MAT 2534 Discrete Math

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1 Logic

1.1 Propositions and logical operations

Definition 1.1.1

A **proposition** is a sentence which is either true or false, but not both.

Example 1.1.2

- "$1 + 2 = 3$" is a true proposition.
- "$1 + 2 = 4$" is a false proposition.
- "$x + 2 = 5$" is neither true nor false since x is unspecified. Usually when we are solving for x, we are trying to find an x-value that makes the statement true.

Definition 1.1.3

A **truth value** is a value indicating whether a given proposition is actually true or false.

Remark. A proposition is still a proposition regardless of whether or not its truth value is known, or is a matter of opinion.

Example 1.1.4

- "$1 + 2 = 3$" has truth value `true`.
- "$1 + 2 = 4$" has truth value `false`.
- "$x + 2 = 5$" has indeterminate truth value (as it depends on the value of x).
- "Dave Matthews Band is awesome" has indeterminate truth value as it's technically a matter of opinion (but objectively false).

1.1.1 The conjunction operation

Definition 1.1.5

A **compound proposition** is created by connecting individual propositions with logical operations. A **logical operation** combines propositions using particular composition rules.

symbol	English translation
$\wedge$	"and"
$\vee$	"or"
$\neg$ or $\sim$	"not"

Remark. "not" should be interpreted generally as negating a statement, which is more commonly how one would use it in English.

The order of operations for these symbols is simply reading them left-to-right, and one can include parentheses to override the order (just like in the usual "PEDMAS" or whatever permutation of those letters you had learned previously).

Definition: Conjunction

If p and q are both statements, then the **conjunction** of p and q is the statement $p \wedge q$, read "p and q." This compound statement is true if both p and q are true, and it is false otherwise. The **truth table** for conjunctions is below.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

To see that this is intuitive, consider the following example.

Example 1.1.6

Let x be a fixed real number and consider the propositions:

$$p: \quad \text{"}x > 0\text{"}$$
$$q: \quad \text{"}x < 5\text{"}$$

Let's fix a couple of different x-values and record the truthfulness of p, q, and $p \wedge q$.

x-value	p $x > 0$	q $x < 5$	$p \wedge q$ $(x < 0) \wedge (x > 5)$
-1	F	F	F
0	F	T	F
4	T	T	T
5	T	F	F
6	T	F	F

1.1.2 Conjunction in English

English is a creative language, which can make it frustrating to parse the logic. Given two propositions p and q, there are a number of ways we can express $p \wedge q$ in English. A non-exhaustive list may be

"p and q"
"p, but q"
"despite p, q"
"although p, q"

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3.1 Sets and subsets

Definition: set,element

A **set** is an unordered list of items, called **elements**. If A is a set and a is an element of the set, we write $a \in A$. If a is not an element of the set, we write $a \notin A$.

Remark. An element of a set cannot appear twice.

We typically write a set using curly braces and listing the elements.

Example 3.1.1

Examples of sets.

1. $S = \{1, 3, 19\}$
2. *[The natural numbers]* $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$
3. *[The empty set]* $\emptyset = \{\}$
4. $T = \{1, 2, \{1, 2\}\}$

3.1.1 More notation related to sets

Definition: empty set

A set with no elements is called the **empty set** (or **null set**) as in denoted $\emptyset$ or sometimes $\{\,\}$.

Definition: finite/infinite set, cardinality

A **finite set** is a set that is either empty or whose elements can be numbered $1, \ldots, n$. An **infinite set** is a set that is not finite. The **cardinality** of a finite set A, denoted $|A|$, is the number of distinct elements in A.

Remark. Discussions of cardinality for infinite sets is left for later.

Definition

The following short-hand notation is used for some commonly-occuring sets.

$\mathbb{N}$	The **natural numbers**: $\{0, 1, 2, 3, \ldots\}$
$\mathbb{Z}$	The **integers**: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Z}^+$	The **positive integers**: $\{1, 2, 3, 4, \ldots\}$
$\mathbb{Q}$	The **rational numbers** (i.e. all possible fractions)
$\mathbb{R}$	The **real numbers**

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