

MAT 2534 Discrete Math

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Fall 2023

Last Updated: July 25, 2023

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1 Logic

1.1 Propositions and logical operations

Definition 1.1.1

A **proposition** is a sentence which is either true or false, but not both.

Example 1.1.2

- “ $1 + 2 = 3$ ” is a true proposition.
- “ $1 + 2 = 4$ ” is a false proposition.
- “ $x + 2 = 5$ ” is neither true nor false since x is unspecified. Usually when we are solving for x , we are trying to find an x -value that makes the statement true.

Definition 1.1.3

A **truth value** is a value indicating whether a given proposition is actually true or false.

Remark. A proposition is still a proposition regardless of whether or not its truth value is known, or is a matter of opinion.

Example 1.1.4

- “ $1 + 2 = 3$ ” has truth value **true**.
- “ $1 + 2 = 4$ ” has truth value **false**.
- “ $x + 2 = 5$ ” has indeterminate truth value (as it depends on the value of x).
- “Dave Matthews Band is awesome” has indeterminate truth value as it’s technically a matter of opinion (but objectively false).

1.1.1 The conjunction operation

Definition 1.1.5

A **compound proposition** is created by connecting individual propositions with logical operations. A **logical operation** combines propositions using particular composition rules.

symbol	English translation
\wedge	“and”
\vee	“or”
\neg or \sim	“not”

Remark. “not” should be interpreted generally as negating a statement, which is more commonly how one would use it in English.

The order of operations for these symbols is simply reading them left-to-right, and one can include parentheses to override the order (just like in the usual “PEDMAS” or whatever permutation of those letters you had learned previously).

Definition: Conjunction

If p and q are both statements, then the **conjunction** of p and q is the statement $p \wedge q$, read “ p and q .” This compound statement is true if both p and q are true, and it is false otherwise. The **truth table** for conjunctions is below.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

To see that this is intuitive, consider the following example.

Example 1.1.6

Let x be a fixed real number and consider the propositions:

$$p: "x > 0"$$

$$q: "x < 5"$$

Let's fix a couple of different x -values and record the truthfulness of p , q , and $p \wedge q$.

x -value	p $x > 0$	q $x < 5$	$p \wedge q$ $(x < 0) \wedge (x > 5)$
-1	F	F	F
0	F	T	F
4	T	T	T
5	T	F	F
6	T	F	F

1.1.2 Conjunction in English

English is a creative language, which can make it frustrating to parse the logic. Given two propositions p and q , there are a number of ways we can express $p \wedge q$ in English. A non-exhaustive list may be

“ p and q ”

“ p , but q ”

“despite p , q ”

“although p , q ”

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3 Sets

3.1 Sets and subsets

Definition: set, element

A **set** is an unordered list of items, called **elements**. If A is a set and a is an element of the set, we write $a \in A$. If a is not an element of the set, we write $a \notin A$.

Remark. An element of a set cannot appear twice.

We typically write a set using curly braces and listing the elements.

Example 3.1.1

Examples of sets.

1. $S = \{1, 3, 19\}$
2. [The natural numbers] $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$
3. [The empty set] $\emptyset = \{\}$
4. $T = \{1, 2, \{1, 2\}\}$

3.1.1 More notation related to sets

Definition: empty set

A set with no elements is called the **empty set** (or **null set**) as in denoted \emptyset or sometimes $\{\}$.

Definition: finite/infinite set, cardinality

A **finite set** is a set that is either empty or whose elements can be numbered $1, \dots, n$. An **infinite set** is a set that is not finite. The **cardinality** of a finite set A , denoted $|A|$, is the number of distinct elements in A .

Remark. Discussions of cardinality for infinite sets is left for later.

Definition

The following short-hand notation is used for some commonly-occurring sets.

\mathbb{N}	The natural numbers : $\{0, 1, 2, 3, \dots\}$
\mathbb{Z}	The integers : $\{\dots, -2, -1, 0, 1, 2, \dots\}$
\mathbb{Z}^+	The positive integers : $\{1, 2, 3, 4, \dots\}$
\mathbb{Q}	The rational numbers (i.e. all possible fractions)
\mathbb{R}	The real numbers

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