MAT1226 Calculus of a Single Variable II

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Preface

These notes were originally written and edited between Spring 2016 and Spring 2019 using the second edition of *Calculus: Early Transcendentals* by J. Stewart. This class is now using the ninth edition and, as one would expect, topic ordering has changed dramatically between editions. I have quickly relabeled the sections in an attempt to match the new edition, but for lack of time, have done little else to update them. This means that, in the current form, certain aspects of these notes might not make sense.

In particular, Chapters 6 and 7 have been swapped between editions:

Chapter	2^{nd} Edition	9^{th} Edition
6	Techniques of Integration	Applications of Integration
7	Applications of Integration	Techniques of Integration

What this means for you, dear reader, is that certain examples currently in Chapter 6 may rely on techniques that aren't covered until Chapter 7. This will, of course, be updated in future versions of these notes, but for now, it's safe to assume that these notes are only an 80% accurate representation of the actual course lecture notes.

You've been warned.

0 Review of Antiderivatives

0.1 Antiderivatives

Definition

Given a function $f(x)$, any function $F(x)$ satisfying $\frac{d}{dx}F(x) = f(x)$ is called an **antiderivative** of $f(x)$.

Remark. Antiderivatives are unique *only up to a constant.* That is, if $F(x)$ is an antiderivative, then for any real number C, $\frac{d}{dx}[F(x) + C] = f(x) + 0 = f(x)$, and so $F(x) + C$ is also an antiderivative of $f(x)$.

Below is a table of common functions and their antiderivatives.

Function $f(x)$	**Antiderivative** $F(x)$
x^n, $n \neq -1$	$\frac{1}{n+1}x^{n+1}$
e^x	e^x
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x)$	$\tan(x)$
$\csc^2(x)$	$-\cot(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\csc(x)\cot(x)$	$-\csc(x)$

Function $f(x)$	**Antiderivative** $F(x)$
$\frac{1}{x}$	$\ln\lvert x\rvert$
n^x, $n > 0$	$\frac{1}{\ln(n)}n^x$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$-\frac{1}{\sqrt{1-x^2}}$	$\arccos(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$-\frac{1}{1+x^2}$	$\text{arccot}(x)$
$\frac{1}{x\sqrt{x^2-1}}$	$\text{arcsec}(x)$
$-\frac{1}{x\sqrt{x^2-1}}$	$\text{arccsc}(x)$

0.2 The Integral

Definition

Let $f(x)$ be a function with antiderivative $F(x)$. The **indefinite integral** of $f(x)$, denoted $\int f(x)\,dx$, is defined as the general antiderivative of $f(x)$; that is $\int f(x)\,dx = F(x) + C$, where C is any real number.

Definition

If $f(x)$ is defined on the interval (a, b), the **definite integral** of $f(x)$, denoted $\displaystyle\int_a^b f(x)\,dx$, is defined as the (signed) area between the the graph of the function $y = f(x)$ and the x-axis.

Remark. The indefinite integral is a *function* (or a family of functions); the definite integral is a *real number*.

Proposition 0.2.1: Properties of Integrals

Let f, g be integrable functions and λ some real number. Then

- $\displaystyle\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\displaystyle\int \lambda f(x)\,dx = \lambda \int f(x)\,dx$
- $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

0.3 The Fundamental Theorem of Calculus

Theorem 0.3.1: Fundamental Theorem of Calculus, ver. 1

If F is continuous on the interval $[a, b]$ and F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Theorem 0.3.2: Fundametal Theorem of Calculus, ver. 2

If f is continuous on the interval $[a, b]$, then the function F defined by

$$F(x) = \int_a^x f(t)\,dt, \qquad a \leq x \leq b$$

is an antiderivative of f, i.e. $\frac{d}{dx}F(x) = f(x)$ whenever $a < x < b$.

The moral of the story is that differentiation and integration and are, in a sense, “inverse” procedures (results may differ up to some additive constant).

4 Applications of Differentiation

4.4 Indeterminate Forms and L'Hospital's Rule

4.4.1 0/0 and ∞/∞ Indeterminate Forms

We have seen limits like the following

$$\lim_{x\to\infty}\frac{e^x}{5^x} \quad \text{and} \quad \lim_{x\to 1}\frac{x^2-1}{x-1}.$$

We are unable to simply plug-in the limits as we would end up with "$\frac{\infty}{\infty}$" and "$\frac{0}{0}$", respectively which are both undefined. However, we were able to still find the limits, and what's more, both limits were very different! This reaffirms what we've known - that infinity doesn't quite behave like other real numbers, and division by 0 is equally as ill-behaved.

Definition: Indeterminate Forms

Let $f(x)$ and $g(x)$ be functions and consider the limit

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

If $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$, then we say this limit is an **indeterminate of type $\frac{0}{0}$.** Similarly, if $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$, then we say that this limit is an **indeterminate form of type $\frac{\infty}{\infty}$.**

For limits of these types, we had various methods of evaluating the limits. For example, in the case of rational functions, we factored and canceled terms. However, such tricks may not work for other "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" indeterminate forms, like

$$\lim_{x\to 0}\frac{\cos x-1}{2^{x+1}-2} \quad \text{or} \quad \lim_{x\to\infty}\frac{\ln x}{x+7}.$$

Now that we have derivatives in our toolbox, we can make use of the following theorem.

Theorem 4.4.1: L'Hospital's Rule

Suppose f and g are differentiable functions with $g'(x)\neq 0$ near a (except possibly at a). Suppose also that

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Proof. We'll prove the *very* special case where we have an indeterminate form of type $\frac{0}{0}$, $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$; the proof of the theorem in full generality is much more difficult and can be found in the textbook.

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} && \text{(since } f(a) = g(a) = 0\text{)} \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} \cdot \frac{\frac{1}{x-a}}{\frac{1}{x-a}} \\
&= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \\
&= \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} \\
&= \frac{f'(a)}{g'(a)} \\
&= \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} && \text{(since } f', g' \text{ are continuous)} \\
&= \lim_{x\to a} \frac{f'(x)}{g'(x)}.
\end{aligned}$$

□

Remark. L'Hospital's rule applies for one-sided limits and limits at infinity as well.

Example 4.4.2

Evaluate $\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$.

We first see that this is an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's rule.

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Remark. Notationally, the author likes to use $\overset{LH}{=}$ to indicate when l'Hospital's rule has been applied. This notation is not at all standard.

Example 4.4.3

Evaluate $\lim\limits_{t\to 0} \dfrac{\sin t}{t}$.

We first see that this is an indeterminate form of type $\frac{0}{0}$, so we can apply l'Hospital's rule.

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{LH}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Example 4.4.4

Evaluate $\lim\limits_{x\to 1} \dfrac{\arctan x - \frac{\pi}{4}}{x-1}$.

Recall that $\arctan 1 = \frac{\pi}{4}$, so we indeed have an indeterminate form of type $\frac{0}{0}$, and thus we can apply l'Hospital's rule.

$$\lim_{x\to 1} \frac{\arctan x - \frac{\pi}{4}}{x-1} \overset{LH}{=} \lim_{x\to 1} \frac{\frac{1}{1+x^2}}{1} = \frac{1}{2}.$$

Example 4.4.5

Evaluate $\lim\limits_{x\to -\infty} \dfrac{x^2}{e^{-x}}$.

We first see that this is an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's rule.

$$\lim_{x\to -\infty} \frac{x^2}{e^{-x}} \overset{LH}{=} \lim_{x\to -\infty} \frac{2x}{-e^{-x}}.$$

Once again, this new limit is an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's rule again.

$$\lim_{x\to -\infty} \frac{2x}{-e^{-x}} \overset{LH}{=} \lim_{x\to -\infty} \frac{2}{e^{-x}} = 0.$$

Remark. L'Hospital's rule **can only be applied** to indeterminate forms of types $\frac{0}{0}$ and $\frac{\infty}{\infty}$; for other indeterminate forms (which we'll get to later).

To see why the indeterminate form is an important hypothesis, consider the following (non)example:

Example 4.4.6

Evaluate $\lim\limits_{\theta\to \pi^-} \dfrac{\sin\theta}{1-\cos\theta}$ by applying l'Hospital's rule blindly. Find the correct limit without l'Hospital's rule.

Blindly applying l'Hospital's to the following limit,

$$\lim_{\theta\to \pi^-} \frac{\sin\theta}{1-\cos\theta} \overset{LH}{=} \lim_{\theta\to \pi^-} \frac{\cos\theta}{\sin\theta} = -\infty.$$

However, **this is wrong.** To see why, note that $\dfrac{\sin\theta}{1-\cos\theta}$ is actually continuous at $\theta = \pi$, so in fact we have

$$\lim_{\theta\to \pi^-} \frac{\sin\theta}{1-\cos\theta} = \frac{\sin\pi}{1-\cos\pi} = \frac{0}{1-(-1)} = 0.$$

4.4.2 $0 \cdot \infty$ and ∞-∞ indeterminate forms

Definition

Let f and g be functions and consider the limit

$$\lim_{x \to a} f(x)g(x).$$

If $f(x) \to 0$ and $g(x) \to \pm\infty$ as $x \to a$, then we say that this limit is an **indeterminate form of type $0 \cdot \infty$**.

To deal with limits of this form, the technique is usually to rewrite $f(x)g(x)$ as either $\dfrac{f(x)}{1/g(x)}$ (resulting in an indeterminate form of type $\frac{0}{0}$) or $\dfrac{g(x)}{1/g(x)}$ (resulting in an indeterminate form of type $\frac{\infty}{\infty}$) and then applyin l'Hospital's rule.

Example 4.4.7

Evaluate $\lim\limits_{x \to \infty} e^{-x}\sqrt{x}$.

First we see that $e^{-x} \to 0$ and $\sqrt{x} \to \infty$, so indeed we have an indeterminate form of type $0 \cdot \infty$. Rewriting

$$e^{-x}\sqrt{x} = \frac{\sqrt{x}}{e^x},$$

we now have an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's rule.

$$\lim_{x \to \infty} e^{-x}\sqrt{x} = \lim_{x \to \infty} \frac{\sqrt{x}}{e^x} \overset{LH}{=} \lim_{x \to \infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} \lim_{x \to \infty} \frac{1}{2e^x\sqrt{x}} = 0.$$

Definition

Let f and g be functions and consider the limit

$$\lim_{x \to a} f(x) - g(x).$$

If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then we say that have an **indeterminate form of type $\infty - \infty$.**

The technique for solving these limits is to convert the difference into a quotient, often by rationalizing or finding a common denominator, and then applying l'Hospital's rule.

Example 4.4.8

Evalute $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{1 - e^{-x}}\right)$.

By rewriting as a fraction with a common denominator, we have that

$$\frac{1}{x} - \frac{1}{1-e^{-x}} = \frac{1-e^{-x}-x}{x-xe^{-x}},$$

which is indeterminate of type $\frac{0}{0}$. So, applying l'Hospital's rule,

$$\lim_{x\to 0^+} \frac{1-e^{-x}-x}{x-xe^{-x}} \overset{LH}{=} \lim_{x\to 0^+} \frac{e^{-x}-1}{1-e^{-x}+xe^{-x}}.$$

This is again an indeterminate form of type $\frac{0}{0}$. So, applying l'Hospital's rule again,

$$\lim_{x\to 0^+} \frac{e^{-x}-1}{1-e^{-x}+xe^{-x}} \overset{LH}{=} \lim_{x\to 0^+} \frac{-e^{-x}}{e^{-x}+e^{-x}-xe^{-x}} = -\frac{1}{2}.$$

4.4.3 $\mathbf{0^0}$, $\mathbf{1^\infty}$, and $\mathbf{\infty^0}$ indeterminate forms

Definition

Let f and g be functions and consider the limit

$$\lim_{x\to a} [f(x)]^{g(x)}.$$

If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then we say that have an **indeterminate form of type $\mathbf{0^0}$.**
If $f(x) \to 1$ and $g(x) \to \infty$ as $x \to a$, then we say that have an **indeterminate form of type $\mathbf{1^\infty}$.**
If $f(x) \to \infty$ and $g(x) \to 0$ as $x \to a$, then we say that have an **indeterminate form of type $\mathbf{\infty^0}$.**

There are two equivalent techniques to handle these types of indeterminate forms (and in fac. The first is to take a natural logarithm of the limit. That is, let $L = \lim\limits_{x\to a} [f(x)]^{g(x)}$. Then, since the logarithmic functions are continuous,

$$\ln L = \ln\Big(\lim_{x\to a} [f(x)]^{g(x)}\Big) = \lim_{x\to a} \ln\big([f(x)]^{g(x)}\big) = \lim_{x\to a} g(x) \cdot \ln(f(x)).$$

Once you have solved for this limit on the right (usually by rewriting into "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" and applying L'Hospital's rule), simply "undo" this natural logarithm by raising the function into the exponent with base e; i.e.

$$L = e^{\ln L} = e^{\lim_{x\to a} g(x)\cdot \ln(f(x))}.$$

The other technique is to appeal to the fact that $[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$, that is,

$$\lim_{x\to a} [f(x)]^{g(x)} = \lim_{x\to a} e^{g(x)\ln(f(x))}.$$

Given that exponential functions are equivalent, this just amounts to solving for the limit of $g(x)\ln(f(x))$, and so the technique is basically equivalent to the first one; either should work.

Example 4.4.9

Evaluate $\lim_{x\to 0^+} x^x$.

Recall that, because the natural log is continuous, we have

$$\lim_{x\to a} \ln(f(x)) = \ln\left(\lim_{x\to a} f(x)\right).$$

We see that this limit is an indeterminate form of type 0^0. So, let L be the limit. Taking a natural logarithm of the limit, we get

$$\ln L = \ln\left(\lim_{x\to 0^+} x^x\right) = \lim_{x\to 0^+} x\ln x.$$

This is now indeterminate of type $0\cdot\infty$, so we rewrite $x\ln x = \frac{\ln x}{1/x}$ to put change it to an indeterminate form of type $\frac{\infty}{\infty}$, whence we can apply l'Hospital's Rule.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} \overset{LH}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

But we're not quite done yet. Notice that we just showed that $\ln L = 0$, but we wanted to find the value of L. From here we deduce that the value of the limit is $L = 1$.

Example 4.4.10

Evaluate $\lim_{x\to\infty} \left(1+\frac{7}{x}\right)^x$.

We see that this is an indeterminate form of type 1^∞. Just as last time, let L be the value of the limit. Taking a natural logarithm of the limit, we get

$$\ln L = \ln\left(\lim_{x\to\infty}\left(1+\frac{7}{x}\right)^x\right) = \lim_{x\to\infty} \ln\left(\left(1+\frac{7}{x}\right)^x\right) = \lim_{x\to\infty} x\ln\left(1+\frac{7}{x}\right).$$

As $x\to\infty$, $\ln\left(1+\frac{7}{x}\right)\to 0$, so we now have an indeterminate form of type $0\cdot\infty$. Rewriting as $\frac{\ln\left(1+\frac{7}{x}\right)}{1/x}$, we get an indeterminate form of type $\frac{\infty}{\infty}$, and can thus apply l'Hospital's rule.

$$\lim_{x\to\infty} x\ln\left(1+\frac{7}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{7}{x}\right)}{1/x} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{1+7/x}\cdot\frac{-7}{x^2}}{-1/x^2} = \lim_{x\to\infty} \frac{-\frac{7}{x^2+7x}}{-1/x^2} = \lim_{x\to\infty} \frac{7x^2}{x^2+7x} = 7.$$

So now we have that $\ln L = 7$, and thus $L = e^7$.

Example 4.4.11

Evaluate $\lim_{x\to\infty} x^{1/x}$.

We notice that this is indeterminate of type ∞^0, so can rewrite $x^{1/x} = e^{(1/x)\ln x}$. Since exponential functions are continuous, taking the whole limit amounts to taking a limit of the exponent. So, let $M = \lim_{x\to\infty} \dfrac{1}{x}\ln x$, which we recognize as an indeterminate form of type $0 \cdot \infty$. Rearranging it as $\frac{\ln x}{x}$, which is indeterminate of type $\frac{\infty}{\infty}$, and so we can apply l'Hospital's Rule.

$$M = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x} \stackrel{LH}{=} \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

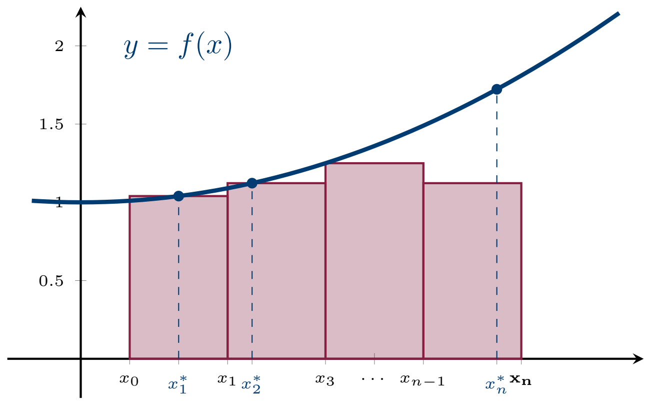
So then our entire limit is

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{(1/x)\ln x} = e^M = e^0 = 1.$$

5 Integrals

5.2 The Definite Integral

To approximate the area under the curve $y = f(x)$ on the interval $[a, b]$, we can first divide up this interval into n evenly-spaced intervals $[x_{i-1}, x_i]$. In each interval, we can choose some specific x-value, call it x_i^*, and draw the rectangle with base $[x_{i-1}, x_i]$ and height $f(x_i^*)$.



Definition: Riemann Sum

The **Riemann sum** is the sum of the rectangles approximating the area under a curve,

$$\sum_{i=1}^{n} f(x_i^*)\Delta x,$$

where x_i^* is any point in the interval $[x_{i-1}, x_i]$. If we choose x_i^* to always be the left (resp. right) endpoint, we call this a **left** (resp. **right**) **Riemann sum**.

Definition

If f is defined on $[a, b]$, then the **definite integral of f from a to b** is the number

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided this limit exists. If the limit exists, we say that f is **integrable on** $[a, b]$. Here $\int$ is called the **integral sign**, f is called the **integrand**, and a and b are called the **limits of integration** (in particular, a is called the **lower limit** and b is called the **upper limit**). As before, dx is just a differential but it doesn't have much meaning by itself – think of it as a bookend for the integral notation.

Proposition 5.2.1

If f is continuous or has only finitely many holes/jump discontinuities on $[a, b]$, then f is integrable on $[a, b]$, that is $\int_a^b f(x)\,dx$ exists.

This is great, because it says that it's easier to be integrable than it is to be differentiable. Moreover, almost every function we've worked with in this class satisfies these properties. The downside is that we allow functions that are less well-behaved, and so computing integrals can require a bit more effort.

Example 5.2.2

Express the following as a definite integral on the given interval:

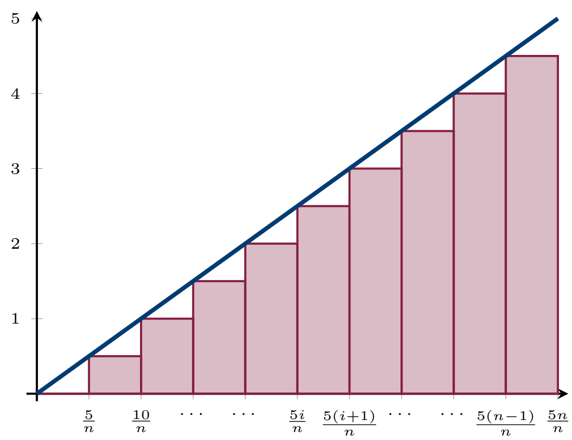
$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{4i}{n}}\cdot\frac{4}{n} \text{ on } [0,4]$$

$$\int_0^4 \sqrt{x}\,dx$$

Example 5.2.3

Evaluate the following definite integral via a limit of Riemann sums.

$$\int_0^5 x\,dx$$



We use a left Riemann sum. Here $f(x)=x$, $\Delta x=\frac{5-0}{n}=\frac{5}{n}$, $x_0=0$, and $x_i=x_0+i\cdot\Delta x=\frac{5i}{n}$. So,

$$\int_0^5 x\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{5i}{n}\right)\left(\frac{5}{n}\right)=\lim_{n\to\infty}\frac{25}{n^2}\cdot\left(\sum_{i=1}^{n}i\right)=\lim_{n\to\infty}\frac{25n^2+25n}{2n^2}=\frac{25}{2}.$$

And indeed, $\frac{25}{2}$ is the area we got by just knowing about the area of a triangle of base 5 and height 5.

Proposition 5.2.4: Properties of Definite Integrals

Let $f(x)$, $g(x)$ be integrable functions on $[a, b]$ and c a real number. Then

1. $\displaystyle\int_a^b c\,dx = c(b-a)$
2. $\displaystyle\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$
3. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

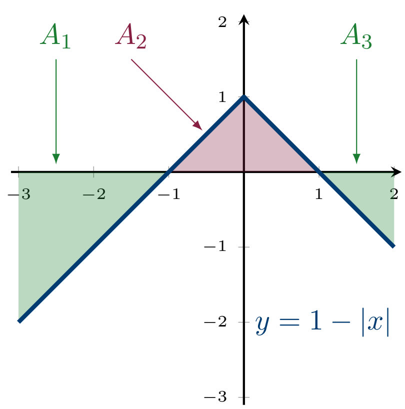
Proof. The proof of each property follows easily from properties of summations and the algebra of limits. Only the last one seems kind of strange, but indeed it comes from the fact that when looking at $\displaystyle\int_a^b$, we have $\Delta x = \dfrac{b-a}{n}$ and when looking at $\displaystyle\int_b^a$, we have $\Delta x = \dfrac{a-b}{n} = -\dfrac{b-a}{n}$. $\square$

When $f(x)$ is positive on $[a, b]$, the definite integral $\int_a^b f(x)\,dx$ represents the area above the x-axis and under the curve $y = f(x)$. When $f(x)$ takes both positive and negative values on $[a, b]$, the definite integral $\int_a^b f(x)\,dx$ represents the net area (that is, the area above x-axis and below the curve $y = f(x)$, minus the area below the x-axis and above the curve $y = f(x)$).

Example 5.2.5

Evaluate the following integrals by interpreting it in terms of area: $\displaystyle\int_{-3}^{2}(1-|x|)\,dx$

Drawing this out, we see that we just have the area of red triangle minus the area of the green triangles.



So,

$$\int_{-3}^{2}(1-|x|)\,dx = A_1 + A_2 + A_3 = -\frac{1}{2}(2)(2) + \frac{1}{2}(2)(1) - \frac{1}{2}(1)(1) = -\frac{3}{2}.$$

This next result tells us how we can combine integrals on adjacent intervals.

Proposition 5.2.6

Suppose $f(x)$ is integrable on $[a,c]$ and $a \le b \le c$. Then certainly f is integrable on both $[a,b]$ and $[b,c]$ and

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^b f(x)\,dx.$$

Example 5.2.7

Suppose $\displaystyle\int_0^{27} f(x)\,dx = 10$ and $\displaystyle\int_0^{15} f(x)\,dx = 3$. Find $\displaystyle\int_{15}^{27} f(x)\,dx$.

By Proposition 5.2.6,

$$\begin{aligned}\int_0^{15} f(x)\,dx + \int_{15}^{27} f(x)\,dx &= \int_0^{27} f(x)\,dx\\ 3 + \int_{15}^{27} f(x)\,dx &= 10\\ \int_{15}^{27} f(x)\,dx &= 7\end{aligned}$$

Example 5.2.8

Suppose $\displaystyle\int_0^{4} f(x)\,dx = 2$, $\displaystyle\int_4^{6} f(x)\,dx = 3$, and $\displaystyle\int_0^{6} g(x)\,dx = 9$, find

$$\int_0^6 \left(3f(x) - g(x) + 1\right)\,dx.$$

$$\begin{aligned}\int_0^6 \left(3f(x) - g(x) + 1\right)\,dx &= \int_0^6 3f(x)\,dx - \int_0^6 g(x)\,dx + \int_0^6 1\,dx\\ &= 3\int_0^6 f(x)\,dx - \int_0^6 g(x)\,dx + \int_0^6 1\,dx\\ &= 3\left(\int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx\right) - \int_0^6 g(x)\,dx + \int_0^6 1\,dx\\ &= 3(2+3) - 9 + 6 = 12.\end{aligned}$$

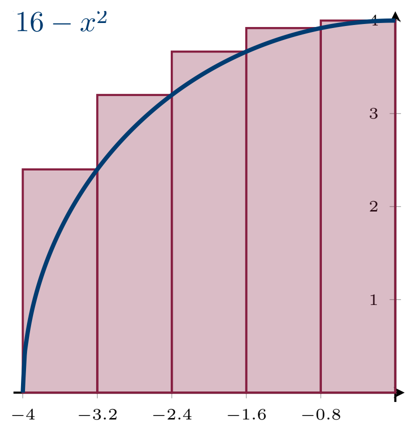
Example 5.2.9

Approximate the following definite integral using a right Riemann sum with 5 rectangles.

$$\int_{-4}^{0} \sqrt{16 - x^2}\, dx$$

We have here that $f(x) = \sqrt{16 - x^2}$, $\Delta x = \frac{0-(-4)}{5} = 0.8$, and

$$x_0 = -4, \quad x_1 = -3.2, \quad x_2 = -2.4, \quad x_3 = -1.6, \quad x_4 = -0.8, \quad x_5 = 0.$$



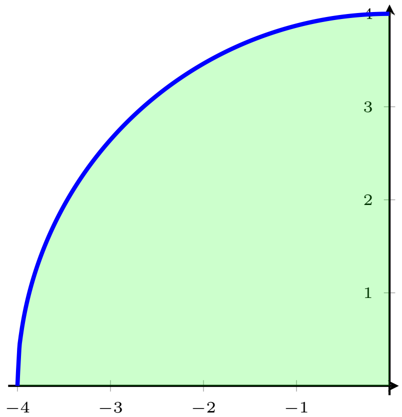
So, the sum of the areas of these rectangles is

$$\begin{aligned}\sum_{i=1}^{5} f(x_i)\,\Delta x &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x + f(x_5)\,\Delta x \\ &\approx (2.4)\,0.8 + (3.2)\,0.8 + (3.3667)\,0.8 + (3.919)\,0.8 + (4)\,0.8 + \\ &\approx 13.748\end{aligned}$$

Example 5.2.10

Evaluate the definite integral in Example 5.2.9 by interpreting it in terms of area.

Notice that, if $y = \sqrt{16 - x^2}$, then $y^2 = 16 - x^2$, and thus $x^2 + y^2 = 16$, so the graph traces out the top half of the circle of radius 4. Drawing this out, we see that we're just looking at the area under one quarter of this circle.



With this in mind, the area is quite simply

$$\int_{-4}^{0} \sqrt{16 - x^2}\, dx = \frac{1}{4}\pi(4)^2 = 4\pi.$$

Since Riemann sums can be very hard to get a hold of explicitly and certain curves don't bound areas that are easily computed via geometric means, it's at least useful to be able to bound the value of a definite integral.

Proposition 5.2.11: Definite Integral Bounds

Let f, g be integrable on $[a, b]$ and let m, M be constants.

a. If $f(x) \geq 0$, then

$$\int_a^b f(x)\, dx \geq 0.$$

b. If $f(x) \geq g(x)$, then

$$\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx.$$

c. If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then

$$m(b-a) = \int_a^b m\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b M\, dx = M(b-a)$$

Proof. For part (a), since $f(x) \geq 0$, in the Riemann sum we must have that for each x_i^* in the partition, the corresponding rectangle has area satisfying $f(x_i^*)\Delta x \geq 0$. This inequality is unchanged in the limit. Parts (b) and (c) are nearly identical arguments as well. □

Example 5.2.12

Use the Proposition 5.2.11 to estimate (i.e. find bounds for) the value of $\displaystyle\int_{\pi/4}^{\pi/3} \tan(x)\, dx$.

Since $\tan(x)$ is increasing on the interval $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$, we could apply part 3 of Proposition 5.2.11

using $m = \tan\left(\frac{\pi}{4}\right) = 1$ and $m = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ as upper bounds. Hence

$$0.262 \approx \frac{\pi}{12} = \int_{\pi/4}^{\pi/3} 1\,dx \leq \int_{\pi/4}^{\pi/3} \tan(x)\,dx \leq \int_{\pi/4}^{\pi/3} \sqrt{3}\,dx = \frac{\pi\sqrt{3}}{12} \approx 0.453$$

Going a bit further, we could also use the fact that $\tan(x)$ is concave up in this interval and take the line between $\left(\frac{\pi}{4}, 1\right)$ and $\left(\frac{\pi}{3}, \sqrt{3}\right)$ as an upper bound and apply part 2 of Proposition 5.2.11; explicitly, this is the line given by $y = \frac{12(\sqrt{3}-1)}{\pi}\left(x - \frac{\pi}{4}\right) + 1$. Hence

$$0.262 \approx \frac{\pi}{12} = \int_{\pi/4}^{\pi/3} 1\,dx \leq \int_{\pi/4}^{\pi/3} \tan(x)\,dx \leq \int_{\pi/4}^{\pi/3} \frac{12(\sqrt{3}-1)}{\pi}\left(x - \frac{\pi}{4}\right) + 1\,dx = \frac{\pi}{24}(1+\sqrt{3}) \approx 0.357.$$

Any estimate in this range is actually pretty good. The actual area is

$$\frac{\ln(2)}{2} \approx 0.346$$

5.5 The Substitution Rule

Example 5.5.1: Warm Up

Find $\dfrac{d}{dx}\left[\left(3x^2-5\right)^8\right]$.

Using the chain rule, we have

$$\frac{d}{dx}\left[\left(3x^2-5\right)^8\right] = 8\left(3x^2-5\right)^7 \cdot 6x.$$

Recall that the chain rule says

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

where $g(x)$ is your "inner function" and $f(x)$ is your "outer function". Recall also that, if $u = g(x)$ is a differentiable function, then in the language of differentials, we have $du = g'(x)\,dx$.

The following rule combines these two concepts in a way that is exactly analogous to the chain rule for differentiation.

Theorem 5.5.2: Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I, and if f is continuous on I, then

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

Example 5.5.3

Using the substitution rule, evaluate $\displaystyle\int 8\left(3x^2-5\right)^7 \cdot 6x\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function".

$$\begin{aligned} u &= g(x) = 3x^2 - 5. \\ du &= g'(x)\,dx = 6x\,dx. \end{aligned}$$

Hence, by the substitution rule,

$$\begin{aligned} \int 8\left(3x^2-5\right)^7 \cdot 6x\,dx &= \int 8u^7\,du \\ &= u^8 + C \\ &= \left(3x^2-5\right)^8 + C \qquad \text{(substitute back in for } u\text{)}. \end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Example 5.5.4

Using the substitution rule, evaluate $\int e^{-2x}\, dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function".

$$\begin{aligned} u &= g(x) = -2x \\ du &= g'(x)\, dx = -2\, dx \quad \Rightarrow \quad -\frac{1}{2}\, du = dx. \end{aligned}$$

Hence, by the substitution rule,

$$\begin{aligned} \int e^{-2x}\, dx &= \int e^{u} \left(-\frac{1}{2}\right)\, du \\ &= -\frac{1}{2} \int e^{u}\, du \\ &= -\frac{1}{2} e^{u} + C \\ &= -\frac{1}{2} e^{-2x} + C && \text{(substitute back in for } u\text{).} \end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Example 5.5.5

Using the substitution rule, evaluate $\int \frac{(\ln x)^2}{x}\, dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function".

$$\begin{aligned} u &= g(x) = \ln x \\ du &= g'(x)\, dx = \frac{1}{x}\, dx. \end{aligned}$$

Hence, by the substitution rule,

$$\begin{aligned} \int \frac{(\ln x)^2}{x}\, dx &= \int u^2\, du \\ &= \frac{1}{3} u^3 + C \\ &= \frac{1}{3} (\ln x)^3 + C && \text{(substitute back in for } u\text{).} \end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Example 5.5.6

Using the substitution rule, evaluate $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function".

$$u = g(x) = x^2 + 1 \tag{5.5.1}$$

$$du = g'(x)\,dx = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}\,du = x\,dx$$

This gives us

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{x^2}{\sqrt{u}}\left(\frac{1}{2}\right)\,du$$

But what do we do with the x^2 term? Well notice that we can rearrange Equation 5.5.1 to get $x^2 = u - 1$, so

$$\begin{aligned}
&= \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,du \\
&= \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)\,du \\
&= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C \\
&= \frac{1}{3}u^{3/2} - u^{1/2} + C \\
&= \frac{1}{3}\left(x^2+1\right)^{3/2} - \left(x^2+1\right)^{1/2} + C \qquad \text{(substitute back in for } u\text{).}
\end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Example 5.5.7

Using the substitution rule, evaluate $\int \tan(x)\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function". But where can this come from? First we recall that $\tan x = \frac{\sin x}{\cos x}$ and let

$$u = g(x) = \cos x$$

$$du = g'(x)\,dx = -\sin x\,dx \quad \Rightarrow \quad -du = \sin x\,dx$$

Hence, by the substitution rule,

$$\begin{aligned}
\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \\
&= -\int \frac{du}{u}
\end{aligned}$$

$$
\begin{aligned}
&= -\ln|u| + C \\
&= -\ln|\cos x| + C \qquad\qquad \text{(substitute back in for } u).
\end{aligned}
$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Theorem 5.5.8: Substituion Rule for Definite Integrals

If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$
\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du
$$

Proof. Let F be an antiderivative for f. Then $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ by the substitution rule. So, we have

$$
\int_a^b f(g(x)) \cdot g'(x)\,dx = F(g(x))\big|_a^b = F(g(b)) - F(g(a))
$$

and

$$
\int_{g(a)}^{g(b)} f(u)\,du = F(u)\big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)),
$$

whence the definite integrals must be equal. □

Example 5.5.9

Evaluate $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

Let

$$
\begin{aligned}
u &= g(x) = \frac{1}{x} \\
du &= g'(x)\,dx = -\frac{1}{x^2}\,dx \quad \Rightarrow \quad -du = \frac{1}{x^2}\,dx.
\end{aligned}
$$

Our new endpoints then become

$$
\begin{aligned}
u(1) &= g(1) = 1 \\
u(2) &= g(2) = \frac{1}{2}.
\end{aligned}
$$

Thus, applying the substitution rule, we have

$$
\begin{aligned}
\int_{x=1}^{x=2} \frac{e^{1/x}}{x^2}\,dx &= \int_{u=1}^{u=1/2} e^u(-du) \\
&= -\int_1^{1/2} e^u\,du
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{1/2}^{1} e^u \, du \\
&= e^u \Big|_{1/2}^{1} \\
&= e^1 - e^{1/2} = e - \sqrt{e}.
\end{aligned}
$$

Example 5.5.10

Evaluate $\displaystyle\int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}} \, dx$

Let

$$
\begin{aligned}
u &= \arcsin x \\
du &= \frac{1}{\sqrt{1-x^2}} \, dx.
\end{aligned}
$$

Our new endpoints then become

$$
\begin{aligned}
u(0) \arcsin(0) &= 0 \\
u\left(\frac{1}{2}\right) = \arcsin\left(\frac{1}{2}\right) &= \frac{\pi}{6}.
\end{aligned}
$$

Thus, applying the substitution rule, we have

$$
\begin{aligned}
\int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}} \, dx &= \int_0^{\pi/6} u \, du \\
&= \frac{1}{2} u^2 \Big|_0^{\pi/6} \\
&= \frac{1}{2}\left(\frac{\pi}{6}\right)^2 - \frac{1}{2}(0)^2 = \frac{\pi^2}{72}.
\end{aligned}
$$

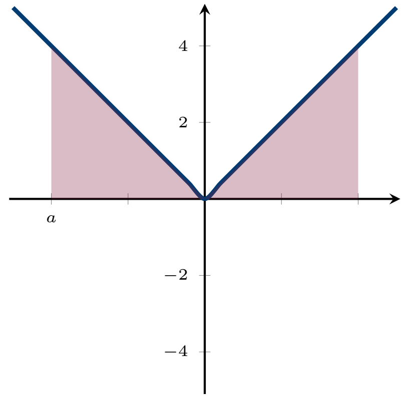
Definition 5.5.11

A function f is **even** if $f(-x) = f(x)$, and f is **odd** if $f(-x) = -f(x)$, where x is any real number in the domain.

Example 5.5.12

Demonstrate that $f(x) = |x|$ is an even function.

Notice that, for some positive real number a, the integral $\displaystyle\int_{-a}^{a} f(x) \, dx$ is represented by the picture below.



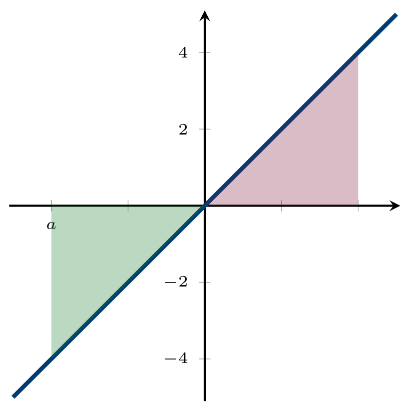
The shaded regions to the left and right of the y-axis appear equal, so the area under the curve $y = f(x)$ over the interval $[-a, a]$ is double the area found over the interval $[0, a]$. In other words,

$$\int_{-a}^{a} |x|\, dx = 2\int_{0}^{a} |x|\, dx.$$

Example 5.5.13

Demonstrate that $f(x) = x$ is an odd function.

Notice that, for some positive real number a, the integral $\int_{-a}^{a} f(x)\, dx$ is represented by the picture below.



Notice that the shaded regions to the left and right of the y-axis are equal, but have opposite sign since area under a curve is "negative". So the area under the curve $y = f(x)$ over the interval $[-a, 0]$ is effectively cancels the area over the interval $[0, a]$. In other words,

$$\int_{-a}^{a} x\, dx = 0.$$

The same sort of symmetry applies in general to even and odd function.

Theorem 5.5.14: Integrating Even & Odd Functions Over Symmetric Intervals

Suppose f is continuous on $[-a, a]$.

1. If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

2. If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Proof. First notice that, from the properties outlines in Proposition 5.2.4,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \tag{5.5.2}$$

For the first integral (with bounds 0 and $-a$), let

$$\begin{aligned} u &= -x, \\ du &= -dx. \end{aligned}$$

Our new bounds are

$$\begin{aligned} u(0) &= 0 \\ u(-a) &= -(-a) = a. \end{aligned}$$

Then

$$\begin{aligned} -\int_{0}^{-a} f(x)\,dx &= -\int_{0}^{a} f(-u)(-du) \\ &= \int_{0}^{a} f(-u)\,du \\ &= \int_{0}^{a} f(-x)\,dx, && \text{(since } u \text{ was a dummy variable)} \end{aligned}$$

so from Equation 5.5.2, we can write

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \tag{5.5.3}$$

1. If f is even, we have $f(-x) = f(x)$, so Equation 5.5.3 yields

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

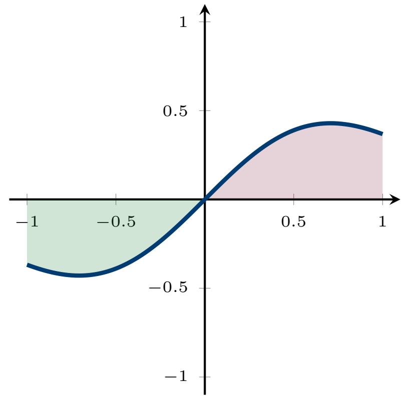
2. If f is odd, we have $f(-x) = -f(x)$, so Equation 5.5.3 yields

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0.$$

□

Example 5.5.15

Evaluate the integral $\int_{-1}^{1} xe^{-x^2}\, dx$.



From the graph, it looks like $f(x) = xe^{-x^2}$ is odd. Indeed

$$\int_{-1}^{1} xe^{-x^2}\, dx = 0.$$

6 Applications of Integration

6.1 Area Between Curves

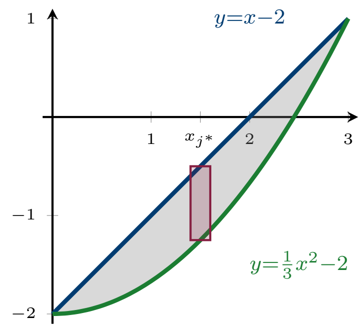
INSERT DISCUSSION HERE ABOUT POSITIVE CASE WHERE $f(x) \geq g(x)$

Example 6.1.1

$f(x) = x + 2$ and $g(x) - x + 2$ on $[0, 2]$. Note that area is triangle of base 4 and height 2.

Example 6.1.2

Find the area bounded between the curves $y = x - 2$ and $y = \dfrac{x^2}{3} - 2$.



First we find our limits of integration by setting the two functions equal to one another and solving for x:

$$\begin{aligned} x-2 &= \frac{x^2}{3}-2 \\ \frac{x^2}{3}-x &= 0 \\ \frac{1}{3}x(x-3) &= 0 \quad \Rightarrow \quad x=0,3 \end{aligned}$$

Since $y=x-2$ is the "top" function, we have

$$\begin{aligned} \int_0^3 \left((x-2)-\left(\frac{x^2}{3}-2\right)\right)\,dx &= \int_0^3 \left(x-\frac{x^2}{3}\right)\,dx \\ &= \left[\frac{x^2}{2}-\frac{x^3}{9}\right]_0^3 = \frac{3}{2}. \end{aligned}$$

NOW DISCUSS WHAT HAPPENS WHEN $f(x) \leq g(x)$.

Conclude that

Definition: Area Between Curves

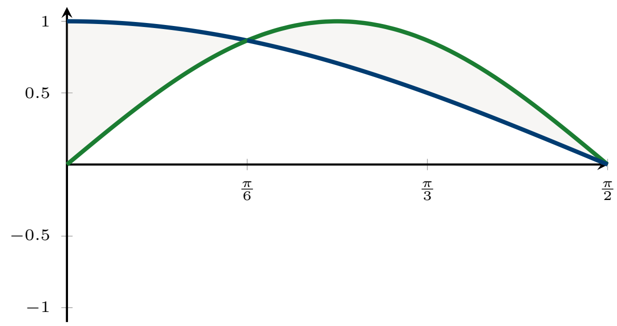
The **area between the curves** $y=f(x)$ and $y=g(x)$ on the interval $[a,b]$ is the *positive number*

$$A=\int_a^b |f(x)-g(x)|\,dx.$$

Remark. Although the definition works fine with the absolute value, in practice we'll need to actually look at cases when $f(x) \geq g(x)$ and $g(x) \geq f(x)$.

Example 6.1.3

Find the area between the two curves $y = \cos x$ and $y = \sin 2x$ on the interval $[0, \frac{\pi}{2}]$.



We have to consider two separate areas. Note that the curves intersect at $x = \frac{\pi}{6}$.

$$\begin{aligned} A_1 &= \int_0^{\pi/6} (\cos x - \sin(2x))\ dx \\ &= \left[\sin x + \frac{1}{2}\cos(2x)\right]_0^{\pi/6} \\ &= \frac{1}{4}. \end{aligned}$$

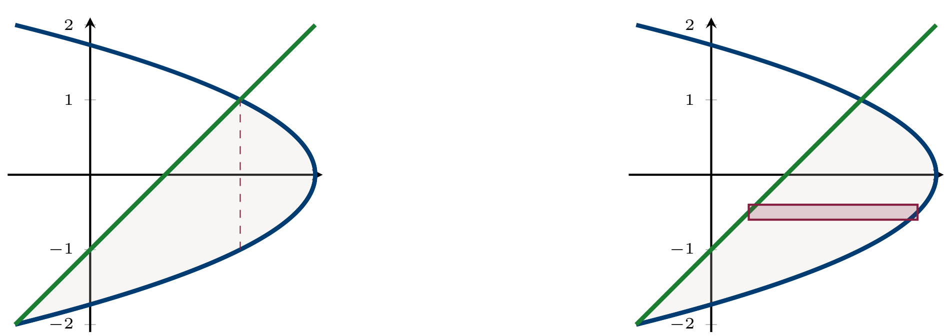
and

$$\begin{aligned} A_2 &= \int_{\pi/6}^{\pi/2} (\sin(2x) - \cos x)\ dx \\ &= \left[-\frac{1}{2}\cos(2x) - \sin x\right]_{\pi/6}^{\pi/2} \\ &= \frac{1}{4}. \end{aligned}$$

And so the entire area is $A = A_1 + A_2 = \frac{1}{2}$.

Example 6.1.4

Find the area between the curves $y^2 = 3 - x$ and $y = x - 1$.



Notice in the image on the left that we would have to have two separate integrals, because the "top" function would change at $x = \frac{1}{2}$. Instead, we'll draw horizontal rectangles and just integrate with respect to y, since the "top" ("right") and "bottom" ("left") functions do not change in this direction. This also saves us from the hassle of integrating square roots of functions. We proceed by solving each equation for x (in terms of y) and setting them equal to find the points of intersection:

$$\begin{aligned} 3 - y^2 &= y + 1 \\ y^2 + y - 2 &= 0 \\ (y+2)(y-1) &= 0 \\ \Rightarrow \quad y &= -2, 1. \end{aligned}$$

So, the area between these two curves is

$$\begin{aligned} \int_{-2}^{1} \left((3 - y^2) - (y+1)\right)\, dx &= \int_{-2}^{1} \left(-y^2 - y + 2\right)\, dx \\ &= \left[-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y\right]_{-2}^{1} = \frac{9}{2}. \end{aligned}$$

6.1.1 Gini Index

In this particular section, we're going to examine income inequality, and so we take the following very narrow definition.

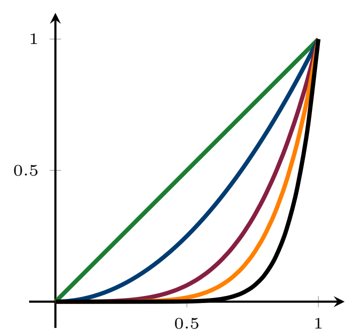
Definition

A **Lorenz Curve** is a curve in the plane which contains the point $(x\%, y\%)$ if $x\%$ of the population has $y\%$ of the total income.

Remark. The Lorenz Curve, say $y = L(x)$, has the following properties:

- $L(0) = 0$
- $L(1) = 1$
- $L'(x) > 0$
- $L''(x) \geq x$

In a perfectly egalitarian society, $L(x) = x$, and at the other extreme, in a perfectly totalitarian society (one singular person has all of the income), $L(x) = \begin{cases} 0 & \text{when } x < 1 \\ 1 & \text{when } x = 1 \end{cases}$. In practice, a Lorenz curve comes from trying to fit a finite amount of sampled data, so it will be somewhere in between these extremes.



The greater the area between the Lorenz curve $L(x)$ and the perfectly egalitarian line x, the greater the income inequality in a particular population. As such

Definition

For a particular Lorenz curve, $L(x)$, the **Gini Index** or **Gini Coefficient** is given by $2\int_0^1 x - L(x)\,dx$.

Example 6.1.5

What is the Gini coefficient for a perfectly egalitarian society?

$L(x) = x$, so we have

$$G = 2\int_0^1 x - L(x)\,dx = 2\int_0^1 x - x\,dx = 2\int_0^1 0\,dx = 0.$$

Example 6.1.6

What is the Gini coefficient for a perfectly totalitarian society?

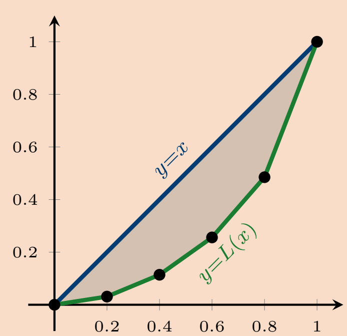
$L(x) = \begin{cases} 0 & \text{when } x < 1 \\ 1 & \text{when } x = 1 \end{cases}$. Since this function is constantly 0 on $(0, 1)$, we have

$$G = 2\int_0^1 x - L(x)\,dx = 2\int_0^1 x - 0\,dx = 2\int_0^1 x\,dx = x^2\Big|_0^1 = 1.$$

Example 6.1.7

The U.S. Census Bureau provided the following data for income distribution in the U.S. in 2016.

Percentage of Population	Percentage of Income
0	0
20	3.1
40	11.4
60	25.6
80	48.5
100	100

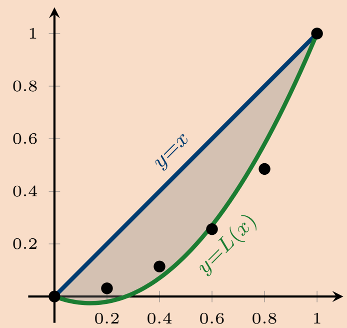


Assuming the Lorenz curve as a piecewise linear function (shown above), what is the Gini index?

Example 6.1.8

The U.S. Census Bureau provided the following data for income distribution in the U.S. in 2016.

Percentage of Population	**Percentage of Income**
0	0
20	3.1
40	11.4
60	25.6
80	48.5
100	100

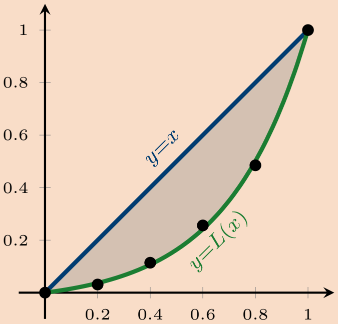


Assuming the Lorenz curve is approximated by the quadratic function $L(x) = 1.374x^2 - 0.374x$ (shown above), find the Gini index.

Example 6.1.9

The U.S. Census Bureau provided the following data for income distribution in the U.S. in 2016.

Percentage of Population	Percentage of Income
0	0
20	3.1
40	11.4
60	25.6
80	48.5
100	100



Assuming the Lorenz curve is approximated by the exponential function $L(x) = \dfrac{1-e^{3.268x}}{1-e^{3.268}}$ (shown above), find the Gini index.

6.2 Volumes

Recall our strategy for finding the area of a shape in the plane (between $x = a$ and $x = b$) was to slice it up via equally-spaced parallel lines (and in this picture, vertical lines).

Each slice was approximately a rectangle with area

$$(\text{height at } x)(\Delta x).$$

Provided the height of this could be measured as a function of x, $h(x)$, then this area was approximated by the Riemann sum and corresponding limit

$$\lim_{n\to\infty} \sum_{j=1}^{n} h(x_j^*)\Delta x = \int_a^b h(x)\, dx$$

Let's employ the same strategy for a 3-D object - slice it up via equally-spaced parallel planes (in this picture, vertical planes). Each slice now has volume approximately

$$(\text{cross-sectional area at } x)(\Delta x)$$

. Provided the area of each cross-section can be measured as a function of x, $A(x)$, then this volume is approximated by

Taking a limit, we obtain the following

Definition

Let S be a solid lying between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x (through x and perpendicular to the x-axis), is $A(x)$, where A is an integrable function, then the **volume** of S is

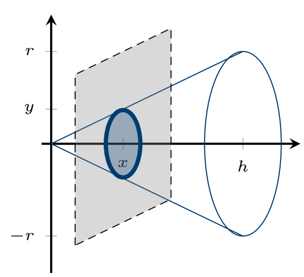
$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)\, dx$$

Remark. The definition above also holds for slicing things up with horizontal planes, making the obvious changes to allow for one to integrate with respect to y instead of x.

Let's check that this agrees with our knowledge of volumes (at least, I assume you've seen this volume equation before)

Example 6.2.1

What is the volume of a right circular cone of height h and radius r?



Notice that the cross-sectional area through x is a circle of radius y, so the area is $A(x) = \pi y^2$. In order to integrate A with respect to x, we need to write y as a function of x. Notice that by similar triangles,

$$\frac{r}{h} = \frac{y}{x} \quad \Rightarrow y = \frac{rx}{h}.$$

Thus the volume is given by

$$\begin{aligned} V = \int_0^h \pi y^2\, dx &= \int_0^h \pi \left(\frac{rx}{h}\right)^2 dx \\ &= \frac{\pi r^2}{h^2} \int_0^h x^2\, dx \\ &= \frac{\pi r^2}{h^2} \left[\frac{1}{3}h^3 - 0\right] \\ &= \frac{1}{3}\pi r^2 h. \end{aligned}$$

This is exactly the volume of a cone that we all remember from the countless conical tank related rates problems.

The previous example affirms that our strategy for finding the volume is indeed correct. So now we can use it to find the volume of objects whose volumes don't know *a priori*.

Example 6.2.2

Find the volume of a spherical cap with radius r and height h.

At each point x, the corresponding cross-section has radius $y = \sqrt{r^2 - x^2}$. This means that each cross-sectional area is

$$A(x) = \pi y^2 = \pi(r^2 - x^2).$$

Since our spherical cap has height h, we integrate from $x = r - h$ to r, which gives us

$$\begin{aligned} V &= \int_{r-h}^{r} A(x)\, dx \\ &= \int_{r-h}^{r} \pi(r^2 - x^2)\, dx \\ &= \left[\pi r^2 x - \frac{\pi}{3}x^3\right]_{r-h}^{r} \\ &= \pi r^3 - \frac{\pi}{3}r^3 - \pi r^2(r-h) + \frac{\pi}{3}(r-h)^3 \\ &= \pi r^3 - \frac{\pi}{3}r^3 - \pi r^3 + \pi r^2 h + \frac{\pi}{3}r^3 - \pi r^2 h + \pi r h^2 - \frac{\pi}{3}h^3 \\ &= \frac{\pi}{3}h^2(3r - h). \end{aligned}$$

Example 6.2.3

Find the volume of the solid bounded by $y = 4 - x^2$ and the x-axis whose horizontal cross-sections are all squares.

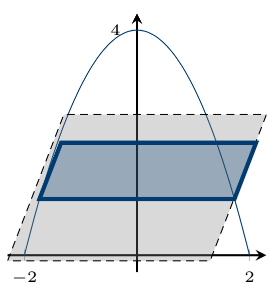


Figure 6.2.1: Cross-sectional Area

Figure 6.2.2: Solid Region

For each y, cross-sectional area here is a square with side length $2x = 2\sqrt{4-y}$. The area function is thus

$$A(y) = (2x)^2 = 4(4 - y) = 16 - 4y.$$

Notice that we'll integrate from $y = 0$ to 4. Thus the volume of this solid is given by

$$\begin{aligned} V &= \int_0^4 A(y)\, dy \\ &= \int_0^4 16 - 4y\, dy \end{aligned}$$

$$
\begin{aligned}
&= \left[16y - 2y^2\right]_0^4 \\
&= 64 - 32 \\
&= 32.
\end{aligned}
$$

6.2.1 Solids of Revolution – The Disk/Washer Method

REDO THIS SECTION WITH pgfplots SOLIDS, SEE ALSO https://tex.stackexchange.com/questions/34302/surface-of-revolution

Solids with rotational symmetry are particularly nice to try to find volumes for. We motivate this idea with the following example.

Example 6.2.4

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ between $x = 0$ and $x = 1$.

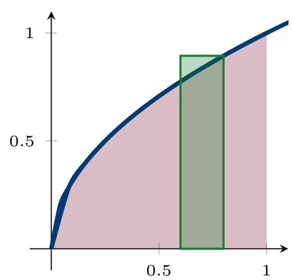


Figure 6.2.3: Cross-sectional Area

Figure 6.2.4: Solid Region

Notice that the disk-shaped slice in the 3-d picture is the same as the disk formed by rotating the rectangle in the 2-D picture. So $A(x) = \pi(f(x))^2$ in this case.

$$
V = \int_0^1 A(x)\,dx = \int_0^1 \pi(\sqrt{x})^2\,dx = \int_0^1 \pi x\,dx = \frac{\pi}{2}x^2\big|_0^1 = \frac{\pi}{2}
$$

Example 6.2.5

Find the volume of the solid obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the x-axis.

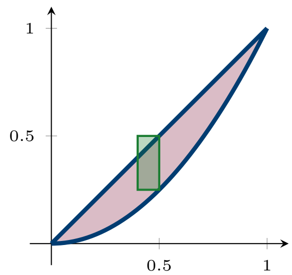


Figure 6.2.5: Cross-sectional Area

Figure 6.2.6: Solid Region

At each point x, the corresponding cross-section is a washer with inner radius x^2 and outer radius x. This means that each cross-sectional area is

$$A(x) = \pi x^2 - \pi(x^2)^2.$$

Since $y = x$ and $y = x^2$ intersect at both $x = 0$ and $x = 1$, we integrate from $x = 0$ to 1, which gives us

$$\begin{aligned} V &= \int_0^1 A(x)\,dx \\ &= \int_0^1 \pi x^2 - \pi x^4\,dx \\ &= \left[\frac{\pi}{3}x^3 - \frac{\pi}{5}x^5\right]_0^1 \\ &= \frac{\pi}{3} - \frac{\pi}{5} \\ &= \frac{2\pi}{15}. \end{aligned}$$

Example 6.2.6

Find the volume of a solid obtained by rotating the region bounded by the curves $y = x$ and $y = \sqrt{x}$ about the line y-axis.

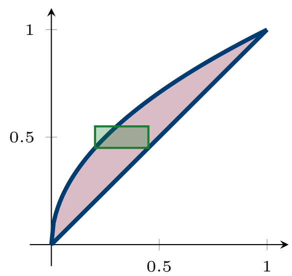


Figure 6.2.7: Cross-sectional Area

Figure 6.2.8: Solid Region

Example 6.2.7

Find the volume of a solid obtained by rotating the region bounded by the curves $y = x$ and $y = \sqrt{x}$ about the line $x = 2$.

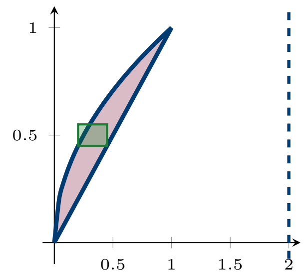


Figure 6.2.9: Cross-sectional Area

Figure 6.2.10: Solid Region

Notice that each horizontal cross section (now we're looking to integrate with respect to y) is a washer with inner radius $x = 2 - y$ and outer radius $x = 2 - y^2$, and thus the area is given by

$$A(y) = \pi(2 - y^2)^2 - \pi(2 - y)^2 = 4\pi - 4\pi y^2 + \pi y^4 - 4\pi + 4\pi y - \pi y^2 = \pi y^4 - 5\pi y^2 + 4\pi y.$$

Since $y = x$ and $y = \sqrt{x}$ intersect at both $y = 0$ and $y = 1$, we integrate from $y = 0$ to 1, which gives us

$$\begin{aligned} V &= \int_0^1 A(y)\,dy \\ &= \int_0^1 \pi y^4 - 5\pi y^2 + 4\pi y\,dy \\ &= \left[\frac{\pi}{5}y^5 - \frac{5\pi}{3}y^3 + 2\pi y^2\right]_0^1 \\ &= \frac{\pi}{5} - \frac{5\pi}{3} + 2\pi = \frac{8\pi}{15}. \end{aligned}$$

Example 6.2.8

Set up the integral representing the volume of a torus with major radius R and minor radius r. Assume $R > r$.

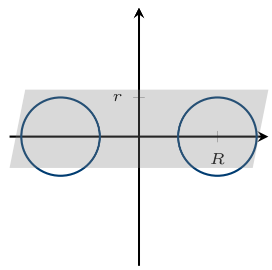


Figure 6.2.11: Cross-sectional Area

Figure 6.2.12: Solid Region

Once again, the horizontal cross-section is a washer with outer radius $R + \sqrt{r^2 - y^2}$ and inner radius $R - \sqrt{r^2 - y^2}$, so the cross-sectional area is

$$A(y) = \pi \left(R + \sqrt{r^2 - y^2}\right)^2 - \pi \left(R - \sqrt{r^2 - y^2}\right)^2$$

Notice that we'll integrate from $y = -r$ to r. Thus the volume of this torus is given by

$$\begin{aligned} V &= \int_{-r}^{r} A(y)\,dy \\ &= \int_{-r}^{r} \pi \left(R + \sqrt{r^2 - y^2}\right)^2 - \pi \left(R - \sqrt{r^2 - y^2}\right)^2 \,dy. \end{aligned}$$

6.3 Volumes by Cylindrical Shells

If we slice our solid of revolution parallel to the axis of rotation, we end up with tiny cylindrical shells with infinitesimal thickness. Notice that the cylinders in question are

VOLUME OF CYLINDRICAL SHELL

Example 6.3.1

Find the volume of the right circular cone with height h and base radius r.

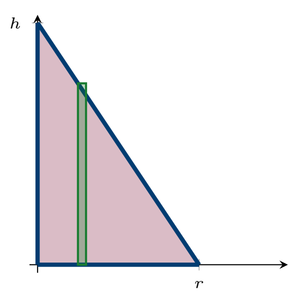


Figure 6.3.1: Cross-sectional Area

Figure 6.3.2: Solid Region

Our shell has radius given by $R(x) = x$ and height given by $H(x) = (h - \frac{h}{r}x)$. So

$$
\begin{aligned}
V = \int_0^2 A(x)\,dx &= \int_0^2 2\pi x(2x - x^2)\,dx \\
&= 2\pi \int_0^2 2x^2 - x^3\,dx \\
&= 2\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2
\end{aligned}
$$

$$= \frac{8\pi}{3}.$$

Example 6.3.2

Find the volume of the solid generated by rotating the region bounded by $y = 2x$ and $y = x^2$ about the y-axis.

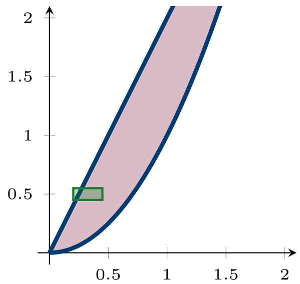


Figure 6.3.3: Cross-sectional Area

Figure 6.3.4: Solid Region

To slice parallel will involve integrating with respect to x. The cylinder will have area $2\pi rh$, so at a specific point x, we get that $r = x$ and the height is the difference of the functions $h = 2x - x^2$, hence at each x, our area is $A(x) = 2\pi x(2x - x^2)$. Our limits of integration are

$$\begin{aligned} 2x &= x^2 \\ x^2 - 2x &= 0 \\ x(x-2) &= 0 \\ \Rightarrow \quad x &= 0, 2. \end{aligned}$$

Thus, we integrate over all of these areas and get

$$\begin{aligned} V = \int_0^2 A(x)\,dx &= \int_0^2 2\pi x(2x - x^2)\,dx \\ &= 2\pi \int_0^2 2x^2 - x^3\,dx \\ &= 2\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 \\ &= \frac{8\pi}{3}. \end{aligned}$$

Example 6.3.3

Find the volume of the solid generated by rotating the region bounded by $y = 2x$, $y = x^2$ about the x-axis. (Use the method of cylindrical shells)

Figure 6.3.5: Cross-sectional Area

Figure 6.3.6: Solid Region

Since cylindrical shells are parallel to the x-axis, we'll be integrating with respect to y. Our functions, rewritten in terms of y, are

$$y = 2x \Rightarrow x = \frac{1}{2}y$$
$$y = x^2 \Rightarrow x = \sqrt{y}$$

At each fixed value y, our cylindrical shell will have radius y and height $\sqrt{y} - \frac{1}{2}y$. So, the shell's area is $A(y) = 2\pi y\left(\sqrt{y} - \frac{1}{2}y\right)$. The limits of integration will be from 0 to 4. Thus, the volume of this solid is

$$\begin{aligned} V = \int_0^4 A(y)\,dy &= \int_0^4 2\pi y\left(\sqrt{y} - \frac{1}{2}y\right)\,dy \\ &= 2\pi \int_0^4 y^{3/2} - \frac{1}{2}y^2\,dy \\ &= 2\pi\left[\frac{2}{5}y^{5/2} - \frac{1}{6}y^3\right]_0^4 \\ &= \frac{64\pi}{15}. \end{aligned}$$

Example 6.3.4

Using cylindrical shells, find the volume of the region bounded by $y = \frac{9x}{\sqrt{1+x^3}}$, $x = 0$, $y = 0$, and $x = 2$, rotated about the y-axis.

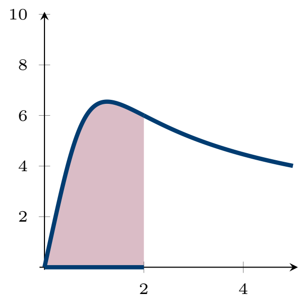


Figure 6.3.7: Cross-sectional Area

Figure 6.3.8: Solid Region

Notice that if we try to use a washer, we'll have to use two separate integrals. So instead we use the shell method, integrating with respect to x. Our radius will thus be x and our height will be $\frac{9x}{\sqrt{1+x^3}}$. Our limits of integration are 0 to 2. Thus, the area of each cylindrical shell will be $A(x) = 2\pi x\left(\frac{9x}{\sqrt{1+x^3}}\right)$. Thus, the volume is

$$\begin{aligned} V = \int_0^2 A(x)\,dx &= \int_0^2 2\pi x\left(\frac{9x}{\sqrt{1+x^3}}\right)\,dx \\ &= 18\pi\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx. \end{aligned}$$

Using the substituion

$$\begin{aligned} u &= 1 + x^3, & du &= 3x^2dx, \\ u(0) &= 1, & u(2) &= 9, \end{aligned}$$

we get

$$\begin{aligned} 18\pi\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx &= 6\pi\int_1^9 \frac{du}{\sqrt{u}} \\ &= 6\pi\left[2\sqrt{u}\right]_1^9 \\ &= 24\pi. \end{aligned}$$

Remark. Beyond simplifying our lives a bit by only requiring one integral, it turns out that there is no way (with only elementary functions) to solve for x in $y = \frac{9x}{\sqrt{1+x^3}}$, so using the disk/washer method is effectively impossible.

Example 6.3.5

Using cylindrical shells, find the volume of the solid generated by rotating the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$, about the line $y = -5$.

Figure 6.3.9: Cross-sectional Area

Figure 6.3.10: Solid Region

Using cylindrical shells, We see that our limits of integration will be 0 to 3. The radius will be $y - (-5) = y + 5$, and the height of these cylinders will be $9 - y^2$. Thus the area of each cylinder will be $A(y) = 2\pi(y + 5)(9 - y^2)$. Hence, our volume is

$$\begin{aligned} V = \int_0^3 A(y)\,dy &= \int_0^3 2\pi(y+5)(9-y^2)\,dy \\ &= 2\pi \int_0^3 (-y^3 - 5y^2 + 9y + 45)\,dy \\ &= 2\pi \left[-\frac{1}{4}y^4 - \frac{5}{3}y^3 + \frac{9}{2}y^2 + 45y \right]_0^3 \\ &= \frac{441\pi}{2}. \end{aligned}$$

Example 6.3.6

Set up the definite integral for the volume of the solid generated by rotating the region bounded by $y = -x^2 + 6x - 8$, $y = 0$, about the y-axis.

Figure 6.3.11: Cross-sectional Area

Figure 6.3.12: Solid Region

From the graph, we see that the shell method is easier. From the y-axis, the radius is x and the height is $x^2 + 6x - 8$, so the area is given by $A(x) = 2\pi x(x^2 + 6x - 8) = 2\pi(x^3 + 6x^2 - 8x)$. Hence our volume is

$$\begin{aligned} V = \int_2^4 A(x)\,dx &= 2\pi \int_2^4 x^3 + 6x^2 - 8x\,dx \\ &= 2\pi \left[\frac{1}{4}x^4 + 2x^3 - 4x^2\right]_2^4 \\ &= 248\pi. \end{aligned}$$

Example 6.3.7

Set up the definite integral for the volume of the solid generated by rotating the region bounded by $y = \sin(\sqrt{x})$, $y = 0$, $x = 0$, $x = \pi^2$, about the y-axis.

Figure 6.3.13: Cross-sectional Area

Figure 6.3.14: Solid Region

From the graph, we see that the shell method is much easier. From the y-axis, the radius is x and the height is $\sin(\sqrt{x})$, so the area is $A(x) = 2\pi x \sin(\sqrt{x})$. Thus the volume is

$$\begin{aligned} V = \int_0^{\pi^2} A(x)\,dx &= 2\pi \int_0^{\pi^2} x \sin(\sqrt{x})\,dx \\ &= 4\pi \int_{u=0}^{u=\pi} u \sin(u)\,du && \left(\text{substitution } u = x^{1/2},\ du = \frac{1}{2}x^{-1/2}\,dx\right) \\ &= 4\pi \left[\sin(u) - u\cos(x)\right]_0^{\pi} && (\text{integrating by parts}) \\ &= 4\pi^2. \end{aligned}$$

6.4 Work

Newton's Second Law of Motion establishes the following familiar equation

$$F = ma \qquad \text{(Force} = \text{mass} \times \text{acceleration)}$$

which is usually measured in Newtons (N) or in pounds (lb). Often times we may need to consider the acceleration due to gravity, which is $9.81\,\mathrm{m/s^2}$ or $32\,\mathrm{ft/s^2}$.

Assuming that F is constant and an object is moving in a straight line at distance d, then the work done in moving the object is given by

$$W = Fd \qquad \text{(Work} = \text{force} \times \text{distance)}$$

which is usually measured in Joules (J) or in foot-pounds (ft-lb).

When the force is constant, it's easy to find the work done, but what if the force is not constant? Well, we can approximate the work done by breaking up our distance d into smaller subintervals, treating the force as approximately constant on each subinterval, and then summing those together. To get an exact answer, we take a limit over more and more subintervals. Well, this is exactly a procedure of Riemann sums, so we get the following

Definition

The **work done in moving an object from** a **to** b is

$$W = \int_a^b F(x)\,dx,$$

where $F(x)$ is the variable force acting on the object.

The definition above is actually slightly misleading in that it suggests you always should "chop up" the distance traveled. In practice, since we're moving in a straight line, it can be much easier to write mass $=$ density $\times$ volume and chop up the volume (in principle, the volume is just a product of 3 distances).

6.4.1 Regular ol' Work

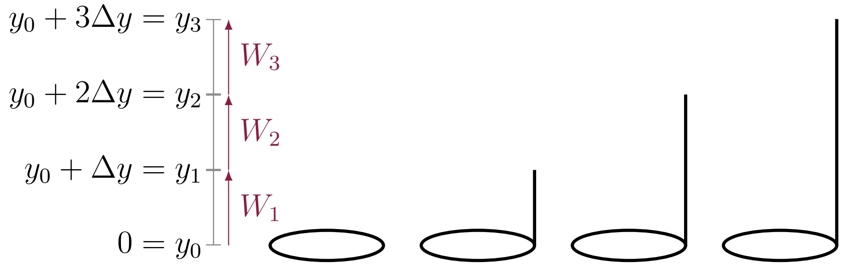
Example 6.4.1

A variable force of $f(x) = 5x^{-2}$ lb moves an object along a straight line when the object is x ft from the origin. Calculate the work done moving the object from $x = 1$ ft to $x = 10$ ft.

$$W = \int_1^{10} f(x)\,dx = \int_1^{10} 5x^{-2}\,dx = \left[-5x^{-1}\right]_1^{10} = \frac{9}{2}\text{ft-lb}.$$

Example 6.4.2

A 15 ft chain weighing 3 lb per foot is lying coiled on the ground. How much work is required to raise one end of the chain to a height of 15 ft?



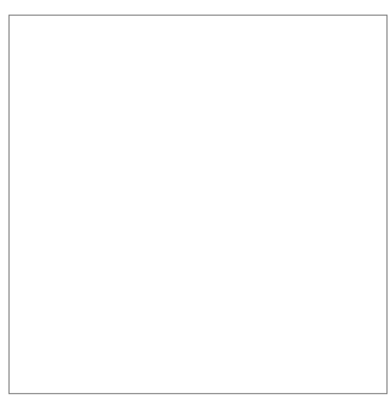
Notice that the force changes with every foot of chain lifted. So at height y_i, there are y_i ft of chain off the ground, which means our force function is $F(y_i) = 3y_i$ lb. The work done in moving this rope from height y_i to height y_{i+1} is approximately $W_i \approx F(y_i)\Delta y = 3y_i\Delta y$. Forming the Riemann sum and taking the limit gives us the following integral:

$$W = \int_0^{15} 3y\,dy = \left[\frac{3}{2}y^2\right]_0^{15} = \frac{675}{2}\text{ ft-lb.}$$

Example 6.4.3

A 200 lb cable is 100 ft long cable and is hanging from the top of a very building (no part is touching the ground).

1. How much work is required to lift the cable to the top of the building?
2. How much work is required to pull up only 20 feet of cable?



6.4.2 Springs (Boing)

Hooke's Law

The force required to hold stretch a spring x units beyond its natural equilibrium point is given by

$$f(x) = kx,$$

where k is a constant called the **spring constant**.

Remark. This applies to both stretching and compressing springs.

Example 6.4.4

A force of 100 lb is required to stretchs a spring 6 in from its natural length of 2 ft. Find the work done (in ft-lb) in stretching the spring an additional 3 in.

First we'll do some conversions: $6\,\text{in} = 0.5\,\text{ft}$ and $3\,\text{in} = 0.25\,\text{ft}$. Now using Hooke's law, $F(x) = kx$, we first have to solve for the spring constant k.

$$100 = k(0.5) \quad \Rightarrow k = 200,$$

and thus the force function for this spring is given by $F(x) = 200x$. So, to stretch it an *additional* 3 in, we need to integrate from 6 in to 9 in:

$$W = \int_{1/2}^{3/4} 200x\,dx = \left[100x^2\right]_{1/2}^{3/4} = \frac{125}{4} = 31.25\,\text{ft-lb}.$$

6.4.3 Moving Water (Drip)

When an object is liquid, the mass can be hard to measure, but volumes are not. So Newton's Second Law can be reinterpreted as

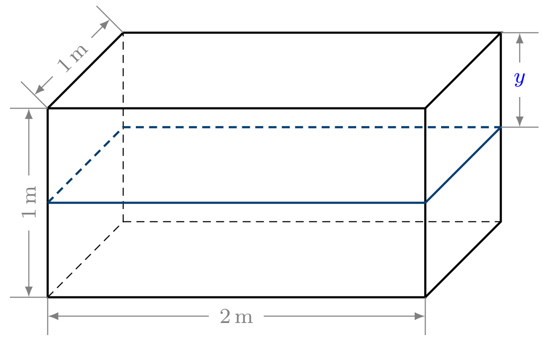
$$F = \rho V g \qquad \text{(density} \times \text{volume} \times \text{gravity's acceleration).}$$

Given the particular liquid, ρ and g are constant. Below the table shows the units for water.

Units	ρg
SI	$(1000\,\mathrm{kg/m^3})\,(9.8\,\mathrm{m/s^2}) = 9800\,\mathrm{N/m^3}$
US Customary	$62.5\,\mathrm{lb/ft^3}$

Example 6.4.5

An aquarium in the shape of a rectangular prism 2 m long, 1 m wide, and 1 m deep is full of water. How much work is done when pumping all of the water out of the top of the tank? How much work is done when pumping only half of the water out?



Each "slice" of water is dy thick, and has volume $2dy\,\mathrm{m}^3$. Recall that, in SI units, water weighs $9800\,\mathrm{kg/(m^2s^2)}$, and so the incremental force function is $9800 \cdot 2dy = 19600dy$. Moving each "slice" of water up y units to the top of the tank until the tank is empty gives us the following:

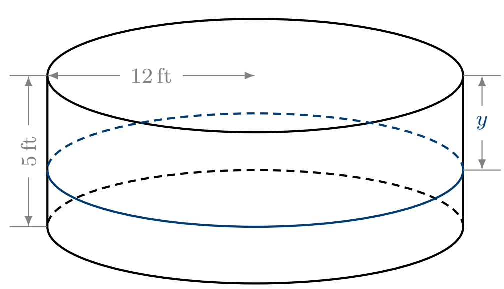
$$W = \int_0^1 9800(y)(2)\,dy = 19600\int_0^1 y\,dy = 19600\left[\frac{1}{2}y^2\right]_0^1 = 9800\,\mathrm{J}.$$

To only empty half of the tank, notice that y varies from 0 to 1/2, so we get that the amount of work done is

$$W = 19600\int_0^{1/2} y\,dy = 19600\left[\frac{1}{2}y^2\right]_0^{1/2} = 2450\,\mathrm{J}.$$

Example 6.4.6

A circular swimming pool has a diameter of 24 ft, sides of 5 ft in height, and a water level of 4 ft. How much work is required to pump all of the water out of the top of pool?

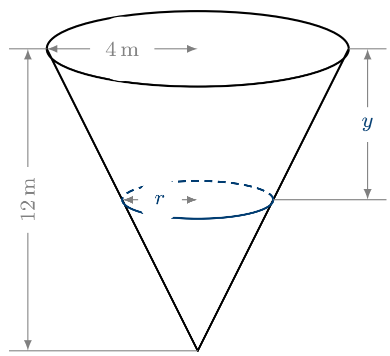


Each "slice" of water is dy thick, and has volume $\pi(12)^2dy = 144\pi dy\,\mathrm{ft}^3$. Recall that water weighs 62.5 lb/ft^3, and so the incremental force function is given by $62.5(144\pi)dy = 9000\pi dy$. Note that y starts at 1 because the water level is 1 ft below the top of the pool. Moving each "slice" of water up y units to the top of the tank until the tank is empty gives us the following:

$$W = \int_1^5 9000\pi y\,dy = 9000\pi \int_1^5 y\,dy = 9000\pi \left[\frac{1}{2}y^2\right]_1^5 = 108,000\pi \text{ ft-lb.}$$

Example 6.4.7

An tank shaped like an inverted right-angled circular cone, with height 12 m and base radius 4 m, is full of water. Water is pumped out through the top of the tank with a hose until the water level is 4 m high. How much work is done in pumping the water out?

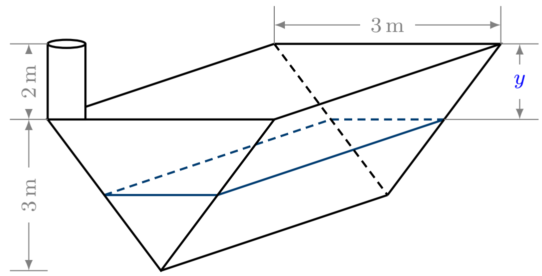


By similar triangles, the "slice" of water shown in the diagram has radius $r = 4(12-y)/12$. So, given a thickness of dy, each "slice" has volume $\pi r^2 dy = (\pi/9)(12-y)^2 dy$. Recall that, in SI units, water weighs 9800 kg/(m^{2}s^2), and so the incremental force function is $(9800\pi/9)(12-y)^2 dy$. Note that y stops at 8 because we want to leave 4 m of water in the tank. Moving each "slice" of water up y units to the top of the tank until the tank has only 4 m of water gives us the following

$$W = \frac{9800\pi}{9}\int_0^8 (12-y)^2 y\,dy = \frac{9800\pi}{9}\left[\frac{1}{4}y^4 - 8y^3 + 72y^2\right]_0^8 = \frac{5,017,600\pi}{3} \text{ J} \approx 5.2544 \times 10^6 \text{ J.}$$

Example 6.4.8

A triangular trough 3 m high, 3 m wide, and 8 m long is full of water. Water is pumped out of the top of the trough through a 2 m tall spigot. How much work is required to empty the trough?

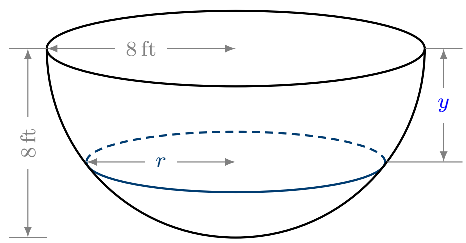


By similar triangles, the width w of the water "slice" is $w = 3 - y$, and since each slice is dy thick and 8 m long, we have that the volume of each slice is given by $8(3 - y)dy$. Recall that, in SI units, water weighs $9800\,\text{kg}/(\text{m}^2\text{s}^2)$, and so the incremental force function is $9800(8)(3-y)dy$. Moving each "slice" up $y + 2$ meters (y for the tank, 2 for the spigot) until the tank is empty, we have that the work is given by

$$W = 78400 \int_0^3 (3 - y)(y + 2)\, dy = 78400 \left[-\frac{1}{3}y^3 + \frac{1}{2}y^2 + 6y\right]_0^3 = 1,058,400\,\text{J}.$$

Example 6.4.9

A hemispherical tank with radius 8 ft is filled with water. Find the amount of work done in pumping all of the water out of the tank.



We can form a right triangle with height y, base r, and hypotenuse 8. At height y, applying the Pythagorean theorem yields that the radius of each "slice" of water is $r = \sqrt{64 - y^2}$. Each "slice" of water is dy thick, and has volume $\pi(64 - y^2)dy\,\text{ft}^3$. Recall that water weighs $62.5\,\text{lb}/\text{ft}^3$, and so the incremental force function is given by $62.5\pi(64 - y^2)dy$. Moving each "slice" of water up y units to the top of the tank until the tank is empty gives us the following:

$$W = \int_0^8 62.5\pi y(64 - y^2)\, dy = 62.5\pi \left[32y^2 - \frac{1}{4}y^4\right]_0^8 = 64,000\pi\,\text{ft-lb} \approx 201061.92\,\text{ft-lb}.$$

6.5 Average Value of a Function

Recall that the average of n numbers, $y_1, \dots, y_n$ is given by

$$\text{avg} = \frac{y_1 + \cdots + y_n}{n} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

What about an average of infinitely-many points?

Let f be an integrable function and suppose we want to find its average value over the interval $[a, b]$. We begin by partitioning $[a, b]$ into n subintervals of size $\Delta x = \frac{b-a}{n}$ and sampling a point from each subinterval x_i^* (so that $y_i = f(x_i^*)$). Through some algebraic manipulation, the above equation then becomes

$$\text{avg} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{\Delta x}{b-a}\sum_{i=1}^{n} f(x_i^*) = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Taking the limit as $n \to \infty$ leads us to the following definition:

Definition: Average Value of a Function

If f is integrable on $[a, b]$, then the **average value of f on** $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 6.5.1

Find the average value of $f(x) = 3x^2 + 8x$ on the interval $[-1, 2]$.

Intuitively, there will always be values of the function f that are *lower* than the average, and some that will be *higher*, and if f is continuous, then the Intermediate Value Theorem suggests there should be some c in $[a, b]$ for which $f(c) = f_{\text{avg}}$. Have I got good news for you, dear reader!

Theorem 6.5.2: Mean Value Theorem for Integrals

Suppose f is continuous on $[a, b]$. Then there is some c where $a < c < b$ satisfying

$$f(c) = f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx,$$

or equivalently,

$$\int_a^b f(x)\, dx = f(c)(b-a).$$

Example 6.5.3

Find a value of c on the interval $[-1, 2]$ such that $f_{\text{avg}} = f(c)$.

7 Techniques of Integration

7.1 Integration by Parts

Example 7.1.1: Warm up

Evaluate $\frac{d}{dx}[xe^x]$.

Using the product rule, we have

$$\frac{d}{dx}[xe^x] = e^x + xe^x.$$

Recall that the product rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In the language of differentials, letting $u = f(x)$ and $v = g(x)$ be differentiable functions, we have that $du = f'(x)\,dx$, $dv = g'(x)\,dx$, and

$$d(uv) = v\,du + u\,dv \quad \Rightarrow \quad u\,dv = d(uv) - v\,du.$$

The following integration technique is then completely analogous to the product rule for derivatives.

Proposition 7.1.2: Integration by Parts

Let $u = f(x)$, $v = g(x)$ be differentiable functions. Then

$$\int u\,dv = uv - \int v\,du. \tag{7.1.1}$$

Remark. Unfortunately, we we typically use u for the substitution method and u, v for Integration by Parts. It should be noted that these u's are wholly unrelated as there is no function substitution happening in Integration by Parts.

The procedure for applying Integration by Parts is as follows:

1. Choose $u = f(x)$, $dv = g'(x)\,dx$ so that $g'(x)\,dx$ is easy to integrate by itself.
2. Find $du = f'(x)$ and $v = \int g'(x)\,dx$.
3. Substitute into Equation 7.1.1 and solve.
4. Apply steps 1-3 again if needed.

Example 7.1.3

Evaluate $\int (e^x + xe^x)\,dx$.

We'll first split this into two separate integrals and solve for the one we already know.

$$\int (e^x + xe^x)\ dx = \int e^x\, dx + \int xe^x\, dx = e^x + \int xe^x\, dx.$$

We'll apply Integration by Parts to solve the rightmost integral. Choose

$$\begin{aligned} u &= x, & du &= dx \\ dv &= e^x\, dx, & v &= e^x. \end{aligned}$$

Substituting into the Integration by Parts formula, we have

$$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C.$$

Thus the original integral becomes

$$\int (e^x + xe^x)\ dx = e^x + (xe^x - e^x) + C = xe^x + C,$$

which is exactly what we would expect to get from Example 7.1.1.

Example 7.1.4

Evaluate $\displaystyle\int x \cos x\, dx$.

First choose,

$$\begin{aligned} u &= x, & du &= dx, \\ dv &= \cos x\, dx, & v &= \sin x. \end{aligned}$$

Then, substituting into the Integration by Parts formula, we have

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + C.$$

What if we had chosen u and v differently in the previous example? Then we would have

$$\begin{aligned} u &= \cos x, & du &= -\sin x\, dx, \\ dv &= x\, dx, & v &= \frac{1}{2}x^2, \end{aligned}$$

and plugging into our Integration by Parts formula gives us

$$\int x \cos x\, dx = \frac{1}{2}x^2 \cos x + \int \frac{1}{2}x^2 \sin x\, dx,$$

and this rightmost integral is even harder to integrate than what we started with.

This suggests to us that, if using integration by parts and one of the functions in the integrand is a polynomial, it might be easiest to choose u to be that polynomial.

Example 7.1.5

Evaluate $\displaystyle\int \arctan x\, dx$.

There aren't a lot of choices for u and dv. Choose

$$\begin{aligned} u &= \arctan x, & du &= \frac{1}{1+x^2}\, dx, \\ dv &= dx, & v &= x. \end{aligned}$$

Substituting into the Integration by Parts formula, we have

$$\int \arctan x\, dx = x \arctan x - \int \frac{x}{1+x^2}\, dx.$$

How do we evaluate this right-most integral? Use the substitution method. Let

$$\begin{aligned} w &= 1 + x^2 \\ dw &= 2x\, dx \quad \Rightarrow \quad \frac{1}{2}\, dw = x\, dx, \end{aligned}$$

So then we have

$$\begin{aligned} \int \arctan x\, dx = x \arctan x - \int \frac{x}{1+x^2}\, dx &= x \arctan x - \frac{1}{2} \int \frac{dw}{w} \\ &= x \arctan x - \frac{1}{2} \ln|w| + C \\ &= x \arctan x - \frac{1}{2} \ln\left(1 + x^2\right) + C. \end{aligned}$$

In some cases, it may be necessary to use integration by parts multiple times.

Example 7.1.6

Evaluate $\displaystyle\int x^2 e^x\, dx$.

Choose

$$\begin{aligned} u &= x^2, & du &= 2x\, dx, \\ dv &= e^x\, dx, & v &= e^x. \end{aligned}$$

Substituting into the Integration by Parts formula, we have

$$\int x^2 e^x\, dx = x^2 e^x - \int 2x e^x\, dx = x^2 e^x - 2 \int x e^x\, dx.$$

We repeat the process for the right-most integral. Choose

$$\begin{aligned} \tilde{u} &= x, & d\tilde{u} &= dx, \\ d\tilde{v} &= e^x\, dx, & \tilde{v} &= e^x. \end{aligned}$$

Substituting into our Integration by Parts formula, we have

$$\begin{aligned}\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx &= x^2 e^x - 2 \left(x e^x - \int e^x \, dx \right) \\ &= x^2 e^x - 2 \left(x e^x - e^x \right) + C \\ &= x^2 e^x - 2 x e^x + 2 e^x + C.\end{aligned}$$

When integrating by parts involving a natural logarithm, it's almost always best to let u be the natural logarithm as then du becomes a rational function.

Example 7.1.7

Evaluate $\displaystyle\int x^4 \ln x \, dx$.

Since we don't know $\int \ln x \, dx$, choose

$$\begin{aligned} u &= \ln x, & du &= \frac{1}{x} \, dx, \\ dv &= x^4 \, dx, & v &= \frac{1}{5} x^5. \end{aligned}$$

Substituting into the Integration by Parts formula, we have

$$\begin{aligned}\int x^4 \ln x \, dx &= \frac{1}{5} x^5 \ln x - \int \frac{1}{5} x^4 \, dx \\ &= \frac{1}{5} x^5 \ln x - \frac{1}{25} x^5 + C.\end{aligned}$$

Sometimes you may encounter a situation where you have to repeat the integration by parts and end up with the original integral.

Example 7.1.8

Evaluate $\displaystyle\int e^x \sin x \, dx$.

Choose

$$\begin{aligned} u &= \sin x, & du &= \cos x \, dx, \\ dv &= e^x \, dx, & v &= e^x. \end{aligned}$$

Substituting into the Integration by Parts formula, we have

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx.$$

Now choose

$$\begin{aligned} \tilde{u} &= \cos x, & d\tilde{u} &= -\sin x, \\ d\tilde{v} &= e^x \, dx, & \tilde{v} &= e^x. \end{aligned}$$

Again, substituting into the Integration by Parts formula gives us

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx.$$

Putting this all together, we have

$$\begin{aligned}
\int e^x \sin x \, dx &= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right] \\
&= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \\
\Rightarrow \quad 2 \int e^x \sin x \, dx &= e^x \sin x - e^x \cos x + C \\
\Rightarrow \quad \int e^x \sin x \, dx &= \frac{1}{2} \left(e^x \sin x - e^x \cos x \right) + C.
\end{aligned}$$

Proposition 7.1.9: Integration by Parts for Definite Integrals

Let $u = f(x)$, $v = g(x)$ be differentiable functions and suppose $f'(x)$ and $g'(x)$ are continuous on the interval $[a, b]$. Then

$$\int_a^b u \, dv = uv|_a^b - \int_a^b v \, du. \tag{7.1.2}$$

Example 7.1.10

Evaluate $\displaystyle\int_0^{\pi/2} x \sin x \, dx$.

Choose

$$\begin{aligned}
u &= x, & du &= dx, \\
dv &= \sin x \, dx & v &= -\cos x.
\end{aligned}$$

Then substituting into Equation 7.1.2, we have

$$\begin{aligned}
\int_0^{\pi/2} x \sin x \, dx &= -x \cos x|_0^{\pi/2} - \int_0^{\pi/2} -\cos x \, dx \\
&= \int_0^{\pi/2} \cos x \, dx \\
&= \sin x|_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1.
\end{aligned}$$

Sometimes we need to use substitution first, and then integration by parts.

Example 7.1.11

Evaluate $\displaystyle\int_0^{\sqrt{\pi}} t^3 \cos\left(t^2\right)\, dt$.

For substitution, choose

$$\begin{aligned} x &= t^2 \\ dx &= 2t\,dt \quad \Longrightarrow \quad \frac{1}{2}dx = t\,dt. \end{aligned}$$

and our limits become

$$\begin{aligned} x(0) &= 0 \\ x(\sqrt{\pi}) &= \pi, \end{aligned}$$

Then

$$\begin{aligned} \int_0^{\sqrt{\pi}} t^3 \cos\left(t^2\right)\, dt &= \int_0^{\sqrt{\pi}} t^2 \cos\left(t^2\right) \cdot t\,dt \\ &= \frac{1}{2}\int_0^{\pi} x\cos x\,dx \end{aligned}$$

and this is the same integral from Example 7.1.4, so we get

$$= \frac{1}{2}\left(x\sin x + \cos x\right)\Big|_0^{\pi} = \frac{1}{2}\left(\cos(\pi) - \cos(0)\right) = -1.$$

Example 7.1.12

Evaluate $\displaystyle\int_1^2 \ln(6x+2)\,dx$.

Choose

$$\begin{aligned} u &= \ln(6x+2), & du &= \frac{6}{6x+2}\,dx, \\ dv &= dx, & v &= x. \end{aligned}$$

Then, substituting into our Integration by Parts formula, we have

$$\begin{aligned} \int_1^2 \ln(6x+2)\,dx &= x\ln(6x+2)\big|_1^2 - \int_1^2 \frac{6x}{6x+2}\,dx \\ &= 2\ln(14) - \ln(8) - \int_1^2 \frac{6x}{6x+2}\,dx. \end{aligned}$$

To solve this remaining integral, we'll need to use substitution, so choose

$$\begin{aligned} w &= 6x+2, \quad \Rightarrow \quad 6x = w-2 \\ dw &= 6\,dx \quad \Rightarrow \quad \frac{1}{6}dw = dx, \end{aligned}$$

and it follows that our new limits are

$$\begin{aligned} w(1) &= 6(1) + 2 = 8, \\ w(2) &= 6(2) + 2 = 14. \end{aligned}$$

So, using substitution,

$$\begin{aligned} \int_1^2 \ln(6x+2)\,dx &= 2\ln(14) - \ln(8) - \int_1^2 \frac{6x}{6x+2}\,dx. \\ &= 2\ln(14) - \ln(8) - \int_8^{14} \frac{w-2}{w}\left(\frac{1}{6}dw\right) \\ &= 2\ln(14) - \ln(8) - \frac{1}{6}\int_8^{14}\left(1 - \frac{2}{w}\right)dw \\ &= 2\ln(14) - \ln(8) - \frac{1}{6}\left(w - 2\ln w\right)\Big|_8^{14} \\ &= 2\ln(14) - \ln(8) - \frac{1}{6}\left(14 - 2\ln(14) - 8 + 2\ln(8)\right) \\ &= \frac{7}{3}\ln(14) - \frac{4}{3}\ln(8) - 1. \end{aligned}$$

7.2 Trigonometric Integrals

7.2.1 Sines and Cosines

Example 7.2.1

Evalute $\displaystyle\int \sin^2(x)\cos(x)\,dx$

Easy u-substitution.

Example 7.2.2

Evalute $\displaystyle\int \sin(x)\cos^2(x)\,dx$

Easy u-substitution.

Example 7.2.3

Evalute $\displaystyle\int \sin(x)\cos(x)\,dx$

Easy u-substitution.

Example 7.2.4

Evalute $\displaystyle\int \sin^2(x)\cos^2(x)\,dx$

Umm...

We'll recall the following useful trigonometric identities: The Pythagorean Identities

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1,\\ 1+\cot^2\theta &= \csc^2\theta,\\ \tan^2\theta + 1 &= \sec^2\theta,\end{aligned}$$

and the Power-Reducing Formulae

$$\begin{aligned}\sin^2\theta &= \frac{1-\cos(2\theta)}{2},\\ \cos^2\theta &= \frac{1+\cos(2\theta)}{2}.\end{aligned}$$

Example 7.2.5

Evaluate $\displaystyle\int \sin^3 x\,dx$.

We can't apply the substitution rule to this integral as there is no cosine term, and it's not completely obvious how we might integrate this by parts (if it is even possible), so instead we'll use one of the Pythagorean identities above to rewrite the integral.

$$\int \sin^3 x\,dx = \int \sin^2 x \cdot \sin x\,dx = \int (1-\cos^2 x)\sin x\,dx.$$

Now we see we can apply the substitution rule by choosing

$$\begin{aligned} u &= \cos x \\ du &= -\sin x\,dx, \end{aligned}$$

whence

$$\begin{aligned} \int \sin^3 x\,dx &= \int (1-\cos^2 x)\sin x\,dx \\ &= -\int (1-u^2)\,du \\ &= -u + \frac{1}{3}u^3 + C \\ &= -\cos x + \frac{1}{3}\cos^3 x + C \end{aligned}$$

Example 7.2.6

Evaluate $\displaystyle\int \cos^2 x\,dx$.

Using the Pythagorean identity here doesn't simplify anything, so we'll instead use one of the power reducing formulae:

$$\begin{aligned} \int \cos^2 x\,dx &= \int \frac{1+\cos(2x)}{2}\,dx \\ &= \frac{1}{2}\int 1+\cos(2x)\,dx \\ &= \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C \qquad \text{(For the } \cos(2x) \text{ part use a substitution } u = 2x\text{)} \\ &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C \end{aligned}$$

General strategy for handling integrals of the form $\displaystyle\int \sin^m x \cos^n x\,dx$

(i) If m is odd, save one $\sin x$ factor and use $\sin^2 x = 1-\cos^2 x$ to express the remaining factors in terms of $\cos x$. Then use substitution with $u = \cos x$.

(ii) If n is odd, save one $\cos x$ factor and use $\cos^2 x = 1-\sin^2 x$ to express the remaining factors in terms of $\sin x$. Then use substitution with $u = \sin x$.

(iii) If m and n are both even, use a power-reducing formula and proceed with either (i) or (ii).

Example 7.2.7

Evaluate $\displaystyle\int \sin^3(5x)\cos^2(5x)\,dx$.

We use the Pythagorean identity $\sin^2(5x) = 1 - \cos^2(5x)$, and then apply the substitution $u = \cos(5x)$, $du = -5\sin(5x)\,dx$.

$$\begin{aligned}
\int \sin^3(5x)\cos^2(5x)\,dx &= \int \sin(5x)\left(1-\cos^2(5x)\right)\cos^2(5x)\,dx \\
&= -\frac{1}{5}\int (1-u^2)u^2\,du \\
&= -\frac{1}{5}\int u^2 - u^4\,du \\
&= -\frac{1}{15}u^3 + \frac{1}{25}u^5 + C \\
&= -\frac{1}{15}\cos^3(5x) + \frac{1}{25}\cos^5(5x) + C.
\end{aligned}$$

Example 7.2.8

Evaluate $\displaystyle\int \sin^4 x\,dx$.

Noting that $\sin^4 x = (\sin^2 x)^2$, we can apply a power-reducing formula.

$$\begin{aligned}
\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx &= \int \left(\frac{1-\cos(2x)}{2}\right)^2\,dx \\
&= \frac{1}{4}\int 1 - 2\cos(2x) + \cos^2(2x)\,dx \\
&= \frac{1}{4}\int 1 - 2\cos(2x) + \left(\frac{1+\cos(4x)}{2}\right)\,dx \quad \text{(power-reducing formula)} \\
&= \frac{1}{4}\int 1 - 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x)\,dx \\
&= \frac{1}{4}\int \frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\,dx \\
&= \frac{1}{4}\left[\frac{3}{2}x - \sin(2x) + \frac{1}{8}\sin(4x)\right] + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C
\end{aligned}$$

7.2.2 Tangents and Secants

Before we begin, we should actually figure out the antiderivaties for $\tan(x)$ and $\sec(x)$.

Proposition 7.2.9

$$\int \tan(x)\,dx = \ln|\cos(x)| + C \int \sec(x)\,dx \qquad = \ln|\sec(x)+\tan(x)| + C$$

Proof. (1) Rewrite $\tan(x) = \sin(x)/\cos(x)$ and take $u = \cos(x)$.
(2) Multiply and divide integrand by $(\sec(x)+\tan(x))$. Take $u = \sec(x)+\tan(x)$. □

General strategy for handling integrals of the form $\int \sec^m x \tan^n x\,dx$

(i) If m is even, save one $\sec^2 x$ factor and use $\sec^2 x = \tan^2 x + 1$ to express the remaining factors in terms of $\tan x$. Then use substitution with $u = \tan x$.

(ii) If m and n are both odd, save one $\tan x$ factor and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$. Then use substitution with $u = \sec x$.

(iii) If m is odd and n is even, turn and run, or get creative with integration by parts.

Example 7.2.10

Evaluate $\int \sec^4 x \tan^2 x\,dx$.

We use the Pythagorean identity $\sec^2 x = 1 + \tan^2 x$ and then apply the substitution $u = \tan x$, $du = \sec^2 x\,dx$.

$$\begin{aligned}\int \sec^4 x \tan^2 x\,dx &= \int \sec^2 x(1+\tan^2 x)\tan^2 x\,dx \\ &= \int (1+u^2)u^2\,du \\ &= \int u^2 + u^4\,du \\ &= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\ &= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C.\end{aligned}$$

Example 7.2.11

Evaluate $\int \sec^9 x \tan^5 x\,dx$.

We use Pythagorean identity $\tan^2 x = \sec^2 x - 1$ and then apply the substitution $u = \sec x$, $du = \sec x \tan x\,dx$.

$$\int \sec^9 x \tan^5 x\,dx = \int \sec^9 x \tan x(\tan^4 x)\,dx$$

$$
\begin{aligned}
&= \int \sec^9 x \tan x (\tan^4 x)\, dx \\
&= \int \sec^9 x \tan x (\tan^2 x)^2\, dx \\
&= \int \sec^9 x \tan x (\sec^2 x - 1)^2\, dx \\
&= \int u^8 (u^2 - 1)^2\, du \\
&= \int u^8 (u^4 - 2u^2 + 1)\, du \\
&= \int u^{12} - 2u^{10} + u^8\, du \\
&= \frac{1}{13} u^{13} - \frac{2}{11} u^{11} + \frac{1}{9} u^9 + C \\
&= \frac{1}{13} \sec^{13} x - \frac{2}{11} \sec^{11} x + \frac{1}{9} \sec^9 x + C.
\end{aligned}
$$

Example 7.2.12

Evaluate $\int \sec^3 x\, dx$.

We integrate by parts with

$$
\begin{aligned}
u &= \sec x, & du &= \sec x \tan x, \\
dv &= \sec^2 x\, dx, & v &= \tan x.
\end{aligned}
$$

So

$$
\begin{aligned}
\int \sec^3 x\, dx &= \sec x \tan x - \int \tan^2 x \sec x\, dx \\
&= \sec x \tan x - \int (sec^2 x - 1) \sec x\, dx \\
&= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx \\
&= \sec x \tan x - \ln|\sec x + \tan x| + \int \sec^3 x\, dx \\
2 \int \sec^3 x\, dx &= \sec x \tan x - \ln|\sec x + \tan x| + C \\
\Rightarrow \int \sec^3 x\, dx &= \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C.
\end{aligned}
$$

7.2.3 Cotangents and Cosecants

Nah.

Proposition 7.2.13

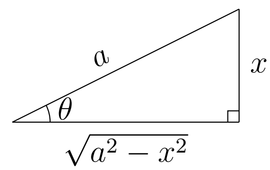
$$\int \cot(x)\,dx = -\ln|\sin(x)| + C \int \csc(x)\,dx \qquad = -\ln|\csc(x)+\cot(x)| + C$$

Proof. Exercise to the reader. Same strategy is as Proposition 7.2.9. □

7.3 Trigonometric Substitutions

7.3.1 Indefinite Integrals

Suppose we're asked to integrate something of the form $\sqrt{a^2 - x^2}$, where a is some constant real number. None of our techniques so far can be applied directly to this. However, we can think about a, x, and $\sqrt{a^2 - x^2}$ as sitting on a right triangle as follows:



Thinking about it this way, we have that $\sin\theta = \frac{x}{a}$, or rather that $x = a\sin\theta$. Using this substitution, we can rewrite the expression

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a\cos\theta \qquad \text{(since } 0 < \theta < \tfrac{\pi}{2}\text{)}$$

which is something we *do* know how to integrate. This same procedure gives us a way to handle expressions $\sqrt{x^2 + a^2}$ and $\sqrt{x^2 - a^2}$ as well.

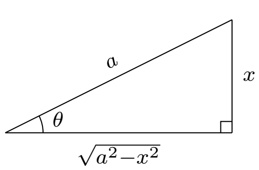
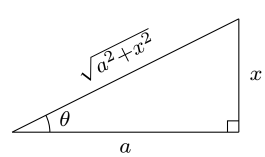
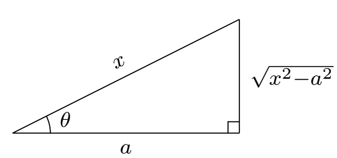
Expression	**Expression**	**Expression**
$\sqrt{a^2 - x^2}$	$\sqrt{a^2 + x^2}$	$\sqrt{x^2 - a^2}$
Substitution	**Substitution**	**Substitution**
$x = a\sin\theta$ $dx = a\cos\theta\, d\theta$	$x = a\tan\theta$ $dx = a\sec^2\theta\, d\theta$	$x = a\sec\theta$ $dx = a\sec\theta\tan\theta\, d\theta$
Trig Identity	**Trig Identity**	**Trig Identity**
$1 - \sin^2\theta = \cos^2\theta$	$\tan^2\theta + 1 = \sec^2\theta$	$\sec^2\theta - 1 = \tan^2\theta$
Reference Triangle	**Reference Triangle**	**Reference Triangle**
		

Table 7.3.1: Trigonometric Substitutions

Example 7.3.1

Perform a trigonometric substitution for the integral $\displaystyle\int \frac{dt}{t^2\sqrt{t^2-16}}$. Evaluate.

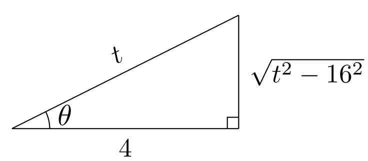
From Table 7.3.1, we use the substitution

$$\begin{aligned} t &= 4\sec\theta \\ dt &= 4\sec\theta\tan\theta\, d\theta \end{aligned}$$

which yields

$$\begin{aligned} \int \frac{dt}{t^2\sqrt{t^2-16}} &= \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\sec^2\theta-16}}\,d\theta \\ &= \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\tan^2\theta}}\,d\theta \\ &= \frac{1}{16}\int \frac{d\theta}{\sec\theta} \\ &= \frac{1}{16}\int \cos\theta\, d\theta \\ &= \frac{1}{16}\sin\theta + C. \end{aligned}$$

Now, we still need to get our answer back in terms of t. To do this, we fill out the relevant reference triangle



And from this triangle, we see that $\sin\theta = \dfrac{\sqrt{t^2-16}}{t}$, hence

$$\int \frac{dt}{t^2\sqrt{t^2-16}} = \frac{1}{16}\frac{\sqrt{t^2-16}}{t} + C.$$

Example 7.3.2

Perform a trigonometric substitution for the integral $\displaystyle\int x^3\sqrt{1-x^2}\,dx$ and draw the reference triangle. Do not evaluate.

From Table 7.3.1, we use the substitution

$$\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta\, d\theta \end{aligned}$$

which yields

$$\begin{aligned}\int x^3\sqrt{1-x^2}\,dx &= \int \sin^3\theta\sqrt{1-sin^2\theta}\,(\cos\theta)\ d\theta\\ &= \int \sin^3\theta\sqrt{\cos^2\theta}\,(\cos\theta)\ d\theta\\ &= \int \sin^3\theta\cos^2\theta\,d\theta.\end{aligned}$$

Example 7.3.3

Perform a trigonometric substitution for the integral $\int \frac{t^5}{\sqrt{t^2+2}}\,dt$ and draw the reference triangle. Do not evaluate.

From Table 7.3.1, we use the substitution

$$\begin{aligned}t &= \sqrt{2}\tan\theta\\ dt &= \sqrt{2}\sec^2\theta\,d\theta\end{aligned}$$

which yields

$$\begin{aligned}\int \frac{t^5}{\sqrt{t^2+2}}\,dt &= \int \frac{(\sqrt{2}\tan\theta)^5}{\sqrt{2\tan^2\theta+2}}\left(\sqrt{2}\sec^2\theta\right)\,d\theta\\ &= \int \frac{8\tan^5\theta\sec^2\theta}{\sqrt{2}\sec\theta}\,d\theta\\ &= 4\sqrt{2}\int \tan^5\theta\sec\theta\,d\theta.\end{aligned}$$

7.3.2 Definite Integrals

As usually happens, once we consider definite integrals, there are domain considerations for our functions. In Example 7.3.1 we wrote

$$\begin{aligned}\int \frac{dt}{t^2\sqrt{t^2-16}} &= \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\tan^2\theta}}\,d\theta\\ &= \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta(4\tan\theta)}\,d\theta\\ &= \frac{1}{16}\frac{\sqrt{t^2-16}}{t}+C.\end{aligned}$$

It is straightforward to check that this is indeed the correct antiderivative, but when writing $\sqrt{16\tan^2\theta} = 4\tan\theta$, we implicitly assumed that $\tan(\theta)$ is positive. Looking at the original integrand, the domain of the function is actually $t \le -4$ or $t > 4$, so the $\tan(\theta) < 0$ case should have appeared as well. To make it explicit how the domain considerations come into play, we just try to write the limits of integration.

$$\int_{t=a}^{t=b} \frac{dt}{t^2\sqrt{t^2-16}} = \int_{\theta=\sec^{-1}(a)}^{\theta=\sec^{-1}(b)} \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\tan^2\theta}}\,d\theta$$

Since sin, sec, and tan are not intervertible on their entire domains, we restrict them and take the following conventions:

$\sin\theta$ domain: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sec\theta$ domain: $\left[0, \frac{\pi}{2}\right)$ or $\left(\frac{\pi}{2}, \pi\right]$

$\tan\theta$ domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example 7.3.4

Evaluate $\displaystyle\int_0^9 x^2\sqrt{81-x^2}\,dx.$

Let $x = 9\sin(\theta)$, so then $dx = 9\cos(\theta)\,d\theta$. We now find our new limits of integration

$$\begin{aligned} 0 = x = 9\sin(\theta) &\qquad\Longrightarrow\qquad \theta = 0 \\ 9 = x = 9\sin(\theta) &\qquad\Longrightarrow\qquad \theta = \frac{\pi}{2} \end{aligned}$$

hence

$$\begin{aligned} \int_0^9 x^2\sqrt{81-x^2}\,dx &= \int_{\theta=0}^{\theta=\pi/2} 9^2\sin^2(\theta)\sqrt{81-81\sin^2(\theta)}\,(9\cos\theta)\,d\theta \\ &= \int_{\theta=0}^{\theta=\pi/2} 9^2\sin^2(\theta)\sqrt{81\cos^2(\theta)}\,(9\cos\theta)\,d\theta \\ &= \int_{\theta=0}^{\theta=\pi/2} 9^4\sin^2(\theta)|\cos(\theta)|\cos(\theta)\,d\theta. \end{aligned}$$

Since $\cos(\theta) \geq 0$ for $0 \leq \theta \leq \frac{\pi}{2}$, we have

$$\int_{\theta=0}^{\theta=\pi/2} 9^4\sin^2(\theta)|\cos(\theta)|\cos(\theta)\,d\theta = \int_{\theta=0}^{\theta=\pi/2} 9^4\sin^2(\theta)\cos^2(\theta)\,d\theta$$

which now reduces to an integral we can solve with our trig reduction formulae.

Exercise 7.3.5

Finish Example 7.3.4 and show that the final answer is

$$9^5\left(\frac{\theta}{8} - \frac{1}{32}\sin(4\theta)\right)\Big|_0^{\pi/2} = \frac{9^4\pi}{16}.$$

Example 7.3.6

Evaluate $\displaystyle\int_0^{1/6} \frac{x^5}{(36x^2+1)^{3/2}}\,dx$.

We note that $36x^2+1 = 36\big(x^2+\frac{1}{36}\big) = 36\Big(x^2+\big(\frac{1}{6}\big)^2\Big)$, so we approach with the trig substitution

$$x = \frac{1}{6}\tan(\theta), \qquad dx = \frac{1}{6}\sec^2(\theta)\,d\theta$$

Our new limits of integration will be

$$\begin{aligned} 0 = x &= \frac{1}{6}\tan(\theta) && \implies && \theta = 0 \\ \frac{1}{6} = x &= \frac{1}{6}\tan(\theta) && \implies && \theta = \frac{\pi}{4}. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{1/6} \frac{x^5\,dx}{(36x^2+1)^{3/2}} &= \int_{\theta=0}^{\theta=\pi/4} \frac{\big(\frac{1}{6}\big)^5\tan^5(\theta)\,\big(\frac{1}{6}\sec^2(\theta)\big)\,d\theta}{(\tan^2(\theta)+1)^{3/2}} \\ &= \int_{\theta=0}^{\theta=\pi/4} \frac{\big(\frac{1}{6}\big)^6\tan^5(\theta)\sec^2(\theta)\,d\theta}{(\sec^2(\theta))^{3/2}} \\ &= \int_{\theta=0}^{\theta=\pi/4} \frac{\big(\frac{1}{6}\big)^6\tan^5(\theta)\sec^2(\theta)}{|\sec(\theta)|^3}\,d\theta \end{aligned}$$

Since $\sec(\theta) \geq 0$ when $0 \leq \theta \leq \frac{\pi}{4}$, then $|\sec(\theta)| = \sec(\theta)$. Hence

$$\int_{\theta=0}^{\theta=\pi/4} \frac{\big(\frac{1}{6}\big)^6\tan^5(\theta)\sec^2(\theta)}{|\sec(\theta)|^3}\,d\theta = \int_{\theta=0}^{\theta=\pi/4} \frac{\big(\frac{1}{6}\big)^6\tan^5(\theta)\sec^2(\theta)}{\sec^3(\theta)}\,d\theta.$$

which now can be handled with some trigonometric identities and a substitution like in the previous section.

Exercise 7.3.7

Finish Example 7.3.6 and show that the final answer is

$$\frac{1}{6^6}\left[-\cos(\theta) - 2\sec(\theta) + \frac{1}{3}\sec^3(\theta)\right]_0^{\pi/4} = \frac{1}{6^6}\left[\frac{16}{6} - \frac{11\sqrt{2}}{6}\right]$$

Hint: $\displaystyle\frac{\tan^5(\theta)\sec^2(\theta)}{\sec^3(\theta)} = \left(\frac{\tan^4(\theta)}{\sec^2(\theta)}\right)\sec(\theta)\tan(\theta)$.

Example 7.3.8

Evaluate $\displaystyle\int_{2/5}^{4/5} \frac{\sqrt{25x^2-4}}{x}\,dx$.

We note that

$$25x^2 - 4 = 25\left(x^2 - \frac{4}{25}\right) = 25\left(x^2 - \left(\frac{2}{5}\right)^2\right)$$

hence we take the trig substitution

$$x = \frac{2}{5}\sec\theta \qquad\qquad dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta$$

Our new limits of integration will be

$$\begin{aligned} \frac{2}{5} = x = \frac{2}{5}\sec\theta & \qquad \implies \theta = 0 \\ \frac{4}{5} = x = \frac{2}{5}\sec\theta & \qquad \implies \theta = \frac{\pi}{3} \end{aligned}$$

Hence

$$\begin{aligned} \int_{2/5}^{4/5} \frac{\sqrt{25x^2-4}}{x}\,dx &= \int_0^{\pi/3} \frac{\sqrt{4\sec^2\theta - 4}}{\frac{2}{5}\sec\theta}\left(\tfrac{2}{5}\sec\theta\tan\theta\right)\,d\theta \\ &= \int_0^{\pi/3} \sqrt{4\sec^2\theta - 4}\tan\theta\,d\theta \\ &= \int_0^{\pi/3} \sqrt{4\tan^2\theta}\tan\theta\,d\theta \\ &= \int_0^{\pi/3} 2|\tan\theta|\tan\theta\,d\theta \end{aligned}$$

Since $\tan(\theta) \geq 0$ when $0 < \theta < \pi/3$, then $|\tan\theta| = \tan\theta$, whence

$$\int_0^{\pi/3} 2|\tan\theta|\tan\theta\,d\theta = \int_0^{\pi/3} 2\tan^2\theta\,d\theta$$

which can be evaluated with techniques from the previous section.

Exercise 7.3.9

Finish Example 7.3.8 and show that the final answer is

$$2\Big[\tan(\theta) - \theta\Big]_0^{\pi/3} = 2\left(\sqrt{3} - \frac{\pi}{3}\right).$$

Example 7.3.10

Evaluate $\int_{-4/5}^{-2/5} \frac{\sqrt{25x^2-4}}{x}\, dx$.

Basically everything is the same as in Example 7.3.8 with the exception of the limits of integration. To find those, we have

$$\begin{aligned} \frac{-2}{5} = x = \frac{2}{5}\sec\theta & \qquad\qquad \Longrightarrow \theta = \pi \\ \frac{-4}{5} = x = \frac{2}{5}\sec\theta & \qquad\qquad \Longrightarrow \theta = \frac{2\pi}{3} \end{aligned}$$

Whence

$$\int_{-4/5}^{-2/5} \frac{\sqrt{25x^2-4}}{x}\, dx = \int_{2\pi/3}^{\pi} 2|\tan\theta|\tan\theta\, d\theta$$

Now $\tan\theta \leq 0$ for $\frac{2\pi}{3} \leq \theta \leq \pi$, so $|\tan\theta| = -\tan\theta$ and thus

$$\int_0^{\pi/3} 2|\tan\theta|\tan\theta\, d\theta = \int_0^{\pi/3} -2\tan^2\theta\, d\theta.$$

This can be evaluated with techniques from the previous section.

Exercise 7.3.11

Finish Example 7.3.10 and show that the final answer is

$$2\Big[\theta - \tan\theta\Big]_0^{\pi/3} = 2\left(\frac{\pi}{3} - \sqrt{3}\right).$$

7.4 Partial Fractions

Example 7.4.1: Warm Up

Rewrite $\dfrac{4}{9(x+4)} + \dfrac{1}{9(2x-1)}$ as a single fraction.

To write as a single fraction, we note that the common denominator will be $9(x+4)(2x-1)$.

$$\begin{aligned}\frac{4}{9(x+4)} + \frac{1}{9(2x-1)} &= \frac{4(2x-1)}{9(x+4)(2x-1)} + \frac{(x+4)}{9(x+4)(2x-1)} \\ &= \frac{4(2x-1)+(x+4)}{9(x+4)(2x-1)} \\ &= \frac{x}{(x+4)(2x-1)}\end{aligned}$$

For this integration technique, we'll be going backwards and "uncommonizing" the denominator.

General Strategy for a Partial Fraction Decomposition

Start with a function $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

1. If $\deg p(x) \geq \deg q(x)$, we perform polynomial long division to get $f(x) = s(x) + \dfrac{r(x)}{q(x)}$, where $s(x)$ and $r(x)$ are polynomials, and $\deg r(x) < \deg q(x)$.
2. Factor $q(x)$ into a product of linear factors $(ax+b)$ and irreducible quadratic factors (ax^2+bx+c). (You can always do this. Also note that ax^2+bx+c is irreducible if $b^2-4ac<0$.)
3. Rewrite $\dfrac{r(x)}{q(x)}$ as a sum of rational expressions of the form $\dfrac{A}{ax+b}$ or $\dfrac{Ax+B}{ax^2+bx+c}$, where A and B are constants.
4. Solve for the unknown constants in the numerators of your partial fractions.

We'll look at this in cases.

7.4.1 Case I

Suppose $q(x)$ factors as a product of distinct linear factors, say

$$q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n).$$

We then have

$$\frac{r(x)}{q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$

Example 7.4.2

Find a partial fraction decomposition of $\dfrac{x}{(x+4)(2x-1)}$.

Since the degree of the numerator is smaller than the degree of the denominator, and the denominator is already factored, we can skip steps 1 and 2 above. Thus we have

$$\frac{x}{(x+4)(2x-1)} = \frac{A}{x+4} + \frac{B}{2x-1}.$$

To solve for A and B, we can clear the denominators by multiplying both sides of the equation by $(x+4)(2x-1)$, leaving us with

$$x = A(2x-1) + B(x+4) = 2Ax - A + Bx + 4B = (2A+B)x + (-A+4B).$$

Two polynomials are equal precisely when their coefficients are equal, so we deduce that

$$\begin{aligned} 2A + B &= 1, \\ -A + 4B &= 0. \end{aligned}$$

And using our favorite method for solving systems of linear equations, we have $A = \frac{4}{9}$ and $B = \frac{1}{9}$. So

$$\frac{x}{(x+4)(2x-1)} = \frac{4}{9(x+4)} + \frac{1}{9(2x-1)},$$

which is exactly what we expected to get given Example 7.4.1.

Example 7.4.3

Use partial fractions to evaluate $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\, dx$.

Again we can skip steps 1 and 2 in our partial fraction decomposition, so we get

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1},$$

and clearing denominators yields

$$5x+1 = A(x-1) + B(2x+1) = Ax - A + 2Bx + B = (A+2B)x + (-A+B).$$

so we deduce that

$$\begin{aligned} A + 2B &= 5, \\ -A + B &= 1, \end{aligned}$$

and solving our system, we get that $A = 1$, $B = 2$. So, we can simplify our integral and apply the substitution rule to get

$$\begin{aligned}\int \frac{5x+1}{(2x+1)(x-1)}\,dx &= \int \left(\frac{1}{2x+1} + \frac{2}{x-1}\right)\,dx \\ &= \frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C.\end{aligned}$$

7.4.2 Case II

Suppose $q(x)$ factors as a product of repeated linear factors, say

$$q(x) = (a_1x+b_1)(a_2x+b_2)^k.$$

We then have to have an exponent for every power of the repeated factor from 1 to k, so

$$\frac{r(x)}{q(x)} = \frac{A_1}{a_1x+b_1} + \frac{B_1}{(a_2x+b_2)} + \frac{B_2}{(a_2x+b_2)^2} + \cdots + \frac{B_k}{(a_2x+b_2)^k}.$$

Example 7.4.4

Use partial fractions to evaluate $\displaystyle\int \frac{2x+3}{(x+1)^2}\,dx$.

Again we can skip steps 1 and 2 in our partial fraction decomposition, so we get

$$\frac{2x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2},$$

and clearing denominators yields

$$2x+3 = A(x+1) + B = Ax + (A+B).$$

This gives us the system

$$\begin{aligned}A &= 2, \\ A+B &= 3,\end{aligned}$$

which has solutions $A = 2$, $B = 1$. So we can simplify our integral and apply the substitution rule to get

$$\begin{aligned}\int \frac{2x+3}{(x+1)^2}\,dx &= \int \left(\frac{2}{x+1} + \frac{1}{(x+1)^2}\right)\,dx \\ &= 2\ln|x+1| - \frac{1}{x+1} + C.\end{aligned}$$

7.4.3 Case III

Suppose $q(x)$ factors as a product of distinct irreducible quadratic factors, say

$$q(x) = (a_1x^2 + b_1x + c_1)\cdots(a_nx^2 + b_nx + c_n).$$

We then have to have an exponent for every power of the repeated factor from 1 to k, so

$$\frac{r(x)}{q(x)} = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \cdots + \frac{A_nx + B_n}{(a_nx^2 + b_nx + c_n)}.$$

Example 7.4.5

Use partial fractions to evaluate $\displaystyle\int \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4}\,dx$.

Skipping steps 1 and 2, we note that the denominator factors as $(x^2+1)(x^2+1)$, both of which are irreducible quadratics. We then get

$$\frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} = \frac{x^3 - 2x^2 + x + 1}{(x^2+4)(x^2+1)} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{x^2 + 1},$$

and clearing denominators gives us

$$\begin{aligned} x^3 - 2x^2 + x + 1 &= (Ax + B)(x^2 + 1) + (Cx + D)(x^2 + 4) \\ &= (A + C)x^3 + (B + D)x^2 + (A + 4C)x + (B + 4D), \end{aligned}$$

resulting in the following system of equations

$$\begin{aligned} A + C &= 1, \\ B + D &= -2, \\ A + 4C &= 1, \\ B + 4D &= 1. \end{aligned}$$

The solutions are $A = 1$, $B = -3$, $C = 0$, $D = 1$. So, by simplifying our integral and applying a substitution rule, we get

$$\begin{aligned} \int \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4}\,dx &= \int \left(\frac{x - 3}{x^2 + 4} + \frac{1}{x^2 + 1}\right) dx \\ &= \int \left(\frac{x}{x^2 + 4} - \frac{3}{x^2 + 4} + \frac{1}{x^2 + 1}\right) dx \\ &= \frac{1}{2}\ln|x^2 + 4| - \frac{3}{2}\arctan\left(\frac{x}{2}\right) + \arctan(x) + C. \end{aligned}$$

7.4.4 Case IV

Suppose $q(x)$ factors as product of repeated irreducible quadratic factors, say

$$q(x) = (a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)^k.$$

We then have to have an exponent for every power of the repeated factor from 1 to k, so

$$\frac{r(x)}{q(x)} = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{(a_2x^2 + b_2x + c_2)} + \frac{A_3x + B_3}{(a_2x^2 + b_2x + c_2)^2} + \cdots + \frac{A_{k+1}x + B_{k+1}}{(a_2x^2 + b_2x + c_2)^k}.$$

Example 7.4.6

Use partial fractions to evaluate $\displaystyle\int \frac{x^4+1}{x(x^2+1)^2}\,dx$.

Since the degree of the denominator is larger than that of the denominator, we again get to skip steps 1 and 2. We note also that x^2+1 is an irreducible quadratic. We then get

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2},$$

and clearing denominators yields

$$\begin{aligned} x^4+1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A. \end{aligned}$$

This results in the following system of equations

$$\begin{aligned} A+B &= 1, \\ C &= 0, \\ 2A+B+D &= 0, \\ C+E &= 0, \\ A &= 1, \end{aligned}$$

which has solutions $A=1$, $B=0$, $C=0$, $D=-2$, $E=0$. With this partial fraction decomposition and applying the substitution rule, we get

$$\begin{aligned} \int \frac{x^4+1}{x(x^2+1)^2}\,dx &= \int \left(\frac{1}{x} - \frac{2x}{(x^2+1)^2}\right)dx \\ &= \ln|x| + \frac{1}{x^2+1} + C. \end{aligned}$$

7.4.5 Using a Matrix to Solve Systems of Linear Equations

In the process of performing a partial fraction decomposition, you may end up having to solve for quite a few unknowns, say

$$\begin{aligned}3x^4 + 5x^2 + 17x - 9 = (A + 2B - 3C)x^4 + (2A + D)x^3 + (B - 9C + E)x^2 + \\ + (-A - B - C - D - E)x + (A + C - 19E).\end{aligned}$$

By equating coefficients of the polynomial, we get the following system:

$$\begin{array}{cccccccc} A & +2B & -3C & +0D & +0E & = & 3 \\ 2A & +0B & +0C & +D & +0E & = & 0 \\ 0A & +B & -9C & +0D & +E & = & 5 \\ -A & -B & -C & -D & -E & = & 6 \\ A & +0B & +C & +0D & -7E & = & -9 \end{array}$$

This system may be a nightmare to solve, but by dropping the letters and the equals sign, we can rewrite this as an augmented matrix where the left columns correspond to the coefficients of A, B, C, D, E (in order), and the right column is the answer column, like so

$$M = \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -9 & 0 & 1 & 5 \\ -1 & -1 & -1 & -1 & -1 & 6 \\ 1 & 0 & 1 & 0 & -7 & -9 \end{array}\right)$$

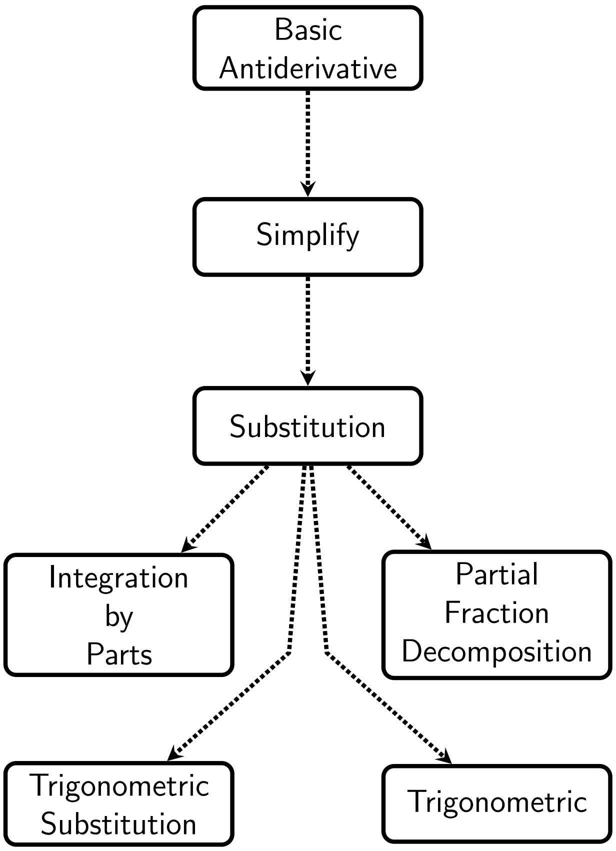
By then putting this matrix into *reduced row echelon form* using the $\boxed{\textbf{rref}}$ command, we get

$$\text{rref}(M) = \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{257}{39} \\ 0 & 1 & 0 & 0 & 0 & \frac{-17}{15} \\ 0 & 0 & 1 & 0 & 0 & \frac{-86}{195} \\ 0 & 0 & 0 & 1 & 0 & \frac{-514}{39} \\ 0 & 0 & 0 & 0 & 1 & \frac{422}{195} \end{array}\right)$$

In this form, we're able to just read off the coefficients in order:

$$A = \tfrac{257}{39}, \quad B = \tfrac{-17}{15}, \quad C = \tfrac{-86}{195}, \quad D = \tfrac{-514}{39}, \quad E = \tfrac{422}{195}.$$

7.5 Strategy for Integration



There's no one-size-fits-all strategy, and often times you may have to do multiple things (a trig substitution may lead to an integration by parts; a partial fraction decomposition might then warrant the use of a substitution). The flow chart above is a suggestion of the order in which to try things.

For each of the following integrals, write down which technique(s) you would use to evaluate them.

A. $\displaystyle\int_0^1 (3x+1)^{\sqrt{2}}\,dx$

Substitution with

$$u = 3x+1, \qquad du = 3\,dx$$

B. $\displaystyle\int \frac{x}{x^4+9}\,dx = \frac{1}{9}\int \frac{x}{\frac{(x^2)^2}{9}+1}\,dx$

Substitution with

$$u = \frac{x^2}{3}, \qquad du = \frac{2}{3}x\,dx$$

C. $\displaystyle\int t\sin(t)\cos(t)\,dt$

Integration by parts with

$$f(t) = t, \qquad g'(t)\,dt = \sin(t)\cos(t)\,dt$$

Finding g requires a u-substitution with

$$u = \sin(t), \qquad du = \cos(t)\,dt.$$

D. $\displaystyle\int_2^4 \frac{(x+2)\,dx}{x^2+3x-4} = \int_2^4 \frac{(x+2)\,dx}{(x-1)(x+4)} = \int_2^4 \frac{(x+2)\,dx}{(x+\frac{3}{2})^2-\frac{25}{4}}$

Partial fraction decomposition with

$$\frac{x+2}{x^2+3x-4} = \frac{A}{x-1} + \frac{B}{x+4}$$

or a trig substitution with

$$x+\frac{3}{2} = \frac{5}{2}\sec(\theta), \qquad dx = \frac{5}{2}\sec(\theta)\tan(\theta)\,d\theta$$

E. $\displaystyle\int \frac{\cos^3(x)}{\csc(x)}\,dx = \int \cos^3(x)\sin(x)\,dx$

Substitution with

$$u = \cos(x), \qquad du = -\sin(x)\,dx$$

F. $\displaystyle\int x\sec(x)\tan(x)\,dx$

Integration by parts with

$$f(x) = x, \qquad g'(x)\,dx = \sec(x)\tan(x)\,dx$$

G. $\displaystyle\int_0^{2\sqrt{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$

Trigonometric substitution with

$$x = \sin(\theta), \qquad dx = \cos(\theta)\,d\theta$$

H. $\displaystyle\int \arctan(\sqrt{x})\,dx$

Integration by parts with

$$f(x) = \arctan(\sqrt{x}), \qquad g'(x)\,dx = dx$$

Followed by substitution

$$u^2 = x, \qquad 2u\,du = dx$$

Followed by some simplification (polynomial long division) or another trig sub

$$u = \tan(\theta), \qquad du = \sec^2\theta\,d\theta$$

I. $\displaystyle\int \frac{\ln(x)}{x\sqrt{1+(\ln x)^2}}\,dx$

Trigonometric substitution with

$$\ln(x) = \tan(\theta), \qquad \frac{1}{x}dx = \sec^2\theta\,d\theta$$

J. $\displaystyle\int \frac{\sec(\theta)\tan(\theta)}{\sec^2(\theta)-\sec(\theta)}\,d\theta$

Substitution with

$$u = \sec(\theta), \qquad = \sec\theta\tan\theta\,d\theta$$

Followed by partial fractions

K. $\displaystyle\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx$

Substitution with

$$u^2 = 1-x^2, \qquad u\,du = -x\,dx$$

Followed by the substitution

$$v = 2-u, \qquad dv = -\,du$$

Then simplify

L. $\displaystyle\int \sqrt{x}e^{\sqrt{x}}\,dx$

Substitution with

$$u^2 = x, \qquad 2u\,du = dx$$

Followed by integration by parts (twice) with

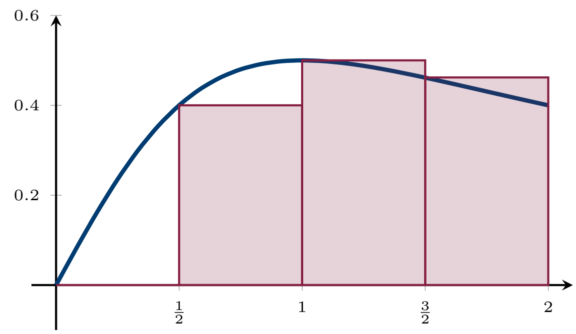
$$f(u) = u^2,\ g'(u)\,du = e^u\,du$$

7.7 Approximate Integration

Back in Chapter 5, we saw ways to approximate indefinite integrals using left, right, and middle Riemann sums. In this section, we'll see two new ways to approximate the area under a curve in ways that are equally about as easy to implement.

Example 7.7.1

Using left Riemann sums, approximate $\int_0^2 \frac{x}{1+x^2}\,dx$ with four rectangles.

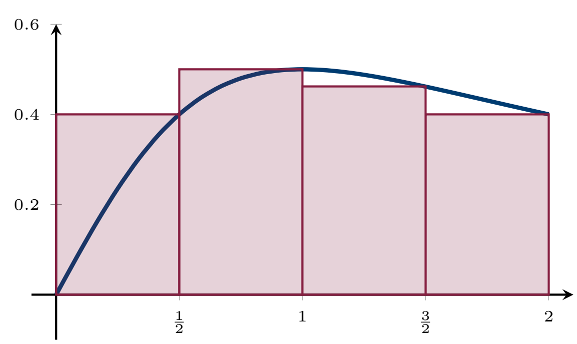


Recall that the left Riemann sum L_4 is given by the formula

$$\begin{aligned} L_4 = \sum_{i=1}^{4} f(x_{i-1})\Delta x = \sum_{i=1}^{4} f(x_{i-1})\left(\frac{2-0}{4}\right) &= f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) \\ &= 0(0.5) + (0.4)(0.5) + (0.5)(0.5) + (0.461538)(0.5) \\ &= 0.680769. \end{aligned}$$

Example 7.7.2

Using right Riemann sums, approximate $\int_0^2 \frac{x}{1+x^2}\,dx$ with four rectangles.

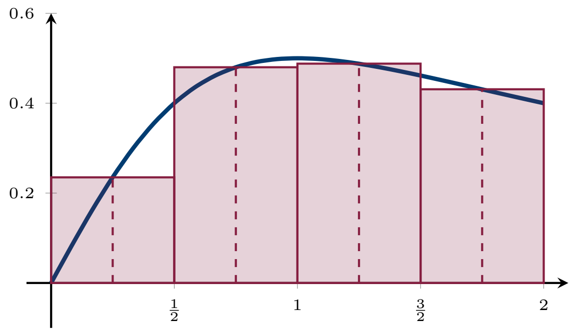


Recall that the right Riemann sum R_4 is given by the formula

$$\begin{aligned} L_4 = \sum_{i=1}^{4} f(x_i)\Delta x &= \sum_{i=1}^{4} f(x_i)\left(\frac{2-0}{4}\right) \\ &= f(0.5)(0.5) + f(1)(0.5) + f(0.5)(0.5) + f(2)(0.5) \\ &= (0.4)(0.5) + (0.5)(0.5) + (0.461538)(0.5) + (0.4)(0.5) \\ &= 0.880769. \end{aligned}$$

Example 7.7.3

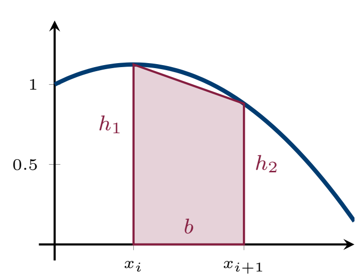
Using middle Riemann sums (AKA, the midpoint rule), approximate $\int_0^2 \frac{x}{1+x^2}\,dx$ with four rectangles.



Recall that the middle Riemann sum M_4 is given by the formula

$$\begin{aligned} M_4 = \sum_{i=1}^{4} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x &= \sum_{i=1}^{4} f\left(\frac{x_i + x_{i-1}}{2}\right)\left(\frac{2-0}{4}\right) \\ &= f(0.25)(0.5) + f(0.75)(0.5) + f(1.25)(0.5) + f(1.75)(0.5) \\ &= (0.235294)(0.5) + (0.48)(0.5) + (0.487805)(0.5) + (0.430769)(0.5) \\ &= 0.816934. \end{aligned}$$

This whole time we've been approximating with rectangles, but that's only because it's a convenient shape for which we already know the area. In a similar vain, we can approximate with trapezoids as well:



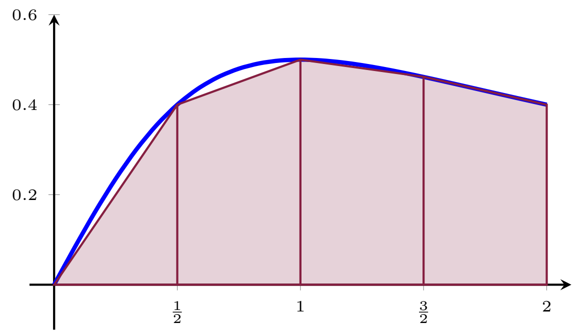
The area of a trapezoid with base b and heights h_1, h_2 is given by $A = \frac{1}{2}(h_1 + h_2)b$. In terms of our function, the area is given by $\frac{1}{2}(f(x_i) + f(x_{i+1}))\Delta x$. If we sum over a n trapezoidal areas of this form, we get an approximation for the area under the curve on the interval $[a, b]$. This gives us

Proposition 7.7.4: Trapezoid Rule

$$\begin{aligned}\int_a^b f(x) \approx T_n &= \frac{1}{2}\left(f(x_0)+f(x_1)\right)\Delta x + \frac{1}{2}\left(f(x_1)+f(x_2)\right)\Delta x + \cdots + \frac{1}{2}\left(f(x_{n-1})+f(x_n)\right)\Delta x \\ &= \frac{1}{2}\Delta x\Big[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\Big].\end{aligned}$$

Example 7.7.5

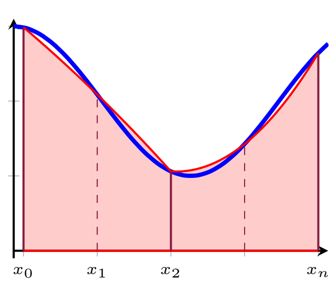
Using the Trapezoid Rule, approximate $\displaystyle\int_0^2 \frac{x}{1+x^2}\,dx$ with four trapezoids.



By the Trapezoid Rule gives the approximation

$$\begin{aligned}T_4 &= \frac{1}{2}\left(\frac{2-0}{4}\right)\Big[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\Big] \\ &= \frac{1}{2}(0.5)\Big[f(0)+2f(0.5)+2f(1)+2f(1.5)+f(2)\Big] \\ &= \frac{1}{2}(0.5)\Big[0+2(0.4)+2(0.5)+2(0.461538)+(0.4)\Big] \\ &= 0.780769.\end{aligned}$$

It is a general fact that for any three distinct, non-collinear points, we can find a unique parabola that passes through all three. This means that, if we break our interval up into an *even number* of smaller intervals $[x_i, x_{i+1}]$, we can find a parabola passing through the points $f(x_{i-1})$, $f(x_i)$, and $f(x_{i+1})$ whose area over the interval $[x_{i-1}, x_{i+1}]$ approximates the area of the curve over this same interval. Visually,



This fact leads us to another approximation method that is easy to implement (but whose derivation is less straightforward – see your book for details). Note that the coefficient pattern is $1, 4, 2, 4, 2, \ldots, 2, 4, 1$.

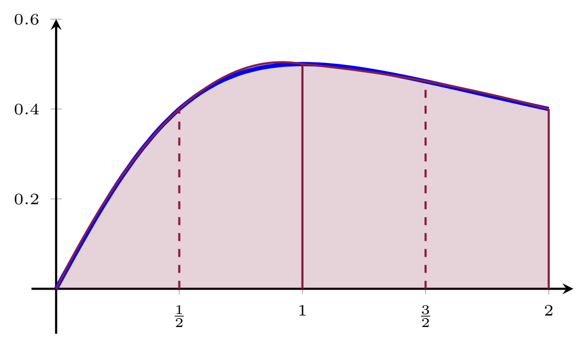
Proposition 7.7.6: Simpson's Rule

For even n,

$$\begin{aligned}&\int_a^b f(x)\,dx \approx S_n \\ &= \frac{1}{3}\Delta x\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 4f(x_{n-1}) + f(x_n)\Big].\end{aligned}$$

Example 7.7.7

Using Simpson's Rule, approximate $\displaystyle\int_0^2 \frac{x}{1+x^2}\,dx$ with four intervals.

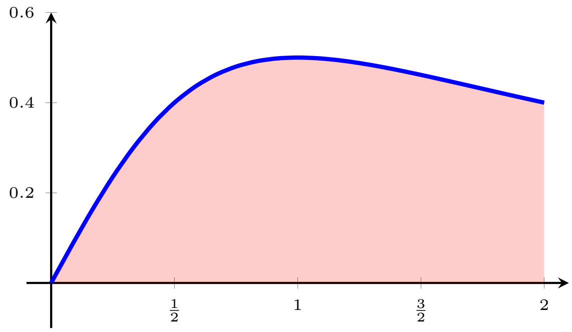


Applying Simpson's Rule, we have

$$\begin{aligned}S_4 &= \frac{1}{3}\Delta x\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\Big] \\ &= \frac{1}{3}\left(\frac{2-0}{4}\right)\Big[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)\Big] \\ &= \frac{1}{3}(0.5)\left[0 + 4(0.4) + 2(0.5) + 4(0.461538) + (0.4)\right] \\ &= 0.807692.\end{aligned}$$

Example 7.7.8

Using the Fundamental Theorem of Calculus, compute $\displaystyle\int_0^2 \frac{x}{1+x^2}\,dx$ exactly. Compare this answer with the previous approximations. Which is most accurate?



Using the substitution

$$\begin{aligned} u &= 1 + x^2 \\ du &= 2x\,dx \\ u(0) &= 1, \\ u(2) &= 5, \end{aligned}$$

we get

$$\int_0^2 \frac{x}{1+x^2}\,dx = \int 1^5 \frac{1}{2} \cdot \frac{du}{u} = \frac{1}{2}\ln(5) - \frac{1}{2}\ln(1) = \frac{1}{2}\ln(5) \approx 0.804719$$

Method	**Approximation**	**Error**
Left Riemann Sum	0.680769	0.123950
Right Riemann Sum	0.880769	0.076050
Middle Riemann Sum	0.816934	0.012215
Trapezoid Rule	0.780769	0.023950
Simpson's Rule	0.807692	0.002973
Exact Value	0.804719	0.000000

Simpson's rule is by far the most accurate approximation.

7.8 Improper Integrals

Definition 1. The integral $\int_a^b f(x)\,dx$ is **improper** if either the integrand is infinite on the interval $[a,b]$ or if the interval is infinite.

As with just every other time we've encountered the infinite in this class, we'll use limits to handle it.

7.8.1 Type I: Infinite Interval

Definition: Improper Integrals of Type I

(a) If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx,$$

provided this limit exists.

(b) If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx,$$

provided this limit exists.

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists, and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent for any real number a, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

Example 7.8.1

Evaluate $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$.

We cannot actually evaluate this integral at infinity. However, we can evaluate it on the interval $[1,t]$ and let $t \to \infty$. So

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx \\ &= \lim_{t\to\infty} \left(-\frac{1}{x}\right)\Bigg|_1^t \\ &= \lim_{t\to\infty} \left(-\frac{1}{t}+1\right) = 1.\end{aligned}$$

Example 7.8.2

Determine whether the integral converges or diverges: $\displaystyle\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx.$

We'll need the substitution $u = x - 2$ and $du = dx$. So, then $u(3) = 1$ and $u(b) \to \infty$ as $b \to \infty$.

$$\begin{aligned}\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx = \lim_{b\to\infty}\int_3^b \frac{1}{(x-2)^{3/2}}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{u^{3/2}}\,du \\ &= \lim_{b\to\infty}\int_1^b u^{-3/2}\,du \\ &= \lim_{b\to\infty} -2u^{-1/2}\Big|_1^b \\ &= \lim_{b\to\infty} -2b^{-1/2} + 2 \\ &= 0 + 2 = 2.\end{aligned}$$

The integral converges.

Example 7.8.3

Evaluate $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ if it converges. Otherwise state that it does not converge.

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty} \ln x\Big|_1^b \\ &= \lim_{b\to\infty} \ln b - \ln 1 \\ &= \infty + 0 = \infty.\end{aligned}$$

The integral diverges.

Example 7.8.4

For which values of p does the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

As we saw in Example 7.8.3, when $p = 1$, the integral does not converge. So let's see about the case where $p \neq 1$.

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx \\ &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx\end{aligned}$$

$$= \lim_{b\to\infty} \frac{1}{-p+1} x^{(-p+1)} \Big|_1^b$$
$$= \lim_{b\to\infty} \frac{1}{-p+1} \left(b^{(-p+1)} - 1\right).$$

We see that $\lim_{b\to\infty} b^{(-p+1)}$ converges precisely when $-p+1 < 0$, which is precisely when $p > 1$. Thus the integral converges for all $p > 1$.

Example 7.8.5

Evaluate $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$.

We'll first approach the indefinite integral with the substitution $u = -x^2$, $du = -2x\,dx$:

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C.$$

Thus we can break up the integral at some arbitrary point in the interval $(-\infty, \infty)$, say at $x = 0$.

$$\begin{aligned}
\int_{-\infty}^{\infty} xe^{-x^2}\,dx &= \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx \\
&= \lim_{a\to-\infty}\int_a^0 xe^{-x^2}\,dx + \lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx \\
&= \lim_{a\to-\infty}\left(-\frac{1}{2} + \frac{1}{2}e^{-a^2}\right) + \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}\right) \\
&= \left(-\frac{1}{2} + 0\right) + \left(0 + \frac{1}{2}\right) = 0.
\end{aligned}$$

7.8.2 Type II: Discontinuous Integrand

Definition: Improper Integrals of Type II

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

provided this limit exists.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

provided this limit exists.

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, a real number in (a, b), and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 7.8.6

Evaluate $\displaystyle\int_0^3 \frac{1}{(x-2)^2}\,dx$.

Notice that the integrand is undefined at $x = 2$, so we'll need to split it up here.

$$\begin{aligned}\int_0^3 \frac{1}{(x-2)^2}\,dx &= \int_0^2 \frac{1}{(x-2)^2}\,dx + \int_2^3 \frac{1}{(x-2)^2}\,dx \\ &= \lim_{b\to 2^-}\int_0^b \frac{1}{(x-2)^2}\,dx + \lim_{a\to 2^+}\int_a^3 \frac{1}{(x-2)^2}\,dx.\end{aligned}$$

Now, with the substitution $u = x - 2$ and $du = dx$, we have that $u(0) = -2$, $u(2) = 0$ and $u(3) = 1$, so

$$\begin{aligned}&= \lim_{b\to 0^-}\int_{-2}^b \frac{1}{u^2}\,du + \lim_{a\to 0^+}\int_a^1 \frac{1}{u^2}\,du \\ &= \lim_{b\to 0^-}\left(-b^{-1} + (-2)^2\right) + \lim_{a\to 0^+}\left(-1 + a^1\right).\end{aligned}$$

Since neither of these limits converge, the integral is divergent.

Example 7.8.7

Evaluate $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$.

The integrand is undefined at $x = 1$, so we'll need to split it up here.

$$\begin{aligned}\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx &= \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx \\ &= \lim_{b\to 1^-}\int_0^b \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{a\to 1^+}\int_a^9 \frac{1}{\sqrt[3]{x-1}}\,dx.\end{aligned}$$

Now, with the substitution $u = x - 1$ and $du = dx$, we have that $u(0) = -1$, $u(1) = 0$ and $u(9) = 8$, so

$$\begin{aligned}&= \lim_{b\to 0^-}\int_{-1}^b u^{-1/3}\,dx + \lim_{a\to 0^+}\int_a^8 u^{-1/3}\,dx \\ &= \lim_{b\to 0^-}\left[\frac{3}{2}u^{2/3}\right]_{-1}^b + \lim_{a\to 0^+}\left[\frac{3}{2}u^{2/3}\right]_a^8 \\ &= \lim_{b\to 0^-}\left(\frac{3}{2}b^{2/3} - \frac{3}{2}\right) + \lim_{a\to 0^+}\left(6 - \frac{3}{2}a^{2/3}\right) \\ &= -\frac{3}{2} + 6 = \frac{9}{2}.\end{aligned}$$

Example 7.8.8

Evaluate $\displaystyle\int_0^2 x\ln(x)\,dx$.

$$\int_0^2 x\ln(x)\,dx = \lim_{t\to 0^+}\int_t^2 x\ln(x)\,dx$$

Using integration by parts with

$$f(x)=\ln(x) \qquad f'(x)=\frac{1}{x}\,dx$$
$$g'(x)\,dx = x\,dx \qquad g(x)=\frac{1}{2}x^2$$

we get

$$\begin{aligned}
\lim_{t\to 0^+}\int_t^2 x\ln(x)\,dx &= \lim_{t\to 0^+}\left[\frac{1}{2}x^2\ln(x)\right]_t^2 - \int_t^2 \frac{1}{2}x\,dx \\
&= \lim_{t\to 0^+}\left[\frac{1}{2}x^2\ln(x)\right]_t^2 - \left[\frac{1}{4}x^2\right]_t^2 \\
&= \lim_{t\to 0^+}\left[2\ln(2) - \frac{1}{2}t^2\ln(t)\right] - \left[1-\frac{1}{4}t^2\right] \\
&= \lim_{t\to 0^+}\left[2\ln(2) - \frac{1}{2}\frac{\ln(t)}{\frac{1}{t^2}}\right] - \left[1-\frac{1}{4}t^2\right] \\
&\overset{L'H}{=} \lim_{t\to 0^+}\left[2\ln(2) - \frac{1}{2}\frac{\frac{1}{t}}{-\frac{2}{t^3}}\right] - \left[1-\frac{1}{4}t^2\right] \\
&= \lim_{t\to 0^+}\left[2\ln(2) + \frac{1}{4}t^2\right] - \left[1-\frac{1}{4}t^2\right] \\
&= 2\ln(2) - 1
\end{aligned}$$

Example 7.8.9

Evaluate $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$.

7.8.3 A Comparison Test for Improper Integrals

Before trying to evaluate or approximate an improper integral, it's important to know whether or not the integral even converges.

Theorem 7.8.10: Integral Comparison Tests

Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
2. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Example 7.8.11

Does $\displaystyle\int_3^\infty \frac{1}{x^2 \ln x}\,dx$ converge or diverge?

Notice that for $x \geq 3$, we have

$$\begin{aligned} 1 &\leq \ln x \\ x^2 &\leq x^2 \ln x \\ \frac{1}{x^2} &\geq \frac{1}{x^2 \ln x}. \end{aligned}$$

So, we'll apply our comparison test with $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x^2 \ln x}$.
We know from Example **??** that $\int_3^\infty f(x)\,dx$ converges, so then it must be that $\int_3^\infty g(x)\,dx$ converges as well.

Example 7.8.12

Does $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$ converge or diverge?

Since $e^{-x} \geq 0$ for all x, we have

$$\frac{2+e^{-x}}{x} \geq \frac{2}{x}.$$

So, we'll apply our comparison test with $f(x) = \frac{2+e^{-x}}{x}$ and $g(x) = \frac{2}{x}$.
We know from Example **??** that $\int_1^\infty g(x)\,dx$ diverges, so then it must be that$\int_1^\infty f(x)\,dx$ diverges also.

8 Further Applications of Integration

8.3 Applications to Physics and Engineering

8.3.1 Hydrostatic Pressure and Force

Definition: Pressure

The **pressure** P on an object with area A is defined as the force F per unit area:

$$P = F/A.$$

Units: SI = Pascals (Pa), US Customary = pounds-per-square-foot (lb/ft^2)

Recall that force/weight of a liquid is given by

$F = \rho g V$ where V is the volume and $\rho g = 9800\,\text{kg}/(\text{m}^2\text{s}^2)$ (in SI units) and $\rho g = 62.5\,\text{lb}/ft^3$ in 'Murican units.

Example 8.3.1

If a tank in the shape of a rectangular prism with base $2\,\text{m} \times 1\,\text{m}$ and height $7\,\text{m}$ is full of water, how much pressure is the water exerting on the bottom of the tank? What if the tank is only half full?

The base of the box has area $2 \times 1 = 2\,\text{m}^2$, so to figure out this out, we just have to compute the weight (force) of the water and divide is by 2. We know that the weight of water is given by $\rho g V$ where V is the volume and $\rho g = 9800\,\text{kg}/(\text{m}^2\text{s}^2)$. The tank has volume $2 \times 1 \times 7 = 14\,\text{m}^3$, so the pressure on the bottom of the tank is

$$P = 9800 \times \frac{14}{2}\,\text{Pa}.$$

If we consider the pressure P exerted by a column of water of height h on an object of area A, then we get that

$$P = \frac{F}{A} = \frac{\rho g V}{A} = \frac{\rho g A h}{A} = \rho g h$$

Fact (Physics Fact). At every point in a liquid, the *pressure is the same in all directions.*

This fact allows us to compute pressure on *vertical surfaces* as well, but now it's not so straightforward because the pressure exerted toward the top of the tank is different from the pressure exerted at the bottom of the tank.

Example 8.3.2

Consider the tank from the previous problem and suppose the water is full. How much force is exercited on the $1\,\text{m} \times 7\,\text{m}$ side?

Consider a small *Deltay* strip of the wall y units from the bottom of the tank. The pressure on this strip is roughly constant

$$P(y) = \rho g(7 - y)$$

The area of this slice of wall is approximately

$$A(y) = 1 \times \Delta y = \Delta y.$$

So rearranging $P = F/A$ to $F = PA$, we get that

$$F(y) = P(y)A(y) = \rho g(7-y)\Delta y.$$

The amount of hydrostatic force on the entire wall is thus the "sum" (read: integral) of all of the y-values where there is water

$$F = \int_0^7 \rho g(7-y)\,dy =$$

Remark. As you can imagine, the shapes you're finding areas for might be noticeably more complicated; the strategies involved will be similar

8.3.2 Centers of Mass

Definition: G

iven a thin plate of any shape (and uniform thickness), the **center of mass** or **centroid** is the point P on the plate where the the plate would balance horizontally.

picture

Fact (Physics Fact - Law of the Lever). If two masses m_1, m_2 are attached to a rod (of negligible mass) on a fulcrum at distances d_1, d_2 away from the fulcrum, then the rod will balance precisely when

$$m_1 d_1 = m_2 d_2$$

We can use this fact to try to find the center of mass for a 2-d object. Suppose m_1 and m_2 are located on the x-axis at x_1, x_2, respectively. Then the center of mass, $\overline{x}$, satisfies

$$m_1(\overline{x} - x_1) = m_2(x_2 - \overline{x}) \Longrightarrow \overline{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

In general, if we have n masses, $m_1, \ldots, m_n$, then the center of mass is located at

$$\overline{x} = \frac{m_1x_1 + \cdots + m_nx_n}{m_1 + \cdots + m_n} = \frac{M_x}{M}$$

where $M_o = \sum_{i=1}^n m_i x_i$ and $M = \sum_{i=1}^n m_i$.

Definition

M_o is called the **moment of the system** and M is the **total mass.**

Remark. When in 2-dimensions, we want to think about finding a fulcrum's location in each dimension. In that case, we need to specify what axis the moment is about. In the above work, the masses were all lying in the x-axis, and the fulcrum could be thought of as a line perpendicular to the x-axis. So the moment of the system could be considered the moment about the y-axis.

Now, suppose our object is defined as the region between the curves $y = f(x)$ and $y = g(x)$ (with $f(x) > g(x)$ on some interval $[a, b]$ and has uniform density ρ.

If we partition this region into a bunch of rectangles (call them R_x) of width Δx and height $f(x)-g(x)$, then each rectangle has area $[f(x) - g(x)]\Delta x$ and mass $\rho[f(x) - g(x)]\Delta x$. For sufficiently small Δx, this rectangle is essentially a line, and thus the center of mass is located at $(x, \frac{1}{2}[f(x) + g(x)]$.

The moment of this rectangle about the y-axis is the product of the mass and the distance to the y-axis (which is just x),

$$M_y(R_x) = x\rho[f(x) - g(x)]\Delta x.$$

If we add up all the moments for each rectangle (and take a limit, turning the sum into an integral) we obtain

$$M_y = \rho \int_a^b x[f(x) - g(x)]\, dx$$

Similarly, the moment of this rectangle about the x-axis is the product of the mass and the distance to the x-axis (which is $\frac{1}{2}f(x) + \frac{1}{2}g(x)$), so

$$M_x(R_x) = \left(\frac{1}{2}[f(x) + g(x)]\right)\rho[f(x) - g(x)]\Delta x.$$

If we add up all of these moments (and take a limit, turning the sum into an integral) we obtain

$$M_x = \frac{\rho}{2}\int_a^b [f(x)]^2 - [g(x)]^2\, dx.$$

Lastly we need to divide by the total mass of the object, which is ρ times the area of the object, $\int_a^b [f(x) - g(x)]\, dx$. This gives us

Definition: Center of Mass/Centroid

Let $\mathfrak{R}$ be an object defined as the region between two curves $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$. If $f(x) > g(x)$, the **center of mass of $\mathfrak{R}$** or **centroid of $\mathfrak{R}$** is located at $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{M_y}{M} = \frac{\int_a^b x[f(x) - g(x)]\,dx}{\int_a^b [f(x) - g(x)]\,dx} \qquad \text{and} \qquad \overline{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2}[f(x)^2 - g(x)^2]\,dx}{\int_a^b [f(x) - g(x)]\,dx}$$

Example 8.3.3: Sanity Check

Let $\mathfrak{R}$ be the circle of radius 2 centered at $(3, 1)$. Show that $\mathfrak{R}$'s center of mass is located at $(3, 1)$. (In general, the center of mass of a circle is at the center of the circle).

Let $f(x) = 1 + \sqrt{4 - (x-3)^2}$ and $g(x) = 1 - \sqrt{4 - (x-3)^2}$.

Notice that, *FOR THESE PARTICULAR FUNCTIONS*, we have that $\frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right) = [f(x) - g(x)]$. Hence

$$\overline{y} = \frac{\int_a^b \frac{1}{2}[f(x)^2 - g(x)^2]\,dx}{\int_a^b [f(x) - g(x)]\,dx} = \frac{\int_a^b [f(x) - g(x)]\,dx}{\int_a^b [f(x) - g(x)]\,dx} = 1.$$

Finding $\overline{x}$ won't be so straightforward. We first compute

$$\int_1^5 x[f(x) - g(x)]\,dx = \int_1^5 x[2\sqrt{4 - (x-3)^2}]\,dx$$

Using the trig substitution

$$\begin{aligned} x - 3 &= 2\sin\theta & \qquad x = 1 &\Rightarrow \theta = -\frac{\pi}{2} \\ dx &= 2\cos\theta\,d\theta & \qquad x = 5 &\Rightarrow \theta = \frac{\pi}{2} \end{aligned}$$

we get

$$\begin{aligned} \int_1^5 x[2\sqrt{4 - (x-3)^2}]\,dx &= \int_{-\pi/2}^{\pi/2} (2\sin\theta + 3)\left[2\sqrt{4 - 4\sin^2\theta}\right] \cdot 2\cos\theta\,d\theta \\ &= 8\int_{-\pi/2}^{\pi/2} 2\sin\theta\cos^2\theta + 3\cos^2\theta\,d\theta \\ &= 8\int_{-\pi/2}^{\pi/2} 2\sin\theta\cos^2\theta\,d\theta + 8\int_{-\pi/2}^{\pi/2} 3\cos^2\theta\,d\theta \end{aligned}$$

Since $2\sin\theta\cos^2\theta$ is an odd function, the first integral evaluates to 0. After applying a power reduction formula to the second one, we're left with

$$8\int_{-\pi/2}^{\pi/2} 3\cos^2\theta\,d\theta = 8\left[\frac{3}{2}\theta + \frac{3}{4}\sin(2\theta)\right]_{-\pi/2}^{\pi/2} = 12\pi$$

hence

$$\overline{x} = \frac{\int_a^b x[f(x) - g(x)]\,dx}{\int_a^b [f(x) - g(x)]\,dx} = \frac{12\pi}{4\pi} = 3.$$

Example 8.3.4

Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.

Let $f(x) = x$ and $g(x) = x^2$. These curves intersect at $(0, 0)$ and $(1, 1)$ and $f(x) > g(x)$ when $0 < x < 1$.

8.5 Probability

8.5.1 Finite Probabilities

Suppose we have a ball with 3 balls in it

$$\{\text{red, blue, green}\}$$

If I pull one ball out at random, the probability that I pull out a red ball is 1/3. In fact, the same is true for any ball color I specify. Succinctly,

$$P(\{\text{red}\}) = \frac{1}{3}, \qquad P(\{\text{blue}\}) = \frac{1}{3}, \qquad P(\{\text{green}\}) = \frac{1}{3}.$$

Now suppose I add one red ball, two blue balls, and four green balls, making 10 balls total.

$$\{\text{red, red, blue, blue, blue, green, green, green, green, green}\}$$

If I pull one ball out at random, the probability that I pull out a red ball is now 2/10. The probabilities for each color choice are now

$$P(\{\text{red}\}) = \frac{2}{10}, \qquad P(\{\text{blue}\}) = \frac{3}{10}, \qquad P(\{\text{green}\}) = \frac{5}{10}.$$

One thing to notice is that, no matter how many of each color I have, or even how many colors I have, it is always the case that these probabilities sum to 1:

$$\sum_{\text{color}} P(\text{color}) = 1.$$

Now we complicate things slightly: what if I want to know the probability that I pull out a red ball OR a blue ball from the 10-ball set? Well, given that 5 of the 10 balls are either red or blue, we expect a 5/10 chance. From this, observation, we get that probabilities are additive

$$P(\{\text{red, blue}\}) = P(\{\text{red}\}) + P(\{\text{blue}\}) = \frac{2}{10} + \frac{3}{19} = \frac{5}{10}.$$

We now ask what happens if there are infinitely-many balls of infinitely-many different colors (which, for simplicity, we'll call the colors x_i). Naively, now it seems like every probability is zero. However, it could also be the case that every other ball has the same color, something like

$$\{x_1 = \text{red}, x_2, \text{red}, x_3, \text{red}, x_4, \text{red}, x_5, \text{red}, x_6, \text{red}, \ldots\}$$

and so now it seems more like $P(\text{red}) = \frac{1}{2}$.

Of course, we can't count all of these balls, so we rely on some function f to tell us how these probabilities should be distributed among each of the colors. Such a function should have the properties that

- $f(x_i) \geq 0$
- $\sum_{i=0}^{\infty} f(x_i) = 1$.

Such a function is called a **(discrete) probability distribution**. Just as in the finite case, we still have the additive property:

$$P(\{x_m, \ldots, x_n\}) = \sum_{i=m}^{n} P(\{x_i\}) = \sum_{i=m}^{n} f(\{x_i\}).$$

8.5.2 Continuous Probabilities

One can imagine making this entire thing continuous (instead of only focusing on positive integers). In this case, we have

Definition

A **(continuous) probability distribution** is a function f with the following properties:

- $f(x) \geq 0$ for each **outcome** or **continuous random variable** x, and
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Now, if we want to ask about a probability of a range of outcomes, say $a \leq X \leq b$, then we get

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx.$$

Indeed, this need not be finite.

$$P(X \geq a) = \int_a^{\infty} f(x)\,dx \qquad \text{and} \qquad P(X \leq b) = \int_{-\infty}^{b} f(x)\,dx.$$

Remark. There are a number of common types of continuous probability distributions.

- The **normal distribution** (with mean μ and standard deviation σ): $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
- The **uniform distribution** on the interval $[A, B]$: $f(x) = \begin{cases} \frac{1}{B-A} & \text{when } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$
- The **exponential distribution** is: $f(x) = \begin{cases} 0 & \text{when } x < 0 \\ ke^{-kx} & \text{when } x \geq 0 \end{cases}$

Example 8.5.1

For which value of A is $f(x) = \begin{cases} Ax(5-x) & \text{when } 0 \leq x \leq 5, \\ 0 & \text{otherwise} \end{cases}$ a probability density function?

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty} 0\,dx + \int_0^5 Ax(5-x)\,dx + \int_5^{\infty} 0\,dx \\
&= 0 + A\int_0^5 5x - x^2\,dx + 0 \\
&= A\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5 \\
&= A\left(\frac{125}{2} - \frac{125}{3}\right) = A\left(\frac{125}{6}\right)
\end{aligned}$$

Since a probability distribution requires that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$, we deduce that $A = \frac{6}{125} = 0.048$.

Example 8.5.2: Uniform Density

A traffic light remains red for 30 seconds at a time. You arrive (at random) at the light and find that it is red. Use an appropriate uniform density function to find the probability that you will have to wait *at least* 10 seconds before the light turns green.

The desired probability distribution is

$$f(x) = \begin{cases} \frac{1}{30} & \text{when } 0 \leq x \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

The desired probability is thus

$$P(10 \leq X) = \int_{10}^{\infty} f(x)\,dx = \int_{10}^{30} \frac{1}{30}\,dx + \int_{30}^{\infty} 0\,dx = \frac{20}{30} + 0 = \frac{2}{3}.$$

Example 8.5.3: Exponential Density

Let X be a random variable that measures the duration of phone calls in Roanoke and suppose that a probability density function for X is given by

$$f(x) = \begin{cases} 0 & \text{when } x < 0 \\ \frac{1}{2}e^{-x/2} & \text{when } x \geq 0 \end{cases} \qquad \text{where } x \text{ is measured in minutes.}$$

What is the probability that a randomly selected call will last for less than 3 minutes?

$$\begin{aligned} P(x \leq 3) = \int_{-\infty}^{3} f(x)\,dx &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{3} \frac{1}{2}e^{-x/2}\,dx \\ &= 0 + \left[-e^{-x/2}\right]_0^3 = 1 - e^{-3/2} \approx 77.7\% \end{aligned}$$

10 Parametric Equations and Polar Coordinates

10.1 Parametric Curves

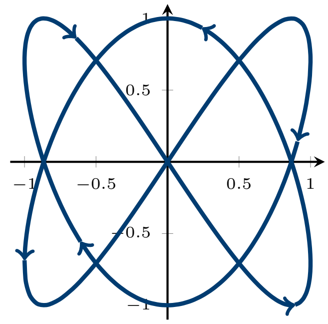


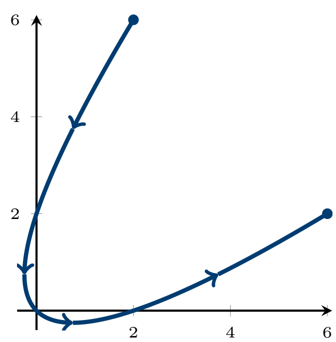
Figure 10.1.1: A parametric curve C given by $x = \sin(2t)$, $y = \sin(3t)$.

The graph of the curve above fails to be a function of the form $y = f(x)$, because it fails the vertical line test, but it may be a reasonable path for an object to travel (maybe a weight attached to a spring attached to a pendulum, or maybe a bee's flight path), so we'd like to be able to model it.

Suppose x and y are both functions of a third variable, t (called a **parameter**), with $x = f(t)$ and $y = g(t)$ (called **parametric equations**). We can then plot the points $(x, y) = (f(t), g(t))$ in the coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ traces out a curve C (called a **parametric curve**).

Example 10.1.1

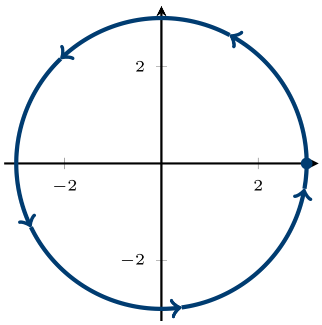
Sketch the curve given by $x = t^2 + t$, $y = t^2 - t$, $-2 \leq t \leq 2$. Indicate with an arrow the direction in which the curve is traced as t increases.



t	(x, y)
-2	$(2, 6)$
-1	$(0, 2)$
0	$(0, 0)$
1	$(2, 0)$
2	$(6, 2)$

Example 10.1.2

Sketch the curve given by $x = 3\cos t$, $y = 3\sin t$, for $0 \le t \le 2\pi$. Indicate with an arrow the direction in which the curve is traced as t increases.



t	(x, y)
0	$(3, 0)$
$\frac{\pi}{2}$	$(0, 3)$
π	$(-3, 0)$
$\frac{3\pi}{2}$	$(0, -3)$
2π	$(3, 0)$

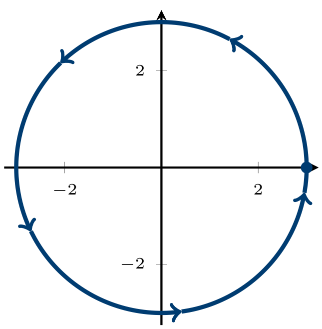
The shape appears to be a circle of radius 3. And indeed, we can see this is the case be eliminating the parameter:

$$x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9(\sin^2 t + \cos^2 t) = 9,$$

so the equation is exactly that of a circle of radius 3.

Example 10.1.3

Sketch the curve given by $x = 3\cos(2t)$, $y = 3\sin(2t)$, for $0 \le t \le \pi$. Indicate with an arrow the direction in which the curve is traced as t increases.



t	(x, y)
0	$(3, 0)$
$\frac{\pi}{2}$	$(0, 3)$
π	$(-3, 0)$
$\frac{3\pi}{2}$	$(0, -3)$
2π	$(3, 0)$

Once again, the shape appears to be a circle of radius 3. And indeed, we can see this is the case be eliminating the parameter:

$$x^2 + y^2 = 9\cos^2(2t) + 9\sin^2(2t) = 9,$$

so the equation is exactly that of a circle of radius 3.

What this example illustrates is that *curves are not uniquely parametrized.* If thinking about a particle traveling along these curves, the particle in the second example completes the curve in half the amount of time (or rather, travels twice as fast).

Because curves are not uniquely parametrized, it may be easier to visualize the curve by eliminating the parameter and obtaining a Cartesian equation of the curve.

Example 10.1.4

Eliminate the parameter from

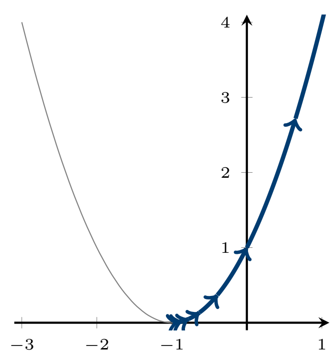
$$x = e^t - 1,\, y = e^{2t}$$

to find a Cartesian equation of the curve. Then sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

By rearranging the first equation as $e^t = x + 1$, we have

$$y = e^{2t} = (e^t)^2 = (x+1)^2,$$

which is a parabola. One thing to keep in mind is that $e^t > 0$ for all t, so the range of x-values is the interval $(1, \infty)$. Since x is increasing as t increases, the arrows trace the curve from left to right.



Example 10.1.5

Eliminate the parameter from

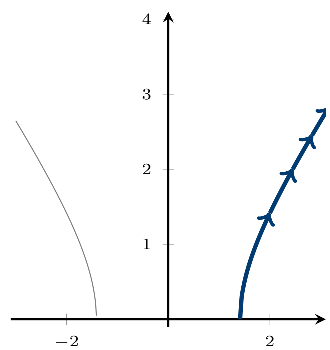
$$x = \sqrt{t+1},\, y = \sqrt{t-1}$$

to find a Cartesian equation of the curve. Then sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

By rearranging the first equation as $t = x^2 - 1$ and the second as $t = y^2 + 1$ we have

$$x^2 - 1 = y^2 + 1 \quad \Rightarrow \frac{x^2}{2} - \frac{y^2}{2} = 1,$$

which is a hyperbola. One thing to keep in mind is that the arguments for $x(t)$ and $y(t)$ are only defined for $t \geq 1$, so the range of x-values is the interval $(2, \infty)$ and the range of y-values is $(0, \infty)$. Since x is increasing as t increases, the arrows trace the curve from left to right.



10.2 Calculus with Parametric Curves

10.2.1 Slopes and Tangent Lines

Suppose that f and g are differentiable functions and we have the parametric curve C given by $x(t) = f(t)$ and $y(t) = g(t)$ (where y can also be expressed as a differentiable function of x). If we want to find the tangent line at a point (x, y) on C, we need to find $\frac{dy}{dx}$. By an application of the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

which rearranges to

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \quad \text{if } \frac{dx}{dt} \neq 0. \tag{10.2.1}$$

Remark. While $\frac{dy}{dx}$ has a simple expression in terms of parametric derivatives, it is not quite so straightforward to find higher derivatives. To find $\frac{d^2y}{dx^2}$ we rewrite Equation 10.2.1 as

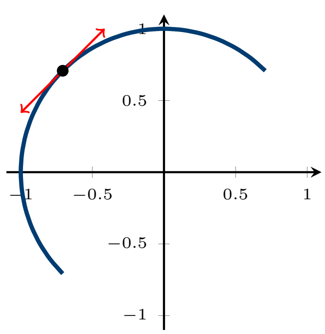
$$\frac{d}{dx}(y) = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(y)}{\frac{dx}{dt}}$$

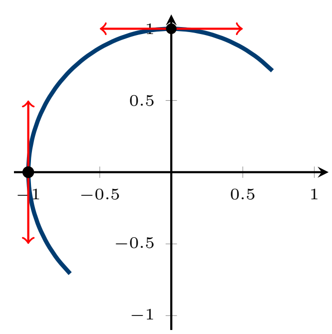
and then replacing y with $\frac{dy}{dx}$ yields

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}. \tag{10.2.2}$$

Example 10.2.1

Find the equation of the tangent line through the point where $t = \frac{3}{4}$ of the parametric curve given by $x = \cos(\pi t)$, $y = \sin(\pi t)$, $\frac{1}{4} \leq t \leq \frac{5}{4}$. At what point is the slope horizontal? At what point is the slope vertical?





We have that $\frac{dx}{dt} = -\pi \sin(\pi t)$ and $\frac{dy}{dt} = \pi \cos(\pi t)$, so the slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{t=\frac{3}{4}} = \frac{\pi \cos\left(\frac{3\pi}{4}\right)}{-\sin\left(\frac{3\pi}{4}\right)} = 1$$

and passes through the point $\big(x(\frac{3}{4}), y(\frac{3}{4})\big) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, so the equation for this line is

$$y = x + \sqrt{2}.$$

Recall that the tangent line is horizontal if the slope is 0, i.e., if $\frac{dy}{dt}\big|_t = 0$, and vertical if $\frac{dx}{dt}\big|_t = 0$. Thus we have a horizontal tangent line when $t = \frac{1}{2}$ and a vertical tangent line when $t = 1$.

Example 10.2.2

For the parametric curve given by $x = \sqrt{t}$, $y = \frac{1}{4}(t^2 - 4)$, $t \geq 0$, find the slope and concavity at the point $(2, 3)$.

We have that $\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$ and $\frac{dy}{dt} = \frac{t}{2}$. Note also that $(x(t), y(t)) = (2, 3)$ when $t = 4$. So, the first derivative is

$$\frac{dy}{dx} = \frac{\frac{t}{2}}{\frac{1}{2\sqrt{t}}} = t^{3/2}.$$

and the slope at the point where $t = 4$ is

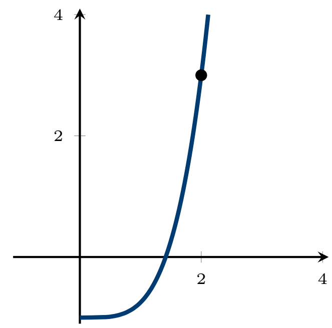
$$\left.\frac{dy}{dx}\right|_{t=4} = 4^{3/2} = 8.$$

The second derivative is

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[t^{3/2}\right]}{\frac{1}{2\sqrt{t}}} = \frac{\frac{3\sqrt{t}}{2}}{\frac{1}{2\sqrt{t}}} = 3t,$$

and the concavity at this point where $t = 4$ is thus

$$\left.\frac{d^2y}{dx^2}\right|_{t=4} = 3(4) = 12$$



Example 10.2.3

The *prolate cycloid* given by $x = 2t - \pi \sin t$ and $y = 2 - \pi \cos t$ crosses itself at the point $(0, 2)$. Find the equations of both tangent lines at this point.

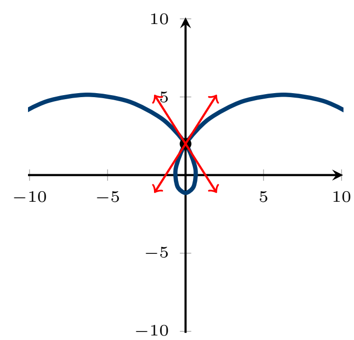
We find that $(x(t), y(t)) = (0, 2)$ when $t = -\frac{\pi}{2}$ and $t = \frac{\pi}{2}$. We have that $\frac{dx}{dt} = 2 - \pi \cos t$ and $\frac{dy}{dx} = \pi \sin t$. The slope at $t = -\frac{\pi}{2}$ is thus

$$\left.\frac{dy}{dx}\right|_{t=-\pi/2} = \frac{\pi \sin\left(-\frac{\pi}{2}\right)}{2 - \pi \cos\left(-\frac{\pi}{2}\right)} = -\frac{\pi}{2}$$

and the equation of the tangent line here is $y = -\frac{\pi}{2}x + 2$. The slope at $t = \frac{\pi}{2}$ is

$$\left.\frac{dy}{dx}\right|_{t=-\pi/2} = \frac{\pi \sin\left(\frac{\pi}{2}\right)}{2 - \pi \cos\left(\frac{\pi}{2}\right)} = \frac{\pi}{2}$$

and the equation of the tangent line here is $y = \frac{\pi}{2}x + 2$.



10.2.2 Arc Length

We saw how to compute arc length in Section 7.4. We can equivalently discuss arc length in terms of parametric functions thanks to the following theorem

Theorem 10.2.4

If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ where f' and g' are continuous on $[\alpha, \beta]$ and C is transversed exactly once on this interval, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Proof. See the text. □

Example 10.2.5

Let $r > 0$ be some real number and consider the curve given by $x = r\cos t$, $y = r\sin t$. Find the arc length of this curve for $0 \le t \le 2\pi$.

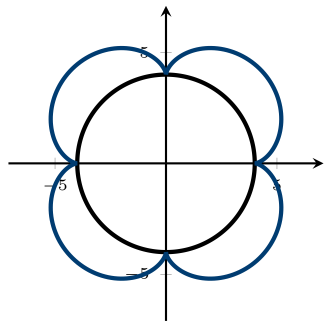
Applying the arc length formula,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\ &= \int_0^{2\pi} r\, dt \\ &= [rt]_0^{2\pi} = 2\pi r. \end{aligned}$$

This is just the formula for the circumference of a circle of radius r, as expected.

Example 10.2.6

A circle of radius 1 rolls around the circumference of a larger circle of radius 4. The epicycloid traced by a point on the circumference of the smaller circle is given by $x = 5\cos t - \cos(5t)$, $y = 5\sin t - \sin(5t)$. Find the distance traveled by the point in one complete trip about the larger circle.



We appeal to symmetry and integrate the first quadrant's curve only, and then multiply the answer by 4:

$$
\begin{aligned}
L &= 4\int_0^{\pi/2} \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt \\
&= 4\int_0^{\pi/2} \sqrt{[-5\sin t + 5\sin(5t)]^2 + [5\cos t - 5\cos(5t)]^2} \\
&= 4\int_0^{\pi/2} \sqrt{25 - 50\sin t\sin(5t) + 25 - 50\cos t\cos(5t)}\, dt \\
&= 20\int_0^{\pi/2} \sqrt{2 - 2\sin t\sin(5t) - 2\cos t\cos(5t)}\, dt \\
&= 20\int_0^{\pi/2} \sqrt{2 - 2\cos(4t)}\, dt && \text{(angle sum identity)} \\
&= 20\int_0^{\pi/2} \sqrt{4\sin^2(2t)}\, dt && \text{(double-angle identity)} \\
&= 20\int_0^{\pi/2} 2\sin(2t)\, dt \\
&= -20\left[\cos(2t)\right]_0^{\pi/2} = 40.
\end{aligned}
$$

10.2.3 Areas

Recall that the area under a curve $y = F(x)$ from $x = a$ to $x = b$ is given by

$$A = \int_a^b y\,dx = \int_a^b F(x)\,dx.$$

If our curve is parametrized as $x = f(t)$, $y = g(t)$, and we have that $\alpha \leq t \leq \beta$ with $a = f(\alpha)$ and $b = f(\beta)$, then

$$dx = f'(t)\,dt$$

and the substitution rule for definite integrals gives us the area under a parametric curve as

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt.$$

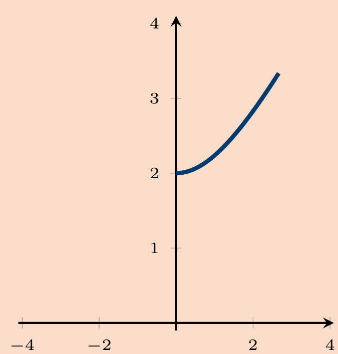
If instead we have that $a = f(\beta)$ and $b = f(\alpha)$ then the area under this parametric curve is

$$\int_\beta^\alpha g(t)f'(t)\,dt.$$

It's worth noting that we have to make some additional assumptions for this area to be unambiguous and for the integral to work. In particular, we need to make sure that the curve is traced out only once on this interval (otherwise we could be adding/subtracting extra area that isn't really there), and we also need to make sure that the curve doesn't fail the vertical line test on this interval.

Example 10.2.7

Find the area below the parametric curve $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$, $1 \leq t \leq 3$.



Since $x'(t), y'(t)$ are not 0 on this interval, the curve never traces itself out more than once. Plotting the portion of curve above, we can see that it also passes the vertical line test. So we integrate according to the definition above. Since $x'(t) = 1 + \frac{1}{t^2}$, we get

$$\begin{aligned} A &= \int_1^3 \left(t + \frac{1}{t}\right)\left(1 + \frac{1}{t^2}\right) dt \\ &= \int_1^3 t + \frac{2}{t} + \frac{1}{t^3}\,dt \end{aligned}$$

$$= \left[\frac{1}{2}t^2 + 2\ln|t| - \frac{1}{2t^2}\right]_1^3$$
$$= \frac{40}{9} + \ln 9 \approx 6.6417.$$

If we relax the condition that our curve C pass the vertical line test and assume that C is a closed loop, then we can actually compute the area enclosed by the loop.

Example 10.2.8

Compute the area of the loop enclosed by the curve $x = t^3 - 4t$, $y = 2t^2$ on the interval $-2 \leq t \leq 2$.

Notice that the "upper half" of this curve is traversed from left to right as t increases. This means that integrating from $t = -2$ to $t = 2$ will result in a positive area. So, computing $x'(t) = 2t^2 - 4$, we get

$$A = \int_{-2}^{2} (2t^2)(2t^2 - 4)\,dt$$
$$= \int_{-2}^{2} 4t^4 - 8t^2\,dt$$
$$= \left[\frac{4}{5}t^5 - \frac{8}{3}t^3\right]_{-2}^{2}$$
$$= \frac{128}{15} \approx 8.5333$$

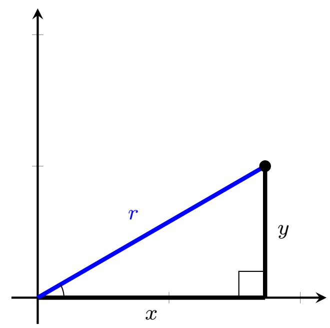
Example 10.2.9

Compute the area of the circle of radius 1, centered at $(x, y) = (2, 2)$, using the parameterization $x = 2 + \cos(t)$, $y = 2 + \sin(t)$.

This exercise is left to the reader. We note, however, that the "upper half" of the circle is traversed backwards (that is, x is decreasing), and so you should choose your limits of integration accordingly.

10.3 Polar Coordinates

Let (x, y) be some point in the Cartesian plane and let r be the length of the line segment from the origin $(0, 0)$ to (x, y). Also, let θ be the angle from the positive x-axis to this line segment (traversing counter-clockwise). It's not hard to see that, when (x, y) is not the origin, this r-value and this θ-value are unique to this (x, y) point, so rather than refer to the point in terms of the ordered pair (x, y), we could refer to them in terms of (r, θ). This is the basis for **polar coordinates**, which is a very useful re-parametrization of the Cartesian plane. So how do we pass between them?



From trigonometry, we see that we have the following relationships

$$x = r\cos\theta, \qquad r = \sqrt{x^2 + y^2},$$
$$y = r\sin\theta, \qquad \tan\theta = \frac{y}{x}.$$

Remark. It's tempting to use $\theta = \arctan\left(\frac{y}{x}\right)$, but since arctan only has range $(-\frac{\pi}{2}, \frac{\pi}{2})$, you have to be aware of the quadrant in which your point (x, y) lies and may have to add multiples of π to get the correct angle. If you want a function with less "guess-work" and a range of $(0, 2\pi)$, then you'd use a function usually called "$\arctan_2$ where

$$\theta = \pi + \arctan_2\left(\frac{y}{x}\right) = \pi + 2\arctan\left(\frac{y}{\sqrt{x^2 + y^2} + x}\right).$$

Example 10.3.1

Convert the point $(1, \frac{\pi}{2})$ from polar coordinates to Cartesian coordinates.

We have that

$$x = r\cos\theta = 1\cos\left(\frac{\pi}{2}\right) = 0,$$
$$y = r\sin\theta = 1\sin\left(\frac{\pi}{2}\right) = 1,$$

so, in Cartesian coordinates, we have $(0, 1)$.

Example 10.3.2

Convert the point $(-\sqrt{3}, -1)$ from Cartesian coordinates to polar coordinates.

We have that

$$r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2,$$
$$\tan\theta = \frac{-1}{-\sqrt{3}} \Rightarrow \theta = \frac{7\pi}{6},$$

so, in polar coordinates, we have $\left(2, \frac{7\pi}{6}\right)$.

Definition

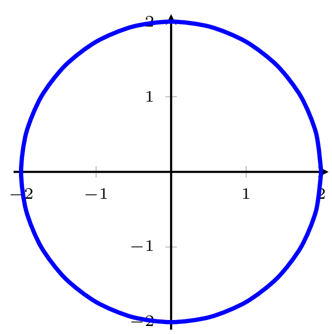
A **polar function** is a function of the form $r = f(\theta)$.

We graph the polar function $r = f(\theta)$ is the same we might graph $y = f(x)$: plot all values (r, θ) for which $r = f(\theta)$.

Example 10.3.3

Sketch a graph of the function $r = 2$.

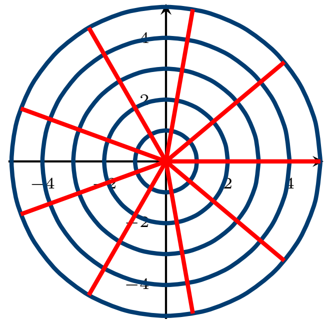
θ	$x = r\cos(\theta)$	$y = r\sin(\theta)$
0	2	0
$\frac{\pi}{4}$	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{2}$	0	2
$\frac{3\pi}{4}$	$-\sqrt{2}$	$\sqrt{2}$
π	-2	0
$\frac{5\pi}{4}$	$-\sqrt{2}$	$-\sqrt{2}$
$\frac{3\pi}{2}$	0	-2
$\frac{7\pi}{4}$	$\sqrt{2}$	$-\sqrt{2}$



Example 10.3.4

Sketch a graph of the curve $\theta = \frac{\pi}{3}$

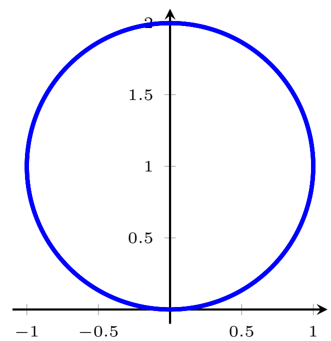
What we see is that fixing the r-coordinate results in concentric circles, and fixing the θ coordinate results in straight lines through the origin. As such, we can construct the the polar coordinate grid:



Example 10.3.5

Sketch a graph of the polar function $r = 2\sin\theta$. Find a Cartesian equation for this curve.

θ	$x = r\cos(\theta)$	$y = r\sin(\theta)$
0	0	0
$\frac{\pi}{4}$	1	1
$\frac{\pi}{2}$	0	2
$\frac{3\pi}{4}$	-1	1
π	0	0
$\frac{5\pi}{4}$	1	1
$\frac{3\pi}{2}$	0	2
$\frac{7\pi}{4}$	-1	1



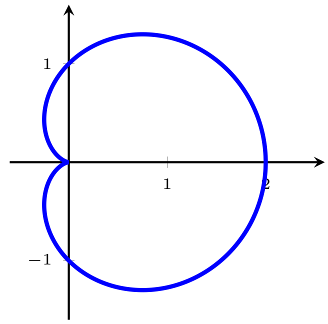
$$\begin{aligned} r &= 2\sin\theta \\ r^2 &= 2r\sin\theta \\ r^2 &= 2y \\ x^2 + y^2 &= 2y \\ x^2 + y^2 - 2y + 1 &= 1 \\ x^2 + (y-1)^2 &= 1 \end{aligned}$$

we get the equation for the circle of radius 1, centered at $(0, 1)$.

Example 10.3.6

Sketch a graph of the polar function $r = 1 + \cos\theta$.

θ	$x = r\cos(\theta)$	$y = r\sin(\theta)$
0	2	0
$\frac{\pi}{4}$	$\frac{1}{2} + \frac{\sqrt{2}}{2}$	$\frac{1}{2} + \frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	0	1
$\frac{3\pi}{4}$	$\frac{1}{2} - \frac{\sqrt{2}}{2}$	$-\frac{1}{2} + \frac{\sqrt{2}}{2}$
π	0	0
$\frac{5\pi}{4}$	$\frac{1}{2} - \frac{\sqrt{2}}{2}$	$\frac{1}{2} - \frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	0	-1
$\frac{7\pi}{4}$	$\frac{1}{2} + \frac{\sqrt{2}}{2}$	$-\frac{1}{2} - \frac{\sqrt{2}}{2}$



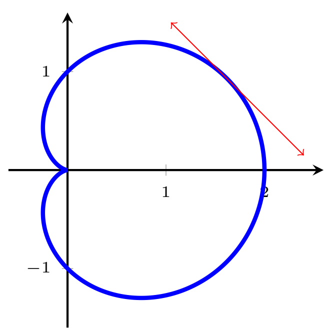
10.3.1 Tangents to Polar Curves

If $r = f(\theta)$, then we can regard θ as a parameter and we get

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}[r\sin\theta]}{\frac{d}{d\theta}[r\cos\theta]} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Example 10.3.7

Find the slope of the tangent line to $r = 1 + \cos\theta$ when $\theta = \frac{\pi}{6}$.

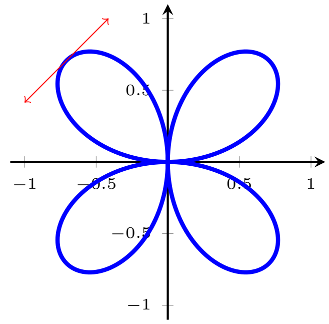


We have that $\frac{dr}{d\theta} = -\sin\theta$, so the slope of the tangent line when $\theta = \frac{\pi}{6}$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{-\sin(\frac{\pi}{6})\sin(\frac{\pi}{6}) + (1+\cos(\frac{\pi}{6}))\cos(\frac{\pi}{6})}{-\sin(\frac{\pi}{6})\cos(\frac{\pi}{6}) - (1+\cos(\frac{\pi}{6}))\sin(\frac{\pi}{6})} = -1.$$

Example 10.3.8

Find the equation of the tangent line to $r = \sin(2\theta)$ at $\theta = \frac{-\pi}{4}$.



We have that $\frac{dr}{d\theta} = 2\cos(2\theta)$, so the slope of the tangent line when $\theta = -\frac{\pi}{4}$ is

$$\left.\frac{dy}{dx}\right|_{\theta=-\pi/4} = \frac{2\cos(-\frac{\pi}{2})\sin(-\frac{\pi}{4}) + \sin(-\frac{\pi}{2})\cos(-\frac{\pi}{4})}{2\cos(-\frac{\pi}{2})\cos(-\frac{\pi}{4}) - \sin(-\frac{\pi}{2})\sin(-\frac{\pi}{4})} = 1.$$

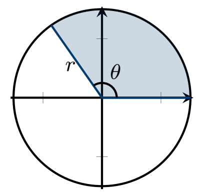
When $\theta = -\frac{\pi}{4}$, we have that $(x, y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Thus, the line with slope 1 passing through the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ has equation

$$y = x + \sqrt{2}$$

10.4 Areas and Lengths in Polar Coordinates

10.4.1 Area in Polar Coordinates

A sector of a circle of radius r spanned by angle θ has area $\underline{A = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2}r^2\theta}$.



Just as we could use rectangles to approximate the area under the curve of a function in cartesian coordinates, we can use sectors to approximate the region enclosed by a polar curve.

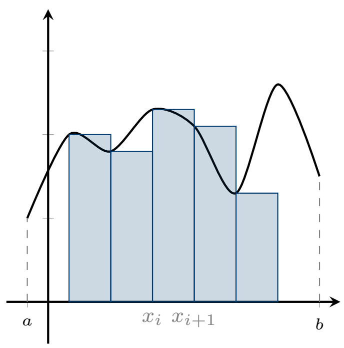


Figure 10.4.1: Approximating the area bounded by a Cartesian curve $y = f(x)$

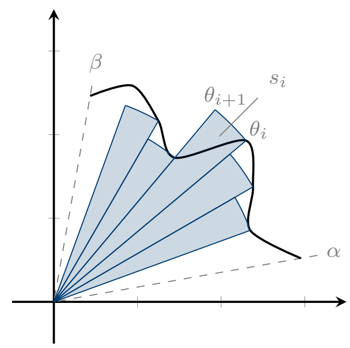


Figure 10.4.2: Approximating the area bounded by a polar curve $r = f(\theta)$

Given the polar curve $r = f(\theta)$ from $\theta = \alpha$ to β, we approximate the region with n sectors, labeled s_i (where $i = 1 \ldots n$), and we can choose the angle spanned by each sector to be $\Delta\theta = \theta_{i+1} - \theta_i = \frac{\beta-\alpha}{n}$. The area of each sector s_i is $\underline{\frac{1}{2}[f(\theta_i)]^2\Delta\theta}$, and thus the approximate area of the polar region is

$$A \approx \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2\Delta\theta = \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2 \left(\frac{\beta-\alpha}{n}\right)$$

As n increases, the size of each sector decreases and our approximation gets better and better. Thus, we take a limit as $n \to \infty$ to get that our exact area is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(\theta_i)]^2\theta_i = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 \, d\theta.$$

Example 10.4.1: Sanity check

Find the area of the circle of radius R using polar coordinates.

A circle of radius R is defined by the polar equation $f(\theta) = R$, for $0 \leq \theta \leq 2\pi$. The area spanned by this curve on this interval is

$$A = \int_0^{2\pi} \frac{1}{2}[f(\theta)]^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}R^2 \, d\theta = \left[\frac{1}{2}R^2\theta\right]_0^{2\pi} = \pi R^2.$$

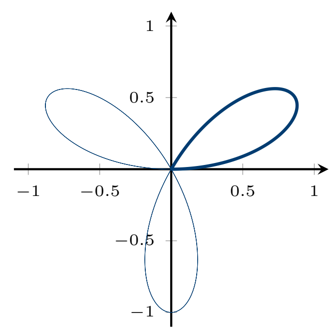
This is good; the formula for the area enclosed by a polar curve agrees with our intuition.

Example 10.4.2

Find the area of one leaf of the rose $r = \sin(3\theta)$.

Notice that $0 \leq \theta \leq \frac{\pi}{3}$ traces out one leaf of the rose. We thus compute the area

$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2} \left[\sin(3\theta)\right]^2 \, d\theta \\ &= \frac{1}{4} \int_0^{2} 1 - \cos(6\theta) \, d\theta \\ &= \frac{1}{4} \left[1 - \frac{1}{6}\sin(6\theta)\right]_0^{\pi/3} \\ &= \frac{\pi}{12}. \end{aligned}$$



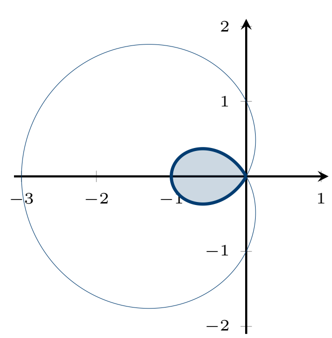
We always want to "sweep" the area counter-clockwise, so it's important that we choose our limits α and β so that $\alpha \leq \theta \leq \beta$.

Example 10.4.3

Find the area of the inner loop of the limaçon $r = 1 - 2\cos\theta$.

The curve passes through itself at the origin, which corresponds to $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. However, when we integrate, we need to choose the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ to make sure we compute the correct region (compare Figure 10.4.4 and Figure 10.4.3).
As such, we have that the area is given by

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2} \left(1 - 2\cos\theta\right)^2 \, d\theta \\ &= \int_{-\pi/3}^{\pi/3} \frac{1}{2} - 2\cos\theta + 2\cos^2\theta \, d\theta \\ &= \int_{-\pi/3}^{\pi/3} \frac{1}{2} - 2\cos\theta + \left(1 + \cos(2\theta)\right) \, d\theta \\ &= \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\sin(2\theta)\right]_{-\pi/3}^{\pi/3} \\ &= \pi - \frac{3\sqrt{3}}{2}. \end{aligned}$$



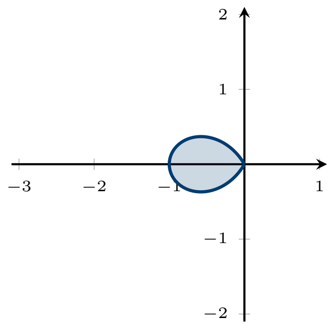


Figure 10.4.3: Area when integrating from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$.

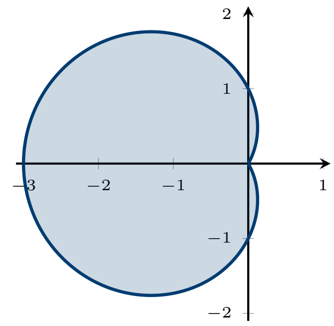
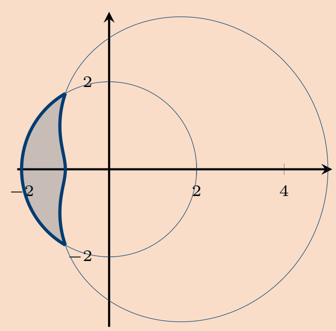


Figure 10.4.4: Area when integrating from $\frac{\pi}{3}$ to $\frac{5\pi}{3}$.

Just as in the case of Cartesian coordinates, the area in between two polar curves is the area of the outer polar region minus the area of the inner polar region. We note that we again have to verify that the angle measures in our limits of integration are correct for each curve separately.

Example 10.4.4

Find the area of the region inside the curve $r = 2$ and outside the curve $r = 3 + 2\cos\theta$.



The two curves intersect at $(r, \theta) = (2, \frac{2\pi}{3})$ and $(r, \theta) = (2, \frac{4\pi}{3})$. The area inside $r = 2$ from this region is given by

$$A_1 = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(2)^2 \, d\theta = \frac{4\pi}{3}$$

and the area inside $r = 3 + 2\cos\theta$ from this region is given by

$$A_2 = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}\left(3 + 2\cos\theta\right)^2 \, d\theta = -\frac{11\sqrt{2}}{2} + \frac{11\pi}{3}.$$

Therefore the area of the region between the two curves is

$$A = A_1 - A_2 = -\frac{7\pi}{3} + \frac{11\sqrt{2}}{2}.$$

10.4.2 Arc Length in Polar Coordinates

Recall that for a parametrized curve $(x(t), y(t))$, the length of the curve on the interval $a \leq t \leq b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In the case where $t = \theta$ and $r = f(\theta)$, we have

$$\begin{aligned} x(\theta) &= r\cos\theta = f(\theta)\cos\theta \\ y(\theta) &= r\sin\theta = f(\theta)\sin\theta \end{aligned}$$

and by the product rule,

$$\begin{aligned} \frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ \frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta. \end{aligned}$$

After working with the algebra and canceling a few terms, we get that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{dt}\right)^2 = [f'(\theta)]^2 + [f(\theta)]^2,$$

and thus, in polar coordinates, arc length is given by

$$L = \int_a^b \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\, d\theta.$$

Example 10.4.5

Compute the arc length of the curve $r = 2 - 2\cos\theta$ for $0 \leq \theta \leq 2\pi$.

We have that $r' = 2\sin\theta$, and thus the arc length integral is

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{[r]^2 + [r']^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4 - 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{8 - 8\cos\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{16\sin^2\left(\frac{\theta}{2}\right)}\, d\theta \\ &= \int_0^{2\pi} 4\sin\left(\frac{\theta}{2}\right)\, d\theta \\ &= \left[-8\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi} \end{aligned}$$

$$= 16.$$

10.4.3 A fun application of polar coordinates

Note: This portion formally relies on some techniques from Calculus III. I will attempt to give intuition for each of the steps and omit formal justification.

Example 10.4.6

Compute $\displaystyle\int_0^\infty e^{-x^2}\,dx$.

We saw before in Example **??** that it was very difficult to exactly evaluate the error function and we generally had to resort to approximations. However, if we want to evaluate $\operatorname{erf}(x)$ as $x \to \infty$, we can get an exact value with only some cleverness. First we'll suppose that

$$J = \int_0^\infty e^{-x}\,dx$$

is a real number. Since it is real, we can square it (and since x is just a dummy variable, there's no issue with using y for one of the integrals). What's more, since integrals are linear, we can move the constant I inside of the integral.

$$J^2 = J\int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-x^2} J\,dx = \int_0^\infty e^{-x^2}\left(\int_0^\infty e^{-y^2}\,dy\right)\,dx.$$

Since e^{-x^2} is not a function of y, it's effectively a constant when integrating with respect to y, so we can move it inside the integral

$$\begin{aligned} J^2 &= \int_0^\infty e^{-x^2}\left(\int_0^\infty e^{-y^2}\,dy\right)\,dx \\ &= \int_0^\infty \left(\int_0^\infty e^{-x^2}e^{-y^2}\,dy\right)\,dx \\ &= \int_0^\infty \left(\int_0^\infty e^{-(x^2+y^2)}\,dy\right)\,dx. \end{aligned}$$

Notice now that we're integrating over all pairs (x, y) where $0 \le x < \infty$ and $0 \le y < \infty$. This is exactly the first quadrant (QI) in the Cartesian plane. In polar coordinates, this plane is described by all pairs (r, θ) where $0 \le r < \infty$ and $0 \le \theta \le \frac{\pi}{2}$. Since $e^{-x^2-y^2} = e^{-r^2}$, it seems reasonable that we might want to change to polar coordinates.
The main technical issue is then figuring out how to replace dx and dy correctly so that we can work with dr and $d\theta$. As it turns out, $dx\,dy = r\,dr\,d\theta$, and the idea behind it is this: an infinitesimal rectangle in the plane with sides dx and dy has infinitesimal area $dA = dx \cdot dy$, and if you try to describe this same area in polar coordinates, you end up getting that $dA = r \cdot dr \cdot d\theta$.

Thus

$$\begin{aligned}
J^2 &= \int_0^\infty \left(\int_0^\infty e^{-(x^2+y^2)}\, dy \right)\, dx \\
&= \int_0^{\pi/2} \left(\int_0^\infty e^{-r^2} r\, dr \right)\, d\theta \\
&= \int_0^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \right]_0^\infty \, d\theta \\
&= \int_0^{\pi/2} \frac{1}{2}\, d\theta \\
&= \left[\frac{1}{2}\theta \right]_0^{\pi/2} \\
&= \frac{\pi}{4} \\
\Rightarrow I &= \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

11 Sequences, Series, and Power Series

11.1 Sequences

Definition

A **sequence** of real numbers is the image of a function $a : \mathbb{N} \to \mathbb{R}$, but can be thought of as an ordered list of real numbers

$$\{a_1, a_2, a_3, \ldots\},$$

We often denote the sequence as $\{a_n\}_{n=1}^{\infty}$ or just $\{a_n\}$ (some authors replace the curly braces with parentheses instead).

Frequently, you will see sequences written in one of three different ways: using the aforementioned notation, giving a defining function for a_n, or explicitly writing out the terms of the sequence. We also note that n does not necessarily have to start at 1.

Example 11.1.1

The following sequential descriptions are equivalent:

$$\left\{\frac{1}{n^2}\right\}_{n=3}^{\infty} \qquad a_n = \frac{1}{n^2},\, n \geq 3 \qquad \left\{\frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \ldots, \frac{1}{n^2}, \ldots\right\}.$$

For the most part, it's useful to find an explicit description for each n^{th} term, as in the first or second way of writing it.

Example 11.1.2

Given the sequence $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$, find an explicit description for a_n.

Notice that the numerator and denominator both increase by 1 each time, and that the denominator is always 1 more than the numerator. Thus, we can write

$$a_n = \frac{n}{n+1},\, n \geq 1 \quad \text{or} \quad \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$$

Example 11.1.3

Given the sequence $\left\{\frac{-1}{2}, \frac{2}{3}, \frac{-3}{4}, \frac{4}{5}, \ldots\right\}$, find an explicit description for a_n.

Notice this is the same sequence as in Example 11.1.2, except when n is odd, we have that the term is negative, and when n is even, we have that the term is positive. Thus, we can use the fact that $(-1)^n$ is negative for n is odd and positive for when n is even, which gives us

$$a_n = \frac{(-1)^n n}{n+1},\, n \geq 1 \quad \text{or} \quad \left\{\frac{(-1)^n n}{n+1}\right\}_{n=1}^{\infty}.$$

Since sequences are just discrete functions, it may be useful to see what sorts of things we can do with them. Much like we've seen before, we can discuss limits of sequences.

Definition

A sequence $\{a_n\}$ is **convergent** if there exists a real number L so that

$$\lim_{n\to\infty} a_n = L.$$

If a sequence is not convergent, then we say that the sequence is **divergent**.

The following theorem tells us that we can deal with limits of sequences by using many of our previous techniques for functions on the real numbers:

Theorem 11.1.4

If $f : \mathbb{R} \to \mathbb{R}$ with $f(n) = a_n$ (where n is an integer) and $\lim_{x\to\infty} f(x) = L$, then

$$\lim_{n\to\infty} a_n = L.$$

As a result of this limit correspondence, we have the following statements about convergent sequences:

Proposition 11.1.5

Let $\{a_n\}$, $\{b_n\}$ be convergent sequences and c a constant. Then,

1. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty} c \cdot a_n = c \cdot \left(\lim_{n\to\infty} a_n\right)$
3. $\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\pm \lim_{n\to\infty} b_n\right)$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$.
5. $\lim_{n\to\infty} (a_n)^p = \left(\lim_{n\to\infty} a_n\right)^p$ if $p > 0$ and $a_n > 0$.

With these rules, we can prove the following useful fact

Proposition 11.1.6

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

The Squeeze Theorem can also be adapted for sequences

Theorem 11.1.7: Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Example 11.1.8

Find the limit of the sequence, if it converges. $\left\{\dfrac{37n+16}{9n-42}\right\}_{n=5}^{\infty}$.

We can appeal to the limit laws in Proposition **??**:

$$\begin{aligned}\lim_{n\to\infty}\frac{37n+16}{9n-42} &= \lim_{n\to\infty}\frac{n\left(37+\frac{16}{n}\right)}{n\left(9-\frac{42}{n}\right)}\\ &= \lim_{n\to\infty}\frac{\left(37+\frac{16}{n}\right)}{\left(9-\frac{42}{n}\right)}\\ &= \frac{\lim_{n\to\infty}\left(37+\frac{16}{n}\right)}{\lim_{n\to\infty}\left(9-\frac{42}{n}\right)}\\ &= \frac{\lim_{n\to\infty}37+\lim_{n\to\infty}\frac{16}{n}}{\lim_{n\to\infty}9-\lim_{n\to\infty}\frac{42}{n}}\\ &= \frac{37}{9}.\end{aligned}$$

Example 11.1.9

Find the limit of the sequence, if it converges. $\left\{\dfrac{(-1)^n}{n^2}\right\}_{n=1}^{\infty}$.

Since $\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$ and $\displaystyle\lim_{n\to\infty}\frac{1}{n^2}=0$, then the given sequence also converges to 0.

Example 11.1.10

Find the limit of the sequence, if it converges. $\{2\arctan(n)\}_{n=0}^{\infty}$.

Since $f(x)=2\arctan(x)$ agrees with a_n for each n, we have that

$$\lim_{n\to\infty}2\arctan(n)=\lim_{n\to\infty}2\arctan(x)=\pi.$$

The sequences we've seen so far are defined explicitly. Sometimes, however, we may see sequences defined recursively.

Example 11.1.11: Fibonacci Sequence

The well-known **Fibonacci sequence** $\{F_n\}$ is defined recursively by

$$F_0=0,\quad F_1=1,\quad F_n=F_{n-1}+F_{n-2}\text{ for }n\geq 3.$$

Write the first 12 terms of the sequence This gives us

$$\{0,1,1,2,3,5,8,13,21,34,55,89,\ldots\}.$$

Exercise 11.1.12: Challenge Exercise

Given the Fibonacci sequence in Example 11.1.11, evaluate the limit

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}}.$$

Example 11.1.13

Find an explicit formula for each term in the given recursive sequence:

$$a_0 = 1, \quad a_n = 2a_n \text{ for } n \geq 1.$$

Writing out the sequence, we have

$$\{1, 2, 4, 8, 16, 32, 64, 128, \ldots\},$$

which we recognize as the sequence where $a_n = 2^n$.

11.2 Series

Definition

The sum of all of the terms in a sequence $\{a_n\}_{n=1}^{\infty}$ is called a **series**, and is denoted $\displaystyle\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

How do we make sense of talking about the value of an infinite series? As usual, we'll use limits.

Definition

The **partial sums** for the series $\displaystyle\sum_{n=1}^{\infty} a_n$ are

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_k &= \sum_{n=1}^{k} a_n. \end{aligned}$$

For lack of better terminology on the author's part, we'll call s_k the $\boldsymbol{k}^{\textbf{th}}$ **partial sum**, as it is the sum of all terms up to (and including) the k^{th} term.

Note: if the series starts at some number $i \neq 1$ then we still take

$$s_k = a_i + \cdots + a_{k-1} + a_{k-1} + a_k = \sum_{n=i}^{k} a_n$$

for the purposes of notational simplicity. The common definition in the literature is to define the k^{th} partial sum as the sum of the first k terms. We will write "the sum of the first k terms" whenever necessary to avoid ambiguity.

Definition

Let s_k denote the k^{th} partial sum of this series $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\{s_k\}_{k=1}^{\infty}$ the corresponding sequence of partial sums. We say that this series is **convergent** if there exists a real number s so that

$$\lim_{k\to\infty} s_k = s.$$

If this series converges, we write

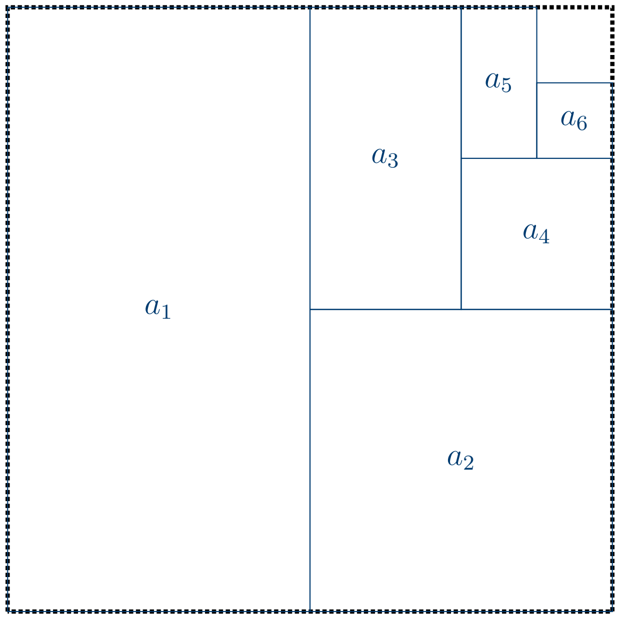
$$\sum_{n=1}^{\infty} a_n = s.$$

We call s the **sum of the series**. If the sequence of partial sums $\{s_k\}$ does not converge, we say that the series is **divergent**.

Example 11.2.1: Zeno's Paradox

Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$.

Let $a_n = \frac{1}{2^n}$. First we draw the following picture to illustrate the idea. Suppose the outer square has area 1:



Intuitively, we expect that this series should evaluate to 1. Now let's explore the sequence of partial sums.

$$
\begin{aligned}
s_1 &= \frac{1}{2} \\
s_2 = s_1 + \frac{1}{4} &= \frac{3}{4} \\
s_3 = s_2 + \frac{1}{8} &= \frac{7}{8} \\
\vdots s_k = s_{k-1} + \frac{1}{2^k} &= \frac{2^k - 1}{2^k}.
\end{aligned}
$$

So

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{2^k - 1}{2^k} = \lim_{k\to\infty} \frac{2^k}{2^k} - \frac{1}{2^k} = 1 + 0 = 1.$$

Definition: Geometric Series

A **geometric series** is a series of the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n,$$

where $a \neq 0$ is some constant and r is called the **common ratio**. The variable a doesn't have a name, but it is useful to think of it as the first term in a geometric series.

For a geometric series, we can obtain a nifty formula for the partial sum s_k:

$$\begin{aligned} s_k &= a + ar + ar^2 + \cdots + ar^k \\ rs_k &= ar + ar^2 + ar^3 + \cdots + ar^{k+1} \end{aligned}$$

Then

$$\begin{aligned} s_k - rs_k &= a - ar^{k+1} \\ s_k(1-r) &= a(1-r^{k+1}) \\ \Rightarrow s_k &= \sum_{n=0}^{k} ar^n = \frac{a(1-r^{k+1})}{1-r}. \end{aligned}$$

If $|r| < 1$, then $\lim\limits_{k\to\infty} |r|^k = 0$, so $\lim_{k\to\infty} r = 0$ by Proposition **??**, and it is easy to see that, for the partial sums above,

$$\lim_{k\to\infty} s_k = \frac{a}{1-r}.$$

We can also see that the series diverges when $|r| > 1$, and the case when $|r| = 1$ is handled in your book. These combined lead us to the following result

Theorem 11.2.2: Geometric Series Test

If $|r| < 1$, the geometric series $\displaystyle\sum_{n=0}^{\infty} ar^n$ converges, and $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

If $|r| \geq 1$, the geometric series $\displaystyle\sum_{n=0}^{\infty} ar^n$ diverges.

Example 11.2.3

Find the value of the geometric series where $a = 5$ and $r = \frac{1}{2}$, if it converges. If the series diverges, clearly state that it diverges.

The series converges since $|r| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1$. Thus, by the previous theorem, we have

$$\sum_{n=0}^{\infty} 5\left(\frac{1}{2}\right)^n = \frac{5}{1-\frac{1}{2}} = 10.$$

Example 11.2.4

Find the value of the geometric series $\sum_{k=1}^{\infty} \frac{\pi^{k+1}}{e^k}$, if it converges.

First we need to find the common ratio. Notice that we can rewrite the series slightly as

$$\sum_{k=1}^{\infty} \frac{\pi \cdot \pi^k}{e^k} = \sum_{k=1}^{\infty} \pi \left(\frac{\pi}{e}\right)^k.$$

In this form, we have that $r = \frac{\pi}{e}$. Since $|r| > 1$, then by the Geometric Series Test 11.2.2 the series diverges.

Example 11.2.5

Find the value of the geometric series $\sum_{n=5}^{\infty} 10\left(\frac{1}{2}\right)^n$, if it converges.

We see that $r = \frac{1}{2}$, and so by the Geometric Series Test 11.2.2, this series converges. Although we'd like to use the formula in the geometric series test to determine the value, we have to be careful because that series starts at $n = 0$ and ours starts at $n = 5$. Recalling that a was the first term in our series, this means that our first term is $a = 10\left(\frac{1}{2}\right)^5 = \frac{5}{16}$, and so we have that

$$\sum_{n=5}^{\infty} 10\left(\frac{1}{2}\right)^n = \frac{\frac{5}{16}}{1-\frac{1}{2}} = \frac{\frac{5}{16}}{\frac{1}{2}} = \frac{5}{8}.$$

An alternative method to solving this problem is to do what's called an "index shift" to make the series start at 0. Begin by letting $m = n - 5$. Then when $n = 5$, $m = 0$, and as $n \to \infty$, $m \to \infty$ as well. So we get

$$\sum_{n=5}^{\infty} 10\left(\frac{1}{2}\right)^n = \sum_{m=0}^{\infty} 10\left(\frac{1}{2}\right)^{m+5} = \sum_{m=0}^{\infty} 10\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^m = \sum_{m=0}^{\infty} \frac{5}{16}\left(\frac{1}{2}\right)^m,$$

and this means that $a = \frac{5}{16}$ again, so we get the same sum with this index shifting approach as well.

One reason we like convergent series so much is because the following result

Proposition 11.2.6: Algebra of Series

If $\sum a_n$, $\sum b_n$ are convergent series and c is a constant, then $\sum ca_n$ and $\sum(a_n \pm b_n)$ are convergent series and

1. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$,
2. $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$.

Example 11.2.7

Find the sum of the following series, if it converges. $\displaystyle\sum_{m=1}^{\infty} \frac{6+6^m}{8^m}$.

We notice that

$$\sum_{m=1}^{\infty} \frac{6}{8^m} = \frac{\frac{6}{8}}{1-\frac{1}{8}} = \frac{6}{7}$$

and

$$\sum_{m=1}^{\infty} \frac{6^m}{8^m} = \frac{\frac{6}{8}}{1-\frac{6}{8}} = 3,$$

by since both converge, by Proposition 11.2.6, we have

$$\sum_{m=1}^{\infty} \frac{6+6^m}{8^m} = \sum_{m=1}^{\infty} \frac{6}{8^m} + \sum_{m=1}^{\infty} \frac{6^m}{8^m} = \frac{6}{7} + 3 = \frac{27}{7}.$$

Example 11.2.8: Telescoping

Find the sum of the following series, if it converges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$

This sequence is not obviously a geometric series, so we'll have to approach by a sequence of partial sums. Notice, however, that we can apply partial fractions to the summand:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} &= \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\end{aligned}$$

And so, each k^{th} partial sum can be written

$$s_k = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k+1} - \frac{1}{k+2}\right) = \frac{1}{2} - \frac{1}{k+2},$$

and so

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+1} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{1}{2} - \frac{1}{k+2} = \frac{1}{2} + 0 = \frac{1}{2}.$$

Up to this point, we've been able to explicitly calculate the sum for these various series. In general, the best we can hope to do is to show that the series converges at all (at which point we can use a computer to approximate the value numerically). Taking our motivation from convergent improper integrals, we can quickly conclude the following:

Theorem 11.2.9: Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Suppose $\lim_{n\to\infty} a_n = L \neq 0$; without loss of generality, we suppose that L is positive. By definition of a limit, there must be infinitely many terms (say, for all $n > N$, where N is some fixed number) for which $0 < \frac{L}{2} < a_n < \frac{3L}{2}$. Since

$$\sum_{n=N}^{\infty} \frac{L}{2} < \sum_{n=N}^{\infty} a_n < \sum_{n=N}^{\infty} \frac{3L}{2}$$

and these outer two series diverge, then so does the series $\sum_{n=N}^{\infty} a_n$. □

Note that this theorem only tests for *divergence*, not necessarily *convergence*. It's entirely possible that there's a sequence $\{a_n\}$ out there for which $a_n \to 0$ but the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 11.2.10

Determine whether or not the following series converges. $\displaystyle\sum_{n=0}^{\infty} \frac{7^n}{3^{(n+41)}}$

Although we recognize that this is a geometric series, we can also see that

$$\lim_{n\to\infty} \frac{7^n}{3^{(n+41)}} = \infty,$$

and so by the previous theorem (Theorem 11.2.9), the whole series diverges.

Example 11.2.11: Harmonic Series

Determine whether or not the following series converges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}.$

As a heuristic, we expect this to behave similarly to $\int_1^{\infty} \frac{1}{x}\,dx$, and thus suspect it might diverge. To see that we are correct, we need to consider the limit of partial sums. But in fact, we can restrict our attention to only some of the partial sums: s_2, s_4, s_8, s_{16}, etc.

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2} \\
s_8 &= \cdots > 1 + \frac{3}{2} \\
s_{16} &= \cdots > 1 + \frac{4}{2} \\
&\vdots
\end{aligned}$$

$$s_{2^n} = \cdots > 1 + \frac{n}{2}$$

So, as $n \to \infty$, we see also that $s_{2^n} \to \infty$, and thus the series above (called the **harmonic series**) diverges.

Remark. The harmonic series demonstrates that the divergence test is not strong enough to test for convergence.

11.3 The Integral Test and Estimate of Sums

Suppose we can think about our series as a Riemann sum of some continuous function f. If f is decreasing and positive, then a left Riemann sum will always provide an overestimate, and a right Riemann sum will always provide an underestimate. Interpreting our series as a right Riemann sum, we have that

$$0 \leq \sum_{n=k}^{\infty} a_n \leq \int_k^{\infty} f(x)\, dx$$

and if the integral converges, then it seems reasonable that the series must as well (since the a_n's are all positive, there's no possible oscillatory behavior in the sequence of partial sums).

Remark. We could just as easily consider left Riemann sum, as the total value of the series would differ by exactly one rectangle's worth of area, as the left- and right Riemann sums are just translated versions of one another. However, the right Riemann sum has the nicer formula: if $a_n = f(n)$, then

$$\int_k^{\infty} f(x)\, dx \approx \sum_{n=k}^{\infty} f(n) = \sum_{n=k}^{\infty} a_n.$$

Theorem 11.3.1: Integral Test

Suppose f is a continuous, positive, decreasing function on $[k, \infty)$ and let $a_n = f(n)$ for all $n \geq k$.

1. If $\displaystyle\int_k^{\infty} f(x)\, dx$ converges, then $\displaystyle\sum_{n=k}^{\infty} a_n$ converges.
2. If $\displaystyle\int_k^{\infty} f(x)\, dx$ diverges, then $\displaystyle\sum_{n=k}^{\infty} a_n$ diverges.

Example 11.3.2

Determine if the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is convergent or divergent.

Use the function $f(x) = \frac{1}{x \ln x}$. For x in the interval $[2, \infty)$, this function is positive. As well, as x increases, the denominator increases, and so the output decreases. So, we determine convergence of the following integral:

$$\begin{aligned}\int_2^\infty \frac{1}{x\ln x}\,dx &= \lim_{t\to\infty}\int_2^t \frac{1}{x\ln x}\,dx \\ &= \lim_{t\to\infty}\left[\ln(\ln x)\right]_2^t && \text{(substitution of } u=\ln x\text{)} \\ &= \lim_{t\to\infty}\ln(\ln t)-\ln(\ln 2)) \\ &= \infty.\end{aligned}$$

So since the integral diverges, the series diverges.

Example 11.3.3

Determine if the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n-1}}$ is convergent or divergent.

(Integral Test)

p-Series: Let $k = 2n-1$. As $n\to\infty$, then so does $k\to\infty$. Also, when $n=1, k=1$. So we have

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n-1}} = \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$$

and this is now clearly a p-series with $p=\frac{1}{2}$, which is divergent by the

Now that we have this integral test, combining it with the results of Example **??** gives us

Theorem 11.3.4: p-Series Test

The **p-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

Example 11.3.5

Determine if the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ converges or diverges.

We have that this series is a p-series with $p = \frac{1}{2}$. So by the p-series test, this series diverges.

Example 11.3.6

Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$ is convergent or divergent.

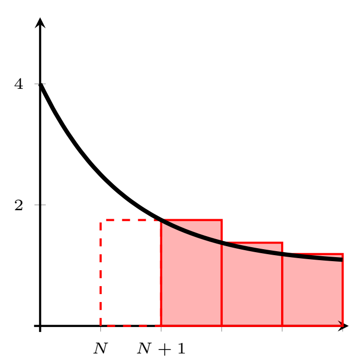
We already know that this series diverges, but now we show that we can apply a p-series test to it.
Let $k = 2n - 1$. As $n \to \infty$, then so does $k \to \infty$. Also, when $n = 1, k = 1$. So we have

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

and this is now clearly a p-series with $p = \frac{1}{2}$, which is divergent by the p-series test 11.3.4.

11.3.1 Integral Test - Remainder Estimates

Suppose $\sum_{n=1}^{\infty} a_n$ converges by the integral test and we used a computer to add up the first $N = 10000$ terms of this series. How accurate is our numerical value?



Notice from the picture above that

$$\begin{array}{ccccc}
\displaystyle\int_{N+1}^{\infty} f(x)\,dx & < & \displaystyle\sum_{n=N+1}^{\infty} a_n & < & \displaystyle\int_{N}^{\infty} f(x)\,dx \\
\displaystyle\sum_{n=0}^{N} a_n + \int_{N+1}^{\infty} f(x)\,dx & < & \displaystyle\sum_{n=0}^{N} a_n + \sum_{n=N+1}^{\infty} a_n & < & \displaystyle\sum_{n=0}^{N} a_n + \int_{N}^{\infty} f(x)\,dx \\
\displaystyle\sum_{n=0}^{N} a_n + \int_{N+1}^{\infty} f(x)\,dx & < & \displaystyle\sum_{n=0}^{\infty} a_n & < & \displaystyle\sum_{n=0}^{N} a_n + \int_{N}^{\infty} f(x)\,dx
\end{array}$$

So these integrals provide a bound as to how far off our estimate is, i.e., the **error** of our approximation.

Example 11.3.7

Compute $\displaystyle\sum_{n=1}^{10} \frac{1}{n^3}$ and estimate the error.

$$\sum_{n=1}^{10} \frac{1}{n^3} = \frac{19,164,113,947}{16,003,008,000} \approx 1.19753$$

$$\int_{11}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{242} \approx 0.00413$$
$$\int_{10}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{200} \approx 0.005$$

So we get that

$$1.19253 = \sum_{n=1}^{\infty} \frac{1}{n^3} < 1.20253$$

Remark. For any number s, the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^s}$, denoted $\zeta(s)$, is a very well-studied series called the **Riemann Zeta Function**. The example above features $\zeta(3)$, known as Apéry's constant, which has a value of 1.20205690315959...

11.4 The Comparison Tests

11.4.1 Direct Comparison Test

The following result is also an adaptation of the comparison test for integrals Integral Comparison Tests.

Theorem 11.4.1: (Direct) Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
2. If $\sum b_n$ is divergent and $b_n \leq a_n$ for all n, then $\sum a_n$ is also divergent.

Remark. It's worth noting that the positivity condition above just has to be true for sufficiently large n values. If the first 30 terms in both series are negative, and terms 31 and on are all positive, then one can apply the comparison test to $\displaystyle\sum_{n=31}^{\infty} a_n$ and $\displaystyle\sum_{n=31}^{\infty} b_n$. (Since the first 30 terms only contribute a finite amount to the value of the series, they don't affect divergence.)

Proof. Let $\{A_k\}$ and $\{B_k\}$ be the sequence of partial sums for $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$, respectively. Since each a_i and b_i are positive, then both of the sequences of partial sums are (1) positive and (2) increasing.

If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then this limit provides a upper bound for the sequence $\{A_k\}$, and by the Monotone Convergence Theorem, $\{A_k\}$ converges.

If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\lim_{k\to\infty} A_k \to \infty$ diverges, hence $\{B_k\}$ diverges as well. $\square$

When using the (Direct) Comparison Test, it's good to keep in mind some relatively simple series which have easy convergence/divergence conditions, namely:

- geometric series
- p-series

Example 11.4.2

Use the (Direct) Comparison Test to determine whether or not the following series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2n}$

We have that, for all n,

$$\frac{3}{2n} = \frac{3}{2} \cdot \frac{1}{n} > \frac{1}{n}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then by statement 2 in the comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2n}$ diverges also.

Example 11.4.3

Use the (Direct) Comparison Test to determine whether or not the following series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n^2}$

We have that, for all n,

$$\frac{2}{3n^2} = \frac{2}{3} \cdot \frac{1}{n^2} < \frac{1}{n^2}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by statement 1 in the comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n^2}$ converges also.

Example 11.4.4

Determine whether or not the following series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n + 1}$

We compare this series with the convergent series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$.

Example 11.4.5

Determine whether or not the following series converges or diverges. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

We compare this series with the divergent series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$.

11.4.2 Limit Comparison Test

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converge and diverge, respectively, indicate that convergence is very intricately related to the growth rate of a function - just "how fast" do the terms tend to 0? This next result says that if two series have the same growth rate, then they share convergence/divergence.

Theorem 11.4.6: Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where $0 < c < \infty$, then either both series converge or both series diverge.

Proof. For sufficiently large n, we have that $\frac{a_n}{b_n}$ is very close to c, say with a tolerance $\pm\varepsilon$ (for some really small number ε):

$$c - \varepsilon < \frac{a_n}{b_n} < c + \varepsilon$$

This rearranges to

$$(c - \varepsilon)b_n < a_n < (c + \varepsilon)b_n$$

If $\sum b_n$ converges, then so does $\sum (c + \varepsilon)b_n$, so by the comparison test, a_n converges too.
If $\sum b_n$ diverges, then so does $\sum (c - \varepsilon)b_n$, so by the comparison test, a_n diverges too. □

Example 11.4.7

Determine whether or not the following series converges or diverges. $\sum_{n=4}^{\infty} \frac{3}{2n-7}$

We do the limit comparison with the divergent series $\sum_{n=4}^{\infty} \frac{1}{n}$.

Example 11.4.8

Determine whether or not the following series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

We do the limit comparison with the convergent series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$.

Example 11.4.9

Determine whether or not the following series converges or diverges. $\sum_{n=1}^{\infty} \frac{n+1}{n^3 + n}$

We do the limit comparison with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

11.4.3 Estimating Sums

In practice, we can only compute a finite number of terms, so it's good to know how far off our finite sum is from the value of the entire series.

Suppose $\sum a_n$ and $\sum b_n$ converge by the **??**. Since, for sufficiently large N, we have that

$$0 \leq \sum_{n=N+1}^{\infty} a_n < \sum_{n=N+1}^{\infty} b_n$$

then by computing the first N terms, we get that

$$\sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n < \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} b_n$$

and with any luck we can say something about the value of the right-most series. In particular, if $\sum b_n$ is geometric, we can compute it exactly, or if it is a p-series, we can use the **??** to estimate its value.

Example 11.4.10

Compute the first 20 terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ and estimate the error involved.

11.5 Alternating Series

Definition: Alternating Series

An **alternating series** is a series of the form

$$\sum_{n=k}^{\infty}(-1)^n b_n \quad \text{or} \quad \sum_{n=k}^{\infty}(-1)^{n+1} b_n,$$

where b_n is a nonnegative number.

Example 11.5.1

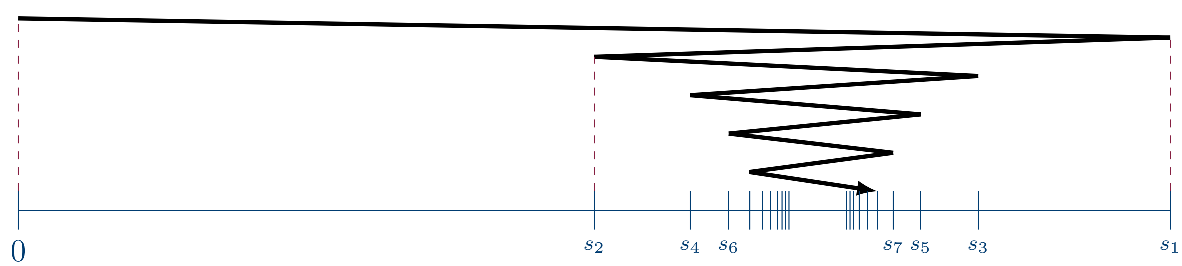
Summing the terms of the sequence in Example **??**, we have an alternating series.

Example 11.5.2: Alternating Harmonic Series

he **alternating harmonic series** is given by

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots .$$

Give a non-rigorous justification as to why this series converges.



The figure above shows what is happening with the sequence of partial sums in the alternating harmonic series. The sequence $\{s_{\text{odd}}\}$ is decreasing, the sequence $\{s_{\text{even}}\}$ is increasing, and $s_{\text{odd}} > s_{\text{even}}$. So by the monotone convergence theorem, both sequences must converge, and they appear to be converging to the same limit.

Exercise 11.5.3

Show that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = \ln(2)$.

Theorem 11.5.4: Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^n b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n+1} b_n$$

satisfies both

1. $b_n \geq b_{n+1} \geq 0$ for all n (i.e. the sequence $\{b_n\}$ is decreasing), and
2. $\lim_{n\to\infty} b_n = 0$,

then the series is convergent.

Example 11.5.5

Test the alternating harmonic series for convergence or divergence.

We have

$$b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1},$$

so we satisfy the first condition of the alternating series test. Also,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0,$$

so we satisfy the second condition of the alternating series test. Therefore the alternating harmonic series converges.

Example 11.5.6

Does the following alternating series converge? $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3n}{4n-1}$.

Let $b_n = \dfrac{3n}{4n-1}$. Although b_n is decreasing, we have that $\lim_{n\to\infty} b_n = \dfrac{3}{4} \neq 0$. This means that the **??** does not apply, but by the **??**, this series does not converge.

11.5.1 Estimation of Alternating Sums

11.5.2 Absolute/Conditional Convergence

Because of alternating series, we can talk about varying strengths of convergence of a series.

Definition

A series $\sum a_n$ is said to be **absolutely convergent** if $\sum |a_n|$ converges, and is said to be **conditionally convergent** if it converges, but not absolutely.

Example 11.5.7

Does the alternating harmonic series converge? And if so, conditionally or absolutely?

We've seen that the alternating harmonic series converges. However, since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n},$$

we have that the alternating harmonic series *does not* converge absolutely. Therefore it converges conditionally.

Proposition 11.5.8

If a series converges absolutely, then it converges.

Proof. Suppose $\sum a_n$ converges absolutely. Then, by definition, $\sum |a_n|$ converges. So since $0 \leq a_n + |a_n| \leq 2|a_n|$, by the **??**, $\sum (a_n + |a_n|)$ also converges. So since $a_n = a_n + |a_n| - |a_n|$, we have that

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

which converges as its a sum/difference of two convergent series. $\square$

Example 11.5.9

Does the following series converge? $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$.

Although cosine oscillates, this series is not actually an alternating series, so we cannot apply the **??**. One thing we can do, however, is check to see if a series converges absolutely. Indeed, $|\cos(n)| \leq 1$ for all n, so we apply the **??** to a convergent p-series

$$\sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and therefore our series converges absolutely (hence is a convergent series).

11.5.3 Rearrangement

You may be wondering why the word "conditional" appears when describing types of convergence – what exactly is the condition being referenced? That condition is the *order of the terms*. In other words, for a given series, the order of the terms can dramatically affect the value of the series.

Example 11.5.10: Rearranging the Alternating Harmonic Series

Let

$$L = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

and observe by the beginning of this section that $L \neq 0$ (in fact, $L = \ln(2)$).

Let's rearrange the terms slightly in the alternating harmonic series:

$$\begin{aligned}
&\underbrace{1 - \frac{1}{2}}_{\frac{1}{2}} - \frac{1}{4} + \underbrace{\frac{1}{3} - \frac{1}{6}}_{\frac{1}{6}} - \frac{1}{8} + \underbrace{\frac{1}{5} - \frac{1}{10}}_{\frac{1}{10}} - \frac{1}{12} + \cdots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right) = \frac{L}{2}
\end{aligned}$$

11.6 The Ratio and Root Tests

Both the Ratio Test and Root Test are based on the following observations for geometric series: If $a_n = ar^n$ (for some constant a and ratio r), then

$$\frac{a_{n+1}}{a_n} = r \qquad \text{and} \qquad \sqrt[n]{ar^n} = \sqrt[n]{a}r$$

so since convergence of geometric series is entirely based on the ratio, one hopes that these two expressions (or the limits, anyway) can dictate convergence of any series.

11.6.1 Ratio Test

Theorem 11.6.1: Ratio Test

1. If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum a_n$ is absolutely convergent (hence, convergent).
2. If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series $\sum a_n$ is divergent.
3. If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ or the limit does not exist, then the test is inconclusive about the convergence or divergence of the series $\sum a_n$.

The ratio test nearly always the go-to when there are exponents involving n and/or factorials.

Remark. For purposes of simplifying fractions, it may behoove you to remember that

$$(n+1)! = (n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = (n+1)\cdot n!.$$

Example 11.6.2

Determine if the following series converges or diverges. $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{2^n}$

We have that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)!}{2^{n+1}}\cdot\frac{2^n}{n!}\right| \\
&= \lim_{n\to\infty}\left|\frac{n+1}{2}\right| > 1.
\end{aligned}$$

So, by the Ratio Test, our series diverges.

Example 11.6.3

Determine if the following series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{(\ln 3)^k}$

We have that

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{\frac{(k+1)^3}{(\ln 3)^{k+1}}}{\frac{k^3}{(\ln 3)^k}}\right| \\ &= \lim_{k\to\infty}\left|\frac{(k+1)^3}{(\ln 3)^{k+1}}\cdot\frac{(\ln 3)^k}{k^3}\right| \\ &= \lim_{k\to\infty}\left|\frac{(k+1)^3}{(\ln 3)k^3}\right| = \frac{1}{\ln 3} < 1.\end{aligned}$$

So, by the Ratio Test, our series converges absolutely (hence the series converges).

Example 11.6.4

Determine if the following series converges or diverges. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2 2^n}{5^n}$

We have that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(n+1)^2 2^{n+1}}{5^{n+1}}}{\frac{n^2 2^n}{5^n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)^2 2^{n+1}}{5^{n+1}}\cdot\frac{5^n}{n^2 2^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{2(n+1)^2}{5n^2}\right| = \frac{2}{5} < 1.\end{aligned}$$

So, by the Ratio Test, our series converges absolutely (hence the series converges).

Example 11.6.5

Determine if the following series converges or diverges. $\displaystyle\sum_{p=1}^{\infty} \frac{(2p)!}{(p!)^2}$

We have that

$$\begin{aligned}
\lim_{p\to\infty}\left|\frac{a_{p+1}}{a_p}\right| &= \lim_{p\to\infty}\left|\frac{\frac{(2p+2)!}{[(p+1)!]^2}}{\frac{(2p)!}{(p!)^2}}\right| \\
&= \lim_{p\to\infty}\left|\frac{(2p+2)!}{[(p+1)!]^2}\cdot\frac{(p!)^2}{(2p)!}\right| \\
&= \lim_{p\to\infty}\left|\frac{(2p+2)(2p+1)}{(p+1)^2}\right| = 4 > 1.
\end{aligned}$$

So, by the Ratio Test, our series diverges.

Example 11.6.6

Determine if the following series converges or diverges. $\displaystyle\sum_{p=1}^{\infty} \frac{(2p)!}{(p!)^2}$

We have that

$$\begin{aligned}
\lim_{p\to\infty}\left|\frac{a_{p+1}}{a_p}\right| &= \lim_{p\to\infty}\left|\frac{\frac{(2p+2)!}{[(p+1)!]^2}}{\frac{(2p)!}{(p!)^2}}\right| \\
&= \lim_{p\to\infty}\left|\frac{(2p+2)!}{[(p+1)!]^2}\cdot\frac{(p!)^2}{(2p)!}\right| \\
&= \lim_{p\to\infty}\left|\frac{(2p+2)(2p+1)}{(p+1)^2}\right| = 4 > 1.
\end{aligned}$$

So, by the Ratio Test, our series diverges.

Example 11.6.7

Determine if the following series converges or diverges. $\displaystyle\sum_{k=1}^{\infty}(-1)^n \frac{n+1}{n^3-1}$

We have that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{n+2}{(n+1)^3-1}}{(-1)^n\frac{n+1}{n^3-1}}\right| \\
&= \lim_{n\to\infty}\left|(-1)\frac{n+2}{(n+1)^3-1}\cdot\frac{n^3-1}{n+1}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+2)(n^3-1)}{((n+1)^3-1)(n+1)}\right| = 1.
\end{aligned}$$

The Ratio Test is inconclusive in this case; however, we see that the series converges by the Alternating Series Test. In fact, using the Limit Comparison Test with $\frac{1}{n^2}$, we get that this series converges absolutely.

11.6.2 Root Test

Theorem 11.6.8: Root Test

1. If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} < 1$, then the series $\sum a_n$ is absolutely convergent (hence, convergent).
2. If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} > 1$, then the series $\sum a_n$ is divergent.
3. If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = 1$ or the limit does not exist, then the test is inconclusive about the convergence or divergence of the series $\sum a_n$.

Proof. If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then there is some number r so that $L < r < 1$ and for all sufficiently large n, $\sqrt[n]{|a_n|} < r$. This is equivalent to saying that $|a_n| < r^n$, so $\sum|a_n| < \sum r^n$ and since $\sum r^n$ is a convergent geometric series, then by the (Direct) Comparison Test, $\sum|a_n|$ converges, hence $\sum a_n$ converges absolutely. $\square$

Strictly-speaking, the Root Test is stronger than the Ratio Test in the following sense:

If $\sum a_n$ is shown to converge/diverge by the Ratio Test, then it must also be convergent/divergent by the Root Test.

The converse is not true, as the next example shows.

Example 11.6.9

Determine whether or not the following series converges. $\displaystyle\sum_{n=0}^{\infty}3^{-n-(-1)^n}$

Notice that

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{-(n+1)+(-1)^{n+1}}}{3^{-n+(-1)^n}} &= 3^{-(n+1)+(-1)^{n+1}+n-(-1)^n} \\
&= 3^{-1+(-1)^{n+1}-(-1)^n} \\
&= 3^{-1+(-1)^{n+1}+(-1)^{n+1}} \\
&= 3^{-1+2(-1)^{n+1}} \\
&= \begin{cases} 3^1 & \text{when } n \text{ is even} \\ 3^{-3} & \text{when } n \text{ is odd} \end{cases}
\end{aligned}$$

So we have that

$$\lim_{\text{even } n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3$$

and

$$\lim_{\text{odd } n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3^{-3}$$

and therefore the limit as $n \to \infty$ does not exist, so the Ratio Test does not apply. However, we can apply the Root Test:

$$\lim_{n\to\infty} |a|^{1/n} = \lim_{n\to\infty} 3^{-1+\frac{1}{n}(-1)^n} = 3^{-1+0} = \frac{1}{3} < 1$$

and thusly conclude that the series is convergent.

11.7 Strategy for Testing Series

For each of the following series, determine if it is absolutely convergent, conditionally convergent, or divergent. It is useful to ask the following questions

(A) What does the Divergence Test say?
(B) Is this a p-series? (p-Series Test)
(C) Is this a geometric series? (Geometric Series Test)
(D) Is it a sum/difference of p-series or geometric series?
(E) Can it be compared to a p-series or geometric series? (Direct/Limit Comparison Test)
(F) Is this an alternating series? (Alternating Series Test)
(G) Does this involve factorials? (Ratio Test)
(H) Is a_n of the form $(b_n)^n$? (Root Test)
(I) Can we apply the Integral Test?

Note that the above ordering only provides a loose suggestion as to the ordering that one might apply the tests. It is arguably much less straightforward to apply the comparison test to a generic series (because it requires cleverness on your part to find a convergent/divergent series for comparison) than just about any other test listed above.

1. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$

3. $\displaystyle\sum_{n=1}^{\infty} n^{-(n+1/n)}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^4}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{3^n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+5n+4}$

10. $\displaystyle\sum_{n=1}^{\infty} \cos(\pi n)e^{-n}$

11.8 Power Series

Definition: Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are the **coefficients** of the series. More generally, a **power series centered at a** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

We note that a power series should always start at $n = 0$. If the series is written to start at $n = k > 0$, then it is assumed that $c_0 = \cdots = c_{k-1} = 0$.

A power series can be thought of as an infinite polynomial. The nice thing about polynomials is that, no matter what x-value we use, the polynomial's value at x is always a finite number. However, this isn't necessarily true for power series - using the wrong x-value could make the series diverge! For this reason, we need to discuss x-values where the series converge/diverge.

Example 11.8.1

For which values of x does the series converge? $\sum_{n=0}^{\infty} x^n$

We see that this looks just like a geometric series with first term $a = 1$ and ratio x. By the geometric series test, this series converges for all x where $|x| < 1$.

Example 11.8.2

For which values of x does the series converge? $\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n}$

We have that

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{(2x-5)^{n+1}}{n+1}}{\frac{(2x-5)^n}{n}} \right| &= \lim_{n\to\infty} \left| \frac{(2x-5)^{n+1}}{n+1} \cdot \frac{n}{(2x-5)^n} \right| \\
&= \lim_{n\to\infty} \left| (2x-5)\frac{n}{n+1} \right| \\
&= |2x-5|.
\end{aligned}$$

By the ratio test, this converges when $|2x-5| < 1$ (i.e., when $2 < x < 3$) and is inconclusive (may possibly converge) when $|2x-5| = 1$ (i.e., when $x = 2, x = 3$). When $x = 2$, our series is the alternating harmonic series, and thus the series converges. When $x = 3$, our series is the harmonic series, which diverges.

The series converges for $2 \leq x < 3$.

Example 11.8.3

For which values of x does the series converge? $\displaystyle\sum_{n=0}^{\infty}(n!)(x-1)^n$

Applying the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{(n+1)!(x-1)^{n+1}}{n!(x-1)^n}\right| = \lim_{n\to\infty}|x-1||n+1|$$

When $x \neq 1$, this limit is infinite, and when $x = 1$, the limit is 0. Thus the series converges precisely when $x = 1$.

Theorem 11.8.4

For given power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Definition

The number R in case 3 above is called the **radius of convergence** of the power series. The interval consisting of all values of x for which the power series converges is called the **interval of convergence**.

Remark. The interval of convergence is not just the open interval $(a-R, a+R)$, but may actually include the endpoints of this interval as well - these have to be tested separately. For example, Example 11.8.2 has interval of convergence $[2,3)$. If a power series converges for all real numbers, we may say it has radius of convergence $R = \infty$, and if a power series converges only at $x = a$, we may say that it has radius of convergence $R = 0$.

Making the connection to polynomials again, since polynomials are defined for every real input, the corresponding polynomial function $p(x)$ has domain $(-\infty, \infty)$. Since power series only produce a real number when they converge, then the corresponding power series function $P(x)$ has the interval of convergence as its domain.

Example 11.8.5

Find the radius and interval of the convergence for the following series: $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$

Notice this is a geometric series with $a = 1$ and $r = \frac{x}{2}$. By the geometric series test, this converges precisely when $\left|\frac{x}{2}\right| < 1$, i.e., when $|x| < 2$. So the radius of convergence is 2 and the interval of convergence is $(-2, 2)$.

Example 11.8.6

Find the interval and radius of convergence for the following series: $\displaystyle\sum_{n=1}^{\infty}\frac{(x-4)^n}{\sqrt[3]{n}}$

Using the ratio test, we have

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{\frac{(x-4)^{n+1}}{\sqrt[3]{n+1}}}{\frac{(x-4)^n}{\sqrt[3]{n+1}}}\right| &= \lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{\sqrt[3]{n+1}}\cdot\frac{\sqrt[3]{n}}{(x-4)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(x-4)\sqrt[3]{n}}{\sqrt[3]{n+1}}\right| \\
&= \lim_{n\to\infty}|x-4|\cdot\left|\frac{\sqrt[3]{n}}{\sqrt[3]{n+1}}\right| \\
&= |x-4|
\end{aligned}$$

converges when this limit is less than 1, and so $|x-4| < 1$ tells us that the radius of convergence is 1. The *open interval of convergence* is thus $(4-1, 4+1) = (3, 5)$. Checking the endpoints, $x = 3$ converges by the alternating series test, and $x = 5$ diverges by the p-series test. So the interval of convergence is $[3, 5)$.

Example 11.8.7

Find the interval and radius of convergence for the following series: $\displaystyle\sum_{n=1}^{\infty}\frac{n^3(x+5)^n}{6^n}$

Using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{(n+1)^3(x+5)^{n+1}}{6^{n+1}}\cdot\frac{6^n}{n^3(x+5)^n}\right| = \left|\frac{x+5}{6}\right|$$

converges when this limit is less than 1, or equivalently, when $|x+5| < 6$, and so the radius of convergence is 6. To find the open interval of convergence, we have $|x+5| < 6$ implies $-11 < x < 1$. When $x = -11$ and 1, the series diverges by divergence test, so the interval of convergence $(-11, 1)$.

Example 11.8.8

Find the interval and radius of convergence for the following series: $\sum_{n=1}^{\infty} \frac{(x-2)^n}{\ln(n+4)}$

Using the Ratio Test, we have

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{\ln(n+5)} \cdot \frac{\ln(n+4)}{(x-2)^n} \right| = |x-2|$$

converges when this limit is less than 1, and so $|x-2| < 1$ tells us that the radius of convergence is 1. The open interval of convergence is thus $(2-1, 2+1) = (1, 3)$. Checking the endpoints, when $x = 1$ the series converges by alternating series test, and when $x = 3$ the series diverges by comparing to the series $\sum \frac{1}{n+4}$. Thus, the interval of convergence is $[1, 3)$.

11.9 Representing Functions as Power Series

We know that, when $|x| < 1$, we have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

What this tells us is that the function $f(x) = \frac{1}{1-x}$ can be approximated to any degree of accuracy (for values of x with $-1 < x < 1$ anyway) by just looking at polynomials $1 + x + x^2 + \cdots + x^n$ for as large n as we require. This is fantastic as polynomials are well-studied and extremely easy to evaluate.

Example 11.9.1

Express $f(x) = \frac{1}{1+8x^3}$ as a power series and find its interval of convergence.

Notice that we have

$$f(x) = \frac{1}{1+8x^3} = \frac{1}{1-(-8x^3)} = \sum_{n=0}^{\infty} (-8x^3)^n = \sum_{n=0}^{\infty} (-1)^n (8x^3)^n.$$

This converges when $|8x^3| < 1$, i.e., when $|x| < \frac{1}{2}$. The interval of convergence is thus $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Example 11.9.2

Find a power series representation that approximates $f(x) = \dfrac{1}{1-x}$ near $x = a \neq 1$.

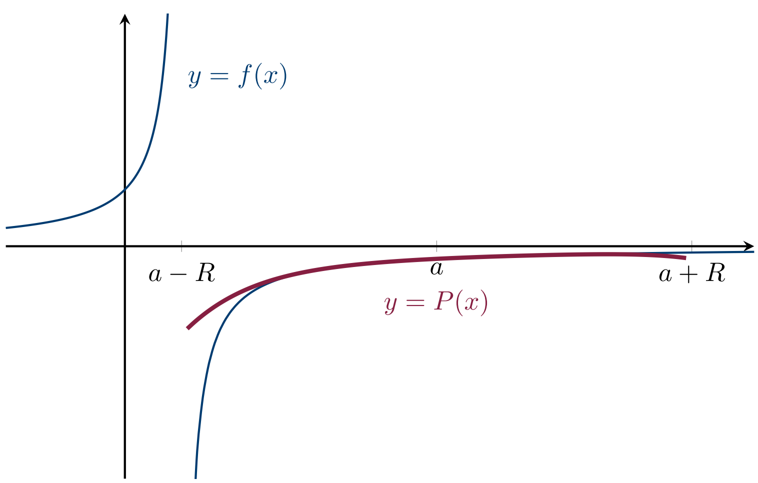
For $a \neq 1$ we can write

$$\frac{1}{1-x} = \frac{1}{1-a+a-x} = \frac{1}{(1-a)-(x-a)} = \frac{1}{1-a} \cdot \frac{1}{1 - \frac{(x-a)}{1-a}}$$

Now this is represented by the power series

$$P(x) = \sum_{n=0}^{\infty} \frac{1}{1-a} \left(\frac{x-a}{1-a}\right)^n$$

which has center a and radius of convergence $R = |1-a|$.



Theorem 11.9.3: Differentiating/Integrating Power Series

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and

1. $\dfrac{d}{dx}f(x) = \sum_{n=0}^{\infty} \dfrac{d}{dx}\left[c_n(x-a)^n\right] = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
2. $\displaystyle\int f(x)\,dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$

Example 11.9.4

Given the power series for $f(x) = \dfrac{1}{1-x}$, use differentiation to express $g(x) = \dfrac{1}{(1-x)^2}$ as a power series. Find the radius of convergence of this new power series.

We notice that

$$f'(x) = \frac{1}{(1-x)^2} = g(x),$$

and so

$$g(x) = \frac{d}{dx}f(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left[x^n\right] = \sum_{n=1}^{\infty} nx^{n-1}.$$

Using the Ratio test, we have

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)x^n}{nx^{n-1}}\right| \\ &= |x|, \end{aligned}$$

which converges when $|x| < 1$, so the radius of convergence is 1, which is exactly the same as the radius of convergence for the series representation of $f(x)$.

Proposition 11.9.5

Given the series with radius of convergence $R > 0$ and the function $f(x)$ defined in the premise for Theorem 11.9.3, the new series obtained from $\frac{d}{dx}[f(x)]$ and $\int f(x)\,dx$ both have radius of convergence R.

Example 11.9.6

Given the power series for $f(x) = \dfrac{1}{1+x}$, find a power series representation for $g(x) = \ln(1+x)$.

Notice that

$$g'(x) = \frac{1}{1+x} = f(x),$$

so we have

$$\begin{aligned} g(x) = \int f(x)\,dx = \int \frac{1}{1-(-x)}\,dx &= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n\right] dx \\ &= \sum_{n=0}^{\infty} \int (-1)^n x^n\,dx \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}. \end{aligned}$$

To determine C, we set $x = 0$ and get that $g(0) = 0 = C$, so our power series representation is just

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

Example 11.9.7

Find a power series representation for $f(x) = \arctan(x)$.

Notice that

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)},$$

so we have that

$$\begin{aligned} f(x) = \int \frac{1}{1-(-x^2)}\,dx &= \int \left[\sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx \\ &= \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}. \end{aligned}$$

To find C, we set $x = 0$ and get that $f(0) = 0 = C$, so the power series representation is just

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

Example 11.9.8

The function $f(x) = \dfrac{4}{(2-x)^2}$ is the derivative of $g(x) = \dfrac{2x}{2-x}$. Find a power series representation for $f(x)$.

$$g(x) = \frac{2x}{2-x} = \frac{2x}{2}\left(\frac{1}{1-\frac{1}{2}x}\right) = \frac{2x}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}x\right)^n = \sum_{n=0}^{\infty}\frac{x^{n+1}}{2^n}.$$

Differentiating this,

$$\begin{aligned} f(x) = \frac{d}{dx}[g(x)] &= \frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{x^{n+1}}{2^n}\right] \\ &= \sum_{n=0}^{\infty}\frac{d}{dx}\left[\frac{x^{n+1}}{2^n}\right] \\ &= \sum_{n=0}^{\infty}\frac{(n+1)x^n}{2^n} \end{aligned}$$

Example 11.9.9

Find a power series representation for $f(x) = e^x$.

We know that know that $f'(x) = f(x) = e^x$. So, we can write

$$e^x = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

$$e^x = f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n x^n\right] = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + c_2 x + c_3 x^2 + \cdots$$

And so,

$$\begin{aligned} c_0 &= c_1 \\ c_1 &= 2c_2 \\ c_2 &= 3c_3 \\ &\vdots \\ c_{n-1} &= nc_n \end{aligned}$$

We have that $1 = f(0) = c_0$, and so using this with the above equalities to compute $c_1 = 1$, $c_2 = \frac{1}{2}$, etc. we see $c_n = \frac{1}{n!}$. So series representation is

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

11.10 Taylor and Maclaurin Series

Suppose that for $|x-a|<R$, $f(x)$ has the power series representation

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

We notice that $f(a) = c_0$, so maybe we can write all of the other coefficients in terms of f. Indeed,

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-)^3 + \cdots$$

Now we have that $f'(a) = c_1$. Again,

$$f''(x) = 2c_2 + 6c_3(x-a) + 24c_4(x-a)^2 + 120c_5(x-a)^3 + \cdots$$

Now we have that $\dfrac{f''(a)}{2} = c_2$. Iterating through consecutive derivatives (and adopting the convention that $f^{(0)}(x) \equiv f(x)$, we get the following relationship for the coefficients:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Definition

The **Taylor series for the function $\boldsymbol{f}$ centered at $\boldsymbol{a}$** is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The **Maclaurin series for the function $\boldsymbol{f}$** is the Taylor series with $a=0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Remark. Our derivation relies on the fact that $f(x)$ has a power series representation. If it does not, it does not need to be the sum of the Taylor series.

Example 11.10.1

Find the Maclaurin series of the function $f(x) = e^x$ and the radius of convergence.

Since $f^{(n)}(x) = e^x$ for all n, $f^{(n)}(0) = 1$ for all n. Thus we get

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This matches what we saw in Example 11.9 By the ratio test, we can see that it has infinite radius of convergence (the series converges for all x-values).

Example 11.10.2

Find the Taylor series expansion of the function $f(x) = \ln(x)$ centered at $a = 2$.

$$\begin{aligned}
f^{(0)}(a) &= \ln a \\
f^{(1)}(a) &= \frac{1}{a} \\
f^{(2)}(a) &= -\frac{1}{a^2} \\
f^{(3)}(a) &= \frac{2}{a^3} \\
f^{(4)}(a) &= -\frac{6}{a^4} \\
&\vdots \\
f^{(n)}(a) &= \frac{(-1)^{n-1}(n-1)!}{a^n} && \text{(for } n > 0\text{)}
\end{aligned}$$

Seeing this pattern, we notice that $c_0 = \ln 2$, but every other coefficient has the form

$$c_n = \frac{f^{(n)}(2)}{n!} = \frac{(-1)^{n-1}(n-1)!}{n!2^n} = \frac{(-1)^{n-1}}{n2^n}.$$

Thus the Taylor series expansion is

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}(x-2)^n$$

Example 11.10.3

Find the Maclaurin series expansion of the function $f(x) = \sin x$.

$$\begin{aligned}
f^{(0)}(0) &= \sin 0 = 0 \\
f^{(1)}(0) &= \cos 0 = 1 \\
f^{(2)}(0) &= -\sin 0 = 0 \\
f^{(3)}(0) &= -\cos 0 = -1 \\
f^{(4)}(0) &= \sin 0 = 0 \\
f^{(5)}(0) &= \cos 0 = 1 \\
&\vdots
\end{aligned}$$

So, we notice that the Maclaurin series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Example 11.10.4

Find the Maclaurin series expansion of the function $g(x) = \cos x$.

Since $g(x) = f'(x)$, we can differentiate the Maclaurin series for $\sin x$ from the previous example to get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Exercise 11.10.5

Let i be the imaginary unit (so $i^2 = -1$). What sort of relationship do you notice about the Maclaurin series expansions for e^{ix}, $\cos x$, and $\sin x$?

Example 11.10.6

Find the Maclaurin series expansion of the binomial $f(x) = (1+x)^k$ for some fixed number k.

$$\begin{aligned}
f^{(0)}(0) &= (1+x)^k = 1 \\
f^{(1)}(0) &= k(1+x)^{k-1} = k \\
f^{(2)}(0) &= k(k-1)(1+x)^{k-2} = k(k-1) \\
f^{(3)}(0) &= k(k-1)(k-2)(1+x)^{k-3} = k(k-1)(k-2) \\
&\vdots \\
f^{(n)}(0) &= k(k-1)(k-2)\cdots(k-n+1).
\end{aligned}$$

So the Maclaurin series for $(1+x)^k$ is

$$\begin{aligned}
(1+x)^k &= 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{6}x^3 + \cdots \\
&= \sum_{n=0}^{\infty} \binom{k}{n} x^n.
\end{aligned}$$

where here $\binom{k}{n}$ is the notation given to the coefficients. We note that when k and n are both positive integers satisfying $k \geq n$, then $\binom{k}{n}$ is exactly the same as what you may have seen in the contexts of combinatorics (and thus $\binom{k}{n} = 0$ for $n > k$).

Example 11.10.7

Find the Maclaurin series for $\dfrac{2\sin(3x)}{x}$.

We can make simple modifications to the Maclaurin series for $\sin x$.

$$
\begin{aligned}
\sin x &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
\sin(3x) &= \sum_{n=0}^{\infty}(-1)^n \frac{(3x)^{2n+1}}{(2n+1)!} \\
2\sin(3x) &= 2\sum_{n=0}^{\infty}(-1)^n \frac{(3x)^{2n+1}}{(2n+1)!} \\
\frac{2\sin(3x)}{x} &= \frac{1}{x}\cdot 2\sum_{n=0}^{\infty}(-1)^n \frac{(3x)^{2n+1}}{(2n+1)!} = 2\sum_{n=0}^{\infty}(-1)^n \frac{3^{2n+1}x^{2n}}{(2n+1)!}
\end{aligned}
$$

To recap, below is a list of particularly useful and common Maclaurin series and radii of convergence.

Function	Maclaurin Series	Radius of Convergence
$\dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$	$R = 1$
$\ln(1+x)$	$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$
e^x	$\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$	$R = \infty$
$\sin x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\arctan x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$(1+x)^k$	$\displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots$	$R = \infty$ if k is a nonnegative integer, $R = 1$ otherwise.

11.11 Applications of Taylor Polynomials

Definition: Taylor Polynomial

The $\boldsymbol{k^{\text{th}}}$ **Taylor polynomial for $\boldsymbol{f(x)}$ centered at $\boldsymbol{a}$** is the k^{th} partial sum of the Taylor series:

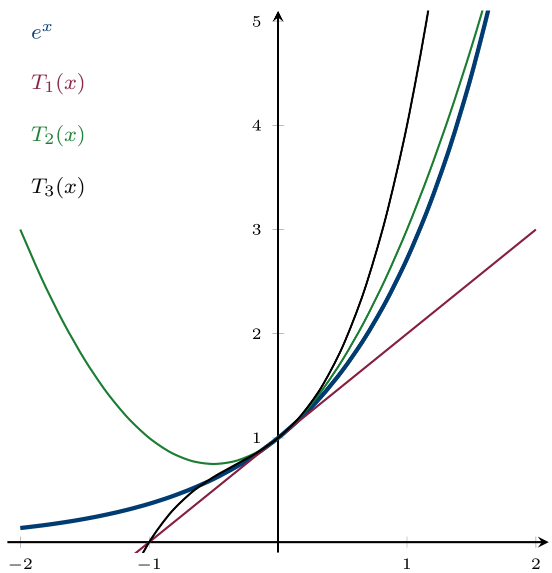
$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Remark. Some will define $T_k(x)$ to be the Taylor polynomial with k nonzero terms, which may be different than our definition.

Notice that when $k = 1$, we have the function

$$T_k(x) = f(a) + f'(a)(x-a).$$

This is just the tangent line approximation of the function f at a! It suggests to us that, as k grows, the k^{th} Taylor polynomial provides us a better and better approximation of our function values when x is close to a. Indeed, if we look at the graph of $y = e^x$ below and a few Taylor polynomials (centered at 0).



Example 11.11.1

Find the first four nonzero terms polynomial to approximate $f(x) = \sin(x)\cos(x)$.

$$\begin{aligned}
&\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\left(\frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\\
&= \frac{x}{0!\,1!} - \frac{x^3}{0!\,3!} - \frac{x^3}{1!\,2!} + \frac{x^5}{0!\,5!} + \frac{x^5}{2!3!} + \frac{x^5}{1!\,4!} - \frac{x^7}{0!7!} - \frac{x^7}{1!\,6!} - \frac{x^7}{2!\,5!} - \frac{x^7}{3!4!} + \cdots\\
&= x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \frac{4}{315}x^7 + \cdots
\end{aligned}$$

Example 11.11.2

Find the first three nonzero terms for the Maclaurin series for $\tan(x)$.

Doing some polynomial long division, we obtain

$$\frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots} = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

Just as with a tangent line approximation, the center a is important, as your estimation becomes less accurate if x and a are very far away.

Definition: Remainder

If $f(x)$ is the sum of its Taylor series and $T_k(x)$ is an approximation, then the difference $R_k(x) = f(x) - T_k(x)$ should tell us the error involved in estimating the value of the function. Indeed, we call $R_k(x)$ the **remainder** of the Taylor series, and $|R_k(x)|$ is the **(absolute value of the) error**.

Example 11.11.3

Use a sixth-degree Taylor polynomial to approximate $\sin(3.15)$. Find the error of the approximation $\sin(3.15) \approx T_6(3.15)$.

Since 3.15 is very close to π, we'll center the Taylor polynomial at $a = \pi$.

$$\begin{aligned}
f^{(0)}(\pi) &= 0\\
f^{(1)}(\pi) &= -1\\
f^{(2)}(\pi) &= 0\\
f^{(3)}(\pi) &= 1\\
f^{(4)}(\pi) &= 0\\
f^{(5)}(\pi) &= -1\\
f^{(6)}(\pi) &= 0\\
\Rightarrow T_6(x) &= -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!}
\end{aligned}$$

So then

$$\sin(3.15) \approx T_6(3.15) = -0.008\,407\,247\,367\,148\,707\ldots$$

Checking with our calculator, we have

$$\sin(3.15) = -0.008\,407\,247\,367\,148\,706\ldots$$

and so the error is $|R_6(3.15)| \approx 1.22258 \times 10^{-16}$.

Now, because the Maclaurin series for $\sin(x)$ converges for all x values, there's no reason we couldn't also approximate this with a Taylor polynomial centered at $a = 0$. So let's do the same example again, this time centering at $a = 0$.

Example 11.11.4

Use a sixth-degree Taylor polynomial to approximate $\sin(3.15)$. Find the error of the approximation $\sin(3.15) \approx T_6(3.15)$.

Since the Maclaurin series for $\sin(x)$ converges for all x values, we'll use the 5^{th} order Taylor polynomial centered at $a = 0$ to approximate $\sin(3.15)$.

$$T_6(x) = x - \frac{x^3}{6} + \frac{x^5}{120}.$$

Thus

$$\sin(3.15) \approx T_6(3.15) = 0.525158$$

And so the error is $|R_6(3.15)| \approx 0.533565$.

While both Taylor series will evaluate $\sin(3.15)$ exactly, by choosing to center our series closer to 3.15, our Taylor polynomials produces a more accurate approximation in fewer terms.

11.11.1 Error Estimation

We'd like to be able to deeper analyze the error of our approximation - can we determine a range of x-values for which our error will be within some prescribed tolerance? Or alternatively, can we find the maximum error on for some range of x-values?

From the Mean Value Theorem, it follows that

$$R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}$$

where c is some number between a and x. If for $|x - a| < R$ (where R is some number) we know that $f^{(k+1)}(x)$ is bounded (i.e., that $|f^{(k+1)}(x)| \leq M$ for all x in the interval $(a - R, a + R)$) then in fact we can write

$$|R_k(x)| = \left|\frac{f^{(k+1)}(c)}{(k+1)!}\right| |x-a|^{k+1} \leq \frac{M}{(k+1)!}|x-a|^{k+1} < \frac{M}{(k+1)!}R^{k+1}$$

Example 11.11.5

We define the (unnormalized) *error function* $\mathrm{erf}_u(x)$ to be

$$\mathrm{erf}_u(x) = \int_0^x e^{-t^2}\,dt$$

Use the first three nonzero terms of the Maclaurin series for e^{-t^2} to approximate $\mathrm{erf}(1)$. What is the maximum error in this estimation?

We have that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\text{whence } e^{-t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}.$$

Now

$$\mathrm{erf}_u(x) = \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{n!(2n+1)}\Bigg|_0^x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}$$

so in particular we have that

$$\mathrm{erf}_u(1) = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \cdots$$

so the first three terms give us the approximation $\mathrm{erf}(1) \approx 1 - \frac{1}{3} + \frac{1}{10} \approx 0.767$. Notice that $\mathrm{erf}_u(1)$ is an alternating series, so the error of this estimation is bounded above by the (absolute value) of the next term in the alternating series: $\frac{1}{42} \approx 0.024$. In fact, the actual error is roughly 0.02.