MAT 4574 Complex Analysis

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Contents

Preface

These notes largely follow along with the course text *Advanced Engineering Mathematics* by Peter V. O'Neil (7th ed). In an attempt to keep things cohesive, chapter and section numbers correspond to those found in the text. There are some discrepancies, however, which I will outline below.

- Theorem numbers are internally consistent, but do not align with the numbers found in the course text. If a theorem in the text is named, I've done my best to retain the naming.
- Sections with letters are comprised of material from the appropriate chapter that I opted to introduce in a different order. For example, 19.A and 20.A both focus on harmonic functions as treated in chapters 19 and 20, respectively. I thought these applications and connections were important enough to deserve their own treatment.

Every effort has been made in these notes to pick examples different from those in the text so that students may have a cornucopia of worked examples to look at. I will reiterate the old adage, however, that "math is not a spectator sport" and that the real learning comes from working through an example, not just reading it over.

The target audience for this course is largely senior undergraduate engineering students, who would be perfectly content to never see the word "proof" ever again. However, this is still a mathematics class and proofs can absolutely contribute to understanding the abstract concepts, and so I've tried to strike a balance and include only proofs (or sketches) which I find relatively simple, clever, or in some way illuminating. Many of these proofs can be found in full in the course text.

Thanks to my many students – Joseph Cunningham, Elijah Gendron, Denys Ovchynnikov, Murphy Smith – for finding the bountiful mistakes and inaccuracies...

19 Complex Numbers and Functions

19.1 Geometry and Arithmetic of Complex Numbers

Definition. A $\boxed{\textit{complex number}}$ is a symbol $x + iy$ or $x + yi$, where x, y are real numbers and i satisfies $i^2 = -1$. The collection of complex numbers is denoted $\mathbb{C}$. Writing $z = x + yi$, we say that x is the $\boxed{\textit{real part}}$ of z, denoted $\mathrm{Re}(z)$, and we say that y is the $\boxed{\textit{imaginary part}}$ of z, denoted $\mathrm{Im}(z)$.

Remark. Some authors use $\Re(z)$ and $\Im(z)$ to denote the real and imaginary parts of z, respectively.

The complex numbers satisfy the following rules of arithmetic:

- *Equality*: $\boxed{a + bi = c + di \text{ if and only if } a = c \text{ and } b = d}$
- *Addition*: $\boxed{(a + bi) + (c + di) = (a + c) + (b + d)i}$
- *Multiplication*: $\boxed{(a + bi)(c + di) = (ac - bd) + (ad + bc)i}$

Remark. If you think of complex numbers as polynomials with indeterminat i, then the arithmetic operations are the same as those for polynomials, with the added simplification of $i^2 = -1$.

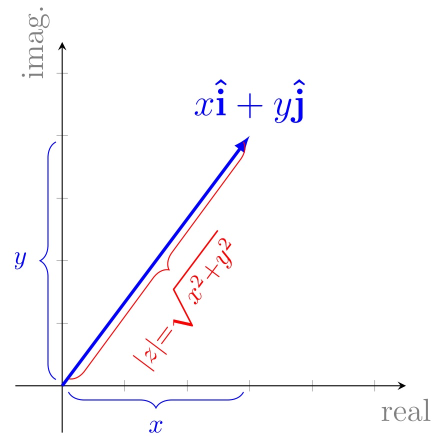
Proposition 19.1.1. *Complex arithmetic has the following familiar properties from arithmetic of the real numbers. For all $u, v, w \in \mathbb{C}$, we have*

- $\boxed{\textit{Associative addition: } u + (v + w) = (u + v) + w}$
- $\boxed{\textit{Commutative addition: } u + v = v + u}$
- $\boxed{\textit{Associative multiplication: } u(vw) = (uv)w}$
- $\boxed{\textit{Commutative multiplication: } uv = vu}$
- $\boxed{\textit{Distributive law: } u(v + w) = uv + uw}$
- $\boxed{w + 0 = w}$
- $\boxed{1w = w}$

Exercise 19.1.1. Prove Proposition 19.1.1.

The Complex Plane

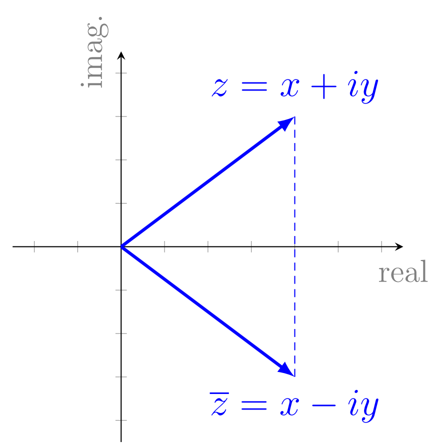
We can identify the complex number $z = x + yi$ with the vector $x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$. Because of this identification, the horizontal axis is known as the $\boxed{\textit{real axis}}$ and the vertical axis is known as the $\boxed{\textit{imaginary axis}}$.



Magnitude and Conjugation

From the vector interpretation of a complex number $z = x+yi$, the following definition is the natural one:

Definition. For a complex number $z = x + yi$, the $\boxed{\textit{magnitude}}$ of z is $|z| = |x + yi| = \sqrt{x^2 + y^2}$. The $\boxed{\textit{(complex) conjugate}}$ of z is $\overline{z} = x - yi$.



Proposition 19.1.2. *For complex numbers z, w, we have the following properties of the conjugate and magnitude:*

1. $\mathrm{Re}(\overline{z}) = \boxed{\mathrm{Re}(z)}$
2. $\mathrm{Im}(\overline{z}) = \boxed{-\mathrm{Im}(z)}$
3. $\overline{\overline{z}} = \boxed{z}$
4. $\overline{z+w} = \boxed{\overline{z}+\overline{w}}$
5. $\overline{(zw)} = \boxed{(\overline{z})(\overline{w})}$
6. *If* $w \neq 0$, *then* $\overline{z}/\overline{w} = \boxed{\overline{(z/w)}}$
7. $|\overline{z}| = \boxed{|z|}$
8. $|zw| = \boxed{|z||w|}$
9. $\mathrm{Re}(z) = \boxed{\dfrac{z+\overline{z}}{2}}$
10. $\mathrm{Im}(z) = \boxed{\dfrac{z-\overline{z}}{2}}$
11. $|z| \geq \boxed{0}$
12. $|z| = 0$ *if and only if* $\boxed{z=0}$
13. $z\overline{z} = \boxed{|z|^2}$

Exercise 19.1.2. Prove Proposition 19.1.2.

Complex conjugation is usually used when computing quotients.

$$\frac{z}{w} = \frac{z}{w}\frac{\overline{w}}{\overline{w}} = \frac{1}{|w|^2}(z\overline{w})$$

Example 19.1.3. Compute the quotient:

$$\frac{7-4i}{3+9i} = \frac{7-4i}{3+9i}\frac{\overline{3+9i}}{\overline{3+9i}} = \frac{(7-4i)(3-9i)}{9+81} = -\frac{15}{90} - \frac{75}{90}i = -\frac{1}{6} - \frac{5}{6}i$$

19.1.1 Argument and Polar Form

Fact (Euler's formula)**.** $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Thinking of $z = x + iy$ as a point in $\mathbb{R}^2$, we can write it in polar coordinates with some radius r and angle θ so that $x = r\cos\theta$ and $y = r\sin\theta$ (and a straightforward computation gives $r = |z|$). Using Euler's formula, we have that

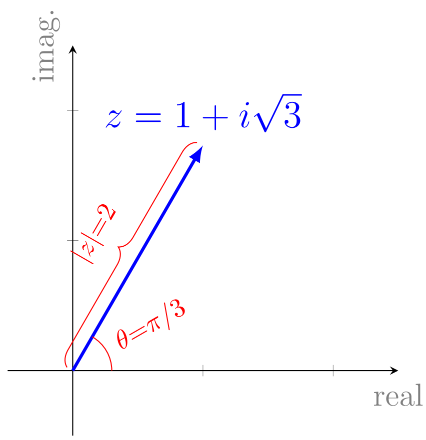
$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Definition. The complex number $z = x + iy = re^{i\theta}$ is said to be in $\boxed{\textit{polar form}}$ with $\boxed{\textit{argument}}$ θ. Sometimes we note this as $\arg(z)$.

If θ is an argument for the complex number z, then so is $\theta + 2k\pi$ for any integer k. As such the argument of z is not unique.

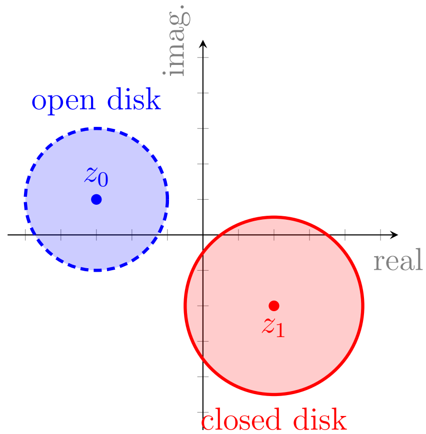
Example 19.1.4. Find the polar form for $z = 1 + i\sqrt{3}$.

Since $|z| = \sqrt{1+3} = 2$ and $\arctan(\sqrt{3}) = \frac{\pi}{3}$, then we can write $z = 2e^{i\pi/3}$. Note, of course, that we could also write $z = 2e^{i(\pi/3+2k\pi)}$ for any integer k since there are infinitely many possible arguments for z.



19.1.2 Disks, Open Sets, Closed Sets

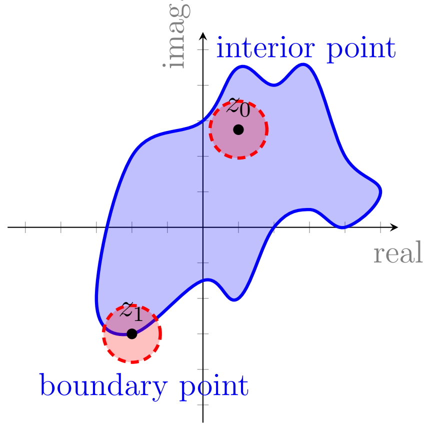
Definition. Let z_0 be a fixed complex number and let r be a positive real number. The $\boxed{\textit{open disk}}$ of radius r centered at z_0 is the set of all z satisfying $\boxed{|z - z_0| < r}$ and the $\boxed{\textit{closed disk}}$ of radius r centered at z_0 is the set of all z satisfying $\boxed{|z - z_0| \leq r}$.



Exercise 19.1.3. Writing $z = x + iy$ and $z_0 = x_0 + iy_0$, convince yourself that these equations look like the familiar equations for open and closed disks in the plane.

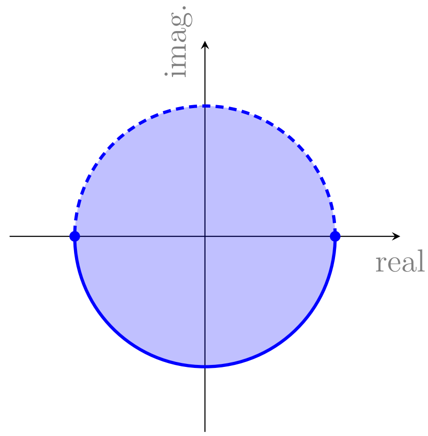
Open disks play the same role for complex analysis as open intervals (a, b) do for calculus. Similarly for closed disks and closed intervals $[a, b]$.

Definition. Let S be a set of complex numbers. A point z is an $\boxed{\textit{interior point}}$ if there is a small positive real number for which the open disk of radius r around z is entirely contained within S. A point z is a $\boxed{\textit{boundary point}}$ if every open disk around z contains both a point in S and a point not in S.



Definition. A set of complex numbers S is $\boxed{\textit{open}}$ if every point in S is an interior point. The set S is $\boxed{\textit{closed}}$ if S contains all boundary points.

Example 19.1.5. A set can be neither open nor closed. Consider the open disk $|z| < 1$ together with the points satisfying $|z| = 1$ and $\mathrm{Im}(z) \leq 0$.



Remark. To head off the question, yes, a set can actually be both open and closed (all of $\mathbb{C}$ is one such set, for example), but understanding when this happens is considerably more subtle and probably best left for office hours or MATH 4324; that is, you don't need to know it for this course.

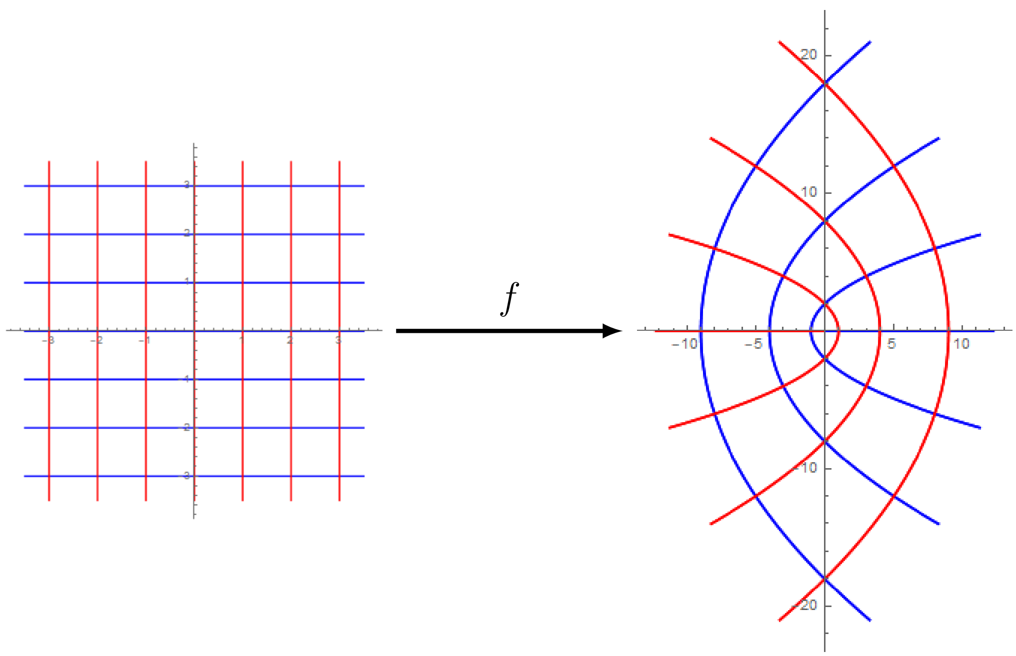
Remark. The definition of a closed set we gave is actually not entirely accurate, but it's perfectly sufficient for this class. We're really only concerned with closed sets that can contain an open disk within them. There are plenty of other closed sets around, like a finite collection of points for instance, but this is maybe best explored in MATH 4324 or my office hours.

19.2 Complex Functions

Definition. A $\boxed{\textit{complex function}}$ is a function whose input is a complex number and whose output is a complex number. We say that the function is *defined on a set* S if we restrict our attention to only input values coming from S.

Remark. Complex functions cannot be visualized quite as easily as the real functions you may be familiar with. A function $f : \mathbb{R} \to \mathbb{R}$ has a one-dimensional input and a one-dimensional output, so we can plot a graph of f on a two-dimensional plane. A function $f : \mathbb{C} \to \mathbb{C}$ has a two-dimensional input and a two-dimensional output, so plotting a graph in the same way would require four dimensions.

Example 19.2.1. The function $f(z) = z^2$ is a function defined on all of $\mathbb{C}$. We can get a feel for the behavior of this function by examining a grid in $\mathbb{C}$ before and after applying f.



19.2.1 Limits, Continuity, and Differentiability

Definition. If f is a complex function, then $f(z)$ has a $\boxed{\textit{limit}}$ L as z approaches z_0 if, for every real number $\varepsilon > 0$, there is a real number $\delta > 0$ such that

$$|f(z) - L| < \varepsilon$$

for every z in S satisfying $0 < |z - z_0| < \delta$.

In words, L is a limit if we can arbitrarily approximate it by $f(z)$ when restricting our focus to z-values in a small disk around z_0.

Unlike single-variable calculus where we can check limits from the left and right, in the complex setting (just like in multivariable calculus), the limit has to exist given *any path from z to z_0*.

Definition. Suppose f is a complex function defined on a set S. For z_0 in S, if $\lim_{z \to z_0} f(z) = f(z_0)$, then f is $\boxed{\textit{continuous at } z_0}$. If f is continuous at every z_0 in S, then we say that f is $\boxed{\textit{continuous on } S}$.

Definition. A set S is $\boxed{\textit{bounded}}$ if there is some positive real number M for which every z in S satisfies $|z| \leq M$. A function f is $\boxed{\textit{bounded}}$ if there is some positive real number K such that $|f(z)| \leq K$ for every z in S.

Definition. The function f is $\boxed{\textit{(complex) differentiable}}$ at z_0 in S if the following limit exists

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad \left(\text{or equivalently} \qquad \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \right).$$

If this limit exists, we denote it limit $f'(z_0)$ or $\left.\dfrac{df}{dz}\right|_{z=z_0}$ and call it the

$\boxed{\textit{(complex) derivative}}$ of f at z_0.

Definition. If f is complex differentiable at every z in S, then we say that f is $\boxed{\textit{(complex) differentiable on } S}$. In particular, when S is an open set, then we may sometimes say that f is $\boxed{\textit{(complex) analytic}}$.

Remark. If you're familiar with the notion of real analytic functions, this definition may seem odd. It turns out that complex differentiation is a *much* stronger notion than real differentiation, so this definition of analyticity is actually equivalent.

Theorem 19.2.2. *If f is differentiable at z_0, then f is continuous at z_0.*

Not every continuous complex function is differentiable, however.

Example 19.2.3. Consider $f(z) = \overline{z}$. If f is differentiable at z, then the following limit must exist when $h \to 0$ along the real axis (so h is real).

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{\overline{z} + h - \overline{z}}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

As well, the limit should exist when $ih \to 0$ along the imaginary axis (so h is real).

$$\lim_{ih\to 0} \frac{f(z+ih) - f(z)}{ih} = \lim_{h\to 0} \frac{\overline{z} - ih - \overline{z}}{ih} = \lim_{h\to 0} \frac{-ih}{ih} = -1$$

But these limits aren't equal, so f is not differentiable at any number z.

19.2.2 The Cauchy–Riemann Equations

Here we're going to see ways to compute complex derivatives (when they exist) without having to use limits.

Let $z = x + iy$. Then $\mathrm{Re}(f(z))$ and $\mathrm{Im}(f(z))$ are real-valued functions of x and y, let's call them $u(x, y)$ and $v(x, y)$, respectively, so we can write:

$$f(z) = u(x, y) + iv(x, y)$$

If f is complex differentiable at $z = x + iy$, then we can compule the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

along two different paths as in Example 19.2.3.

<u>Path 1 (real axis):</u> For a real number h

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h+iy) - f(x+iy)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h, y) + iv(x+h, y) - u(x, y) - iv(x, y)}{h} \\
&= \left(\lim_{h\to 0} \frac{u(x+h, y) - u(x, y)}{h}\right) + i\left(\lim_{h\to 0} \frac{v(x+h, y) - v(x, y)}{h}\right) \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}
\end{aligned}$$

provided the partial derivatives exist.

Path 2 (imaginary axis): In what follows, h is a real number.

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(z+ih)-f(z)}{ih}\\
&=\lim_{h\to 0}\frac{f(x+ih+iy)-f(x+iy)}{ih}\\
&=\lim_{h\to 0}\frac{u(x,y+h)+iv(x,y+h)-u(x,y)-iv(x,y)}{ih}\\
&=\frac{1}{i}\left(\lim_{h\to 0}\frac{u(x,y+h)-u(x,y)}{h}\right)+\left(\lim_{h\to 0}\frac{v(x,y+h)-v(x,y+h)}{h}\right)\\
&=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}
\end{aligned}$$

provided the partial derivatives exist.

Since the two limits above must agree in both their real and imaginary parts, we must have that

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial v}{\partial y},\\ \frac{\partial u}{\partial y}&=-\frac{\partial v}{\partial x}.\end{aligned}\right\}\boxed{\textit{Cauchy–Riemann equations}} \tag{19.2.1}$$

Theorem 19.2.4 (Cauchy–Riemann equations)**.** *Suppose $z=x+iy$ and that $u,v:\mathbb{R}^2\to\mathbb{R}$ are differentiable at (x,y). If $f(z)=u(x,y)+iv(x,y)$ is complex differentiable at z, then u and v satisfy the Cauchy–Riemann equations 19.2.1. Moreover*

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$

Note that the above implication does not guarantee that f is complex differentiable. We actually require a further assumption about u and v for that to be true.

Theorem 19.2.5. *Suppose $z=x+iy$ and that $u,v:\mathbb{R}^2\to\mathbb{R}$ are differentiable at (x,y). If the first partial derivatives of u and v are all continuous at (x,y), then $f(z)=u(x,y)+iv(x,y)$ is complex differentiable at z.*

Example 19.2.6. Cosider the function $f(z) = z^2$ defined on all of $\mathbb{C}$. We have that $f(x+iy) = u(x,y) + iv(x,y)$ where

$$u(x,y) = x^2 - y^2 \qquad \text{and} \qquad v(x,y) = 2xy.$$

Computing the partial derivatives, we have that

$$\frac{\partial u}{\partial x} = 2x \qquad\qquad \frac{\partial u}{\partial y} = -2y$$
$$\frac{\partial v}{\partial x} = 2y \qquad\qquad \frac{\partial v}{\partial y} = 2x$$

These satisfy the Cauchy–Riemann equations and at every point (x,y) the first partial derivatives are all continuous, so f is complex differentiable and

$$\begin{aligned} f'(z) = f'(x+iy) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= 2x + 2iy \\ &= 2z \end{aligned}$$

More generally,

Proposition 19.2.7 (Power Rule). *For every positive integer n, $\dfrac{d}{dz}[z^n] = \boxed{nz^{n-1}}$.*

Exercise 19.2.1. Use the Cauchy–Riemann equations to prove Proposition 19.2.7.

Example 19.2.8. Consider the function $f(z) = z\,\mathrm{Im}(z)$ defined on all of $\mathbb{C}$. We have that $f(x+iy) = u(x,y) + iv(x,y)$ where

$$u(x,y) = xy \qquad \text{and} \qquad v(x,y) = y^2.$$

Computing the partial derivatives, we have that

$$\frac{\partial u}{\partial x} = y \qquad\qquad \frac{\partial u}{\partial y} = x$$
$$\frac{\partial v}{\partial x} = 0 \qquad\qquad \frac{\partial v}{\partial y} = 2y$$

At every point (x,y) the first partial derivatives are all continuous, but the Cauchy–Riemann equations are only satisfied when $x = y = 0$:

$$y = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2y$$
$$x = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

so f is only complex differentiable at $z = 0$.

Remark. This behavior is different than for real derivatives - in the real setting, a function cannot be differentiable at a single point (rather, it must be differentiable within an entire interval around that point). This just lends evidence to the fact that complex differentiability is a much stricter condition than real differentiability.

Proposition 19.2.9 (Algebra of complex derivatives)**.** *Let f, g be functions that are complex differentiable at z, and let c any complex number. The following functions are also complex differentiable at z and familiar properties hold:*

1. $(f \pm g)'(z) = \boxed{f'(z) \pm g'(z)}$
2. $(cf)'(z) = \boxed{cf'(z)}$
3. $(fg)'(z) = \boxed{f'(z)g(z) + g'(z)f(z)}$
4. *For* $g(z) \neq 0$, $\left(\dfrac{f}{g}\right)'(z) = \boxed{\dfrac{f'(z)g(z) - g'(z)f(z)}{[g(z)]^2}}$
5. *If f is differentiable at $g(z)$, then* $(f \circ g)'(z) = \boxed{f'(g(z))g'(z)}$

Exercise 19.2.2. Use the Cauchy–Riemann equations to prove Proposition 19.2.9.

19.3 The Exponential and Trigonometric Functions

Definition. The $\boxed{\textit{complex exponential function}}$ e^z is defined for all $z = x + iy$ as

$$e^z := e^x \cos(y) + ie^x \sin(y)$$

Proposition 19.3.1. $\dfrac{d}{dz}[e^z] = \boxed{e^z}$

Exercise 19.3.1. Use the Cauchy–Riemann equations to prove Proposition 19.3.1.

The following are obvious consequences of the definition of the complex exponential.

Proposition 19.3.2 (Properties of e^z)**.**

1. *$e^0 = \boxed{1}$*
2. *For z, w complex numbers, $e^z e^w = \boxed{e^{z+w}}$*
3. *$e^z \neq 0$ for all complex numbers z*
4. *$e^{-z} = \boxed{1/e^z}$*
5. *For t a real number, $\overline{e^{it}} = \boxed{e^{-it}}$*
6. *For $z = x + iy$ (with x, y real numbers), $\boxed{|e^z| = e^x}$*

Exercise 19.3.2. Prove Proposition 19.3.2.

Example 19.3.3. Find all complex numbers $z = x + iy$ so that $e^z = 2 - 2i$.

Note that $|e^z| = e^x = |2 - 2i| = 2\sqrt{2}$, so we get that $x = \ln(2\sqrt{2})$. Thus, the equation

$$e^z = e^x \cos(y) + ie^x \sin(y) = 2 - 2i$$

implies that

$$2\sqrt{2}\cos(y) = 2 \qquad \text{and} \qquad 2\sqrt{2}\sin(y) = -2.$$

Thus

$$\frac{2\sqrt{2}\sin(y)}{2\sqrt{2}\cos(y)} = \frac{-2}{2} \Longrightarrow \tan(y) = -1$$

and thus $y = \arctan(-1) + 2k\pi = -\frac{\pi}{4} + 2k\pi$ for any integer k. As such, all possible solutions are

$$z = \ln(2\sqrt{2}) - i\left(\frac{\pi}{4} + 2k\pi\right).$$

It's not too hard to see that the complex exponential has period $2k\pi i$ for any integer k. After all, for any complex number z,

$$e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z\left(\cos(2k\pi) + i\sin(2k\pi)\right) = e^z(1) = e^z.$$

What's less obvious is

Theorem 19.3.4. *The only periods of the complex exponential are of the form $2k\pi i$ where k is any integer.*

Proof. Suppose p is some complex number for which $e^{z+p} = e^z$ for every z. Let $z = x + iy$ and $p = a + ib$. Then

$$\begin{aligned} e^{z+p} &= e^z \\ e^{x+a+i(y+b)} &= e^{x+iy} \\ e^{x+a}\left(\cos(y+b) + i\sin(y+b)\right) &= e^x\left(\cos(y) + i\sin(y)\right) \\ e^{a}\left(\cos(y+b) + i\sin(y+b)\right) &= \cos(y) + i\sin(y). \end{aligned}$$

If this is true for every z, then it must be true in the particular case when $y = 0$. So

$$e^a\cos(b) + ie^a\sin(b) = 1 + 0i$$

Since a is real, $e^a > 0$, hence $\sin(b) = 0$ and $\cos(b) > 0$. Thus $b = 2k\pi$ for some integer k. But then $\cos(b) = 1$ and thus $e^a = 1$, which implies that $a = 0$. So

$$p = a + ib = 0 + 2k\pi i.$$

$\square$

Theorem 19.3.4 also shows us why the real exponential isn't periodic – all possible periods are purely imaginary!

Notice that for a real number θ, we have

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{19.3.1}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{19.3.2}$$

Adding and subtracting these two equations yields

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \end{aligned}$$

Since we have defined the complex exponential for all complex numbers, we now have a natural way to extend the definition of some trigonometric functions.

Definition. For any complex number z, we define the $\boxed{\textit{complex cosine function}}$ as

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2},$$

and we define the $\boxed{\textit{complex sine function}}$ as

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Theorem 19.3.5. *Both* $\cos(z)$ *and* $\sin(z)$ *are complex differentiable on all of* $\mathbb{C}$ *(i.e. are analytic on all of* $\mathbb{C}$*). Moreover,*

$$\frac{d}{dz}\sin(z) = \cos(z), \qquad \frac{d}{dz}\cos(z) = -\sin(z).$$

Proof. We will only prove that $\cos(z)$ is complex differentiable, as the proof for $\sin(z)$ is the same *mutatis mutandus*. Letting $z = x + iy$, then we have that $iz = -y + ix$ and $-iz = y - ix$. So

$$\begin{aligned}
\cos(z) &= \frac{1}{2}e^{iz} + \frac{1}{2}e^{-iz} \\
&= \frac{1}{2}e^{-y+ix} + \frac{1}{2}e^{y-ix} \\
&= \frac{1}{2}e^{-y}\left(\cos x + i\sin x\right) + \frac{1}{2}e^{y}\left(\cos x - i\sin x\right) \\
&= \frac{1}{2}\cos(x)\left(e^{y} + e^{-y}\right) + \frac{i}{2}\sin(x)\left(e^{y} - e^{-y}\right).
\end{aligned}$$

Let

$$u(x,y) = \frac{1}{2}\cos(x)\left(e^{y} + e^{-y}\right) \qquad v(x,y) = \frac{i}{2}\sin(x)\left(e^{y} - e^{-y}\right).$$

One can check the u and v satisfy the Cauchy–Riemann equations. Moreover, all of the first partial derivatives of u and v are continuous, so by Theorem 19.2.5, $\cos(z)$ is differentiable. □

Exercise 19.3.3. With u, v as given in the above, check that they do indeed satify the Cauchy–Riemann equations and show that $\frac{d}{dz}\cos(z) = -\sin(z)$.

It's worth noting that the complex trig functions $\cos(z)$ and $\sin(z)$ are unbounded functions (to see this, take the imaginary part of z to be arbitrarily large). As it turns out, all of the familiar behavior of sine and cosine happens only in the real numbers. Precisely,

Theorem 19.3.6. 1. $\sin(z) = 0$ *precisely when* $z = k\pi$ *for any integer* k.

2. $\cos(z) = 0$ *precisely when* $z = \frac{1}{2}(2k+1)\pi$ *for any integer* k.

3. *The only periods of the complex sine and cosine are of the form* $2k\pi$ *where* k *is any integer.*

The remaining trigonometric functions can also be made complex in the obvious ways:

Definition. The complex secant, cosecant, tangent, and cotangent are defined as

$$\sec(z) = \frac{1}{\cos(z)}, \qquad \csc(z) = \frac{1}{\sin(z)},$$
$$\tan(z) = \frac{\sin(z)}{\cos(z)}, \qquad \cot(z) = \frac{\cos(z)}{\sin(z)},$$

provided $\sin(z) \neq 0$ or $\cos(z) \neq 0$ when appropriate.

Theorem 19.3.7. *The remaining complex trigonometric functions are complex differentiable on their domains (i.e. analytic) and have the following complex derivatives:*

$$\frac{d}{dz}\sec(z) = \sec(z)\tan(z), \qquad \frac{d}{dz}\csc(z) = -\csc(z)\cot(z),$$
$$\frac{d}{dz}\tan(z) = \sec^2(z), \qquad \frac{d}{dz}\cot(z) = -\csc^2(z)$$

Proof. That these functions are complex differentiable is an immediate consequence of the differentiability of $\sin(z)$ and $\cos(z)$ in conjunction with Proposition 19.2.9. Since we now know the complex derivatives of sine and cosine, verification of derivatives is a sraightforward computation and is left as an exercise for the reader. $\square$

19.4 The Complex Logarithm

Suppose z is fixed and we are trying to find all w for which $e^w = z$. We can make things a bit easier if we put z into polar form ($z = re^{i\theta}$) and rewrite $w = u + iv$. Now we have

$$e^u e^{iv} = e^{u+iv} = e^w = z = re^{i\theta} \tag{19.4.1}$$

Since $|e^{iv}| = |e^{i\theta}| = 1$, and since e^u and r are both positive, taking magnitudes of both sides gives us

$$e^u = |e^u||e^{iv}| = |r||e^{i\theta}| = r$$

hence $u = \ln(r) = \ln|z|$.

But now Equation 19.4.1 simplifies to

$$e^{iv} = e^{i\theta}.$$

Since the complex exponential has periods of the form $2k\pi i$ for integers k, it must be that $v = \theta + 2k\pi$ for some integer k. As such, the answer to our original equation is

$$w = \ln|z| + i(\theta + 2k\pi)$$

where θ is an argument for z and k is any integer. From this we define

Definition. If z is a nonzero complex number with argument θ, then the *(complex) logarithm*, $\log(z)$, is the set of complex numbers

$$\log(z) := \ln|z| + i(\theta + 2k\pi)$$

where k ranges over all integers.

Note that $\log(z)$ *is not a function* because has infinitely-many outputs for every input.

Remark. In most areas of math, log without any base is interpreted to be "base-e" and whether it is the real or complex natural logarithm is clear from context. In these notes, we'll use ln to distinguish the real natural logarithm from the complex counterpart.

Example 19.4.1. Compute $\log(z)$ where $z = 1 + i\sqrt{3}$.

Since $|z| = 2$ and z has an argument $\theta = \frac{\pi}{3}$, we have that

$$\log(z) = \ln 2 + i\left(\frac{\pi}{3} + 2k\pi\right)$$

As it will be useful in the next section, suppose $z \neq 0$ is a complex number with a chosen argument θ. We then have that

$$\log(z) = \ln|z| + i(\theta + 2k\pi) \qquad \text{(for all integers } k\text{).}$$

It follows that

$$\begin{aligned} e^{\log(z)} = e^{\ln|z| + i(\theta+2k\pi)} &= e^{\ln|z|}e^{i\theta}e^{i2k\pi} \\ &= |z|e^{i\theta}(1) = z \end{aligned}$$

so even though $\log(z)$ has infinitely-many values, $e^{\log(z)}$ is a <u>single complex number.</u>

19.5 Powers

The goal of this section is to give meaning to z^w where both z and w are complex numbers (we already know what happens when w is an integer) and $z \neq 0$.

19.5.1 n^{th} Roots

Definition. Let n be a positive integer. An $\boxed{n^{th} \textit{ root of } z}$ is a complex number $z^{1/n}$ whose n^{th} power is z.

To find n^{th} roots of $z \neq 0$, we begin by putting z into polar form $z = re^{i(\theta+2k\pi)}$ with all possible arguments in the exponent. Then

$$z^{1/n} = r^{1/n} e^{i(\theta+2k\pi)/n}$$

where $r^{1/n}$ is the real n^{th} root of r. Thus for each number $k = 0, \ldots, (n-1)$, we obtain all of the different n^{th} roots of z (and there are exactly n of these):

$$r^{1/n}e^{i\theta/n}, \qquad r^{1/n}e^{i(\theta+2\pi)/n}, \qquad r^{1/n}e^{i(\theta+4\pi)/n}, \qquad \ldots \qquad r^{1/n}e^{i(\theta+2(n-1)\pi)/n}$$

Exercise 19.5.1. Convince yourself that for any $k \geq n$, the number $r^{1/n}e^{i(\theta+2k\pi)/n}$ is already featured in the list above. Hint: What are the periods of the complex exponential?

Example 19.5.1. Compute the fifth roots of $z = 1 + i$.

Since $|z| = \sqrt{2} = 2^{1/2}$ and $\theta = \frac{\pi}{4}$ is an argument for z, the fifth roots of z are

$$2^{1/10}e^{i(\pi/4)/5}, \quad 2^{1/10}e^{i(\pi/4+2\pi)/5}, \quad 2^{1/10}e^{i(\pi/4+4\pi)/5}, \quad 2^{1/10}e^{i(\pi/4+6\pi)/5}, \quad 2^{1/10}e^{i(\pi/4+8\pi)/5},$$

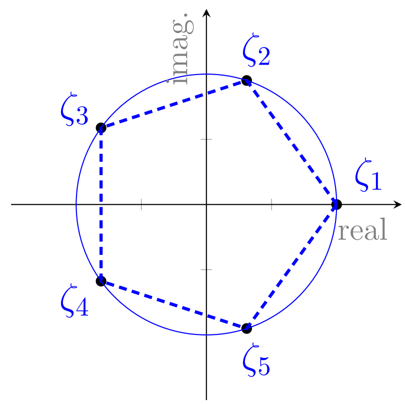
which, when simplified a bit, are

$$2^{1/10}e^{i\pi/20}, \qquad 2^{1/10}e^{9i\pi/20}, \qquad 2^{1/10}e^{17i\pi/20}, \qquad 2^{1/10}e^{25i\pi/20}, \qquad 2^{1/10}e^{33i\pi/20}.$$

Example 19.5.2. An $\boxed{n^{th} \textit{ root of unity}}$ is a complex number $z^{1/n}$ whose n^{th} power is 1, and these are quite ubiquitous throughout different branches of mathematics (for example, in Fourier analysis, in Galois theory, or even in your instructor's geometry PhD thesis). Compute the 5^{th} roots of unity and plot them.

In polar form, $1 = 1e^{0\pi i}$, hence

$$\begin{aligned}
\zeta_1 &= 1 \\
\zeta_2 &= e^{2\pi i/5} \\
\zeta_3 &= e^{4\pi i/5} \\
\zeta_4 &= e^{6\pi i/5} \\
\zeta_5 &= e^{8\pi i/5}
\end{aligned}$$



19.5.2 Rational Exponents

If $\frac{m}{n}$ is a rational number (fully reduced), then we define

$$z^{m/n} := \left(z^m\right)^{1/n}.$$

Remark. Because there are n-many n^{th} roots, there will be n-many values of $z^{m/n}$.

Example 19.5.3. Find $(1+i)^{2/3}$.

We have that

$$(1+i)^{2/3} = \left((1+i)^2\right)^{1/3} = (2i)^{1/3}.$$

Since $2i = 2e^{i\pi/2}$, it follows that the cube roots of $2i$ are

$$2^{1/3}e^{i(\pi/2)/3}, \qquad 2^{1/3}e^{i(\pi/2+2\pi)/3}, \qquad e^{i(\pi/2+4\pi)/3}$$

which, when simplified a bit, are

$$2^{1/3}e^{i\pi/6}, \qquad 2^{1/3}e^{5i\pi/6}, \qquad 2^{1/3}e^{9i\pi/6}.$$

19.5.3 Complex Exponents

Stemming from the discussion at the end of the previous section, we have that $z = e^{\log(z)}$. As such for any complex w, we can unambiguously define

$$z^w := e^{w \log(z)}.$$

If w is not a rational number, then there are infinitely-many values of z^w.

Example 19.5.4. Find i^i.

$i = e^{i\pi/2}$, so $\log(i) = i(\pi/2 + 2k\pi)$ for any integer k. Hence

$$i^i = e^{i \log(i)} = e^{i^2(\pi/2+2k\pi)} = e^{\pi/2+2k\pi}$$

for any integer k.

Remark. Yes. i^i produces real numbers.

19.5.4 Remarks - Log and Arg Functions

As mentioned, given any complex number z, there are infinitely-many possible arguments, and thus logarithms, for z. What this means is that we can't treat $\arg(z)$ and $\log(z)$ as functions. This has some drawbacks, so it's not uncommon for other authors to restrict the possible ranges of values in order to make them into function. Specifically,

$$\begin{aligned} &\mathrm{Arg}(z) = \theta \text{ where } \theta \text{ is an argument of } z \text{ and } -\pi < \theta \leq \pi \\ &\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z) \end{aligned}$$

Your book works very hard to avoid treating them as functions, and I will follow suit for the semester. However, you may find yourself looking to outside sources, so you should be aware of different authors' conventions.

19.A Harmonic Functions

Recall the following definition

Definition. A real-valued function $u(x, y)$ is $\boxed{\textit{harmonic}}$ on an open set S if its second partial derivatives are continuous and it satisfies the *Laplace equation* $\nabla^2 u = 0$ on S, i.e. if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

at every point in S.

As was suggested in the Problems for Section 19.2, there is an intimate relationship with complex differentiability and harmonic functions. Namely,

Theorem 19.A.1. *Suppose f is analytic on a set S. Then the functions u and v satisfying $f(x+iy) = u(x, y) + iv(x, y)$ on S are harmonic on S.*

Proof. If f is analytic, then u and v satisfy the Cauchy–Riemann equations on S. To see that u is harmonic on S,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) \qquad \text{(Cauchy–Riemann Equations)} \\
&= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0.
\end{aligned}$$

The proof that v is harmonic is the same *mutatis mutandis.* □

Definition. If $f = u+iv$ is analytic, then u and v are called $\boxed{\textit{harmonic conjugates}}$.

Remark. Harmonic conjugates are not unique.

Example 19.A.2. Find a harmonic conjugate for $u(x, y) = y$.

Since $\nabla^2 u = 0$ on the whole plane, we need to find a function f that is analytic on all of $\mathbb{C}$ and a real-valued function for which $f(x + iy) = u(x, y) + iv(x, y)$. To do this, we look for a v satisfying the Cauchy–Riemann equations on all of $\mathbb{C}$:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -1 \tag{19.A.1}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0 \tag{19.A.2}$$

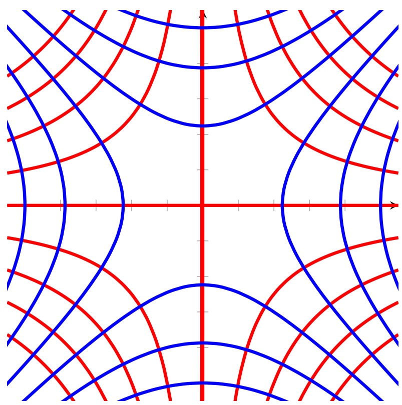
Integrating the first equation with respect to x, we deduce that $v(x, y) = -x + \varphi(y)$ where $\varphi(y)$ is a differentiable function of y. Differentiating with respect to y (and keeping in mind the second equation above), we have

$$\frac{\partial v}{\partial y} = \varphi'(y) = 0$$

in which case $\varphi(y) = K$, a constant. Thus for any constant K, a harmonic conjugate for u is

$$v(x, y) = -x + K.$$

Example 19.A.3. $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ are harmonic conjugates associated the function $f(z) = z^2$. Plotting out the level sets for each (with u blue and v red), we see that the level sets intersect each other perpendicularly.



20 Complex Integration

20.1 The Integral of a Complex Function

Complex functions are integrated over curves, and share much in common with line integrals of vector fields.

20.1.1 Integral Over a Closed Interval

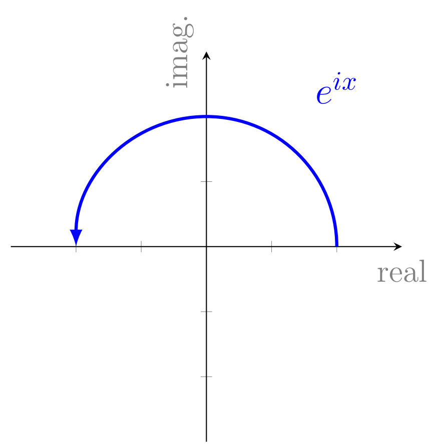
Supose f is a complex function and u, v are functions of one real variable satisfying $f(x) = u(x) + iv(x)$. If $f(x)$ is defined for $a \le x \le b$, then we define

$$\int_a^b f(x)\,dx = \int_a^b u(x)\,dx + i\int_a^b v(x)\,dx.$$

Example 20.1.1. Compute $\int_0^\pi e^{ix}\,dx$.

Since $f(x) = e^{ix} = \cos(x) + i\sin(x)$, we have that $u(x) = \cos(x)$ and $v(x) = \sin(x)$. Hence

$$\begin{aligned}\int_0^\pi e^{ix}\,dx &= \int_0^\pi \cos(x)\,dx + i\int_0^\pi \sin(x)\,dx \\ &= \sin(x)\Big|_{x=0}^{x=\pi} - i\cos(x)\Big|_{x=0}^{x=\pi} \\ &= -ie^{ix}\Big|_{x=0}^{x=\pi} = -i(-1-1) = 2i.\end{aligned}$$



For simplicity later, we note that $\int_a^b e^{ix}\,dx = -i(e^{ib} - e^{ia})$.

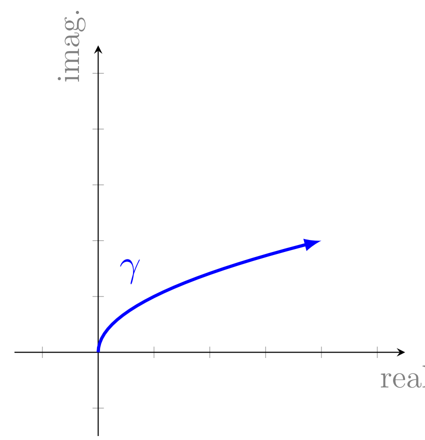
20.1.2 Integral Over a Smooth Curve

Let γ be a smooth curve in the plane with $\gamma(t)$ defined for $a \le t \le b$. Writing $\gamma(t) = x(t) + iy(t)$ (and thus $\gamma'(t) = x'(t) + iy'(t)$), suppose that f is continuous at all points on γ. Then we define

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

Example 20.1.2. Compute $\int_\gamma z\,\mathrm{Re}(z)\,dz$ for $\gamma(t) = t^2 + ti$ where $0 \le t \le 2$.

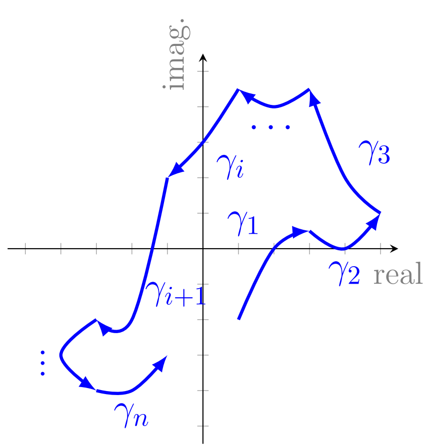
$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_0^2 (t^2+ti)(t^2)(2t+i)\,dt \\
&= \int_0^2 2t^5 - t^3 + 3it^4\,dt \\
&= \int_0^2 2t^5 - t^3\,dt + i\int_0^2 3t^4\,dt \\
&= \left(\frac{1}{3}t^6 - \frac{1}{4}t^4\right)\Bigg|_{t=0}^{t=2} + i\frac{3}{5}t^5\Bigg|_{t=0}^{t=2} \\
&= \frac{52}{3} + \frac{96}{5}i.
\end{aligned}$$



20.1.3 Integral Over a Piecewise Smooth Curve

Definition. A $\boxed{\textit{piecewise smooth curve}}$ γ is a curve comprised of finitely-many smooth curves $\gamma_1, \ldots, \gamma_n$ such that the terminal point of each γ_k is the initial point of γ_{k+1}. A piecewise smooth curve is also sometimes called a $\boxed{\textit{contour}}$. Your book denotes the piecewise smooth curve γ above as $\gamma = \gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_n$

Piecewise smooth curves can have finitely-many corners or cusps.



Definition. Suppose $\gamma = \gamma_1 \oplus \oplus \gamma_2 \oplus \cdots \oplus \gamma_n$ is a piecewise smooth curve defined for $a \leq t \leq b$ (where $\gamma_k(t)$ is defined for $a_k \leq t \leq b_k$, $a = a_1$, and $b = b_k$) and assume f is continuous at all points along the curve. Then the $\boxed{\textit{integral of } f \textit{ over } \gamma}$ is

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_{a_k}^{b_k} f(\gamma_k(t))\gamma_k'(t)\,dt.$$

Sometimes γ is called a *contour* and the integral is a *contour integral.*

Example 20.1.3. Compute $\int_\gamma \overline{z}\,dz$ for $\gamma = \gamma_1 \oplus \gamma_2$, where

$$\begin{aligned} \gamma_1(t) &= 2 + 2ti && (0 \leq t \leq 1), \\ \gamma_2(t) &= 4 - 2t + i(4 - 2t) && (1 < t \leq 2). \end{aligned}$$

We have that

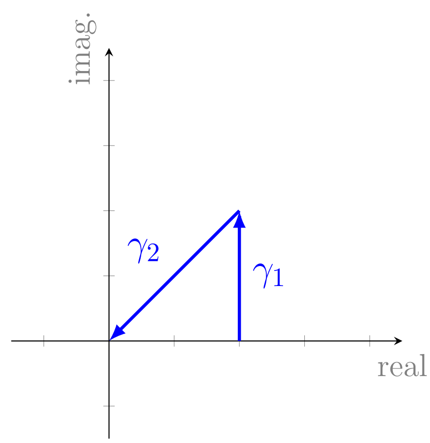
$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = \int_0^1 f(\gamma_1(t))\gamma_1'(t)\,dt + \int_1^2 f(\gamma_2(t))\gamma_2'(t)\,dt.$$

Computing these integrals separately,

$$\begin{aligned} \int_0^1 f(\gamma_1(t))\gamma_1'(t)\,dt &= \int_0^1 (2 - 2ti)(2i)\,dt \\ &= \int_0^1 4t + 4i\,dt = 2 + 4i, \end{aligned}$$

and

$$\begin{aligned} &\int_1^2 f(\gamma_2(t))\gamma_2'(t)\,dt \\ &= \int_1^2 (4 - 2t - 4i + 2ti)(-2 - 2i)\,dt \\ &= \int_1^2 8t - 16\,dt = -4. \end{aligned}$$

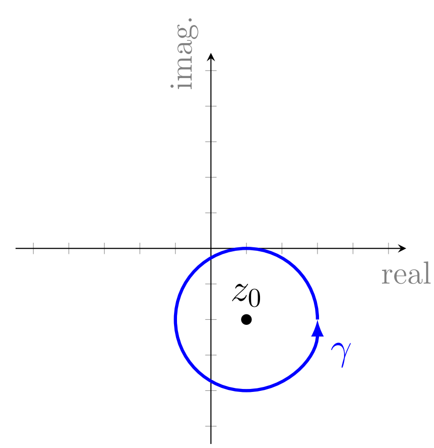


Hence $\int_\gamma f(z)\,dz = (2 + 4i) + (-4) = -2 + 4i.$

Example 20.1.4. For some fixed z_0 and positive real number r, let γ be the circle $|z - z_0| = r$ traversed once in the counterclockwise direction. For each integer $n > 0$, compute $\int_\gamma (z - z_0)^n \, dz$.

We know that the circle of radius r centered at the origin is parameterized by re^{it} for $0 \le t \le 2\pi$, so we can write $\gamma(t) = z_0 + re^{it}$ for $0 \le t \le 2\pi$.

$$\begin{aligned}
\int_\gamma f(z)\, dz &= \int_\gamma (z - z_0)^n \, dz \\
&= \int_0^{2\pi} (\gamma(t) - z_0)^n \gamma'(t)\, dt \\
&= \int_0^{2\pi} (r^n e^{int})(ire^{it})\, dt \\
&= \int_0^{2\pi} ir^{n+1} e^{i(n+1)t}\, dt \\
&= ir^{n+1} \left(\frac{-i}{n+1} \right) e^{i(n+1)t} \Big|_{t=0}^{t=2\pi} = 0.
\end{aligned}$$



Theorem 20.1.5 (Properties of Complex Integrals)**.** *Let f and g be integrable over some (piecewise) smooth curve γ.*

1. $\displaystyle\int_\gamma (f(z) + g(z))\, dz = \boxed{\int_\gamma f(z)\, dz + \int_\gamma g(z)\, dz}$.
2. *For any complex number c,* $\displaystyle\int_\gamma cf(z)\, dz = \boxed{c \int_\gamma f(z)\, dz}$.
3. *Reversing orientation of the curve changes the integral sign. If $\tilde{\gamma}$ is the same curve as γ, traversed in the opposite direction, then*
$$\int_{\tilde{\gamma}} f(z)\, dz = -\int_\gamma f(z)\, dz.$$
4. *There is a version of the fundamental theorem of calculus for complex integrals: Suppose f is continuous on an open set S and F is defined on S with the property that $F'(z) = f(z)$. If γ is a smooth curve in S, defined on an interval $[a, b]$, then*
$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a)).$$

5. $\displaystyle\int_\gamma f(z)\,dz$ *can be written as a sum of two real line integrals: Suppose* γ *is defined on* $[a,b]$*. Then we have that*

$$f(z) = u(x,y) + iv(x,y) \qquad \text{and} \qquad dz = (x'(t) + iy'(t))\,dt.$$

Then

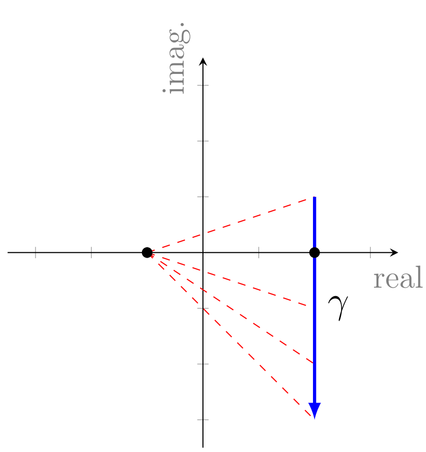
$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b (u(x,y) + iv(x,y))\,(x'(t) + iy'(t))\;dt \\ &= \int_a^b u(x,y)x'(t)\,dt - v(x,y)y'(t),\,dt + iu(x,y)y'(t)\,dt + iv(x,y)x'(t)\,dt \\ &= \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx)\end{aligned}$$

6. *Let* γ *be a smooth curve defined on* $[a,b]$*, let* L *be its length, and let* f *be continuous on* γ*. If* $f(z)$ *is bounded by* M *(i.e.* $|f(z)| \le M$ *for all* z *on* γ*), then*

$$\left|\int_\gamma f(z)\,dz\right| \le ML.$$

With the conditions given in # 4 above, the value of $\displaystyle\int_\gamma f(z)\,dz$ is determined only on the endpoints of γ, i.e., it is *independent of path.* In particular, when γ is a closed curve, then $\displaystyle\int_\gamma f(z)\,dz = 0$ (just like with conservative vector fields). We'll explore this more thoroughly in the next section.

Example 20.1.6. Using Part 6 of the above Theorem (20.1.5), find a bound for $\displaystyle\left|\int_\gamma \frac{1}{1+z}\,dz\right|$ where γ is the straight line segment from $2+i$ to $2-3i$.



We are looking for a number M so that $\left|\frac{1}{z+1}\right| \leq M$ for z-values on γ. Notice that $|z+1|$ is just the distance from a point z to -1, so the largest value of $\left|\frac{1}{z+1}\right|$ occurs when the point on γ is closest to -1. Looking at the picture, it's not hard to see that this happens when γ passes through 2. So we have that, for all z on the curve γ

$$|z+1| \geq |2+1| = 3 \qquad \Longrightarrow \qquad \left|\frac{1}{z+1}\right| \leq \frac{1}{3} = M.$$

Since γ is a line segment with length $L = 4$, then by Theorem 20.1.5, it follows that

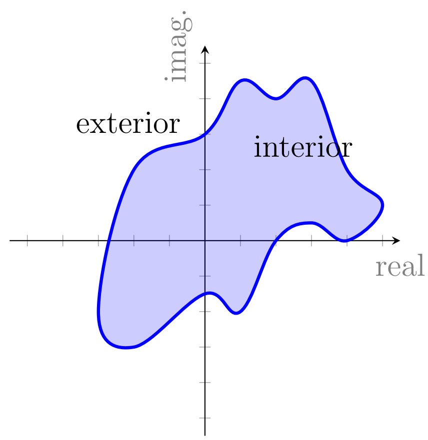
$$\left|\int_\gamma \frac{1}{1+z}\, dz\right| \leq ML = \left(\frac{1}{3}\right)(4) = \frac{4}{3}.$$

20.2 Cauchy's Theorem

We'll begin this section by stating a somewhat obvious (but actually very hard to prove) theorem; we'll just accept it as fact.

Theorem 20.2.1 (Jordan curve)**.** *If γ is a continuous, simple closed curve in the plane, then γ separates the plane into two connected regions - a bounded open set (the interior of γ) and an unbounded open set (the exterior of γ).*

Visually,



We'll introduce a few pieces of terminology for our purposes.

Definition. A *path* (in a set S) is a simple, piecewise smooth curve (that lies in S).

Remark. This definition of a path is not consistent with some other texts. Other authors may only require that a path be continuous, not necessarily piecewise smooth.

Definition. A set S is *connected* if every two points of S are endpoints of some path in S. An open, connected set is called a *domain*.

Remark. This definition of connectedness is not consistent with many other texts. Many other authors would call this "path-connectedness" and reserve the term "connected" for a strictly weaker notion.

Definition. A set S is *simply connected* if every closed path in S encloses only points in S.

Theorem 20.2.2 (Cauchy)**.** *Suppose f is differentiable on a simply connected domain S. Then*

$$\oint_\gamma f(z)\,dz = 0$$

for every closed path γ in S.

We use the symbol $\oint_\gamma$ as a visual reminder that we're integrating over a closed path.

Unless otherwise stated, we always assume closed curves are oriented counterclockwise.

Proof. The general proof is very involved, but a simpler case can be proven quite easily. For $z = x + iy$, write $f(z) = u(x, y) + iv(x, y)$ and suppose u, v are continuous on S with all first partial derivatives continuous on S. Using property 5 from Theorem 20.1.5, we can write

$$\oint_\gamma f(z)\,dz = \oint_\gamma (u\,dx - v\,dy) + i\oint_\gamma (u\,dy + v\,dx)$$

to which we can apply Green's Theorem

$$= \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\,dx\,dy + i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\,dx\,dy$$

and because of the Cauchy–Riemann equations, we have

$$\begin{aligned}&= \iint_S \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)\,dx\,dy + i\iint_S \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right)\,dx\,dy \\ &= 0 + 0 = 0.\end{aligned}$$

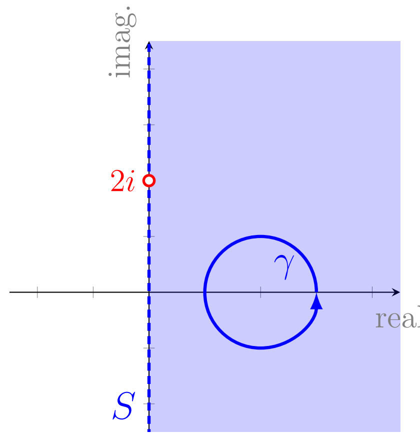
□

Example 20.2.3. Compute $\oint_\gamma \frac{1}{z-2i}\,dz$ where γ is the circle $|z-2|=1$ (traversed once].

$f(z)=\frac{1}{z-2i}$ is differentiable everywhere except at $z=2i$, and we can find a simply connected domain which excudes this point and contains all of γ (for example, take S to be the set of z satisfying $\text{Re}(z)>0$).

By Cauchy's theorem,

$$\int_\gamma \frac{1}{z-2i}\,dz = 0.$$

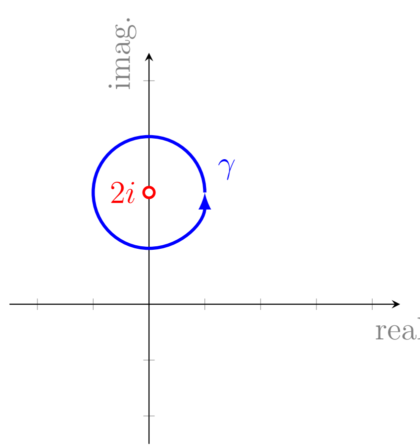


Example 20.2.4. Compute $\oint_\gamma \frac{1}{z-2i}\,dz$ where γ is the circle $|z-2i|=1$ (traversed once].

$f(z)=\frac{1}{z-2i}$ is differentiable everywhere except at $z=2i$. Unfortunately, γ encloses $z=2i$ and thus there are no simply connected open domains containing γ on which f is differentiable. Therefore, Cauchy's Theorem doesn't apply. However, f is still continuous (even differentiable!) along γ, so we can compute this the old fashioned way.

We parameterize $\gamma(t)=2i+e^{it}$ where $0\le t\le 2\pi$. Then

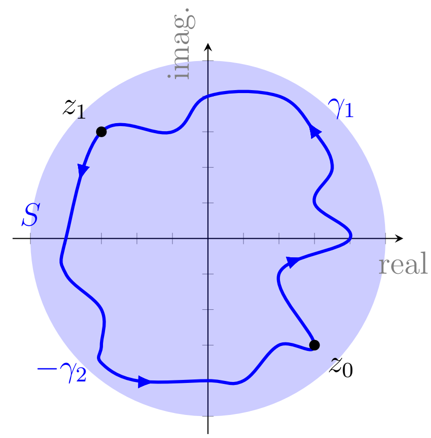
$$\begin{aligned}\int_\gamma \frac{1}{z-2i}\,dz &= \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)-2i}\,dt\\ &= \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt\\ &= \int_0^{2\pi} i\,dt = 2\pi i.\end{aligned}$$



20.3 Consequences of Cauchy's Theorem

20.3.1 Independence of Path

Suppose f is differentiable on a simply connected domain S, and suppose z_0, z_1 are points of S. Let γ_1, γ_2 be paths in S from z_0 to z_1. Writing $-\gamma_2$ to denote the reverse path of γ_2 (that is $-\gamma_2$ is a path from z_1 to z_0), then we have that $\gamma = \gamma_1 \oplus (-\gamma_2)$ is a closed path.



Since γ is a closed path,

$$\begin{aligned} 0 &= \oint_\gamma f(z)\,dz && \text{(Cauchy's Theorem)} \\ &= \int_{\gamma_1} f(z)\,dz + \int_{-\gamma_2} f(z)\,dz \\ &= \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz && \text{(\# 3 in Theorem 20.1.5)} \end{aligned}$$

which implies that

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

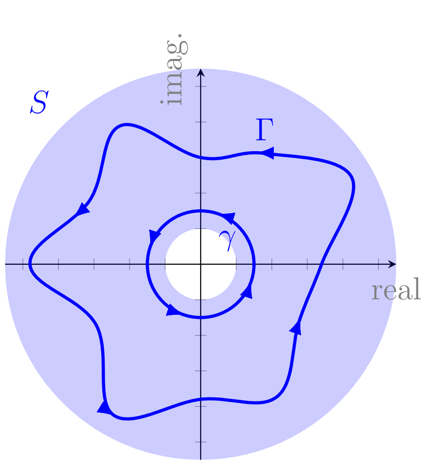
Exercise 20.3.1. Let γ_1 be the straight line from 1 to i given by $\gamma_1(t) = (1-t) + it$ with $0 \le t \le 1$. Let γ_2 be the circular arc from 1 to i given by $\gamma_2(t) = e^{it}$ with $0 \le t \le \frac{\pi}{2}$. Compute $\displaystyle\int_{\gamma_1} f(z)\,dz$ and $\displaystyle\int_{\gamma_2} f(z)\,dz$ with $f(z) = z^2 + 2iz - 3$.

20.3.2 The Deformation Theorem

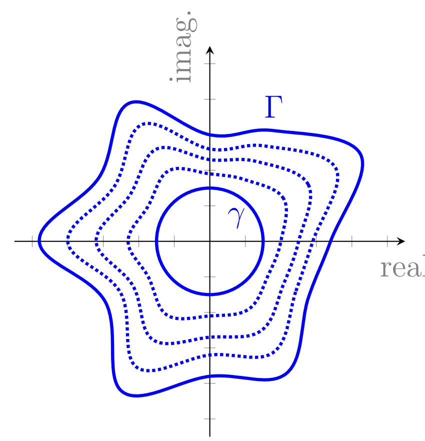
Independence of path suggests that, with minor assumptions, when integrating over a closed path, we can always try to integrate over a more convenient closed path.

Theorem 20.3.1 (Deformation)**.** *Let Γ and γ be closed paths with γ in the interior of Γ. Suppose f is differentiable on a set S containing both paths* <u>*and*</u> *all points in between them. Then*

$$\oint_{\Gamma} f(z)\, dz = \oint_{\gamma} f(z)\, dz.$$

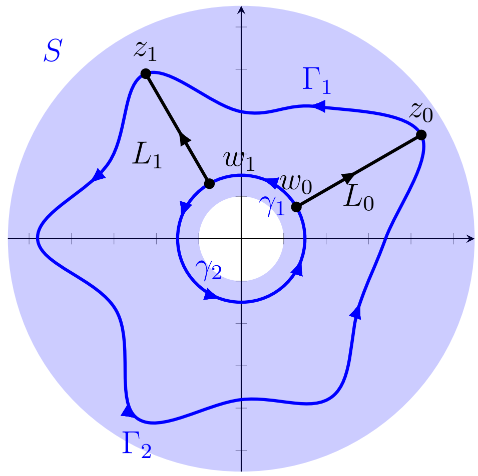


Assumptions of deformation theorem.



Step-by-step, the outer path Γ is being "deformed" and shrinking to the inner path γ

Proof. Let z_0, z_1 be any two distinct points on Γ and let w_0, w_1 be any two distinct points on γ. Let Γ_1 be the path from z_0 to z_1 and Γ_2 the path from z_1 to z_0 so that $\Gamma = \Gamma_1 \oplus \Gamma_2$. Similarly let γ_1 be the path from w_0 to w_1 and γ_2 is the path from w_1 to w_0 so that $\gamma = \gamma_1 \oplus \gamma_2$. Finally, let L_0 be a path from w_0 to z_0, and L_1 a path w_1 to z_1 (without loss of generality, we can choose L_0 and L_1 so that they do not cross). We have the picture below.



By Cauchy's Theorem we have that

$$\oint_{\Gamma_1 \oplus(-L_1)\oplus(-\gamma_1)\oplus L_0} f(z)\,dz = 0,$$
$$\oint_{\Gamma_2 \oplus(-L_0)\oplus(-\gamma_2)\oplus L_1} f(z)\,dz = 0.$$

Recall that

$$\oint_{\Gamma_1 \oplus(-L_1)\oplus(-\gamma_1)\oplus L_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz - \int_{L_1} f(z)\,dz - \int_{\gamma_1} f(z)\,dz + \int_{L_0} f(z)\,dz$$
$$\oint_{\Gamma_2 \oplus(-L_0)\oplus(-\gamma_2)\oplus L_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz - \int_{L_0} f(z)\,dz - \int_{\gamma_2} f(z)\,dz + \int_{L_1} f(z)\,dz$$

So,

$$\begin{aligned}
0 &= \oint_{\Gamma_1 \oplus(-L_1)\oplus(-\gamma_1)\oplus L_0} f(z)\,dz + \oint_{\Gamma_2 \oplus(-L_0)\oplus(-\gamma_2)\oplus L_1} f(z)\,dz \\
&= \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz - \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz \\
&= \int_{\Gamma_1\oplus\Gamma_2} f(z)\,dz - \int_{\gamma_1\oplus\gamma_2} f(z)\,dz \\
&= \oint_{\Gamma} f(z)\,dz - \oint_{\gamma} f(z)\,dz
\end{aligned}$$

hence

$$\oint_{\Gamma} f(z)\,dz = \oint_{\gamma} f(z)\,dz.$$

□

20.3.3 Cauchy's Integral Formula

In Example 20.2.4, we saw that, for $z_0 = 2i$

$$\oint_C \frac{1}{z - z_0}\, dz = 2\pi i,$$

where C was a circle of radius 1 centered at z_0. By the Deformation Theorem, we can conclude that in fact that should be true for *any* closed path γ surrounding z_0 (and in fact, the particular choice of z_0 is also unimportant). So it may be reasonable to ask, what happens if we consider a slightly more general integral like the one below?

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz$$

For simplicity, we'll assume γ lives in a domain, that f is differentiable on that domain, and that γ encloses a simply connected domain containing z_0. Then

$$\begin{aligned}
\oint_\gamma \frac{f(z)}{z - z_0}\, dz &= \oint_\gamma \frac{f(z) - f(z_0) + f(z_0)}{z - z_0}\, dz \\
&= \oint_\gamma \frac{f(z_0)}{z - z_0}\, dz + \oint_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz \\
&= 2\pi i f(z_0) + \oint_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz
\end{aligned}$$

What we want to know is that happens to that second integral. By the deformation theorem, it suffices to consider the case when γ is a small circle around z_0, so for some small radius r, we can parameterize γ as

$$\gamma(t) = z_0 + re^{it} \qquad 0 \le t \le 2\pi.$$

Then

$$\begin{aligned}
\oint_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz &= \int_0^{2\pi} \frac{f(z_0 + re^{it}) - f(z_0)}{re^{it}}\, ire^{it}\, dt \\
&= i \int_0^{2\pi} f(z_0 + re^{it}) - f(z_0)\, dt.
\end{aligned}$$

Since the integrand can be negative, it follows that

$$\left| i \int_0^{2\pi} f(z_0 + re^{it}) - f(z_0)\, dt \right| \le \int_0^{2\pi} |f(z_0 + re^{it}) - f(z_0)|\, dt.$$

Now, by continuity of f at z_0,

$$\lim_{r\to 0} |f(z_0 + re^{it}) - f(z_0)| = 0$$

so this implies that

$$\left| i \int_0^{2\pi} f(z_0 + re^{it}) - f(z_0)\, dt \right| = 0$$

and thus

$$\oint_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0.$$

Theorem 20.3.2 (Cauchy's Integral Formula)**.** *Let f be differentiable on an open set S. Let γ be a closed path in S enclosing only points of S. Then, for any z_0 enclosed by γ,*

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz$$

Example 20.3.3. Let γ be any closed path that does not pass through i and let $f(z) = 85e^{iz^{100}\pi}$. Evaluate $\displaystyle\oint \frac{f(z)}{z - i}\, dz$.

Note: $f(z)$ is differentiable for all $z \in \mathbb{C}$.

Case 1 (γ does not enclose i): Then by Cauchy's Theorem

$$\oint_\gamma \frac{f(z)}{z - i}\, dz = 0$$

Case 2 (γ encloses i): Then by Cauchy's Integral Formula

$$\oint_\gamma \frac{f(z)}{z - i}\, dz = 2\pi i (85e^{i\pi}) = -170\pi i.$$

We can even cleverly use Cauchy's Integral Formula to evaluate some real integrals that would have made us cry in MATH 1226. (It might still make you cry now, but at least it can be solved without resorting to numerical techniques.)

Example 20.3.4. Evaluate $\displaystyle\int_0^{2\pi} e^{\cos(\theta)} \cos(\sin(\theta))\, d\theta$.

Before tackling this head-on, we first examine another contour integral. By Cauchy's Theorem, for any closed path γ that encloses 0,

$$\oint_\gamma \frac{e^z}{z}\, dz = 2\pi i e^0 = 2\pi i.$$

Letting $\gamma(\theta) = e^{i\theta}$ be the unit circle, we have

$$\begin{aligned}\oint_\gamma \frac{e^z}{z}\, dz &= \int_0^{2\pi} \frac{e^{e^{i\theta}}}{e^{i\theta}}\, ie^{i\theta}\, d\theta = i\int_0^{2\pi} e^{e^{i\theta}}\, d\theta \\ &= i\int_0^{2\pi} e^{\cos\theta} e^{i\sin\theta}\, d\theta \\ &= i\int_0^{2\pi} e^{\cos\theta}\left(\cos(\sin\theta) + i\sin(\sin\theta)\right)\, d\theta \\ &= -\int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\, d\theta + i\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\, d\theta\end{aligned}$$

Notice that the imaginary part of this integral is exactly what we set out to solve! So its value must be the same as the imaginary part of $2\pi i$!

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\, d\theta = 2\pi.$$

Remark. You are not expected to just have the brilliant insight and cleverness to use contour integrals in that way; even WolframAlpha resorts to numerical techniques when given that integral. It's just really interesting to see that, with enough ingenuity, even some complicated real integrals can have deceptively simple values.

Theorem 20.3.5 (Cauchy's Integral Formula for Derivatives)**.** *Let f, S, γ, and z_0 be as in Cauchy's integral formula (Theorem 20.3.2). Then for any integer $n \geq 0$*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

Proof. The proof is a bit tedious, but more-or-less comes down to applying Cauchy's integral formula to the definition of the derivative and simplifying the terms. $\square$

Example 20.3.6. Evaluate $\oint_\gamma \frac{e^{z^3}}{(z-i)^3}\,dz$ where γ is a closed path that encloses i.

Let $f(z) = e^{z^3}$. Since γ encloses i and f is differentiable on all of $\mathbb{C}$, we can compute $f''(z)$ and then apply Theorem 20.3.5.

$$f'(z) = 3z^2 e^{z^3}, \qquad \text{and} \qquad f''(z) = (6z + 9z^4)e^{z^3}$$

hence

$$\oint_\gamma \frac{e^{z^3}}{(z-i)^3}\,dz = \frac{2\pi i}{2!} f''(i) = \pi i(6i+9)e^{-i}.$$

Not only does Cauchy's integral formula kind of feel like cheating, it actually has the following completely amazing consequence.

Corollary 20.3.7. *Suppose f is complex analytic on an open set S. Then <u>all</u> derivatives f exist on S.*

This is *very* different from the behavior in real analysis. For example. $f(x) = x^{2/3}$ is differentiable on all of $\mathbb{R}$, but the second derivative does not exist at $x = 0$.

20.3.4 Bounds on Derivatives

The following is also a consequence of Cauchy's Integral Formula for Derivatives.

Theorem 20.3.8. *Suppose f is differentiable on an open set S, suppose $z_0 \in S$, and suppose the closed disk of radius r centered at z_0 is entirely contained within S. If $|f(z)| \leq M$ for all z on the circle bounding the disk, then for any positive integer, we have that*

$$|f^{(n)}(z_0)| \leq \frac{M\,n!}{r^n}.$$

Since complex analytic functions are infinitely differentiable, a fascinating corollary of this theorem is

Theorem 20.3.9 (Liouville (1847))**.** *Suppose f is analytic on all of $\mathbb{C}$. If f is bounded, then f is a constant function.*

This behavior is *definitely* different from real analytic functions. For example, $f(x) = \arctan(x)$ is bounded, is infinitely differentiable on all of $\mathbb{R}$, and every one of its derivatives is bounded.

20.3.5 An Extended Deformation Theorem

What if we wanted to apply Cauchy's Integral Formula at multiple points?

Example 20.3.10. Let γ be the circle of radius $\frac{3}{2}$ centered at $\frac{1}{2}$, and let $f(z) = 3z+4$. Evaluate $\oint_\gamma \frac{f(z)}{z^2 - z}\,dz$.

We begin with a partial fraction decomposition.

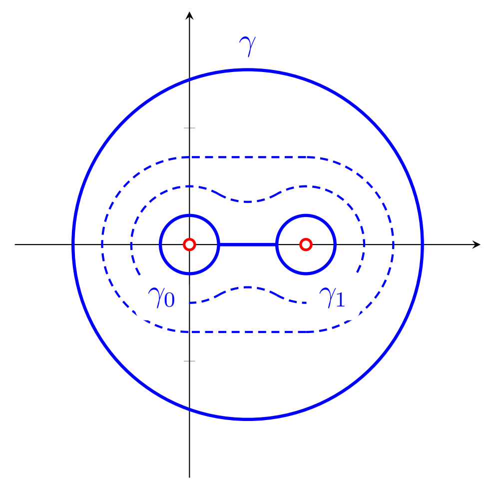
$$\frac{3z+4}{z^2-z} = \frac{A}{z} + \frac{B}{z-1} \qquad \Longrightarrow \qquad 3z+4 = A(z-1) + B(z).$$

We deduce that $A = -4$ and $B = 7$. Hence

$$\oint_\gamma \frac{3z+4}{z^2-z}\,dz = \oint_\gamma \frac{-4}{z}\,dz + \oint_\gamma \frac{7}{z-1}\,dz$$

By the deformation theorem, we can deform γ into the figure drawn on the right. From here it should be clear that the middle "loop" contributes nothing to the value of the integral, so we have

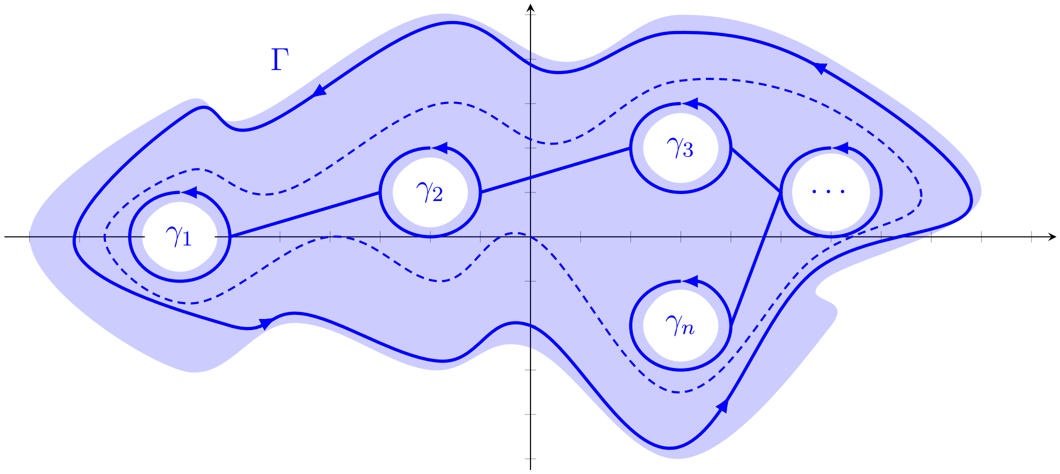
$$\begin{aligned}\oint_\gamma \frac{3z+4}{z^2-z}\,dz &= \oint_{\gamma_0} \frac{-4}{z}\,dz + \oint_{\gamma_1} \frac{-4}{z}\,dz \\ &\quad + \oint_{\gamma_0} \frac{7}{z-1}\,dz + \oint_{\gamma_1} \frac{7}{z-1}\,dz \\ &= -4(2\pi i) + 0 + 0 + 7(2\pi i) = 6\pi i.\end{aligned}$$



Intuitively, the strategy we used to deform the closed path should apply for any number of closed paths. And this is precisely the idea behind the proof of the following theorem.

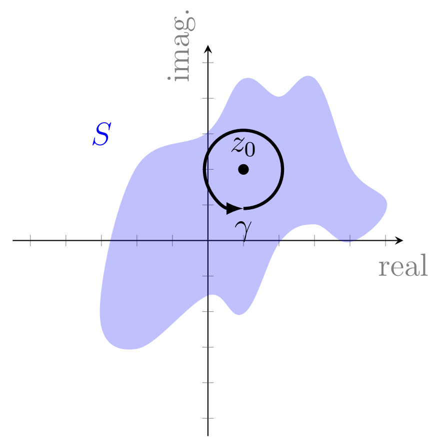
Theorem 20.3.11 (Extended Deformation)**.** *Let Γ be a closed path and let $\gamma_1, \dots, \gamma_n$ be closed paths enclosed by Γ. Assume that no two of $\Gamma, \gamma_1, \dots, \gamma_n$ intersect and no point interior to any γ_i is interior to any other γ_k. Let f be differentiable on an open set containing Γ and each γ_i and all points that are both interior to Γ and exterior to each γ_j. Then*

$$\oint_\Gamma f(z)\,dz = \sum_{k=1}^{n} \oint_{\gamma_k} f(z)\,dz.$$


Γ
γ_1
γ_2
γ_3
$\cdots$
γ_n

20.A Applications to Harmonic Functions

Further exploring the theme of how complex analysis can inform about features of real functions, one may wonder if Cauchy's Integral Formula tells us anything about harmonic function values. To explore this idea, let's suppose f is analytic on an open set S (so $f = u + iv$ with u and v harmonic on S). Let $z_0 = x_0 + iy_0$ be a fixed point in S, and let γ be a small circle of radius r around z_0 and completely contained in S.



Then

$$\begin{aligned}
f(z_0) = u(x_0, y_0) + iv(x_0, y_0) &= \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz \\
&= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\, d\theta \\
&\quad + \frac{i}{2\pi} \int_0^{2\pi} v(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\, d\theta
\end{aligned}$$

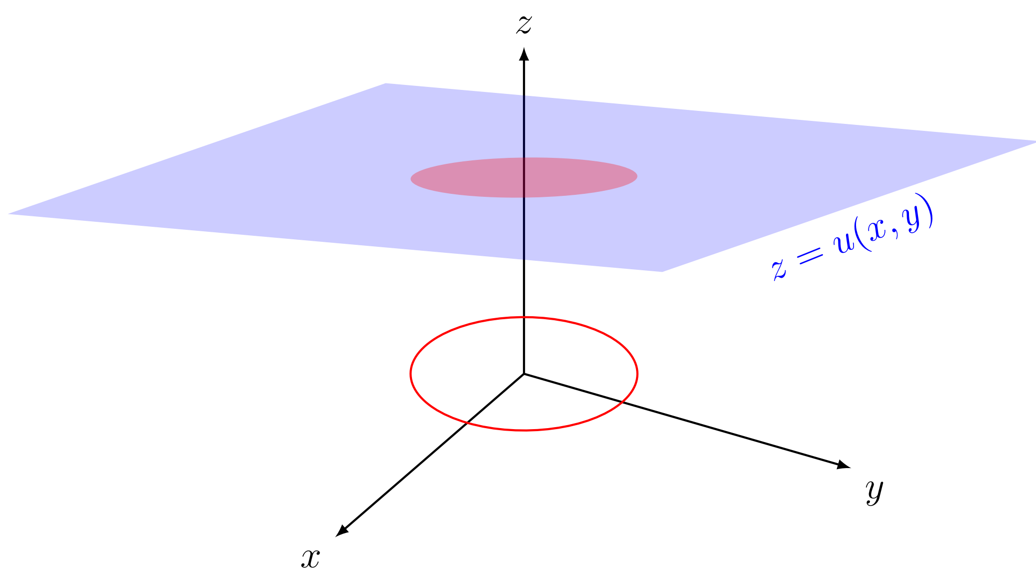
by comparing the real and imaginary parts of these equations, we get the following

Theorem 20.A.1 (The Mean Value Property). *Let u be harmonic on a domain D, let (x_0, y_0) be any point of D, and let C be a circle of radius r in D centered at (x_0, y_0) which encloses only points of D. Then*

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\, d\theta.$$

One implication of this theorem is that the "average value" of a harmonic function lies at the center of the disk enclosed by C.

Example 20.A.2. Let u be the harmonic function given by $u(x, y) = 3x + 5y + 2$ and let C be the circle of radius r about the origin $(x_0, y_0) = (0, 0)$. Geometrically in $\mathbb{R}^3$, $u(x, y)$ is a plane.



If we think about the region sitting above C (and the disk it encloses), we would expect that the average value on that region would be at the center of the disk - afterall, a plane is linear in every direction. To see this explicitly, we compute

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} u(0 + r\cos(\theta), 0 + r\sin(\theta))\ d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} 3r\cos(\theta) + 5r\sin(\theta) + 2\, d\theta \\
&= \frac{1}{2\pi}\left[3r\sin(\theta) - 5r\cos(\theta) + 2\theta\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left[3r(0) - 5r(1) + 2(2\pi) - 3r(0) + 5r(1) - 2(0)\right] \\
&= \frac{1}{2\pi}(4\pi) = 2 = u(0, 0).
\end{aligned}
$$

For a bounded domain D, we write $\overline{D}$ to denote the set D together with all of its boundary points; $\overline{D}$ is a closed set. The familiar Extreme Value Theorem says that if the function $u(x, y)$ is continous on $\overline{D}$, then it must obtain a maximum and minimum value on $\overline{D}$. If $u(x, y)$ is harmonic, then we can say a little bit more.

Theorem 20.A.3 (The Maximum Princple). *Let D be a bounded domain in the plane and suppose u is continuous on $\overline{D}$ and harmonic on D. Then $u(x, y)$ achieves its maximum value at a boundary point of D.*

This is also easy to see in the case of the preceding example.

21 Series Representations of Functions

21.1 Power Series

21.1.1 Sequences and Series of Complex Numbers

We assume familiarity with real sequences and series.

Definition. A *sequence* of complex numbers is an infinite collection $\{z_1, z_2, \ldots, \}$ where z_n is a complex number for every nonnegative integer n. We sometimes denote sequnces $\{z_n\}_{n=0}^{\infty}$ or just $\{z_n\}$.

Definition. Let $\{z_n\}$ be a sequence of complex numbers. For every z_n, there are real numbers x_n, y_n such that $z_n = x_n + iy_n$. The *limit* of $\{z_n\}$ is a complex number $L = a + ib$ where

$$\lim_{n\to\infty} x_n = a \qquad \text{and} \qquad \lim_{n\to\infty} y_n = b.$$

In this case we write $\lim_{n\to\infty} z_n = L$ or possibly just $z_n \to L$.

Definition. Given a sequence of complex numbers $\{c_n\}$, a *series* is a sequence of partial sums

$$\sum_{n=0}^{\infty} c_n = \lim_{k\to\infty} \sum_{n=0}^{k} c_n.$$

If this limit exists, we say that the series *converges*.

Proposition 21.1.1. *For each complex number c_n in the sequence $\{c_n\}$, write $c_n = a_n + ib_n$ for real a_n, b_n. Then $\sum_{n=0}^{\infty} c_n$ converges to a complex number $C = A + iB$ if and only if $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$.*

So convergence of complex series is equivalent to asking about convergence of real series. Although the reader is assumed to be familiar with these convergence tests, we will state them again for complex series.

Theorem 21.1.2 (Divergence Test)**.** *If $lim_{n\to\infty} z_n \neq 0$, then the series $\sum_{n=0}^{\infty} z_n$ diverges.*

Theorem 21.1.3 (Comparison Test). *Suppose* $\sum_{n=0}^{\infty} z_n$ *is a series of complex numbers and* $\sum_{n=0}^{\infty} M_n$ *is a series of real numbers with* $|z_n| \leq M_n$ *for all* n.

1. *If* $\sum_{n=0}^{\infty} M_n$ *converges, then* $\sum_{n=0}^{\infty} z_n$ *converges.*
2. *If* $\sum_{n=0}^{\infty} z_n$ *diverges, then* $\sum_{n=0}^{\infty} M_n$ *diverges.*

Remark. It may be worth noting that the Comparison Test for complex series is slightly different than expected. You may have initially wanted to compare two terms of a series, but given two complex numbers z and w, the inequality $z \leq w$ does not have a meaning (the fancy phrase is that "there is no partial ordering on $\mathbb{C}$ which respects the field structure".) The next best thing, which is how the theorem is stated, is to compare the magnitude of the terms of a real series.

Theorem 21.1.4 (Ratio Test). *Consider the series* $\sum_{n=0}^{\infty} z_n$ *and let* $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$.

1. *If* $L > 1$, $\sum_{n=0}^{\infty} z_n$ *diverges.*
2. *If* $L < 1$, $\sum_{n=0}^{\infty} z_n$ *converges.*

Theorem 21.1.5 (Geometric Series). *The series* $\sum_{n=0}^{\infty} az^n$ *converges if and only if* $|z| < 1$.

Moreover, if $\sum_{n=0}^{\infty} az^n$ *converges, then it converges to* $\frac{a}{1-z}$.

Definition. A series of complex numbers $\sum_{n=0}^{\infty} c_n$ *converges absolutely* if the real series $\sum_{n=0}^{\infty} |c_n|$ converges.

21.1.2 Power Series and Taylor Series

Definition. A $\boxed{\textit{power series}}$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots .$$

The complex numbers c_n are called the $\boxed{\textit{coefficients}}$ of the power series, and z_0 is called the $\boxed{\textit{center}}$ of the power series.

Just as for real power series, one fundamental question is about finding z-values for which the power series converges.

Remark. Just as in the real power series case, our goal is to think about the function $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$. This function is only defined when the output is some finite number, i.e., for z-values where the series converges.

Theorem 21.1.6. *Suppose $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges at $z_1 \neq z_0$ (that is, suppose $\sum_{n=0}^{\infty} c_n(z_1-z_0)^n$ is a convergent series). Then this series converges absolutely for all z-values satisfying*

$$|z-z_0| < |z_1-z_0|.$$

Proof. Because $\sum_{n=0}^{\infty} c_n(z_1-z_0)^n$ converges,

$$\lim_{n\to\infty} c_n(z_1-z_0)^n = 0.$$

This means that we can find a sufficiently large N so that, for all $n \geq N$,

$$|c_n(z_1-z_0)^n| < 1.$$

As such, for all $n \geq N$,

$$|c_n(z-z_0)^n| = \frac{|(z_1-z_0)^n|}{|(z_1-z_0)^n|}|c_n(z-z_0)^n|$$

which rearranges to

$$|c_n(z-z_0)^n| = \frac{|(z-z_0)^n|}{|(z_1-z_0)^n|}|c_n(z_1-z_0)^n| \leq \left|\frac{(z-z_0)^n}{(z_1-z_0)^n}\right|(1) = \left|\frac{z-z_0}{z_1-z_0}\right|^n.$$

When $|z-z_0| < |z_1-z_0|$, then we have that $\left|\dfrac{z-z_0}{z_1-z_0}\right| < 1$, hence the geometric series

$$\sum_{n=1}^{\infty}\left|\frac{z-z_0}{z_1-z_0}\right|^n$$

converges by the Geometric Series Test (21.1.5). By the comparison test (21.1.3), it follows that the series

$$\sum_{n=0}^{\infty}|c_n(z-z_0)^n|$$

converges. As such, the following series converges absolutely:

$$\sum_{n=0}^{\infty}c_n(z-z_0)^n.$$

□

Letting $r = |z_1 - z_0|$, then the equation in the theorem has the form $|z-z_0| < r$, so geometrically, if the series converges on the boundary of a disk of radius r centered at z_0, then it converges absolutely on the interior of that disk.

Definition. The *radius of convergence* R, is the radius of the largest disk around z_0 on which the series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges. The disk $|z-z_0| < R$ is called the *disk of convergence*.

Proposition 21.1.7. *For a given power series, the radius of convergence is unique, and the series diverges outside of this disk (i.e. for z-values satisfying $|z-z_0| > R$.*

Proof. The radius is unique by definition. The series must diverge outside of this disk, for if it didn't, then by Theorem 21.1.6, there would be a disk of larger radius on which the series converged. □

Fact. A power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ always converges at the center z_0.

If a power series converges *only* at the center, then we may write $R = 0$, and if it converges for all complex numbers, we write $R = \infty$.

As with real power series, we can sometimes compute the radius of convergence via the ratio test.

Example 21.1.8. Determine the radius of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{(-5)^n}{n+1}(z-i)^n$.

According to the ratio test, this series converges when

$$\begin{aligned}
1 &> \lim_{n\to\infty} \left| \frac{\dfrac{(-5)^{n+1}}{n+2}(z-i)^{n+1}}{\dfrac{(-5)^n}{n+1}(z-i)^n} \right| \\
&= \lim_{n\to\infty} \left| (-5)\left(\frac{n+1}{n+2}\right)(z-i) \right| \\
&= 5|z-i| \\
\implies \qquad \frac{1}{5} &> |z-i|.
\end{aligned}$$

So the radius of convergence for this power series is $R = \frac{1}{5}$.

Example 21.1.9. Determine the radius of convergence for $\displaystyle\sum_{n=0}^{\infty} n!\,(z-2+3i)^n$.

According to the ratio test, this series converges when

$$\begin{aligned}
1 &> \lim_{n\to\infty} \left| \frac{(n+1)!(z-2+3i)^{n+1}}{n!(z-2+3i)^n} \right| \\
&= \lim_{n\to\infty} n\left|(z-2+3i)\right|.
\end{aligned}$$

When $z \neq 2-3i$ this series diverges, so it has radius of convergence $R = 0$.

Example 21.1.10. Determine the radius of convergence for $\sum_{n=0}^{\infty} \frac{1}{n!} (z+2)^n$.

According to the ratio test, this series converges when

$$\begin{aligned} 1 &> \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}(z+2)^{n+1}}{\frac{1}{n}(z+2)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(z+2)}{n+1} \right| \\ &= 0. \end{aligned}$$

The series converges for all $z \in \mathbb{C}$, So the radius of convergence for this power series is $R = \infty$.

The following theorem is analogous to the familiar version from real analysis.

Theorem 21.1.11 (Differentiation and Integration of Power Series)**.** *Let f be the function given by*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

for z in D, the open disk of convergence.

1. *f is complex differentiable with derivative given by*

$$f'(z) = \sum_{n=0}^{\infty} \frac{d}{dz} c_n (z-z_0)^n = \sum_{n=1}^{\infty} n c_n (z-z_0)^{n-1} \qquad \text{for } z \in D.$$

   Moreover, the power series for $f'(z)$ has the same radius of convergence as f.

2. *If γ is a path within D, then*

$$\int_\gamma f(z)\,dz = \sum_{n=0}^{\infty} c_n \int_\gamma (z-z_0)^n\,dz.$$

Proof. The proof of this is actually longer and less straightforward than one might hope; we can't just quickly apply Cauchy–Riemann. That the derivative and integral are defined the way they are is obvious, but that the sequence of partial sums still converges to the appropriate limit (and with the same disk of convergence) is technical and involves the notion of uniform convergence, which we wont be covering. □

Theorem 21.1.12 (Taylor Expansion)**.** *Suppose f is differentiable on an open disk D of radius R centered at z_0. Then, for $z \in D$,*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where

$$c_n = \frac{f^{(n)}(z_0)}{n!}$$

Proof. See the text. The strategy is effectively

- Apply Cauchy's Integral Formula to rewrite $f(z)$ as a contour integral.
- With clever algebraic manipulations, recognize the integrand as the limit of a convergent geometric series
- Integrate this series using 21.1.11.
- Use Cauchy's Integral Formula for Derivatives to rewrite the coefficients of this series.

□

Definition. The series in Theorem 21.1.12 is called the $\boxed{\textit{Taylor series}}$ of f about z_0 (or $\boxed{\textit{Maclaurin series}}$ in the case that $z_0 = 0$). The coefficients are called the $\boxed{\textit{Taylor coefficients}}$ of f at z_0.

Example 21.1.13. Since $\frac{d}{dz}[e^z] = e^z$, just as in the real case, the Maclaurin expansion of e^z should also look like the Maclaurin series for e^x:

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

Example 21.1.14. Since $\frac{d}{dz}[\sin(z)] = \cos(z)$ and $\frac{d}{dz}[\cos(z)] = -\sin(z)$, just as in the real case, the Maclaurin expansions of $\sin(z)$ and $\cos(z)$ should also look like the Maclaurin series for $\sin(x)$ and $\cos(x)$ (respectively):

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1},$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.$$

21.1.3 Isolated Zeros

Definition. For a function f, a number ζ for which $f(\zeta) = 0$ is called an $\boxed{\textit{isolated zero}}$ if there is an open disk around ζ which contains no other zero for f.

Example 21.1.15. The function $f(z) = \sin(z)$ has an isolated zero at $z = 0$.

Given what we know about the real sine function, the fact that $z = 0$ is an isolated zero for $f(z) = \sin(z)$ certainly *seems* reasonable, but how do we know it's actually the case for the complex sine function? It turns out Taylor series can provide the answer.

Let ζ be a zero for f and consider the Taylor expansion of f in a small disk D around around ζ

$$f(z) = \sum_{n=0}^{\infty} c_n (z - \zeta)^n.$$

If every $c_n = 0$, then $f(z) = 0$ for all $z \in D$, so suppose this isn't the case. Let m be the first value for which $c_m \neq 0$ (that is, $c_0 = c_1 = \ldots = c_{m-1} = 0$). Then we have that

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} c_n (z - \zeta)^n \\
&= \sum_{n=m}^{\infty} c_n (z - \zeta)^n && \text{(first terms all 0)} \\
&= \sum_{k=0}^{\infty} c_{k+m} (z - \zeta)^{k+m} && \text{(where } k = n - m\text{)} \\
&= (z - \zeta)^m \sum_{k=0}^{\infty} c_{k+m} (z - \zeta)^k
\end{aligned}$$

Now let

$$g(z) = \sum_{k=0}^{\infty} c_{k+m} (z - \zeta)^k = c_m + \sum_{k=1}^{\infty} c_{k+m} (z - \zeta)^k.$$

By construction we have that $g(\zeta) = c_m \neq 0$ and

$$f(z) = (z - \zeta)^m g(z).$$

Since f is differentiable on D, then so is g; in particular, g is continuous at ζ, so since $g(\zeta) \neq 0$, then there is a small disk D_g around ζ on which $g(z) \neq 0$ (otherwise we would break the intermediate value theorem). It follows that $f(z) \neq 0$ on this same disk as well, making ζ an isolated zero. So what this says is that

Theorem 21.1.16. *Suppose f is differentiable on a domain S, and let $\zeta \in S$ be a zero of f. Then either*

$$f(z) = 0 \text{ on all of } S,$$

or

$$\zeta \text{ is an isolated zero.}$$

From the theorem and the proof preceding it, what we have is that, if $f(z)$ has a nonzero Taylor coefficient in the series centered at ζ, then ζ is an isolated zero.

Example 21.1.17. Show that $f(z) = \sin(z)$ has an isolated zero at $z = 0$.

By the above commentary, we just need to examine the Taylor coefficients of f centered at 0.

$$c_0 = \frac{f(0)}{0!} = \frac{\sin(0)}{1} = 0c_1 \qquad = \frac{f'(0)}{1!} = \frac{\cos(0)}{1} = 1$$

Since $c_1 \neq 0$, then $z = 0$ must be isolated.

Definition. A point z_0 is said to be a zero of $\boxed{\textit{order } m}$ if f is differentiable at z_0 and the first nonzero coefficient in the Taylor expansion around z_0 is c_m. If z_0 is a zero of order 1, it is sometimes called a $\boxed{\textit{simple zero}}$.

Example 21.1.18. For $f(z) = \sin(z)$, determine the order of the zero $z = 0$.

$z = 0$ has order 1 per our work in Example 21.1.17.

While proving Theorem 21.1.16, we actually proved the following, but we'll state explicitly.

Proposition 21.1.19. *If f is differentiable at z_0, then z_0 is a zero of order m if and only if we can write*

$$f(z) = (z - z_0)^m g(z).$$

where $g(z_0) \neq 0$ and g is differentiable at z_0.

Example 21.1.20. Find the order of the zero $z_0 = 0$ of the function $\varphi(z) = \sin^3(z)$.

Looking at the Taylor expansion of $\sin^3(z)$ about $z_0 = 0$, we have

$$\begin{aligned}\sin^3(z) = (\sin(z))^3 &= \left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)^3 \\ &= z^3 - \frac{1}{2}z^5 + \frac{13}{120}z^7 - \frac{41}{3024}z^9 + \cdots \\ &= z^3\left(1 - \frac{1}{2}z^2 + \frac{13}{120}z^4 - \frac{41}{3024}z^6 + \cdots\right)\end{aligned}$$

so taking $g(z) = 1 - \frac{1}{2}z^2 + \frac{13}{120}z^4 - \frac{41}{3024}z^6 + \cdots$, we have that

$$\sin^3(z) = z^3 g(z)$$

with $g(0) \neq 0$, hence $\varphi(z) = \sin^3(z)$ has a zero of order 3 at $z_0 = 0$.

Since we can write $f(z) = (z - z_0)^n g(z)$ with $g(z_0) \neq 0$, we get the following

Corollary 21.1.21. *Suppose z_0 is a zero of order m of $h(z)$, and that z_0 is a zero of order n of $k(z)$. Then*

1. *At z_0, $h(z)k(z)$ has a zero of order* $m + n$
2. *If $m > n$, then at z_0, $h(z)/k(z)$ has a zero of order* $m - n$.

Proof sketch. Write

$$\begin{aligned}h(z) &= (z - z_0)^m \alpha(z) \\ k(z) &= (z - z_0)^n \beta(z)\end{aligned}$$

Then

$$h(z)k(z) = (z - z_0)^{m+n}\alpha(z)\beta(z)$$

and

$$\frac{h(z)}{k(z)} = \frac{(z - z_0)^m \alpha(z)}{(z - z_0)^n \beta(z)} = (z - z_0)^{m-n}\frac{\alpha(z)}{\beta(z)}$$

□

Remark. The term "zero" in the second item of Corollary 21.1.21 is maybe a bit misleading, because $h(z)/k(z)$ is not even defined at z_0 (and as such, is certainly not 0). The requirement that $m > n$ implies that the limit $L = \lim_{z \to z_0} h(z)/k(z)$ exists.

So what we're actually thinking of is a zero of a *continuous extension* (or an *analytic continuation* of $h(z)/k(z)$ at z_0. Explicitly, for z in a small disk around z_0 where $k(z) \neq 0$, we are looking at a zero of the function

$$\widetilde{(h/k)}(z) := \begin{cases} h(z)/k(z) & \text{when } z \neq z_0, \\ L & \text{when } z = z_0. \end{cases}$$

Remark. Some authors may write $h(z)/k(z)$ to refer to the maximal analytic continuation of the quotient of $h(z)$ and $k(z)$. I will not be adopting this convention, but it is out there.

Example 21.1.22. Find the order of the zero $z_0 = 0$ of the function $f(z) = z^2 \sin^2(z)$.

By the previous theorem, it suffices to find the orders of z^2 and $\sin^2(z)$ independently and add them together.

$\boldsymbol{h(z) = z^2}$		$\boldsymbol{k(z) = \sin^2(z)}$	
n	$h^{(n)}(z_0)$	n	$k^{(n)}(z_0)$
1	$2(0) = 0$	1	$2\sin(0)\cos(0) = 0$
2	2	2	$2\cos^2(0) - 2\sin^2(0) = 2$

Since z_0 is a zero of order 2 for z^2 and order 2 for $\sin^2(z)$, then z_0 is a zero of order $2 + 2 = 4$ for f.

Example 21.1.23. Find the order of the zero $z_0 = \frac{3\pi}{2}$ of the function $f(z) = \dfrac{\cos^3(z)}{z - \frac{3\pi}{2}}$.

By the previous theorem, it suffices to find the orders of $\cos^3(z)$ and $z - \frac{3\pi}{2}$ independently and add them together.

$\boldsymbol{h(z) = z - \frac{3\pi}{2}}$		$\boldsymbol{k(z) = \cos^3(z)}$	
n	$h^{(n)}(z_0)$	n	$k^{(n)}(z_0)$
1	1	1	$-3\sin\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{2}\right) = 0$
		2	$6\sin^2\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{2}\right) - 3\cos^3\left(\frac{3\pi}{2}\right) = 0$
		3	$21\sin\left(\frac{3\pi}{2}\right)\cos^2\left(\frac{3\pi}{2}\right) - 6\sin^3\left(\frac{3\pi}{2}\right) = 6$

Since z_0 is a zero of order 1 for $z - \frac{3\pi}{2}$ and order 3 for $\cos^3(z)$, then z_0 is a zero of order $3 - 1 = 2$ for f.

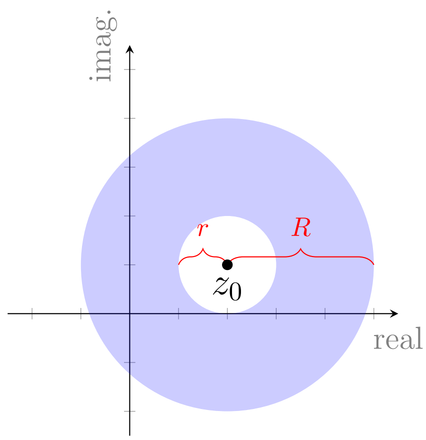
21.2 The Laurent Expansion

If f is differentiable in a small disk around z_0, then it has a Taylor series expansion at z_0. Of course, this isn't always the case, but we can get a different kind of series expansion if f is differentiable *near* z_0 (even if it isn't differentiable at z_0).

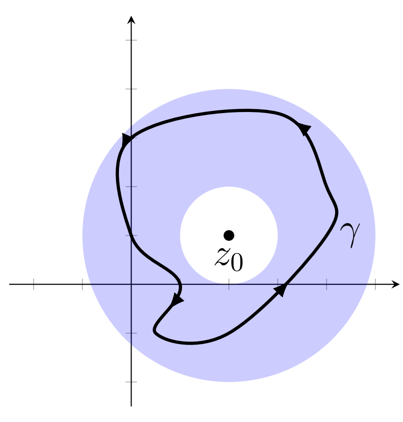
Definition. An $\boxed{\textit{annulus}}$ is the open set between two concentric circles, and can be written as the set of all z satisfying

$$r < |z - z_0| < R.$$

If $r = 0$, we call this a $\boxed{\textit{punctured disk}}$.



Suppose f is differentiable in an annulus centered at z_0 and let γ be a simple closed path in this annulus that encloses z_0.



If we wanted to try to construct Taylor coefficients for this function, they would be of the form

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

Since f is differentiable on an open set containing γ, it *almost* meets the criteria to apply Cauchy's Integral Formula for derivatives (Theorem 20.3.5), in which case the Taylor coefficients would simplify to

$$c_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$

Even though we can't obtain a proper Taylor series or apply Cauchy's Integral Formula in exactly this way, we can show that one does obtain a similar-looking series:

Theorem 21.2.1 (Laurent Expansion)**.** *Suppose f is differentiable in the annulus $r < |z - z_0| < R$ where $0 \le r < R \le \infty$. Then for each z in this annulus, we have*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where, for each integer n,

$$c_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$

for any closed path γ in the annulus enclosing z_0.

Definition. A series as in Theorem 21.2.1 is called a $\boxed{\textit{Laurent series}}$.

Example 21.2.2. Find a Laurent series expansion for $f(z) = e^{1/z}$ around $z_0 = 0$.

Firing up our trusty computer algebra system, one can confirm the following:

$$c_n = \begin{cases} \frac{1}{(-n)!} & \text{if } n \le 0 \\ 0 & \text{if } n > 0 \end{cases}$$

so we have that

$$e^{1/z} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n.$$

We notice that the Laurent series obtained for $e^{1/z}$ looks a lot like the Taylor series for e^z with the modification that $z \mapsto \frac{1}{z}$.

Proposition 21.2.3. *Given a function f that is differentiable in an annulus centered at z_0, the Laurent expansion about z_0 is unique.*

This is great because it means that, no matter how we obtain a Laurent series for our function, it must be the correct one. So often times we'll just manipulate known Taylor series to obtain the correct Laurent expansion (because nobody wants to compute those coefficients by hand if they can help it).

Example 21.2.4. Compute the Laurent series expansion for $f(z) = \dfrac{1}{z-2}$ on the annulus $|z-1| > 1$ (i.e. the punctured plane).

Notice that this annulus is centered at 1, so our Laurent series will be as well. Notice also that this annulus excludes $z = 2$, so in fact $f(z)$ is differentiable on it. Recall that the geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r| < 1$. Since $|z-1| > 1$, we must have that $\left|\frac{1}{z-1}\right| < 1$, hence

$$\begin{aligned}
f(z) = \frac{1}{z-2} &= \frac{1}{(z-1)-1} \\
&= \left(\frac{1}{z-1}\right)\frac{1}{1-\frac{1}{z-1}} \\
&= \left(\frac{1}{z-1}\right)\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n \\
&= \frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \cdots \\
&= \sum_{n=1}^{\infty}\frac{1}{(z-1)^n} \\
&= \sum_{n=-\infty}^{-1}(z-1)^n.
\end{aligned}$$

Example 21.2.5. Compute the Laurent series expansion for $f(z) = \dfrac{2z-1-i}{(z-1)(z-i)}$ about $z = 1$.

Clearly f is not differentiable at 1 or i, so whatever our annulus is must avoid these two points. Using partial fractions

$$\frac{2z-1-i}{(z-1)(z-i)} = \frac{1}{z-1} + \frac{1}{z-i}. \tag{21.2.1}$$

The first term is already a Laurent series about $z = 1$ (it only has one term) and its defined on the annulus $|z-1| > 0$, so we focus only on the second term and aim to use the geometric series again. We can rewrite

$$\frac{1}{z-i} = \frac{1}{(z-1)+(1-i)}. \tag{21.2.2}$$

At this points, we have a couple of options for approach: we can factor out $\frac{1}{z-1}$ or we can factor out $\frac{1}{1-i}$. In both cases we'll work with the geometric series, but the resulting annulus will be different.

(Case 1) We rewrite Equation 21.2.2 as

$$\frac{1}{z-i} = \frac{1}{(z-1)+(1-i)} = \left(\frac{1}{z-1}\right)\frac{1}{1-\frac{i-1}{z-1}}$$

and assuming $\left|\frac{1-i}{z-1}\right| < 1$, from the geometric series this becomes

$$\begin{aligned}
\left(\frac{1}{z-1}\right)\frac{1}{1-\frac{i-1}{z-1}} &= \left(\frac{1}{z-1}\right)\sum_{n=0}^{\infty}\left(\frac{i-1}{z-1}\right)^{n} \\
&= \sum_{n=0}^{\infty}\frac{(i-1)^{n}}{(z-1)^{n+1}} \qquad\qquad (\text{setting } k=-n-1) \\
&= \sum_{k=-\infty}^{-1}\frac{1}{(1-1)^{k+1}}(z-1)^{k}.
\end{aligned}$$

Substituting this into the Equation 21.2.1 we get the Laurent expansion for f

$$f(z) = \frac{1}{z-1} + \sum_{k=-\infty}^{-1}\frac{1}{(1-1)^{k+1}}(z-1)^{k}$$

for z-values satisfying $|z-1| > 0$ and $\left|\frac{1-i}{z-1}\right| < 1$, i.e. on the annulus $|z-1| > \sqrt{2}$.

(Case 2) We rewrite Equation 21.2.2 as

$$\frac{1}{z-i} = \frac{1}{(z-1)+(1-i)} = \left(\frac{1}{1-i}\right)\frac{1}{1-\frac{z-1}{i-1}}$$

and assuming $\left|\frac{z-1}{1-i}\right| < 1$, for the geometric series this becomes

$$\begin{aligned}
\left(\frac{1}{1-i}\right)\frac{1}{1-\frac{z-1}{i-1}} &= \left(\frac{1}{1-i}\right)\sum_{n=0}^{\infty}\left(\frac{z-1}{i-1}\right)^{n} \\
&= \sum_{n=0}^{\infty}\frac{-1}{(i-1)^{n+1}}(z-1)^{n}.
\end{aligned}$$

Substituting this into the Equation 21.2.1 we get the Lauren expansion for f

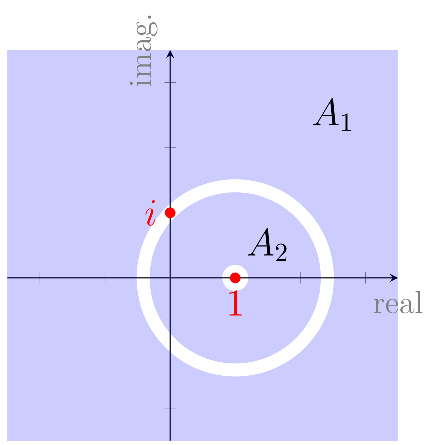
$$f(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{-1}{(i-1)^{n+1}}(z-1)^n$$

for z-values satisfying $|z-1| > 0$ and $\left|\frac{z-1}{1-i}\right| < 1$, i.e. on the annulus $0 < |z-1| < \sqrt{2}$.

Remark. There's not really a cohesive way to write these series succinctly in the form $\sum_{n=-\infty}^{\infty} c_n(z-1)^n$ because the coefficients don't all follow a nice pattern. That's fine. The same is true of Taylor series of real functions. For example, the Taylor expansion of $1 + \cos(x)$ about $x_0 = 0$ is

$$1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Remark. The $\sqrt{2}$ in the annulus isn't too surprising when you think about it geometrically - that's the distance between 1 and i. The fact that we had two possible annuli is then not surprising at all - having an annulus centered at $z = 1$, there were only two possible options for which f could be analytic within the whole annulus (either it avoided all singularities, or it avoided one and not the other). Visually, letting A_1 and A_2 be the annuli in cases 1 and 2, respectively,



22 Singularities and the Residue Theorem

22.1 Singularities

In this section we'll use the Laurent expansion to find and classify points at which complex functions are not differentiable.

Definition. We say that a function f has an *isolated singularity* at z_0 if f is differentiable in an annulus $0 < |z - z_0| < R$, but not at z_0.

Example 22.1.1. $f(z) = \dfrac{1}{z}$ has an isolated singularity at $z = 0.$

22.1.1 Classification of Singularities

Definition. Suppose f has an isolated singularity at z_0. Let the Laurent expasion of $f(z)$ in an annulus $0 < |z - z_0| < R$ be

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

1. z_0 is a *removable singularity* if $c_n = 0$ for every $n < 0$.
2. z_0 is a *pole* of order m if $m > 0$ and $c_{-m} \neq 0$, but $c_{-m-1} = c_{-m-2} = c_{-m-3} = \cdots = 0$.
3. z_0 is an *essential singularity* if $c_{-n} \neq 0$ for infinitely many $n > 0$.

In other words,

1. z_0 is removable if the Laurent expansion is actually a power series.
2. z_0 is a pole of order m if $\dfrac{1}{(z - z_0)^m}$ is the largest power of $\dfrac{1}{z - z_0}$ appearing in the Laurent expansion.
3. z_0 is essential if the Laurent expansion contains infinitely many powers of $\dfrac{1}{z - z_0}$ with nonzero coefficients.

Example 22.1.2. Consider $f(z) = \dfrac{\sin(z)}{z}$, which is analytic on all of $\mathbb{C}$ except at

$z = 0$ (where it isn't defined). The Laurent expansion of f around $z_0 = 0$ is

$$\begin{aligned}\frac{1}{z}\sin(z) &= \frac{1}{z}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}\right)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}\\ &= 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \frac{1}{7!}z^6 + \cdots\end{aligned}$$

Since this is a power series about $z_0 = 0$, $z_0 = 0$ is a removable singularity. As such we can extend f to a function φ which is differentiable at $z_0 = 0$:

$$\varphi(z) = \begin{cases} f(z) & \text{when } z \neq 0 \\ 1 & \text{when } z = 0 \end{cases}$$

Definition. The function φ above is called the $\boxed{\textit{analytic continuation}}$ of f at z_0.

Example 22.1.3. Consider the function $f(z) = \dfrac{1}{(z-i)^5}$. f has an isolated signularity at $z_0 = i$, which is a pole of order 5.
The Laurent expansion of f around $z_0 = i$ is

$$\frac{1}{(z-i)^5} \quad (f \text{ is its own Laurent expansion}).$$

There is no way that f can be extended to be differentiable at $z_0 = i$.

Example 22.1.4. Consider the function $f(z) = \dfrac{\cos(z)}{z^4}$, which is analytic on all of $\mathbb{C}$ except at $z_0 = 0$ where it isn't defined. The Laurent expansion of f about $z_0 = 0$ is

$$\begin{aligned}\frac{1}{z^4}\cos(z) &= \frac{1}{z^4}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}\right)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n-4}\\ &= \frac{1}{z^4} - \frac{1}{2!z^2} + \frac{1}{4!} - \cdots\end{aligned}$$

so $f(z)$ has a pole of order 4, and it cannot be extended to be differentiable at $z_0 = 0$.

Example 22.1.5. The Laurent expansion of $f(z) = e^{1/z}$ about $z_0 = 0$ is

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n} = \cdots + \frac{1}{3! z^3} + \frac{1}{2! z^2} + \frac{1}{z} + 1.$$

f has an essential singularity at $z_0 = 0$ because infinitely many powers of $\frac{1}{z}$ appear in this Laurent expansion.

Definition. A pole of order 1 is called a $\boxed{\textit{simple pole}}$ and a pole of order 2 is called a $\boxed{\textit{double pole}}$.

Let us consider the Laurent expansion of some function f in an annulus $0 < |z - z_0| < R$:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

If f has a pole of order m at z_0, then $c_{-m} \neq 0$ but $c_{-m-1} = c_{-m-2} = \cdots = 0$, so the Laurent expansion is

$$f(z) = \frac{c_{-m}}{(z - z_0)^m} + \frac{c_{-m-1}}{(z - z_0)^{m+1}} + \frac{c_{-m-2}}{(z - z_0)^{m+2}} + \cdots$$
$$(z - z_0)^m f(z) = c_{-m} + c_{-m-1}(z - z_0) + c_{-m-2}(z - z_0^2 + \cdots$$

and so

$$\lim_{z \to z_0} (z - z_0)^m f(z) = c_{-m} \neq 0.$$

As it turns out, the existence of this limit is enough to deduce that a function has a pole of order m at z_0. Explicitly,

Theorem 22.1.6. *Suppose f is differentiable in $0 < |z - z_0| < R$. Then f has a pole of order m at z_0 if and only if*

$$\lim_{z \to z_0} (z - z_0)^m f(z)$$

exists and is nonzero.

22.1.2 Zeroes and Poles

When looking for poles, it seems natural to look for places where the denominator is zero, especially if $f(z) = g(z)/h(z)$ is a quotient of functions.

Lemma 22.1.7. *A function f has a pole of order m at z_0 if and only if, in some annulus $0 < |z - z_0| < R$,*

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

where g is differentiable at z_0 and $g(z_0) \neq 0$.

Proof. If f has a pole of order m, then its Laurent expansion about z_0 is

$$f(z) = \frac{c_{-m}}{(z - z_0)^m} + \frac{c_{-m+1}}{(z - z_0)^{m-1}} + \cdots$$
$$(z - z_0)^m f(z) = c_{-m} + c_{-m+1}(z - z_0) + c_{-m+2}(z - z_0)^2 + \cdots$$

so writing

$$g(z) = c_{-m} + c_{-m+1}(z - z_0) + c_{-m+2}(z - z_0)^2 + \cdots$$

we have that $g(z_0) \neq 0$ and g is defined by its Taylor expansion about z_0 (whence it is differentiable). □

Theorem 22.1.8. *Let $f(z) = g(z)/h(z)$ where g, h are analytic in some open disk about z_0. Suppose that z_0 is a zero of order m for g and a zero of order n for h with $n > m$. Then f has a pole of of order $n - m$ at z_0.*

Proof. From Proposition **??**,we can write

$$g(z) = (z - z_0)^m \tilde{g}(z)$$
$$h(z) = (z - z_0)^n \tilde{h}(z)$$

where $\tilde{g}$ and $\tilde{h}$ are differentiable and nonzero at z_0. It follows that $\tilde{g}/\tilde{h}$ is differentiable and nonzero at z_0 and

$$\frac{g(z)}{h(z)} = \frac{\tilde{g}(z)/\tilde{h}(z)}{(z - z_0)^{n-m}}$$

thus, by the above lemma, $f(z) = g(z)/h(z)$ has a pole of order $n - m$ at z_0. □

Remark. One could adopt the convention that $f(z)$ has a zero of order 0 (and likewise, a pole of order 0) if f is defined at z and if $f(z) \neq 0$. This is not standard to my knowledge, but the calculations do align with the theorem.

Example 22.1.9. Find the order of the pole of $f(z) = \dfrac{e^z - 1}{\sin^7(z)}$ at $z_0 = 0$. The motivated student could compute the Laurent expansion of f about 0 directly to get

$$f(z) = \frac{1}{z^6} + \frac{1}{2}\frac{1}{z^5} + \frac{4}{3}\frac{1}{z^4} + \cdots$$

in which case it is quickly seen that f has a pole of order 6 at $z_0 = 0$. Rather than do this, by looking at the Maclaurin series, it is straightforward to see that $e^z - 1$ has a zero of order 1

$$e^z - 1 = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots$$

and that $\sin^7(z)$ has a zero of order 7

$$\begin{aligned} \sin(z) &= z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots \\ \sin^7(z) &= \left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)^7 \\ &= z^7 - \frac{7}{6}z^9 + \frac{77}{120}z^{11} + \cdots \end{aligned}$$

so by the theorem, $f(z)$ has a pole of order $7 - 1 = 6$ at $z_0 = 0$.

Example 22.1.10. Find all poles of $f(z) = \frac{1}{\left(z - \frac{\pi}{2}\right)^3 \cos^4(z)}$ and their orders.

$\cos^4(z)$ has a zero of order 4 at all odd multiples of $\frac{\pi}{2}$, and $\left(z - \frac{\pi}{2}\right)^3$ has a zero of order 3 at $\frac{\pi}{2}$. So f has a pole of order 7 at $\frac{\pi}{2}$ and a pole of order 4 at all other odd multiples of $\frac{\pi}{2}$.

22.2 The Residue Theorem

The aim of this section is to explore the relationship with singularities, the Laurent expansion, and integrals.

Suppose f is differentiable in the annulus $0 < |z - z_0| < R$ and has an isolated singularity at z_0. Let γ be a closed path in this annulus that encloses z_0. The Laurent expansion for f in this annulus is

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where the coefficients c_n are given by

$$c_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

for all integers n. Notice that when $n = -1$, the $\frac{1}{z-z_0}$ term in the series has coefficient

$$c_{-1} = \frac{1}{2\pi i} \oint_\gamma f(z)\, dz$$

which rearranges to

$$\oint_\gamma f(z)\, dz = 2\pi i c_{-1}.$$

This means that finding that one single coefficient is all that we need to evaluate the integral! Magic!

Definition. The coefficient of $\frac{1}{z-z_0}$ in the Laurent expansion of f about z_0 is called the $\boxed{\textit{residue}}$ of f at z_0 and is denoted $\operatorname{Res}(f, z_0)$.

What we have is that

$$\oint_\gamma f(z)\, dz = 2\pi i \operatorname{Res}(f, z_0),$$

but what if γ enclosed multiple isolated singularities $z_1, \ldots, z_n$?

Around each singularity z_k, we can find a small loop γ_k so that none of the γ_k's intersect and none of the γ_k's enclose any other singularity. By the extended deformation theorem,

$$\oint_\gamma f(z)\, dz = \sum_{k=1}^{n} \oint_{\gamma_k} f(z)\, dz$$

and since each of the integrals on the right can be written in terms of the corresponding residues, the following result is an immediate consequence:

Theorem 22.2.1 (Residue Theorem)**.** *Let γ be a closed path. Suppose f is differentiable on γ and all points enclosed by γ, exceot for $z_1, \ldots, z_n$ which are all of the isolated singularities of f enclosed by γ. Then*

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k).$$

In this way, we see that computing an integral is as efficient as our ability to evaluate these residues. Obviously computing Laurent series by hand is a little time consuming, so we want to find a faster way to obtain the residue.

Proposition 22.2.2. *If f has a simple pole at z_0, then*

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Proof. Since f has a simple pole at z_0, then the Laurent expansion of f in some annulus about z_0 is

$$f(z) = \frac{c_{-1}}{z - z_0} + \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

As such

$$(z - z_0) f(z) = c_{-1} + \sum_{n=0}^{\infty} c_n (z - z_0)^{n+1}$$

and therefore

$$\lim_{z \to z_0} (z - z_0) f(z) = c_{-1}.$$

□

Example 22.2.3. Evaluate $\displaystyle\oint_\gamma \frac{\sin(z)}{z^2}\,dz$ where γ is any closed path enclosing $z_0 = 0$.

Since $z_0 = 0$ is a zero of order 1 for $\sin(z)$ and a zero of order 2 for z^2, then z_0 is a simple pole for $f(z)$ by Theorem 22.1.8. By the preceding theorem, we have that

$$\mathrm{Res}(f, 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{\sin(z)}{z} = 1.$$

Since $z_0 = 0$ is the only singularity of $\dfrac{\sin(z)}{z^2}$, we have

$$\oint_\gamma \frac{\sin(z)}{z^2}\,dz = 2\pi i \operatorname{Res}(f, 0) = 2\pi i.$$

Corollary 22.2.4. *Let $f(z) = \dfrac{h(z)}{g(z)}$ where h is continuous at z_0 and $h(z_0) \neq 0$. Suppose g is differentiable at z_0 and has a simple zero there. Then f has a simple pole at z_0, and*

$$\operatorname{Res}(f, z_0) = \frac{h(z_0)}{g'(z_0)}.$$

Proof. Since $g(z_0) = 0$ and z_0 is a simple pole for f, then

$$\begin{aligned}
\operatorname{Res}(f, z_0) &= \lim_{z\to z_0} (z - z_0) f(z) \\
&= \lim_{z\to z_0} (z - z_0)\frac{h(z)}{g(z)} \\
&= \lim_{z\to z_0} (z - z_0)\frac{h(z)}{g(z) - g(z_0)} \\
&= \lim_{z\to z_0} \frac{h(z)}{\frac{g(z)-g(z_0)}{z-z_0}} \\
&= \lim_{z\to z_0} \frac{h(z_0)}{g'(z_0)}.
\end{aligned}$$

□

Example 22.2.5. Evaluate $\displaystyle\oint_\gamma f(z)\,dz$ where $f(z) = \dfrac{10 - 2iz}{\cos(z)}$ and γ is the circle $\left|z - \frac{\pi}{2}\right| = 1$.

Let $h(z) = 10 - 2iz$ and $g(z) = \cos(z)$. Then $f(z) = \dfrac{h(z)}{g(z)}$ has a simple pole at $z_0 = \frac{\pi}{2}$ and $g\left(\frac{\pi}{2}\right) = 0$. By the corollary,

$$\operatorname{Res}\left(f, \frac{\pi}{2}\right) = \lim_{z\to\pi/2} \frac{10 - 2iz}{-\sin(z)} = \frac{10 - i\pi}{-1} = i\pi - 10.$$

Hence

$$\oint_\gamma \frac{10 - 2iz}{\cos(z)}\,dz = 2\pi i \operatorname{Res}\left(f, \frac{\pi}{2}\right) = 2\pi i(i\pi - 10).$$

What if we want to compute $\mathrm{Res}(f, z_0)$ when z_0 is a pole of order m for f? In an annular neighborhood about z_0, f has the Laurent expansion

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \frac{c_{-m+2}}{(z-z_0)^{m-2}} + \frac{c_{-m+3}}{(z-z_0)^{m-3}} + \cdots$$

Thus

$$(z-z_0)^m f(z) = c_{-m} + c_{-m+1}(z-z_0) + \cdots + c_{-1}(z-z_0)^{m-1} + \cdots .$$

By differentiating $m-1$ times, we obtain

$$\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] = (m-1)!\, c_{-1} + m!\, c_0(z-z_0) + \cdots .$$

and taking the limit as $z \to z_0$, the right-hand side reduces to

$$(m-1)!\, c_{-1}.$$

This yields the following

Theorem 22.2.6 (Residue at a Pole of Order m)**.** *Let f have a pole of order m at z_0. Then*

$$\mathrm{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right].$$

Remark. Note that Proposition 22.2.2 is the special case of this theorem when $m = 1$.

Example 22.2.7. Evaluate $\oint_\gamma f(z)\, dz$ where $f(z) = \dfrac{2iz - \cos(z)}{z(z-i)^3}$ and γ is any closed path that encloses 0 and i.

0 is a simple pole of f, so

$$\mathrm{Res}(f, 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{2iz - \cos(z)}{(z-i)^3} = \frac{-1}{i} = i.$$

i is a pole of order 3, so

$$\begin{aligned}
\operatorname{Res}(f,i) &= \frac{1}{(3-1)!}\lim_{z\to i}\frac{d^{3-1}}{dz^{3-1}}\left[(z-i)^3 f(z)\right] \\
&= \frac{1}{2}\lim_{z\to i}\frac{d^2}{dz^2}\left[\frac{2iz-\cos(z)}{z}\right] \\
&= \frac{1}{2}\lim_{z\to i}\frac{(z^2-2)\cos(z)-2z\sin(z)}{z^3} \\
&= \frac{1}{2}\frac{-3\cos(i)-2i\sin(i)}{-i} \\
&= \sin(i)-\frac{3}{2}i\cos(i).
\end{aligned}$$

So, by the Residue Theorem 22.2.1,

$$\begin{aligned}
\oint_\gamma \frac{2iz-\cos(z)}{z(z-i)^3}\,dz &= 2\pi i(\operatorname{Res}(f,0)+\operatorname{Res}(f,i)) \\
&= 2\pi i\left(i+\sin(i)-\frac{3}{2}i\cos(i)\right) = -2\pi+3\cos(i)+2\pi i\sin(i).
\end{aligned}$$

22.3 Evaluation of Real Integrals

22.3.1 Rational Functions

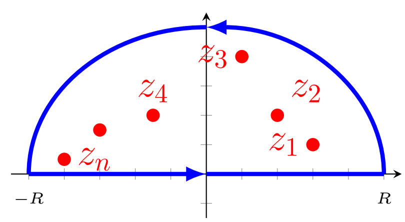
Suppose we're trying to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$$

where p, q are polynomials. To ensure that this integral converges, we'll assume that $\deg(q) \geq \deg(p) + 2$ and that q has no real roots. For simplicity, we'll also assume that p and q have no common roots (so the fraction is fully reduced).

Since q has no real roots and is a polynomial with real coefficients, all of its roots are complex and come in conjugate pairs, $z_1, \overline{z_1}, \ldots, z_n, \overline{z_n}$. Without loss of generality, assume that all of the z_k's live in the upper half plane (i.e. satisfy $\mathrm{Im}(z_k) > 0$)), and thus all of the $\overline{z}_k$'s live in the lower half plane.

For a positive real number R, let γ_R be the upper semi-circle from R to $-R$ and let S_R be the line segment from $-R$ to R. Let Γ_R be loop formed from these two segments, and take R taken large enough that Γ_R encloses all of $z_1, \ldots, z_n$.



Γ is the loop formed from the line segment from $-R$ to R and the upper half circle from R to $-R$.

The z_i's are all of the poles of $f(z) = \dfrac{p(z)}{q(z)}$ in the upper half plane, so by the Residue Theorem

$$\int_{S_R} \frac{p(z)}{q(z)}\,dz + \int_{\gamma_R} \frac{p(z)}{q(z)}\,dz = \oint_{\Gamma_R} \frac{p(z)}{q(z)}\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_n).$$

Since $\mathrm{Im}(z) = 0$ for all z on the segment S_R, the equation above can be rewritten

$$\int_{-R}^{R} \frac{p(x)}{q(x)}\,dx + \int_{\gamma_R} \frac{p(z)}{q(z)}\,dz = \oint_{\Gamma_R} \frac{p(z)}{q(z)}\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_n). \tag{22.3.1}$$

We'll state as a fact the following result (which is the complex analog of a familiar result from calculus):

Fact. If $p(z), q(z)$ are polynomials with $\deg(q) \geq \deg(p)$, then the limit as $|z| \to \infty$ of $\frac{p(z)}{q(z)}$ exists.

Since $\deg(q) \geq \deg(p) + 2$ in our case, then we must have that $z^2 p(z)/q(z)$ is bounded for $|z| \geq R$, say

$$\left|\frac{z^2 p(z)}{q(z)}\right| = |z^2| \left|\frac{p(z)}{q(z)}\right| \leq M \qquad \Longrightarrow \qquad \left|\frac{p(z)}{q(z)}\right| \leq \frac{M}{|z|^2}.$$

Since γ_R has length πR, then it follows from Theorem 20.1.5 that

$$\left|\int_{\gamma_R} \frac{p(z)}{q(z)}\, dz\right| \leq \frac{M}{|z|^2}(\pi R) \leq \frac{M}{R^2}(\pi R)$$

and so when we take the limit as $R \to \infty$, Equation 22.3.1 becomes

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx + 0 = 2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(\frac{p(z)}{q(z)}, z_n\right)$$

Example 22.3.1. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx$.

Approaching the old-fashioned way,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx &= \lim_{a\to\infty} \int_{-a}^{a} \frac{1}{x^2+1}\, dx \\
&= \lim_{a\to\infty} \arctan(a) - \arctan(-a) \\
&= \frac{\pi}{2} + \frac{\pi}{2} = \pi.
\end{aligned}$$

Approaching the new way, we see that $f(z) = \dfrac{p(z)}{q(z)} = \dfrac{1}{z^2+1}$ has a simple pole at $z = -i$ and a simple pole at $z = i$. Only i lies in the upper half plane, so we compute the residue

$$\operatorname{Res}(f, i) = \frac{p(i)}{q'(i)} = \frac{1}{2i}.$$

and from our work above,

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx = 2\pi i \operatorname{Res}(f, i) = \frac{2\pi i}{2i} = \pi.$$

Example 22.3.2. Compute $\int_{-\infty}^{\infty} \frac{1}{x^4+4}\, dz$.

Writing

$$f(z) = \frac{p(z)}{q(z)} = \frac{1}{z^4+4}$$

We see that $f(z)$ has simple poles at $4^{1/4}e^{\pi i/4}$, $4^{1/4}e^{3\pi i/4}$, $4^{1/4}e^{5\pi i/4}$, and $4^{1/4}e^{7\pi i/4}$. Only two of these have positive imaginary parts, and they are

$$\begin{aligned} z_1 &= 4^{1/4}e^{\pi i/4} = 1+i \\ z_2 &= 4^{1/4}e^{3\pi i/4} = -1+i. \end{aligned}$$

Computing residues, we have

$$\begin{aligned} \operatorname{Res}(f, z_1) &= \frac{1}{4z_1^3} = \frac{1}{4(1+i)^3} = -\frac{1+i}{16} \\ \operatorname{Res}(f, z_2) &= \frac{1}{4z_2^3} = \frac{1}{4(-1+i)^3} = \frac{1-i}{16} \end{aligned}$$

whence

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{x^4+4}\, dx &= 2\pi i \left(\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2)\right) \\ &= 2\pi i \left(-\frac{1+i}{16} + \frac{1-i}{16}\right) = \frac{\pi}{4} \end{aligned}$$

22.3.2 Rational Functions Times Cosine or Sine

With p,q,Γ_R,γ_R, and S_R as in the previous subsection, we consider, for a positive real number c, the integral

$$\oint_{\Gamma_R} \frac{p(z)}{q(z)} e^{icz}\, dz$$

By the Residue Theorem,

$$\begin{aligned} 2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(\frac{p(z)}{q(z)} e^{icz}, z_k\right) &= \oint_{\Gamma_R} \frac{p(z)}{q(z)} e^{icz}\, dz \\ &= \int_{\gamma_R} \frac{p(z)}{q(z)} e^{icz}\, dz + \int_{S_R} \frac{p(z)}{q(z)} e^{icz}\, dz \\ &= \int_{\gamma_R} \frac{p(z)}{q(z)} e^{icz}\, dz + \int_{-R}^{R} \frac{p(x)}{q(x)} \cos(x)\, dx + i \int_{-R}^{R} \frac{p(x)}{q(x)} \sin(x)\, dx \end{aligned}$$

Since $|e^{icz}| = 1$, then the same bound argument holds as in the previous subsection. And so, setting $\lim_{R\to\infty}$, the integral over γ_R tends to 0 and we are left with

$$\int_{-R}^{R} \frac{p(x)}{q(x)} \cos(x)\,dx + i\int_{-R}^{R} \frac{p(x)}{q(x)} \sin(x)\,dx = 2\pi i \sum_{k=1}^{n} \text{Res}\left(\frac{p(z)}{q(z)} e^{icz}, z_k\right).$$

As such, but comparing real and imaginary parts, we can solve either of the real integrals on the right via residues.

Example 22.3.3. Let α, β, c be positive real numbers. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(cx)}{(x^2+\alpha^2)(x^2+\beta^2)}\,dx$.

Let $f(z) = \dfrac{e^{icz}}{(z^2+\alpha^2)(z^2+\beta^2)}$, which has simple poles at $x = \pm\alpha i, \pm\beta i$. Computing the residues for the poles in the upper half plane

$$\text{Res}(f, \alpha i) = \frac{e^{-c\alpha}}{2\alpha i(-\alpha^2+\beta^2)}$$
$$\text{Res}(f, \beta i) = \frac{e^{-c\beta}}{2\beta i(-\alpha^2+\beta^2)}$$

We thus have that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin(cx)}{(x^2+\alpha^2)(x^2+\beta^2)}\,dx &= \text{Im}\left(2\pi i\left[\frac{e^{-c\alpha}}{2\alpha i(-\alpha^2+\beta^2)} + \frac{e^{-c\beta}}{2\beta i(-\alpha^2+\beta^2)}\right]\right)\\ &= \text{Im}\left(\pi\frac{e^{-c\alpha}}{\alpha(-\alpha^2+\beta^2)} + \pi\frac{e^{-c\beta}}{\beta(\alpha^2-\beta^2)}\right) = 0.\end{aligned}$$

That it's zero isn't too surprising - the integrand is an odd function.

Example 22.3.4. Let α, β, c be positive real numbers. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(cx)}{(x^2+\alpha^2)(x^2+\beta^2)}\,dx$.

From the work we did in the previous example,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos(cx)}{(x^2+\alpha^2)(x^2+\beta^2)}\,dx &= \text{Re}\left(2\pi i\left[\frac{e^{-c\alpha}}{2\alpha i(-\alpha^2+\beta^2)} + \frac{e^{-c\beta}}{2\beta i(-\alpha^2+\beta^2)}\right]\right)\\ &= \text{Re}\left(\pi\frac{e^{-c\alpha}}{\alpha(-\alpha^2+\beta^2)} + \pi\frac{e^{-c\beta}}{\beta(\alpha^2-\beta^2)}\right)\\ &= \pi\frac{e^{-c\alpha}}{\alpha(-\alpha^2+\beta^2)} + \pi\frac{e^{-c\beta}}{\beta(\alpha^2-\beta^2)}.\end{aligned}$$

22.3.3 Rational Functions of Cosine and Sine

Let $K(x, y)$ be a rational function of x and y (i.e. a quotient of multivariate polynomials in x and y). For example,

$$K(x, y) = \frac{x^3y + 6y^2 - 7xy + 2x}{y^3 - x^2 + y - 9x}.$$

We are interested in evaluating integrals of the form

$$\int_0^{2\pi} K(\cos\theta, \sin\theta)\, d\theta.$$

Let γ be the unit circle about the origin, which we parameterize as $\gamma(\theta) = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$. On this curve, $z = e^{i\theta}$ and $\overline{z} = e^{-i\theta} = \frac{1}{z}$, so we can write

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{and} \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

What's more, we have

$$dz = ie^{i\theta}\, d\theta = iz\, d\theta \qquad \Longrightarrow \qquad d\theta = \frac{1}{iz}\, dz$$

and so

$$\oint_\Gamma K\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)\frac{1}{iz}\, dz = \int_0^{2\pi} K(\cos(\theta), \sin(\theta))\, d\theta. \qquad (22.3.2)$$

As such, we can interpret the integral on the right as the contour integral on the left, and use the Residue Theorem to solve it.

Example 22.3.5. Evaluate $\int_0^{2\pi} \frac{1}{\alpha + \beta\cos(\theta)}\, d\theta$ with $0 < \beta < \alpha$.

With $x = \cos\theta$ and $y = \sin\theta$, the integrand can be thought of as the rational function

$$K(x, y) = \frac{1}{\alpha + \beta x}.$$

Converting into z-coordinates as in Equation 22.3.2, our integrand becomes

$$\begin{aligned} f(z) &= K\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)\frac{1}{iz} \\ &= \frac{1}{\alpha + \frac{\beta}{2}\left(z + \frac{1}{z}\right)}\frac{1}{iz} \\ &= \frac{-2i}{2\alpha z + \beta z^2 + \beta}. \end{aligned}$$

From the quadratic formula, we get that the poles are

$$z = \frac{-2\alpha \pm \sqrt{\alpha^2 - \beta^2}}{\beta}$$

both of which are real because of our assumption that $\alpha > \beta$, however, only

$$z_0 = \frac{-2\alpha + \sqrt{\alpha^2 - \beta^2}}{\beta}$$

is contained within the unit circle. Therefore

$$\begin{aligned}\int_0^{2\pi} \frac{1}{\alpha + \beta\cos(\theta)}\, d\theta &= 2\pi i \operatorname{Res}(f, z_0) \\ &= 2\pi i \frac{-2i}{2\alpha + 2\beta z_0} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2}}.\end{aligned}$$

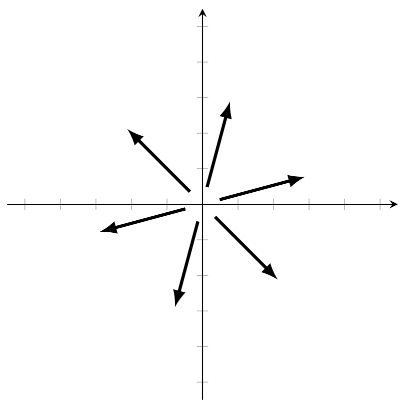
22.A The Point at Infinity and the Riemann Sphere

22.A.1 The Point at Infinity

If f has a pole at z_0, it has no doubt occured that we might consider taking the value of $f(z_0) = \infty$. Let's look at what happens if we try to define $1/0 = \infty$.

If $z \to 0$ (from the right) along the real axis, we might say that $\frac{1}{z}$ approaches "$+\infty$." If $z \to 0$ (from the left) along the real axis, we might say that $\frac{1}{z}$ approaches "$-\infty$. If $z \to 0$ along the imaginary axis, what would we say $\frac{1}{z}$ approaches? "$\pm i\infty$"?

If we want to try to define this limit, it has to agree from all directions, so writing "$\frac{1}{0} = \infty$" implies that we are identifying all of these limits with the same point, which we name ∞. The other implication, of course, is that if we think of a complex number z as growing without bound, then it is necessarily tending to this singular point ∞.



Growing without bound in any direction, the limit point is always the same: ∞

Definition. The *extended complex plane* $\hat{\mathbb{C}}$ is the set $\mathbb{C} \cup \{\infty\}$. This is also referred to as the *Riemann sphere*.

Example 22.A.1. If $f(z) = \dfrac{2z+1}{z-1}$, then $f(1) = \infty$ and $f(\infty) = 2$.

We might consider certain properties of functions defined on all of $\hat{C}$. Notice that the mapping $z \mapsto \frac{1}{z}$ has the effect of interchanging 0 and ∞, so studying the behavior of a function $f(z)$ at ∞ is equivalent to studying the behavior of the function $g(w) := f\left(\frac{1}{w}\right)$ at 0.

As such, we say that

1. $f(z)$ is differentiable at ∞ if $f\left(\frac{1}{w}\right)$ is differentiable (or has a removable singularity) at $w=0$.
2. $f(z)$ has a pole of order m at ∞ if $f\left(\frac{1}{w}\right)$ has a pole of order m at $w=0$.
3. $f(z)$ has an essential singularity at ∞ if $f\left(\frac{1}{w}\right)$ has an essential singularity at $w=0$.

Example 22.A.2. Let $f(z)=\dfrac{2z+1}{z-1}$. Show that f is differentiable at ∞.

f is certainly analytic for all $z\in\mathbb{C}$ except at 1, so we'll look at a Laurent series expansion of f about 0 on the annulus $1<|z|<\infty$:

$$\begin{aligned} f(z) = \frac{2z+1}{z-1} &= \frac{2z+1}{z}\left(\frac{1}{z-\frac{1}{z}}\right) \\ &= \frac{2z+1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \\ &= \cdots + \frac{2z+1}{z^4}+\frac{2z+1}{z^3}+\frac{2z+1}{z^2}+\frac{2z+1}{z} \end{aligned}$$

whence

$$\begin{aligned} f\left(\frac{1}{w}\right) &= \cdots + \frac{\frac{2}{w}+1}{\left(\frac{1}{w}\right)^4}+\frac{\frac{2}{w}+1}{\left(\frac{1}{w}\right)^3}+\frac{\frac{2}{w}+1}{\left(\frac{1}{w}\right)^2}+\frac{\frac{2}{w}+1}{\frac{1}{w}} \\ &= (2+w)+(2+w)w+(2+w)w^2+(2+w)w^3+\cdots \\ &= 2+3w+3w^2+3w^3+\cdots \end{aligned}$$

and thus $f\left(\frac{1}{w}\right)$ has a removable singularity at $w=0$.

Example 22.A.3. Let $f(z)=z^3+2$. Show that f has a pole of order 3 at ∞.

Notice that

$$f\left(\frac{1}{w}\right)=\frac{1}{w^3}+2$$

is a Laurent expansion of $f\left(\frac{1}{w}\right)$ about $w=0$, which is clearly a pole of order 3.

Example 22.A.4. Show that $f(z)=\sin(z)$ has an essential singularity at ∞.

f is analytic for all $z\in\mathbb{C}$, so we'll look at the Laurent (Taylor) series expansion of f about 0.

$$f(z)=\sin(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$$

whence

$$f\left(\frac{1}{w}\right) = \sin\left(\frac{1}{w}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{w}\right)^{2n+1} = \sum_{n=-\infty}^{0} \frac{(-1)^n}{(2n+1)!} w^{2n+1}$$

and thus $f\left(\frac{1}{w}\right)$ has an essential singularity at $w = 0$.

Example 22.A.5. Find all functions that are analytic on all of $\hat{\mathbb{C}}$.

If f has a pole at any z_0, then $\lim_{z \to z_0} f(z) = \infty$. Since f is analytic on all of $\hat{\mathbb{C}}$, then it doesn't have any poles, and f it must be bounded. Since f is bounded on the open set $|z| < 1$, it is constant (by Liouville's theorem), and since f is bounded on the open set $|z| > 1$, it is constant (by Liouville's theorem). By continuity on $\hat{\mathbb{C}}$ (and in particular on $|z| = 1$), these constants must be the same, so f is constant.

Example 22.A.6. Find all functions that have a single pole and are analytic on the rest of $\hat{\mathbb{C}}$.

Suppose f has a pole of order m at some finite point z_0 (so $z_0 \neq \infty$). Then the Laurent expansion of f about z_0 is

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

and it converges for all $z \neq z_0$ (since the only pole is at z_0, f must be bounded on the rest of $\hat{\mathbb{C}}$, including at ∞). For any positive integer n, $c_n(\infty - z_0)^n = \infty$, so it must be that $c_n = 0$ for $n > 0$. As such, f has the form

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{c_{-1}}{(z-z_0)} + c_0.$$

If the pole occurs at $z_0 = \infty$, then $f\left(\frac{1}{w}\right)$ has a pole at 0, so the Laurent series expansion of $f\left(\frac{1}{w}\right)$ about 0 is

$$f\left(\frac{1}{w}\right) = \frac{c_{-m}}{w^m} + \frac{c_{-m+1}}{c^{m-1}} + \cdots + \frac{c_{-1}}{w} + c_0 + c_1 w + c_2 w^2 + \cdots$$

Since $f(z)$ is bounded near 0, then $f\left(\frac{1}{w}\right)$ is bounded near ∞ (i.e. for all sufficiently large $|w|$), and just as last we must have that $c_m = 0$ for $m > 0$. It follows that

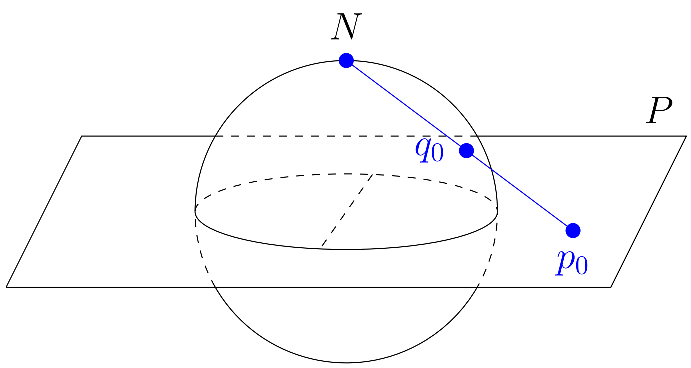
$$f\left(\frac{1}{w}\right) = \frac{c_{-m}}{w^m} + \frac{c_{-m+1}}{c^{m-1}} + \cdots + \frac{c_{-1}}{w} + c_0 \Longrightarrow \qquad f(z) \ = c_0 + c_{-1}z + \cdots + c_{-m}z^m$$

is a polynomial.

22.A.2 Stereographic projection

In $\mathbb{R}^3$, the unit sphere (denoted S^2) is the set $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$. Letting P be any plane through the origin, P divides S^2 into two hemispheres, call one the northern hemisphere and the other the southern hemisphere.

The northern hemisphere contains a point N, the *north pole*, and so we look at the line passing through N and any other point q_0 on the sphere. This line intersects P in a unique point p_0, and this gives us a way of uniquely identifying points on the sphere (minus the north pole) and points in the plane!



Definition. The function that makes this identification is called the $\boxed{\textit{stereographic projection}}$ from N.

Remark. This same procedure gives a stereographic projection from the n-sphere, S^n (as a subset of $\mathbb{R}^{n+1}$) down to an n-dimensional subspace of $\mathbb{R}^{n+1}$. We will only care about it in dimension 2, however.

If we take P to be the x_1x_2-plane (so the set of points $(x_1, x_2, 0)$), then $N = (0, 0, 1)$. And if we identify P with $\mathbb{C}$ (so $(x_1, x_2, 0)$ is identified with $z = x_1 + ix_2$), then we can describe stereographic projection in coordinates:

$$\rho : S^2\text{-}\{N\} \longrightarrow \mathbb{C}$$
$$\rho(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}$$

$$\rho^{-1} : \mathbb{C} \longrightarrow S^2\text{-}\{N\}$$
$$\rho^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{1 + |z|^2}, \frac{2\,\mathrm{Im}(z)}{1 + |z|^2}, \frac{-1 + |z|^2}{1 + |z|^2}\right)$$

We notice a couple of things:

1. ρ can be extended to a map $\tilde{\rho} : S^2 \to \hat{\mathbb{C}}$ by defining $\tilde{\rho}(N) = \infty$.
2. ρ sends the equator of S^2 to the unit circle in $\mathbb{C}$.
3. ρ sends the northern hemisphere of S^2-$\{N\}$ to the exterior of the unit disk: $|z| > 1$.
4. ρ sends the southern hemisphere of S^2 to the interior of the unit disk: $|z| < 1$.
5. The map $z \mapsto 1/z$ corresponds to exchanging the northern and southern hemispheres of S^2.

Suppose

$$Ax_1 + Bx_2 + Cx_3 + D = 0 \tag{22.A.1}$$

is some plane passing through S^2. The distance from the origin to this plane is

$$\sqrt{\frac{D^2}{A^2 + B^2 + C^2}}$$

so to pass through S^2 we must have that $A^2 + B^2 + C^2 > D^2$. The corresponding point in $\mathbb{C}$ thus satisfies

$$\begin{aligned} A\left(\frac{2\operatorname{Re}(z)}{1+|z|^2}\right) + B\left(\frac{2\operatorname{Im}(z)}{1+|z|^2}\right) + C\left(\frac{-1+|z|^2}{1+|z|^2}\right) + D = 0 \\ \implies \qquad 2A\operatorname{Re}(z) + 2B\operatorname{Im}(z) + (C+D)|z|^2 = C - D \end{aligned}$$

If the plane contains $N(0,0,1)$, then from Equation 22.A.1 we deduce that $C+D = 0$. Writing $z = x + iy$, then we have

$$2Ax + 2By = C - D$$

which is the equation of a line. If the plane does not contain $N(0,0,1)$, then $C+D \neq 0$. So

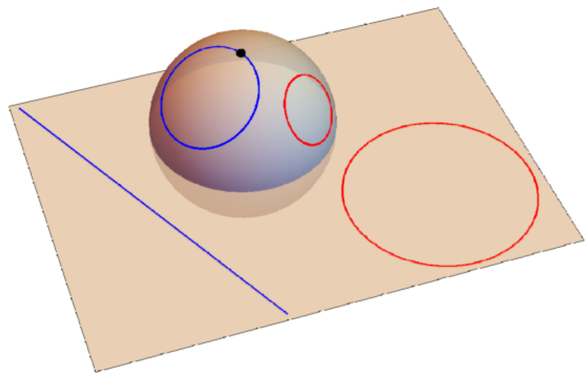
$$\begin{aligned} 2A\operatorname{Re}(z) + 2B\operatorname{Im}(z) + (C+D)|z|^2 &= C - D \\ \frac{2A\operatorname{Re}(z)}{C+D} + \frac{2B\operatorname{Im}(z)}{C+D} + |z|^2 &= \frac{C-D}{C+D}. \end{aligned}$$

Noting that $|z|^2 = (\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2$ and completing the square, we get that this rearranges to

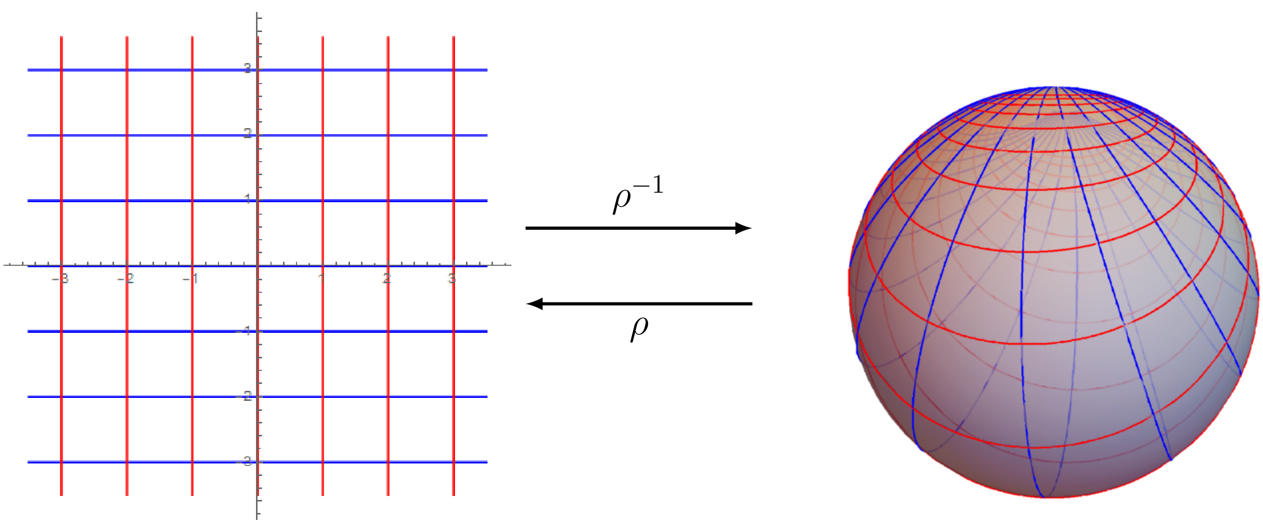
$$\begin{aligned} \left|z + \frac{A+Bi}{C+D}\right|^2 &= \frac{A^2+B^2+C^2-D^2}{(C+D)^2} \\ \left|z + \frac{A+Bi}{C+D}\right| &= \sqrt{\frac{A^2+B^2+C^2-D^2}{(C+D)^2}} \end{aligned}$$

which is the equation of a circle.

Theorem 22.A.7. *Stereographic sends circles on S^2 not passing through N to circles in $\mathbb{C}$. It sends circles on S^2 passing through N to lines in $\mathbb{C}$.*



One more thing that we may notice is that the (inverse) stereographic projection of the grid on $\mathbb{C}$ extends to a "grid" on the sphere minus N, where the circles all still meet at right angles.



Even though it distorts distances, the stereographic projection map preserves angles. One might wonder about other types of complex functions that preserve angles.

23 Conformal Mappings and Applications

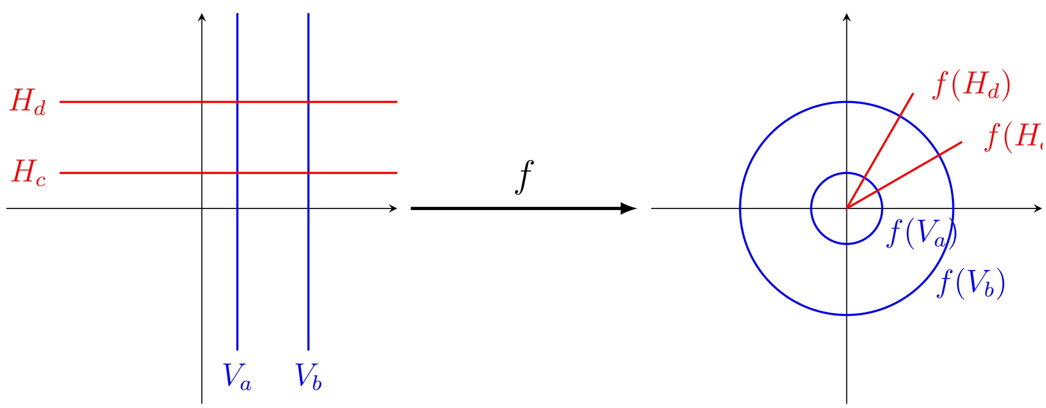
23.1 Conformal Mappings

Given a set S and a function $f : \mathbb{C} \to \mathbb{C}$, we will write $f(S)$ to be the set of all points $f(z)$ where $z \in S$.

Consider the map $f(z) = e^z$. For any real number s, let V_a be the vertical line consisting of points $z \in \mathbb{C}$ for which $\mathrm{Re}(z) = s$. Similarly, for any real number t let H_t be the horizontal line consisting of points $z \in \mathbb{C}$ for which $\mathrm{Im}(z) = t$.

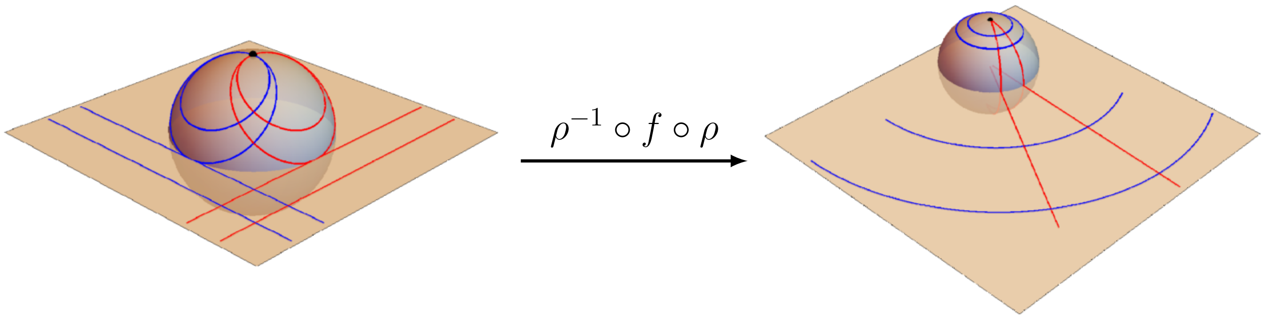
Let $w \in f(V_a)$. Then $|w| = |e^z| = e^{\mathrm{Re}(z)} = e^a$, which is a circle of radius e^a. So $f(V_a)$ is a circle about the origin, and if $b > a$, then $f(V_b)$ is a larger circle centered at the origin.

Since $e^z = e^{\mathrm{Re}(z)} e^{i\,\mathrm{Im}(z)}$, it follows that every $w \in f(H_c)$ has fixed argument c, hence $f(H_c)$ is a ray from the origin (minus the origin) at an angle of c from the positive real axis.

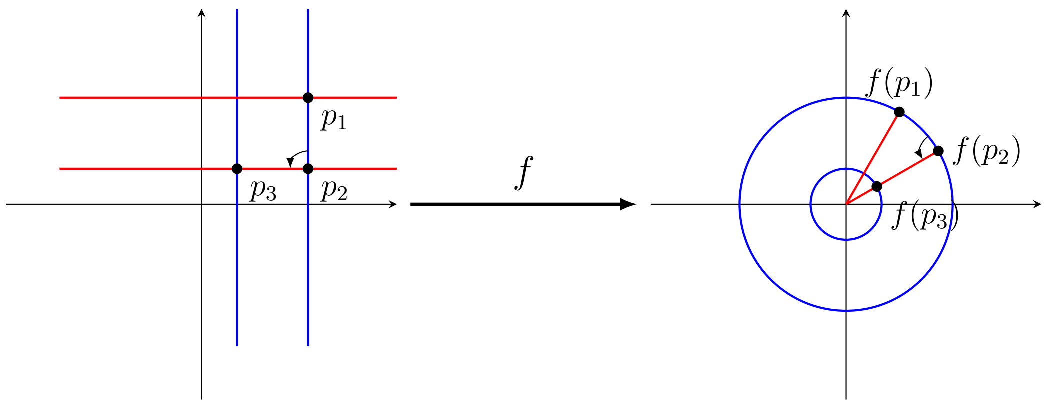


Not-to-scale image of the grid after applying the map $z \mapsto e^z$

For funsies, here's what it all looks like on the sphere.



Notice that $f(z) = e^z$ preserves angles (the right angles in the square grid get sent to right angles on the circular/curvy grid), but it also preserves *orientation*



Definition. A function that preserves both angles and orientations on a domain is said to be *conformal* on this domain. Such a function is usually called a *conformal mapping*.

Theorem 23.1.1 (Conformal Mappings). *Let D_1, D_2 be domains and $f : D_1 \to D_2$. Suppose f is differentiable on D_1 and that $f'(z) \neq 0$ for all $z \in D_1$. Then f is a conformal mapping.*

Proof. Let $p \in D_1$ and let $\gamma(t)$ be a smooth curve in D_1 for which $\gamma(0) = p$. Let $\Gamma(t) = f(\gamma(t))$ be the image of γ under f. The tangent vectors of γ and Γ at p and $f(p)$ (respectively) are related by the chain rule

$$\Gamma'(0) = f'(p)\gamma'(0)$$

so applying a differentiable function to any curve through the point p has the effect of multiplying the tangent vector to that curve by the complex number $f'(p) = re^{i\theta}$. Since all curves through p get multiplied by the same complex number, their arguments are all also changed by the same angle θ. $\square$

Proposition 23.1.2. *If $f : D_1 \to D_2$ is conformal and one-to-one on D_1 (i.e. f^{-1} is well-defined), then $f^{-1} : D_2 \to D_1$ is conformal.*

Proposition 23.1.3. *If $f : D_1 \to D_2$ and $g : D_2 \to D_3$ are conformal, then so is the composition $(g \circ f) : D_1 \to D_3$.*

Given two domains D_1 and D_2, we may want to construct a conformal mapping $f : D_1 \to D_2$, but this can be very difficult. However, there is s relatively simple class of conformal mappings that we can use for convenient domains (disks, half-planes, etc.)

23.1.1 Fractional Linear Transformations

Definition. A $\boxed{\textit{fractional linear transformation}}$ (or $\boxed{\textit{Möbius transformation}}$) is a map of the form

$$M(z) = \frac{az + b}{cz + d}$$

where a, b, c, d are constants and $ad - bc \neq 0$. The latter condition ensures the transformation is invertible, and its inverse is given by the fractional linear transformation

$$M^{-1}(z) = \frac{dz - b}{-cz + a}.$$

Remark. Although your book calls these *bilinear mappings*, I don't think this term is quite as common anymore outside of maybe algebraic geometry (when considering special cases of birational maps). Furthermore, "bilinear" is used in other areas of math as well and the meanings do not overlap, so we'll avoid this terminology.

Let A be the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We can use this matrix to encode the fractional linear transformation with the following notation

$$A \bullet z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet z := \frac{az + b}{cz + d} = M(z).$$

Moreover, the invertibility condition on the fractional linear transformation is exactly the requirement that $\det(A) \neq 0$, i.e., that A is invertible!

Exercise 23.1.1. Let M_1, M_2 be fractional linear transformations and let A_1, A_2 be invertible 2×2 matrices so that $A_1 \bullet z = M_1(z)$ and $A_2 \bullet z = M_2(z)$. Show that

1. $M_1^{-1}(z) = A_1^{-1} \bullet z$ and
2. $(M_2 \circ M_1)(z) = (A_2 A_1) \bullet z$.

Because of the parallels, it becomes very convenient to encode fractional linear transformations into a matrix form. What's more, this connection allows us to freely pass between studying an algebraic object (the set of all invertible 2×2 matrices) and studying a geometric object (functions on the complex plane). This is the subject for another class, but is generally the motivation behind the area of "geometric group theory."

Let's look at the properties of some specific types of fractional linear trasnformations.

Definition. A transformation of the form $M(z) = z + b$ is called a *translation*. It translates z by $\mathrm{Re}(b)$-units horizontally and $\mathrm{Im}(b)$-units vertically. The associated matrix A is the unipotent matrix

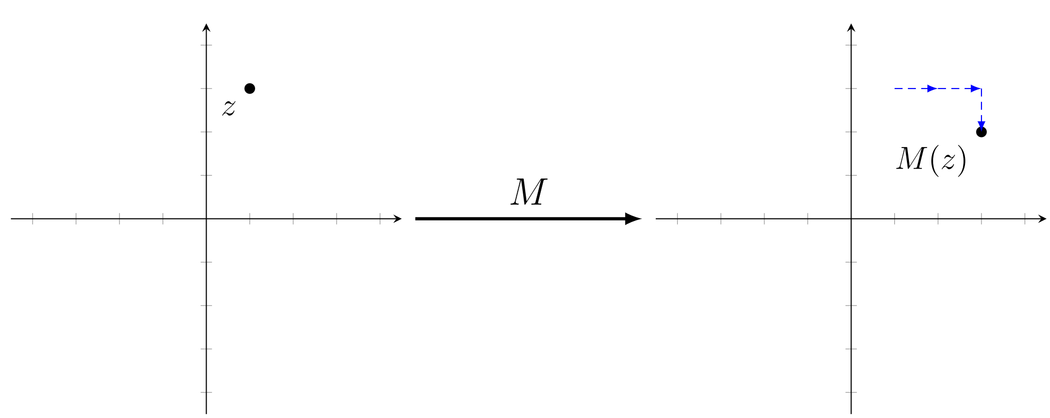
$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

Remark. Technically the matrix could be any invertible matrix of the form

$$\begin{pmatrix} \alpha & \alpha b \\ 0 & \alpha \end{pmatrix},$$

but these are all conjugate, so we take the natural choice having determinant 1.

Example 23.1.4. Consider $M(z) = z + 2 - i$.



Definition. A transformation of the form $M(z) = az$ is called a $\boxed{\textit{(pure) rotation}}$ when $|a| = 1$, and a $\boxed{\textit{(pure) dilation}}$ when a is real. When a is neither of those, then writing it in polar for as $a = re^{i\theta}$ makes it clear that it's a composition of a rotation of angle θ and a dilation with factor r. In any case, the corresponding matrix is the diagonal matrix

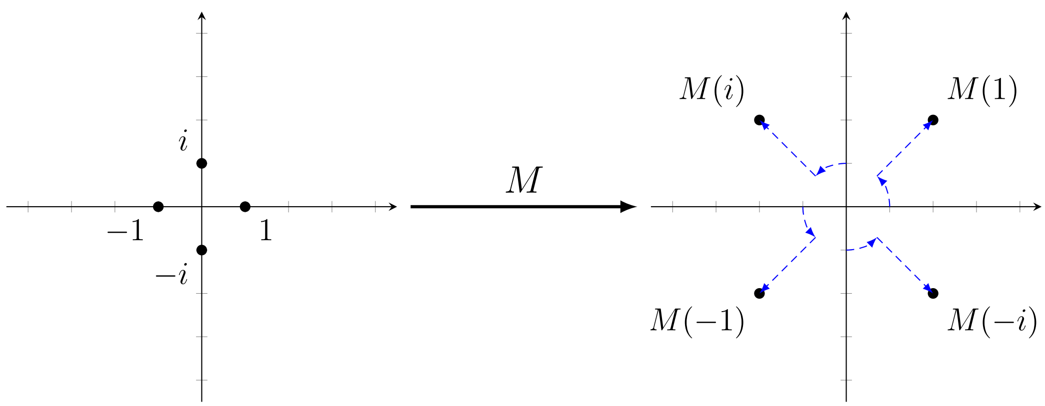
$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark. As before, it's natural to pick the determinant-1 matrix

$$\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}$$

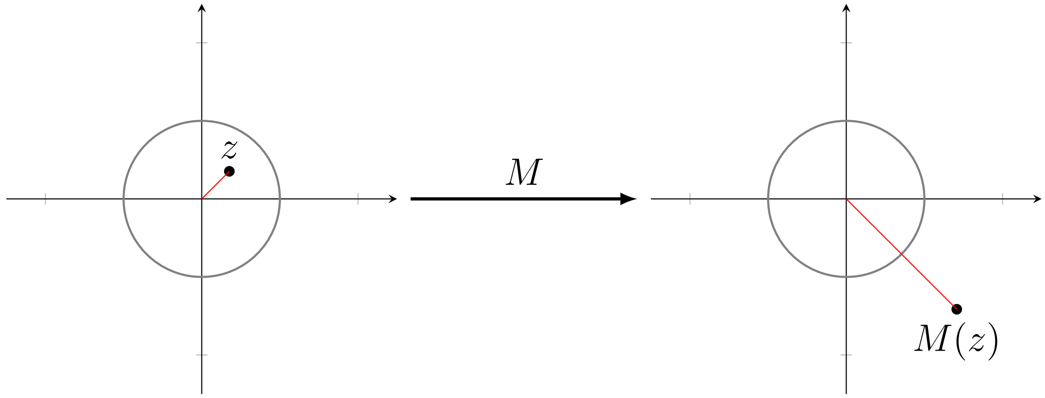
where $\sqrt{a}$ is fixed to be one of $a^{1/2}$.

Example 23.1.5. Consider $M(z) = (2+2i)z$. Since $2+2i = \sqrt{8}e^{i\pi/4}$, we can picture this as a rotation throuh angle $\pi/4$ and then a dilation with a factor of $\sqrt{8}$.



Definition. A transformation of the form $M(z) = \dfrac{1}{z}$ is called an $\boxed{\textit{inversion}}$. The associated matrix is the order-2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

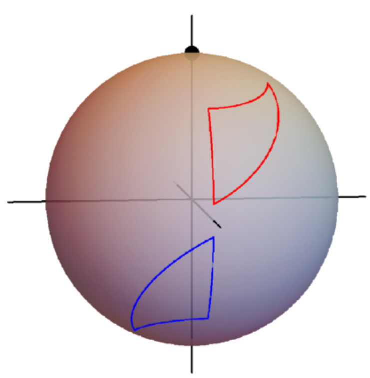


$\arg(M(z)) = -\arg(z)$ and $|M(z)| = |1/z|$

Remark. The associated determinant-1 matrix is

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Inversion is possibly the least obvious of these transformations, but if we use stereographic projeciton to see what's happening on the sphere, it's more intuitive. Let Δ be a triangle in the plane and let Δ' the inversion of Δ. Using ρ^{-1} to visualize on the sphere, we see that we can pass between $\rho^{-1}(\Delta)$ and $\rho^{-1}(\Delta')$ by a rotation of the sphere around the x-axis by an angle π (which makes sense, the inversion map fixes ± 1 and sends every point $e^{i\theta}$ to $e^{-i\theta}$).



Letting R be the rotation of the sphere around the x-axis by an angle of π, we can thus think of invertion as the composition

$$\rho \circ R \circ \rho^{-1}.$$

Theorem 23.1.6. *All fractional linear transformations $M(z) = \dfrac{az+b}{cz+d}$ are a composition of these three types of transformations.*

Proof. When $c = 0$, $M(z)$ is an actual linear transformation, which is clearly a rotation/dilation followed by a translation. Since compositions of fractional linear transformations can be represened by products of the corresponding matrices, when $c \neq 0$, we have

$$\underbrace{\begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix}}_{\text{translate}} \underbrace{\begin{pmatrix} \frac{bc-ad}{c} & 0 \\ 0 & 1 \end{pmatrix}}_{\text{rotate/dilate}} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{invert}} \underbrace{\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}}_{\text{translate}} \underbrace{\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}}_{\text{rotate/dilate}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

□

Theorem 23.1.7. *Fractional linear transformations send lines to lines or circles, and send circles to lines or circles.*

Proof. It's completely obvious that rotations, dilations, and transformations take lines to lines and circles to circles, so it only remains to check inversion. Given that inversion is a composition of stereographic projection (which has the desired property) and a rotation of the sphere (which sends circles on the sphere to circles on the sphere), the inversion has the desired property. □

Theorem 23.1.8 (Three point theorem)**.** *Given any three points z_1, z_2, z_3 in $\mathbb{C}$, any other three points w_1, w_2, w_3 in $\mathbb{C}$, there is a unique fractional linear transformation M for which $M(z_1) = w_1$, $M(z_2) = w_2$, and $M(z_3) = w_3$.*

Proof. Let M_1 and M_2 be the following fractional linear transformations:

$$M_1(z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \qquad \text{and} \qquad M_2(z) = \frac{(z-w_1)(w_2-w_3)}{(z-w_3)(w_2-w_1)}.$$

As a functions on the extended complex plane,

$$\begin{aligned} M_1(z_1) &= 0, & M_2(w_1) &= 0, \\ M_1(z_2) &= 1, & M_2(w_2) &= 1, \\ M_1(z_3) &= \infty, & M_2(w_3) &= \infty. \end{aligned}$$

By setting $M = M_2^{-1} \circ M_1$, we have the desired fractional linear transformation. □

We'll see in the next section how we can use the above technique to pass between domains.

23.2 Construction of Conformal Mappings

One strategy for solving problems is to find the solution on a simple domain (a disk, half-plane, etc) and to use a conformal mapping to pass between them. The following result tells us that this is always possible.

Theorem 23.2.1 (Riemann Mapping Theorem)**.** *Let D_0 be the unit disk and D_1 any domain in $\mathbb{C}$ (that is not all of $\mathbb{C}$). Then there exists a conformal mapping $f : D_0 \to D_1$ that is both one-to-one and onto.*

This major theorem tells us that we can pass between any two domains. Let D_0 be the unit disk and let D_1, D_2 be any two domains (that aren't all of $\mathbb{C}$). Then there exist one-to-one and onto conformal mappings $f_1 : D_0 \to D_1$ and $f_2 : D_0 \to D_2$. Since each of these maps is invertible, we get that $f_2 \circ f_1^{-1}$ is a one-to-one and onto conformal mapping.

Of course, FINDING these conformal maps in practice is generally very hard. Since a conformal mapping must send the boundary of D_1 to the boundary of D_2, one strategy is to try finding a map between the boundaries, and then test to see if the interior points are mapped to interior points. We'll explore this idea in the context of fractional linear transformations.

Example 23.2.2. Find a conformal mapping from the open unit disk to the disk $|z| < 3$.

Clearly the map

$$M(z) = 3z$$

sends the unit circle to the circle of radius 3. If $|z| < 1$, then

$$|M(z)| = |3z| = 3|z| < 3,$$

as desired.

Example 23.2.3. Find a conformal mapping from the open unit disk to the exterior of the disk $|z| > 3$

We know that inversion preserves the circle $|z| = 1$ and that dilation maps this circle conformally onto the circle $|z| = 3$, so we try composing the two

$$M(z) = 3\left(\frac{1}{z}\right).$$

Suppose that $|z| < 1$, then $|1/z| > 1$ and

$$M(z) = \left|\frac{3}{z}\right| = 3\left|\frac{1}{z}\right| > 3,$$

as desired.

Example 23.2.4. Find a conformal mapping from the open unit disk to the open disk $|z - 1| < 3$

We know $M(z) = 3z$ takes the unit disk to the disk of radius 3, so we try composing it with a translation by 1

$$M(z) = (3z) + 1.$$

Indeed, if $|z| < 1$, then

$$|M(z) - 1| = |3z + 1 - 1| = |3z| = 3|z| < 3,$$

as desired.

Example 23.2.5. Find a conformal mapping from the right half-plane to the unit disk.

Let's try the conformal mapping f for which

$$\begin{aligned} f(i) &= 1 \\ f(0) &= i \\ f(-i) &= -1 \end{aligned}$$

As in the proof of Theorem 23.1.8, we can look for two fractional linear transformations on $\hat{\mathbb{C}}$ that send our points to 0, 1, and ∞ and then compose them appropriately. Let

$$M_1(z) = \frac{(z-i)(0+i)}{(z+i)(0-i)} = \frac{iz+1}{-iz+1}, \text{ and } \qquad M_2(z) = \frac{(z-1)(i+1)}{(z+1)(i-1)} = \frac{(1+i)z-(1+i)}{(-1+i)z+(-1+i)}.$$

Now we have that

$$\begin{Bmatrix} i \\ 0 \\ -i \end{Bmatrix} \xrightarrow{M_1} \begin{Bmatrix} 0 \\ 1 \\ \infty \end{Bmatrix} \xrightarrow{M_2^{-1}} \begin{Bmatrix} 1 \\ i \\ -1 \end{Bmatrix}$$

and thus can pick

$$f(z) = (M_2^{-1} \circ M_1)(z) = \frac{iz - i}{-z - 1}$$

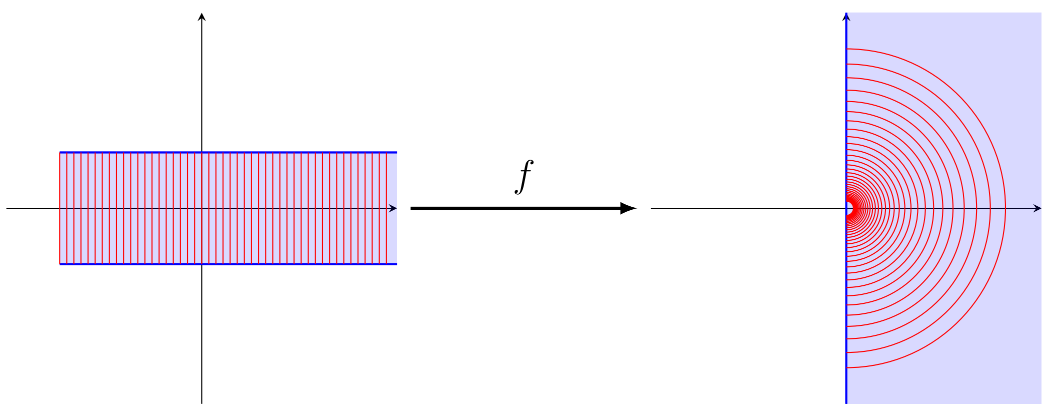
Now we have mapped the boundary of the right half-plane to the boundary of the disk, so it remains to check that the interiors map accordingly. Checking

$$f(1) = \frac{i - i}{-1 - 1} = 0$$

which is in the interior of the disk.

Example 23.2.6. Find a conformal map sending the infinite strip $-\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2}$ to the right half plane

Clearly this cannot be a fractional linear transformation, because there are two boundary lines in the infinite strip and fractional linear transformations send lines to lines (or circles). So instead we look back at the first motivational map we used. Recall that $f(z) = e^z$ sends horizontal lines to rays from the origin (although it technically excludes the origin, this doesn't matter because we're not including the boundary in our map). If $z = x + i\frac{\pi}{2}$ then $f(z) = ie^x$ and if $z = x - i\frac{\pi}{2}$ then $f(z) = -ie^x$, so the boundary of the strip is sent to the imaginary axis (minus the origin). It's quick to check that $f(0) = e^0 = 1$ which is in the right half of the plane.



The complex exponential "opens up" the infinite strip $-\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2}$ like a book into right half plane $\mathrm{Re}(z) > 0$

23.2.1 The Schwarz-Christoffel Transformation

In the proof of Theorem 23.1.1, the crucial intuition was that the derivative of the conformal map could be thought of as a rotation of the plane at a point. As such, we can cook up the derivative of a function that bending the real line into a polygon (and thus the upper half plane into the interior of a polygon), and the conformal mapping will exactly the be integral of this function.

What follows is formally known as a branch cut, but we'll avoid the greater discussion surrounding them and introduce only the salient features.

Writing $z = |z|e^{i\theta}$, let $L : \mathbb{C} \to \mathbb{C}$ be the function defined by

$$L(z) = \ln|z| + i(\theta + 2k\pi)$$

where k is chosen so that $-\frac{\pi}{2} \leq \theta < \frac{3\pi}{2}$. So $L(z)$ is defined to pick out a single value of $\log(z)$.

Fact. $L(z)$ is analytic on $\mathbb{C}$-$\{it \ : \ 0 \leq t < \infty\}$

PICTURE OF ANALYTIC REGION

We're ultimately interested in the upper half plane, so we'll always take our complex numbers to have arguments between 0 and π. For some angle α with $0 < \alpha < \pi$, define the function

$$g_\alpha(z) := e^{-\alpha L(z)}$$

It follows from the chain rule that

Corollary 23.2.7. *For each α as above, g_α is analytic on the same set as L.*

Notice that, since $L(z)$ is just a specific value of the logarithm, we must have that

$$g_\alpha(z) = z^\alpha$$

for any z where $L(z)$ is differentiable. So provided we're willing to accept this slight abuse of notation and the restricted domain, we can think of the map $z \mapsto z^\alpha$ as an analytic function. Since the upper half plane is contained in the domain of analyticity, then z^α is analytic on the upper half plane

Let's look at what happens along the real axis (minus 0) with this mapping. Since

$$x = \begin{cases} |x|e^{i\pi} & \text{when } x < 0 \\ |x| & \text{when } x > 0 \end{cases}$$

then

$$x^\alpha = \begin{cases} |x|^\alpha e^{i\alpha\pi} & \text{when } x < 0 \\ |x|^\alpha & \text{when } x > 0 \end{cases}$$

and so the arguments are

$$\arg(x^\alpha) = \begin{cases} \alpha\pi & \text{when } x < 0 \\ 0 & \text{when } x > 0. \end{cases}$$

As we move along the real axis (from left to right) and pass 0, then g_α has the effect of bending the axis and decreasing the argument by an angle of $\alpha\pi$.

PICTURE of straight line and bent line

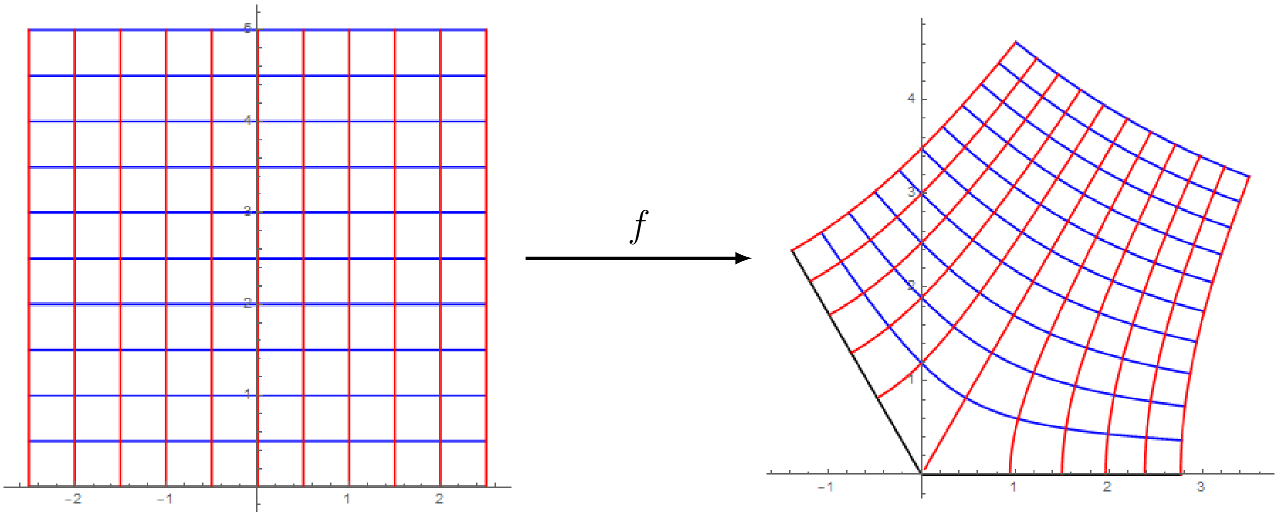
For some fixed angle $\theta \in (-\pi, \pi)$, set $\alpha = -\frac{\theta}{\pi}$. In the case that $\theta > 0$, passing 0 has the effect of bending the axis and increasing the argument by θ.

PICTURE of straight line and bent line

If we let $f(z) = \int_{z_0}^{z} \zeta^\alpha \, d\zeta$ (where $\mathrm{Im}(z_0) \geq 0$), then this makes $f'(z) = z^\alpha$, and as we saw in the proof of Theorem 23.1.1, the argument of $f'(z)$ is precisely the angle by which the tangent vectors are bent.

Remark. You can absolutely take $z_0 = 0$ above; the integral just becomes an improper integral. However, you'll almost always be working with such maps numerically and improper integrals cause computational issues, so it's recommended you choose z_0 with $\mathrm{Im}(z_0) > 0$.

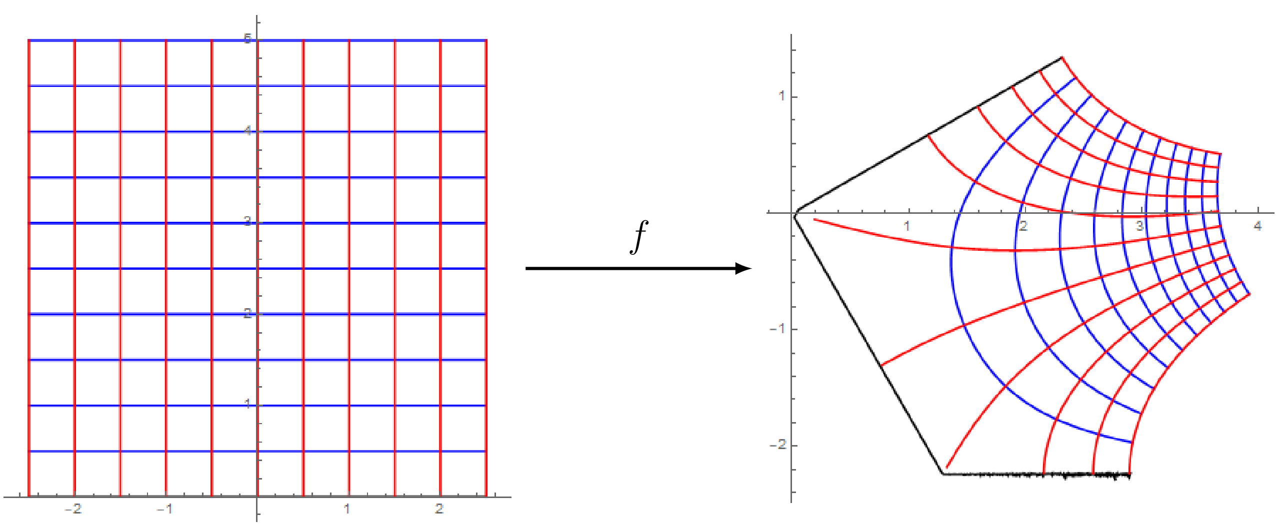
Example 23.2.8. If we set $\alpha = -\frac{\pi}{3}$, then $f(z) = \int_0^z \zeta^{-1/3}\, d\zeta$, and the image of the upper half plane is



Notice now that for any two complex numbers $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$, we have that the argument of $z_1 z_2$ is $\theta_1 + \theta_2$. So if we fix real numbers x_1, x_2 with $x_1 < x_2$ and angles α_1, α_2, then we have that

$$\arg((x - x_1)^{\alpha_1}(x - x_2)^{\alpha_2}) = \begin{cases} \alpha_1 + \alpha_2 & \text{when } x < x_1 \\ \alpha_1 & \text{when } x_1 < x < x_2 \\ 0 & \text{when } x > x_2 \end{cases}$$

Example 23.2.9. Let $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{3}$. If we consider $f(z) = \int_0^z \zeta^{-1/2}(\zeta - 1)^{-1/3}\, d\zeta$, then the image of the upper half plane is



Given any angles $\theta_1, \ldots, \theta_n$ for which $\sum_{k=1}^{n} \theta_k$ and real numbers $x_1 < x_2 < \cdots < x_n$, the function

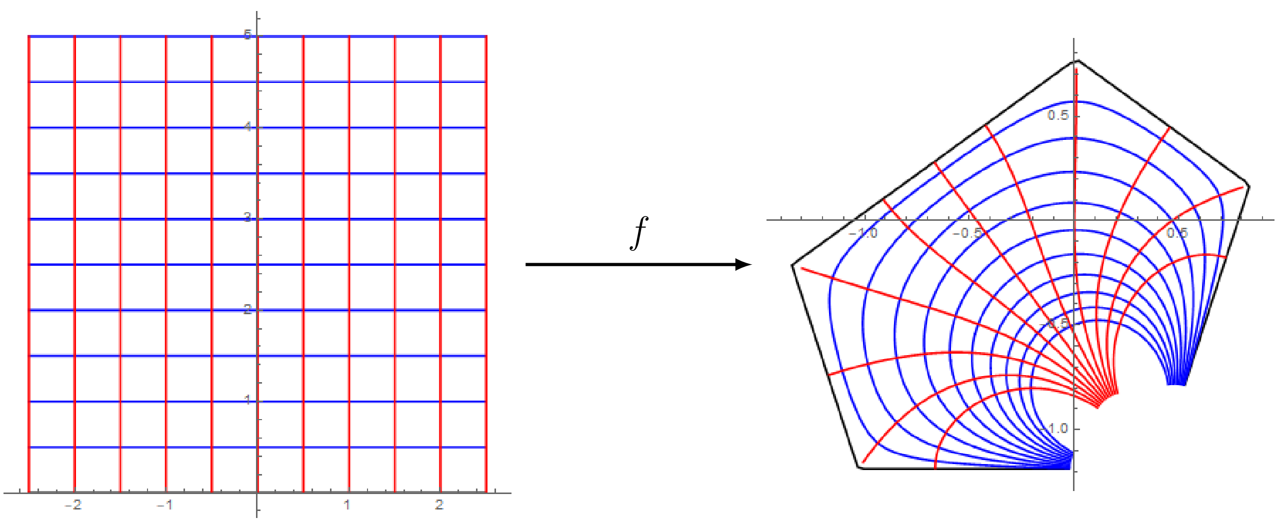
$$f(z) = \int_0^z (\zeta - x_1)^{-\theta_1/\pi} (\zeta - x_2)^{-\theta_2/\pi} \cdots (\zeta - x_{n-1})^{-\theta_{n-1}/\pi} \, d\zeta$$

is a conformal mapping of the upper half plane to a polygon with exterior angles $\theta_1, \ldots, \theta_n$ (the last angle is uniquely determined by the other $n-1$ angles, and with the above map, the point at infinity is sent to the remaining vertex).

Example 23.2.10. A pentagon has exterior angles $\theta_i = \frac{2\pi}{5}$, so taking

$$f(z) = \int_i^z (\zeta + 2)^{-2/5} (\zeta + 1)^{-2/5} (\zeta - 1)^{-2/5} (\zeta - 1)^{-2/5} \, d\zeta$$

then the image of the upper half plane is

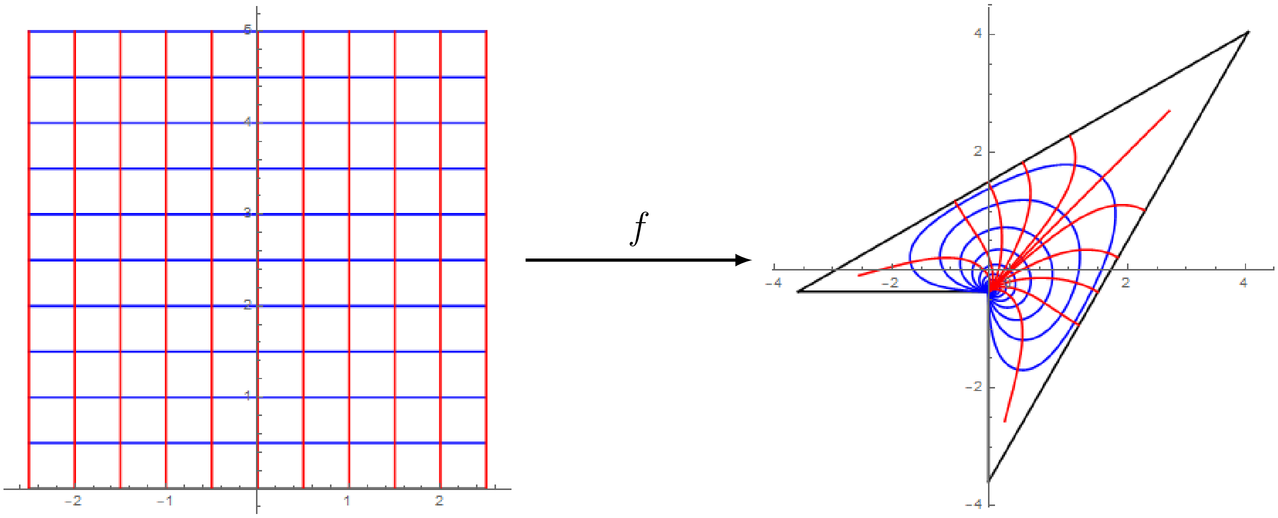


Of course, this construction works as well for non-convex polygons

Example 23.2.11. Consider the quadrilateral with exterior angles $\theta_1 = \theta_2 = \theta_3 = \frac{5\pi}{6}$ and $\theta_4 = -\frac{\pi}{2}$. Taking

$$f(z) = \int_i^z (\zeta + 1)^{-5/6} \zeta^{-5/6} (\zeta - 1)^{-5/6} \, d\zeta$$

then the image of the upper half plane is



A polygon in the plane is not uniquely determined by its angles - it may be scaled, translated, or rotated. Thankfully, each of those are conformal maps!

Theorem 23.2.12 (Schwarz–Christoffel transformation)**.** *Given any polygon P in the plane with exterior angles $\theta_1, \dots, \theta_n$, them for any real numbers $x_1 < x_2 < \cdots < x_{n-1}$, there are complex numbers A and B for which the function*

$$f(z) = A + B \int_0^z (\zeta - x_1)^{-\theta_1/\pi} (\zeta - x_2)^{-\theta_2/\pi} \cdots (\zeta - x_{n-1})^{-\theta_{n-1}/\pi} \, d\zeta$$

is a conformal mapping of the upper half plane to the interior of P.

Example 23.2.13. Find a conformal map from the upper half plane into the square with vertices $0, 1, 1+i, i$.

We know from our previous work that

$$f(z) = \int_0^z (\zeta + 1)^{-1/2} \zeta^{-1/2} (\zeta - 1)^{-1/2} \, d\zeta$$

sends the upper half plane to the interior of a rectangle. So all that remains is for us to find constants A and B so that $F(z) = A + Bf(z)$ modifies the rectangle appropriately. We can do this by finding A and B for which our chosen "bend points" $-1, 0, 1, \infty$ correspond to the necessary vertices. Note that f is a so-called "elliptic integral" and has no closed form solution, so this will all have to be done numerically.

Since $f(0) = 0$, the following correspondence seems natural

$$\text{Bend points } \{-1, 0, 1, \infty\} \leftrightarrow \text{Rectangle } \{i, 0, 1, 1+i\}$$

This yields the following system of equations

$$\left\{\begin{array}{rcl} A+Bf(-1) & = & i \\ A+Bf(0) & = & 0 \\ A+Bf(1) & = & 1 \\ A+Bf(\infty) & = & 1+i \end{array}\right\} \quad\Longrightarrow\quad \left\{\begin{array}{rcl} A+B(2.62206) & \approx & i \\ A & = & 0 \\ A+B(2.62206i) & \approx & 1 \\ A+B(2.62206-2.62206)i & \approx & 1+i \end{array}\right\}$$

hence $A=0$ and $B \approx i/2.62206$. The mapping

$$F(z) = \frac{i}{2.62206} f(z)$$

is thus the desired conformal mapping.