}{(z - w_3)(w_2 - w_1)}.$$

As a functions on the extended complex plane,

$$\begin{aligned} M_1(z_1) &= 0, & \qquad M_2(w_1) &= 0, \\ M_1(z_2) &= 1, & \qquad M_2(w_2) &= 1, \\ M_1(z_3) &= \infty, & \qquad M_2(w_3) &= \infty. \end{aligned}$$

By setting $M = M_2^{-1} \circ M_1$, we have the desired fractional linear transformation. □

We'll see in the next section how we can use the above technique to pass between domains.

7.4 Móbius Transformations, Continued

7.5 The Schwarz-Christoffel Transformation

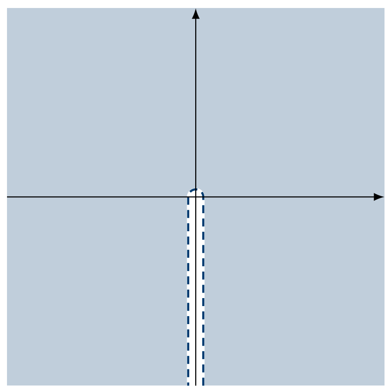
In the proof of Theorem 7.3.1, the crucial intuition was that the derivative of the conformal map could be thought of as a rotation of the plane at a point. As such, we can cook up the derivative of a function that bending the real line into a polygon (and thus the upper half plane into the interior of a polygon), and the conformal mapping will exactly the be integral of this function.

What follows is formally known as a branch cut, but we'll avoid the greater discussion surrounding them and introduce only the salient features.

Writing $z = |z|e^{i\theta}$, let $L : \mathbb{C} \to \mathbb{C}$ be the function defined by

$$L(z) = \ln|z| + i(\theta + 2k\pi)$$

where k is chosen so that $-\frac{\pi}{2} \leq \theta < \frac{3\pi}{2}$. In this way $L(z)$ is defined to pick out a single value of $\log(z)$.
Fact. $L(z)$ is analytic on $\mathbb{C}$ - $\{it \ : \ -\infty < t \leq 0\}$ (that is, the whole complex plane except for the nonpositive imaginary axis).



We're ultimately interested in the upper half plane, so we'll always take our complex numbers to have arguments between 0 and π. For some angle α with $0 < \alpha < \pi$, define the function

$$g_\alpha(z) := e^{-\alpha L(z)}$$

It follows from the chain rule that

Corollary 7.5.1

For each α as above, g_α is analytic on the same set as L.

Notice that, since $L(z)$ is just a specific value of the logarithm, we must have that

$$g_\alpha(z) = z^\alpha$$

for any z where $L(z)$ is differentiable. So provided we're willing to accept this slight abuse of notation and the restricted domain, we can think of the map $z \mapsto z^\alpha$ as an analytic function. Since the upper half plane is contained in the domain of analyticity, then z^α is analytic on the upper half plane

Let's look at what happens along the real axis (minus 0) with this mapping. Since

$$x = \begin{cases} |x|e^{i\pi} & \text{when } x < 0 \\ |x| & \text{when } x > 0 \end{cases}$$

then

$$x^\alpha = \begin{cases} |x|^\alpha e^{i\alpha\pi} & \text{when } x < 0 \\ |x|^\alpha & \text{when } x > 0 \end{cases}$$

and so the arguments are

$$\arg(x^\alpha) = \begin{cases} \alpha\pi & \text{when } x < 0 \\ 0 & \text{when } x > 0. \end{cases}$$

As we move along the real axis (from left to right) and pass 0, then g_α has the effect of bending the axis and decreasing the argument by an angle of $\alpha\pi$.

PICTURE of straight line and bent line

For some fixed angle $\theta \in (-\pi, \pi)$, set $\alpha = -\frac{\theta}{\pi}$. In the case that $\theta > 0$, passing 0 has the effect of bending the axis and increasing the argument by θ.

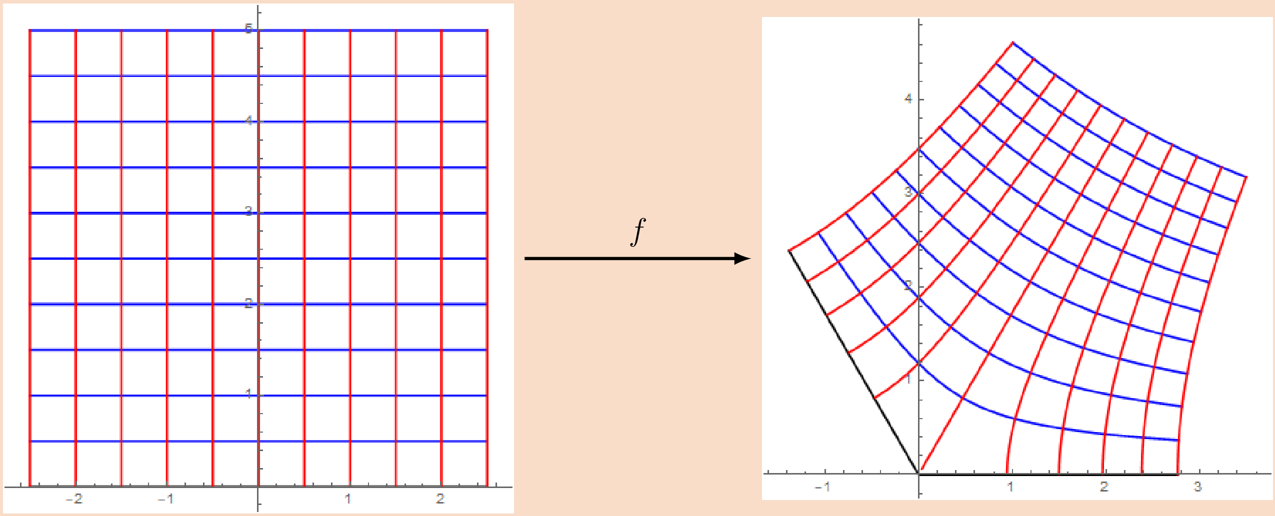
PICTURE of straight line and bent line

If we let $f(z) = \int_{z_0}^{z} \zeta^\alpha \, d\zeta$ (where $\mathrm{Im}(z_0) \geq 0$), then this makes $f'(z) = z^\alpha$, and as we saw in the proof of Theorem 7.3.1, the argument of $f'(z)$ is precisely the angle by which the tangent vectors are bent.

Remark. You can absolutely take $z_0 = 0$ above; the integral just becomes an improper integral. However, you'll almost always be working with such maps numerically and improper integrals cause computational issues, so it's recommended you choose z_0 with $\mathrm{Im}(z_0) > 0$.

Example 7.5.2

If we set $\alpha = -\frac{\pi}{3}$, then $f(z) = \int_0^z \zeta^{-1/3} \, d\zeta$, and the image of the upper half plane is



Notice now that for any two complex numbers $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$, we have that the argument of $z_1 z_2$ is $\theta_1 + \theta_2$. So if we fix real numbers x_1, x_2 with $x_1 < x_2$ and angles α_1, α_2, then we have that

$$\arg((x - x_1)^{\alpha_1}(x - x_2)^{\alpha_2}) = \begin{cases} \alpha_1 + \alpha_2 & \text{when } x < x_1 \\ \alpha_1 & \text{when } x_1 < x < x_2 \\ 0 & \text{when } x > x_2 \end{cases}$$

Example 7.5.3

Let $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{3}$. If we consider $f(z) = \int_0^z \zeta^{-1/2}(\zeta - 1)^{-1/3}\, d\zeta$, then the image of the upper half plane is