LAX-PHILLIPS SCATTERING THEORY AND WELL-POSED LINEAR SYSTEMS: A COORDINATE-FREE APPROACH

JOSEPH A. BALL, PHILIP T. CARROLL, AND YOICHI UETAKE

Abstract. We give a further elaboration of the fundamental connections between Lax-Phillips scattering, conservative input/state/output linear systems and Sz.-Nagy-Foias model theory for both the discrete- and continuous-time settings. In particular, for the continuous-time setting, we show how to locate a scattering-conservative L^2-well-posed linear system (in the sense of Staffans and Weiss) embedded in a Lax-Phillips scattering system presented in axiomatic form; conversely, given a scattering-conservative linear system, we show how one can view the space of finite-energy input-state-output trajectories of the system as the ambient space for an associated Lax-Phillips scattering system. We use these connections to give a simple, conceptual proof of the identity of the scattering function of the scattering system with the transfer function of the input-state-output linear system. As an application we show how system-theoretic ideas can be used to arrive at the spectral analysis of the scattering function.

1. Introduction

It is well known that linear system theory, Lax-Phillips scattering theory and the Sz.-Nagy-Foias model theory for a Hilbert space contraction are intimately related. For the discrete-time case such connections were first noticed in [1, 11, 12]. Surveys of the discrete-time with a slant toward possible generalizations to various multivariable/multidimensional settings appear in [3, 4]; details for some of these more general settings are now appearing (see [5, 8]). The continuous-time version of the Lax-Phillips scattering theory influenced the search for a distributed-parameter system theory, especially in the presence of energy conservation or dissipation (see [13, 19, 20, 2]); these ideas have now matured into the theory of well-posed linear systems (conservative or not) due essentially to Staffans and Weiss—see [26] for a comprehensive treatment. Recent accounts of these connections between well-posed linear systems and Lax-Phillips scattering theory are given in [23, 24, 25].

Our purpose here is to develop further in a more explicit, coordinate-free fashion the connections between continuous-time Lax-Phillips scattering and conservative well-posed linear systems, especially for the continuous-time case. Unlike the treatment in [23, 24], we use a coordinate-free axiomatized formulation of a Lax-Phillips scattering system. We provide an explicit description of the translation-representation operators for the incoming and outgoing spaces and then arrive at an embedded scattering-conservative well-posed linear system (in the sense of Staffans and Weiss as in [26]). Conversely, starting with a scattering-conservative well-posed linear system, we construct the associated Lax-Phillips scattering system as the space of admissible trajectories (with an appropriate norm) of the linear system with the unitary evolution being given by the time shift on trajectories. Various coordinate representations as in [23, 24] then fall out in a natural way from this coordinate-free description. As an application of the connection between the scattering function and transfer function, we characterize the isolated poles of the scattering function (including their multiplicities) in terms of eigenstructure of the cogenerator of the so-called scattering semigroup, for both the discrete-time and the continuous-time settings. Our focus here is exclusively on the scattering and linear-system aspects; a direct treatment of the continuous-time version of the associated Sz.-Nagy-Foias model theory is given in [26, Chapter 11].

In Section 2 we describe the more elementary discrete-time case in a fashion completely analogous to our treatment of the continuous-time case. Section 3 then treats the continuous-time case.

2. Lax-Phillips scattering and conservative linear systems: the discrete-time case

In this section we review the discrete-time case as a more accessible benchmark by which to compare our results for the continuous-time case.

2.1. **Lax-Phillips scattering system: axiomatic form.**

By a *discrete-time Lax-Phillips scattering system* we mean a collection

$$\mathfrak{S} = (\mathcal{U}; \mathcal{K}, \mathcal{G}, \mathcal{G}_*)$$

where $\mathcal{K}$ is a Hilbert space (the *ambient space*), $\mathcal{U}$ is a unitary operator on $\mathcal{K}$, $\mathcal{G}_*$ (the *incoming space*) and $\mathcal{G}$ (the *outgoing space*) are subspaces of $\mathcal{K}$ subject to the following axioms:

(DT-A1) $\mathcal{U}\mathcal{G} \subset \mathcal{G}$ and $\bigcap_{n=0}^{\infty} \mathcal{U}^n \mathcal{G} = \{0\}$,
(DT-A2) $\mathcal{U}^*\mathcal{G}_* \subset \mathcal{G}_*$ and $\bigcap_{n=0}^{\infty} \mathcal{U}^{*n} \mathcal{G}_* = \{0\}$, and
(DT-A3) the *causality condition* $\mathcal{G}_* \perp \mathcal{G}$ holds.

The scattering system $\mathfrak{S}$ is said to be *minimal* if

$$\widetilde{\mathcal{G}}_* + \widetilde{\mathcal{G}} \text{ is dense in } \mathcal{K} \tag{2.1}$$

where we have set

$$\widetilde{\mathcal{G}} = \text{ closure of } \bigcup_{n=0}^{\infty} \mathcal{U}^{*n}\mathcal{G}, \qquad \widetilde{\mathcal{G}}_* = \text{ closure of } \bigcup_{n=0}^{\infty} \mathcal{U}^{n}\mathcal{G}_*. \tag{2.2}$$

We remark that Lax and Phillips usually assumed that

$$\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_* = \mathcal{K} \tag{2.3}$$

but the weaker assumption (2.1) is more natural for our purposes here; this greater flexibility was introduced by Adamjan and Arov [1]. From the assumption (DT-A1) we see that the subspace $\mathcal{E} := \mathcal{G} \ominus \mathcal{U}\mathcal{G}$ is *wandering* for $\mathcal{U}$ (i.e., $\mathcal{U}^n\mathcal{E} \perp \mathcal{U}^m\mathcal{E}$ for $n \neq m$) and that $\mathcal{G}$ has the internal orthogonal decomposition

$$\mathcal{G} = \bigoplus_{n=0}^{\infty} \mathcal{U}^n \mathcal{E}.$$

Similarly, from the assumption (DT-A2) we see that $\mathcal{E}_* := \mathcal{U}\mathcal{G}_* \ominus \mathcal{G}_*$ is also wandering for $\mathcal{U}$ and that $\mathcal{G}_*$ has the internal orthogonal decomposition

$$\mathcal{G}_* = \bigoplus_{n=-\infty}^{-1} \mathcal{U}^n \mathcal{E}_*.$$

As $\mathcal{G}$ and $\mathcal{G}_*$ are assumed to be orthogonal to each other by axiom DT-A3, it is natural to define the *scattering subspace* $\mathcal{H}$ by

$$\mathcal{H} = \mathcal{K} \ominus [\mathcal{G} \oplus \mathcal{G}_*]. \tag{2.4}$$

Then we have the orthogonal decomposition for the ambient space $\mathcal{K}$

$$\begin{aligned} \mathcal{K} = \mathcal{G}_* \oplus \mathcal{H} \oplus \mathcal{G} &= \bigoplus_{n=-\infty}^{-1} \mathcal{U}^n \mathcal{E}_* \oplus \mathcal{H} \oplus \bigoplus_{n=0}^{\infty} \mathcal{U}^n \mathcal{E} \\ &= \bigoplus_{n=-\infty}^{-1} \mathcal{U}^n \mathcal{E}_* \oplus \mathcal{H} \oplus \mathcal{E} \oplus \bigoplus_{n=1}^{\infty} \mathcal{U}^n \mathcal{E}. \end{aligned} \tag{2.5}$$

Since $\mathcal{U}$ is unitary and $\mathcal{E}_*$ and $\mathcal{E}$ are wandering for $\mathcal{U}$, we also have the shifted version of (2.5)

$$\begin{aligned} \mathcal{K} = \mathcal{U}\mathcal{K} &= \bigoplus_{n=-\infty}^{0} \mathcal{U}^n \mathcal{E}_* \oplus \mathcal{U}\mathcal{H} \oplus \bigoplus_{n=1}^{\infty} \mathcal{U}^n \mathcal{E} \\ &= \bigoplus_{n=-\infty}^{-1} \mathcal{U}^n \mathcal{E}_* \oplus \mathcal{E}_* \oplus \mathcal{U}\mathcal{H} \oplus \bigoplus_{n=1}^{\infty} \mathcal{U}^n \mathcal{E}. \end{aligned} \tag{2.6}$$

Canceling the common left and right tails on the right hand sides of (2.5) and (2.6) then leaves us with the two orthogonal decompositions for the same subspace:

$$\mathcal{E}_* \oplus \mathcal{U}\mathcal{H} = \mathcal{H} \oplus \mathcal{E}. \tag{2.7}$$

As we shall see, (2.7) is a key piece of the Lax-Phillips scattering geometry.

We let $i_{\mathcal{E}} \colon \mathcal{E} \to \mathcal{K}$ and $i_{\mathcal{E}_*} \colon \mathcal{E}_* \to \mathcal{K}$ be the inclusion maps with adjoints $i_{\mathcal{E}}^* \colon \mathcal{K} \to \mathcal{E}$ and $i_{\mathcal{E}_*}^* \colon \mathcal{K} \to \mathcal{E}_*$ equal to the orthogonal projections of $\mathcal{K}$ onto $\mathcal{E}$ (respectively $\mathcal{E}_*$) with target space considered equal to $\mathcal{E}$ itself (respectively $\mathcal{E}_*$ itself) rather than $\mathcal{E}$ (respectively, $\mathcal{E}_*$) considered as a subspace of $\mathcal{K}$. In general, we let $\mathbb{Z}$ denote the integers, $\mathbb{Z}_+$ denote the nonnegative integers and $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{Z}_+$ denote the (strictly) negative integers. For $\mathcal{X}$ any coefficient Hilbert space and $\mathbb{S}$ a set, $\ell^2_{\mathcal{X}}(\mathbb{S})$ denotes the space of norm-squared summable $\mathcal{X}$-valued functions on $\mathbb{S}$

$$\ell^2_{\mathcal{X}}(\mathbb{S}) = \left\{ \{u(s)\}_{s \in \mathbb{S}} \colon \|u\|^2_{\ell^2_{\mathcal{X}}(\mathbb{S})} := \sum_{s \in \mathbb{S}} \|u(s)\|^2_{\mathcal{X}} < \infty \right\}.$$

We are mainly interested in the cases $\mathbb{S} = \mathbb{Z}$, $\mathbb{Z}_+$ or $\mathbb{Z}_-$. Given an element $x = \{x(n)\}_{n \in \mathbb{Z}_-}$ of $\ell^2_{\mathcal{X}}(\mathbb{Z}_-)$, we define $x(n) = 0$ for $n \in \mathbb{Z}_+$ and view x as an element of $\ell^2_{\mathcal{X}}(\mathbb{Z})$ as well with a similar convention for $\ell^2_{\mathcal{X}}(\mathbb{Z}_+) \subset \ell^2_{\mathcal{X}}(\mathbb{Z})$. It will also be convenient to define, for any $k \in \mathbb{Z}$, unitary identification maps $i_{\mathcal{X},k} \colon \mathcal{X} \to \ell^2_{\mathcal{X}}(\mathbb{Z})$ by

$$i_{\mathcal{X},k} \colon x \mapsto \{\delta_{n,k} x\}_{n \in \mathbb{Z}} \text{ where } \delta_{n,k} = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise} \end{cases}$$

with adjoint $i^*_{\mathcal{X},k} \colon \ell^2_{\mathcal{X}}(\mathbb{Z}) \to \mathcal{X}$ given by

$$i^*_{\mathcal{X},k} \colon \{x(n)\}_{n \in \mathbb{Z}} \mapsto x(k).$$

In case $k \in \mathbb{Z}_+$ (respectively $k \in \mathbb{Z}_-$), we may view $i_{\mathcal{X},k}$ as mapping $\mathcal{X}$ into $\ell^2_{\mathcal{X}}(\mathbb{Z}_+)$ (respectively, $\ell^2_{\mathcal{X}}(\mathbb{Z}_-)$); the meaning will be clear from the context. We will denote by $\tau \otimes I_{\mathcal{X}}$ the backwards bilateral shift operator on $\ell^2_{\mathcal{X}}(\mathbb{Z})$:

$$\tau\colon \{x(n)\}_{n\in\mathbb{Z}} \mapsto \{x(n+1)\}_{n\in\mathbb{Z}}.$$

The restriction to $\ell^2_{\mathcal{X}}(\mathbb{Z}_-)$ and the compression to $\ell^2_{\mathcal{X}}(\mathbb{Z}_+)$ are denoted by $\tau_- \otimes I_{\mathcal{X}}$ and $\tau_+ \otimes I_{\mathcal{X}}$ respectively:

$$\tau_- \otimes I_{\mathcal{X}}\colon \{x(n)\}_{n\in\mathbb{Z}_-} \mapsto \{x'(n)\}_{n\in\mathbb{Z}_-} \text{ where } \begin{cases} x'(n) = x(n+1) & \text{for } n \le -1, \\ x'(-1) = 0 \end{cases}$$

$$\tau_+ \otimes I_{\mathcal{X}}\colon \{x(n)\}_{n\in\mathbb{Z}_+} \mapsto \{x(n+1)\}_{n\in\mathbb{Z}_+}.$$

We next define *translation representation operators* $\Phi\colon \mathcal{K} \to \ell^2_{\mathcal{E}}(\mathbb{Z})$ and $\Phi_*\colon \mathcal{K} \to \ell^2_{\mathcal{E}_*}(\mathbb{Z})$ by

$$\Phi\colon k \mapsto \{i_{\mathcal{E}}^* \mathcal{U}^{*n} k\}_{n\in\mathbb{Z}}, \qquad \Phi_*\colon k \mapsto \{i_{\mathcal{E}_*}^* \mathcal{U}^{*n} k\}_{n\in\mathbb{Z}}. \tag{2.8}$$

Then Φ and Φ_* are coisometries onto $\ell^2_{\mathcal{E}}(\mathbb{Z})$ (respectively, $\ell^2_{\mathcal{E}_*}(\mathbb{Z})$) with initial space equal to $\widetilde{\mathcal{G}}$ (respectively, to $\widetilde{\mathcal{G}}_*$), and define unitary maps between $\mathcal{G}$ and $\ell^2_{\mathcal{E}}(\mathbb{Z}_+)$ (respectively between $\mathcal{G}_*$ and $\ell^2_{\mathcal{E}_*}(\mathbb{Z}_-)$), i.e.,

$$\Phi(\mathcal{G}) = \ell^2_{\mathcal{E}}(\mathbb{Z}_+), \qquad \Phi_*(\mathcal{G}_*) = \ell^2_{\mathcal{E}_*}(\mathbb{Z}_-),$$

and satisfy the intertwining relations

$$\Phi\mathcal{U} = (\tau^* \otimes I_{\mathcal{E}})\Phi, \qquad \Phi_*\mathcal{U} = (\tau^* \otimes I_{\mathcal{E}_*})\Phi_*, \tag{2.9}$$

i.e., Φ and Φ_* transform the unitary operator $\mathcal{U}$ to the respective shift operators (or discrete-time *translation* operators) $\tau^* \otimes I_{\mathcal{E}}$ and $\tau^* \otimes I_{\mathcal{E}_*}$. For future reference we note that the adjoints are given by

$$\Phi^*\colon \{e_n\}_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} \mathcal{U}^n i_{\mathcal{E}} e_n, \qquad \Phi_*^*\colon \{e_{*n}\}_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} \mathcal{U}^n i_{\mathcal{E}_*} e_{*n}. \tag{2.10}$$

We define the *scattering operator* $\mathbf{S}\colon \ell^2_{\mathcal{E}}(\mathbb{Z}) \to \ell^2_{\mathcal{E}_*}(\mathbb{Z})$ by

$$\mathbf{S} = \Phi_*\Phi^*. \tag{2.11}$$

The role of Axiom (DT-A3) finally enters here: as a consequence of (DT-A3) it follows that $\mathbf{S}\colon \ell^2_{\mathcal{E}}(\mathbb{Z}_+) \to \ell^2_{\mathcal{E}_*}(\mathbb{Z}_+)$. Moreover, from (2.9) we see that $\mathbf{S}$ satisfies the intertwining condition

$$\mathbf{S}(\tau^* \otimes I_{\mathcal{E}}) = (\tau^* \otimes I_{\mathcal{E}_*})\mathbf{S}. \tag{2.12}$$

For $\mathcal{X}$ any coefficient space, we let $L^2_{\mathcal{X}}(\mathbb{T})$ denote the standard L^2-space on the unit circle $\mathbb{T}$ consisting of measurable $\mathcal{X}$-valued functions square-integrable in norm with respect to normalized Lebesgue arc-length measure on $\mathbb{T}$. We abbreviate $L^2_{\mathbb{C}}(\mathbb{T})$ to $L^2(\mathbb{T})$ and note that $\{z^n\colon n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$. The Hardy space $H^2_{\mathcal{X}}(\mathbb{D})$ is the subspace of $L^2(\mathbb{T})$ consisting of functions f in $L^2(\mathbb{T})$ such that the Fourier coefficients

$$\langle f, z^n x\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} \langle f(z), x\rangle_{\mathcal{X}} z^{-n}\, |dz|$$

vanishes for all $n = 0, 1, 2, \dots$ and for all $x \in \mathcal{X}$, or

$$H^2_{\mathcal{X}}(\mathbb{D}) = \left\{ f \in L^2_{\mathcal{X}}(\mathbb{T})\colon f(z) \cong \sum_{n\in\mathbb{Z}_+} f_n z^n \right\}.$$

As the notation suggests, elements f of $H^2_{\mathcal{X}}(\mathbb{D})$ have analytic continuation to the unit disk $\mathbb{D}$ and the norm can alternatively be given as

$$\|f\|^2_{H^2(\mathbb{D})} = \sup_{r<1} \left\{ \frac{1}{2\pi} \int_{\mathbb{T}} \|f(rz)\|^2 \; |dz| \right\}.$$

Similarly, we denote by $H^2(\mathbb{D})^\perp$ the orthogonal complement of $H^2(\mathbb{D})$ described by

$$H^2_{\mathcal{X}}(\mathbb{D})^\perp = \left\{ f \in L^2_{\mathcal{X}}(\mathbb{T})\colon f(z) \cong \sum_{n\in\mathbb{Z}_-} f_n z^n \right\}.$$

We define the Z-transform from $\ell^2_{\mathcal{X}}(\mathbb{Z})$ to $L^2_{\mathcal{X}}(\mathbb{T})$ by

$$\{x(n)\}_{n\in\mathbb{Z}} \mapsto \widehat{x}(z) := \sum_{n\in\mathbb{Z}} x(n) z^n \in L^2_{\mathcal{X}}(\mathbb{T}).$$

The Fourier representation operators $\widehat{\Phi}\colon \mathcal{K} \to L^2_{\mathcal{E}}(\mathbb{T})$ and $\widehat{\Phi}_*\colon \mathcal{K} \to L^2_{\mathcal{E}_*}(\mathbb{T})$ are then defined as the Z-transformed versions for the translation representation operators:

$$\widehat{\Phi}\colon k \mapsto \widehat{\Phi k} = \sum_{n\in\mathbb{Z}} (i^*_{\mathcal{E}} \mathcal{U}^{*n} k) z^n,$$
$$\widehat{\Phi}_*\colon k \mapsto \widehat{\Phi_* k} = \sum_{n\in\mathbb{Z}} (i^*_{\mathcal{E}_*} \mathcal{U}^{*n} k) z^n.$$

from which we deduce the intertwining relations

$$\widehat{\Phi}\mathcal{U} = (M_z \otimes I_{\mathcal{E}})\Phi, \qquad \widehat{\Phi}_*\mathcal{U} = (M_z \otimes I_{\mathcal{E}_*})\Phi_* \tag{2.13}$$

where $M_z\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the multiplication operator $M_z\colon f(z) \mapsto z \cdot f(z)$ (the image of τ after Z-transform: $M_z \widehat{x} = \widehat{\tau x}$ for $x \in \ell^2(\mathbb{Z})$). We let $\widehat{\mathbf{S}}\colon L^2_{\mathcal{E}}(\mathbb{T}) \to L^2_{\mathcal{E}_*}(\mathbb{T})$ be the image of $\mathbf{S}$ after Z-transform:

$$\widehat{\mathbf{S}x} = \widehat{\mathbf{S}} x \text{ for } x \in \ell^2_{\mathcal{E}}(\mathbb{Z}).$$

Then we have

$$\widehat{\mathbf{S}} = \widehat{\Phi}_* \widehat{\Phi}^* \tag{2.14}$$

and $\widehat{\mathbf{S}}$ satisfies intertwining relations

$$\widehat{\mathbf{S}}(M_z \otimes I_{\mathcal{E}}) = (M_z \otimes I_{\mathcal{E}_*})\widehat{\mathbf{S}}. \tag{2.15}$$

Moreover, the causality axiom (DT-A3) translates to the property

$$\widehat{\mathbf{S}}\colon H^2_{\mathcal{E}}(\mathbb{D}) \to H^2_{\mathcal{E}_*}(\mathbb{D}). \tag{2.16}$$

The condition (2.15) implies that necessarily $\widehat{\mathbf{S}}$ has the form

$$\widehat{\mathbf{S}} = M_S\colon f(z) \mapsto S(z) \cdot f(z) \tag{2.17}$$

for a function S in $L^2_{\mathcal{L}(\mathcal{E},\mathcal{E}_*)}(\mathbb{T})$ and then condition (2.16) implies that in fact the multiplier S is in $H^\infty_{\mathcal{L}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$. The multiplier S associated with the Z-transformed version $\widehat{\mathbf{S}}$ of the scattering operator $\mathbf{S}$ we shall call the *scattering function* for the scattering system (often called the *scattering matrix* in the literature). As the scattering operator $\mathbf{S} = \Phi_* \Phi^*$ is the product of two partial isometries, it follows that $\mathbf{S}$ and $\widehat{\mathbf{S}}$ have operator norm at most 1, and hence $\|S\|_\infty \leq 1$, i.e., S is in the *Schur class* $\mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ consisting of analytic $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued functions S on $\mathbb{D}$ with values $S(z)$ having operator norm at most 1 for all $z \in \mathbb{D}$.

Let us say that two scattering systems

$$\mathfrak{S} = (\mathcal{U}; \mathcal{K}, \mathcal{G}, \mathcal{G}_*) \text{ and } \mathfrak{S}' = (\mathcal{U}'; \mathcal{K}', \mathcal{G}', \mathcal{G}'_*)$$

are *unitarily equivalent* if there is a unitary transformation $U\colon \mathcal{K} \to \mathcal{K}'$ such that

$$U\mathcal{U} = \mathcal{U}'U, \qquad U\mathcal{G} = \mathcal{G}', \qquad U\mathcal{G}_* = \mathcal{G}'_*. \tag{2.18}$$

We shall also say that two Schur-class functions $S \in \mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ and $S' \in \mathcal{S}(\mathcal{E}', \mathcal{E}'_*)$ *coincide* if there are unitary operators $\iota\colon \mathcal{E} \to \mathcal{E}_*$ and $\iota_*\colon \mathcal{E}_* \to \mathcal{E}'_*$ so that

$$\iota_* S(z) = S'(z)\iota \text{ for all } z \in \mathbb{D}.$$

It turns out that the scattering function is a complete unitary invariant for a minimal Lax-Phillips scattering system in the sense that *two minimal Lax-Phillips scattering systems $\mathfrak{S}$ and $\mathfrak{S}'$ are unitarily equivalent if and only if their associated scattering functions $S_{\mathfrak{S}}(z)$ and $S_{\mathfrak{S}'}(z)$ coincide.* Moreover, *given any Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, there is an associated functional-model scattering system $\mathfrak{S}_S$ based on the scattering function S so that the scattering function of $\mathfrak{S}_S$ coincides with the preassigned Schur-class function S.* This last statement can be verified by writing down a *model Lax-Phillips scattering system* associated with a given Schur-class function S; we refer to [5] for further details.

2.2. **Lax-Phillips Schäffer-matrix model scattering system.** One can use the translation representation operators Φ and Φ_* to convert the abstract Lax-Phillips scattering system $\mathfrak{S}$ as defined above to a *model Lax-Phillips scattering system* as follows. The subspace

$$\mathcal{H} := \mathcal{K} \ominus [\mathcal{G} \oplus \mathcal{G}_*]$$

is often referred to as the *scattering space* of the scattering system $\mathfrak{S}$ and hence we have the orthogonal decomposition of $\mathcal{K}$:

$$\mathcal{K} = \mathcal{G}_* \oplus \mathcal{H} \oplus \mathcal{G}.$$

In the same spirit as was done with respect to the wandering subspaces $\mathcal{E}$ and $\mathcal{E}_*$, it is convenient to have a notation

$$i_{\mathcal{H}}\colon h \mapsto h$$

for the inclusion map from $\mathcal{H}$ into $\mathcal{K}$, with adjoint $i_{\mathcal{H}}^*\colon \mathcal{K} \to \mathcal{H}$ equal to the the orthogonal projection $P_{\mathcal{H}}$ followed by the inverse inclusion map. As $\Phi_*|_{\mathcal{G}_*}\colon \mathcal{G} \to \ell^2_{\mathcal{E}_*}(\mathbb{Z}_-)$ and $\Phi|_{\mathcal{G}}\colon \mathcal{G}_* \to \ell^2_{\mathcal{E}}(\mathbb{Z}_+)$ are unitary, the operator

$$\mathcal{I}_{\mathcal{K}\to\boldsymbol{\mathcal{K}}} = \begin{bmatrix} \Phi_* P_{\mathcal{G}_*} \\ i_{\mathcal{H}}^* \\ \Phi P_{\mathcal{G}} \end{bmatrix} : \mathcal{K} \to \begin{bmatrix} \ell^2_{\mathcal{E}_*}(\mathbb{Z}_-) \\ \mathcal{H} \\ \ell^2_{\mathcal{E}}(\mathbb{Z}_+) \end{bmatrix} =: \boldsymbol{\mathcal{K}} \tag{2.19}$$

is unitary as an operator from $\mathcal{K}$ to $\boldsymbol{\mathcal{K}} := \ell^2_{\mathcal{E}_*}(\mathbb{Z}_-) \oplus \mathcal{H} \oplus \ell^2_{\mathcal{E}}(\mathbb{Z}_+)$. An elementary computation shows that the the image $\boldsymbol{\mathcal{U}}^*$ of the unitary operator $\mathcal{U}^*$ on the space $\boldsymbol{\mathcal{K}}$, i.e., the unitary operator $\boldsymbol{\mathcal{U}}^*$ on $\boldsymbol{\mathcal{K}}$ determined by the condition $\boldsymbol{\mathcal{U}}^* \mathcal{I}_{\mathcal{K}\to\boldsymbol{\mathcal{K}}} = \mathcal{I}_{\mathcal{K}\to\boldsymbol{\mathcal{K}}} \mathcal{U}^*$, is given by

$$\boldsymbol{\mathcal{U}}^* = \begin{bmatrix} \tau_- \otimes I_{\mathcal{E}_*} & i_{\mathcal{E}_*,-1} C & i_{\mathcal{E}_*,-1} D i^*_{\mathcal{E},0} \\ 0 & A & B i^*_{\mathcal{E},0} \\ 0 & 0 & \tau_+ \otimes I_{\mathcal{E}} \end{bmatrix} : \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \tag{2.20}$$

where $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is the *unitary colligation* given explicitly by

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} i_{\mathcal{H}}^* \mathcal{U}^* \\ i_{\mathcal{E}_*}^* \end{bmatrix} \begin{bmatrix} i_{\mathcal{H}} & i_{\mathcal{E}} \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}. \tag{2.21}$$

For the record, then the adjoint $\boldsymbol{\mathcal{U}}$ of $\boldsymbol{\mathcal{U}}^*$ is given by

$$\boldsymbol{\mathcal{U}} = \begin{bmatrix} \tau_-^* \otimes I_{\mathcal{E}_*} & 0 & 0 \\ C^* i_{\mathcal{E}_*,-1}^* & A^* & 0 \\ i_{\mathcal{E},0} D^* i_{\mathcal{E}_*,-1}^* & i_{\mathcal{E}_0} B^* & \tau_+^* \otimes I_{\mathcal{E}} \end{bmatrix} : \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}. \tag{2.22}$$

Note also that

$$\mathcal{I} \colon \mathcal{G}_* \mapsto \begin{bmatrix} \ell^2_{\mathcal{E}_*}(\mathbb{Z}_-) \\ \{0\} \\ \{0\} \end{bmatrix} =: \boldsymbol{\mathcal{G}}_* \subset \boldsymbol{\mathcal{K}},$$

$$\mathcal{I} \colon \mathcal{H} \mapsto \begin{bmatrix} \{0\} \\ \mathcal{H} \\ \{0\} \end{bmatrix} =: \boldsymbol{\mathcal{H}} \subset \boldsymbol{\mathcal{K}}$$

$$\mathcal{I} \colon \mathcal{G} \to \begin{bmatrix} \{0\} \\ \{0\} \\ \ell^2_{\mathcal{E}}(\mathbb{Z}_+) \end{bmatrix} =: \boldsymbol{\mathcal{G}} \subset \boldsymbol{\mathcal{K}}. \tag{2.23}$$

Then the above analysis shows that the collection

$$\boldsymbol{\mathfrak{S}} = (\boldsymbol{\mathcal{U}}; \boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{G}}_*) \tag{2.24}$$

is a (discrete-time) Lax-Phillips scattering system which is unitarily equivalent (in the sense of (2.18)) to our original scattering system via the unitary map $\mathcal{I} \colon \mathcal{K} \to \boldsymbol{\mathcal{K}}$.

Conversely, if $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H} \oplus \mathcal{E}_*$ is any unitary colligation, we can use the formulas (2.19), (2.22), (2.23) to define a collection $\boldsymbol{\mathfrak{S}}$ as in (2.24). Then one can check, using the assumption that the colligation $\mathbf{U}$ is assumed to be unitary, that $\boldsymbol{\mathfrak{S}}$ is a discrete-time Lax-Phillips scattering system, i.e., the axioms (DT-A1), (DT-A2) and (DT-A3) are all satisfied. We refer to a Lax-Phillips scattering system of this special form (2.24) as a *(discrete-time) model Lax-Phillips scattering system.* The analysis above shows that any Lax-Phillips scattering system $\mathfrak{S}$ is unitarily equivalent to a model Lax-Phillips scattering system $\boldsymbol{\mathfrak{S}} = \boldsymbol{\mathfrak{S}}(\mathbf{U})$ where the unitary colligation $\mathbf{U}$ is determined by (2.21).

Remark 2.1. We remark that the geometry of a Lax-Phillips scattering system is intimately connected with the geometry of the Sz.-Nagy unitary dilation for a contraction operator T on a Hilbert space $\mathcal{H}$. Indeed, given a contraction operator T on a Hilbert space $\mathcal{H}$, introduce the defect operators

$$D_T = (I - T^*T)^{1/2}, \qquad D_{T^*} = (I - TT^*)^{1/2}$$

and the defect spaces

$$\mathcal{D}_T = \text{ closure of Ran } D_T, \qquad \mathcal{D}_{T^*} = \text{ closure of Ran } D_{T^*}$$

and let $\mathbf{U}_T$ be the so-called Halmos unitary dilation of T^*:

$$\mathbf{U}_T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} T^* & D_T|_{\mathcal{D}_T} \\ D_{T^*} & -T|_{\mathcal{D}_T} \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{D}_T \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{D}_{T^*} \end{bmatrix}$$

then the formula (2.22) amounts to the Schäffer matrix representation for the Sz.-Nagy unitary dilation of T (see [22] and [16, Section I.5]).

2.3. **Scattering system to embedded unitary colligation and system trajectories.** Associated with any colligation (unitary or not)

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{E}_* \end{bmatrix}.$$

is a discrete-time linear system

$$\Sigma(\mathbf{U})\colon \left\{ \begin{array}{rcl} x(n+1) & = & Ax(n) + Bu(n) \\ y(n) & = & Cx(n) + Du(n) \end{array} \right. \tag{2.25}$$

where $x(n)$ takes values in the *state space* $\mathcal{X}$, $u(n)$ takes values in the *input space* $\mathcal{E}$ and $y(n)$ takes values in the *output space* $\mathcal{E}_*$ for each $n \in \mathbb{Z}$. If $\mathbf{U}$ is unitary, we say that $\mathbf{U}$ is a *unitary colligation* and that the associated system $\Sigma(\mathbf{U})$ is a (discrete-time) *conservative* linear system. In this case we have *energy balance relations*

$$\|x(n+1)\|^2_{\mathcal{H}} - \|x(n)\|^2_{\mathcal{H}} = \|u(n)\|^2_{\mathcal{E}} - \|y(n)\|^2_{\mathcal{E}_*} \text{ (one-step form)}$$
$$\|x(N+1)\|^2 - \|x(-M)\|^2 = \sum_{n=-M}^{N} \left[\|u(n)\|^2_{\mathcal{E}} - \|y(n)\|^2_{\mathcal{E}_*}\right] \text{ (summed form)} \tag{2.26}$$

for the system as well as for the adjoint system:

$$\|x_*(n+1)\|^2_{\mathcal{H}} - \|x_*(n)\|^2_{\mathcal{H}} = \|u_*(n)\|^2_{\mathcal{E}} - \|y_*(n)\|^2_{\mathcal{E}_*} \text{ (one-step form)}$$
$$\|x_*(N+1)\|^2 - \|x_*(-M)\|^2 = \sum_{n=-M}^{N} \left[\|u_*(n)\|^2_{\mathcal{E}} - \|y_*(n)\|^2_{\mathcal{E}_*}\right] \text{ (summed form)} \tag{2.27}$$

where (u_*, x_*, y_*) is a trajectory of the adjoint system

$$\Sigma(\mathbf{U}^*)_{\text{backward time}}\colon \left\{ \begin{array}{rcl} x_*(n) & = & A^* x_*(n+1) + C^* u_*(n) \\ y_*(n) & = & B^* x_*(n+1) + D^* u_*(n). \end{array} \right. \tag{2.28}$$

We shall assume that we are in the conservative case, so that $\mathbf{U}^* = \mathbf{U}^{-1}$. For this case it happens that (u, x, y) is a system trajectory if and only if $(u_*, x_*, y_*) = (y, x, u)$ is a trajectory of the adjoint system. Assuming that $x(n) = 0$ and $u(n) = 0$ for all $n < -N$ for some $N < \infty$, application of the bilateral Z-transform

$$\{x(n)\}_{n\in\mathbb{Z}} \mapsto \widehat{x}(z) = \sum_{n\in\mathbb{Z}} x(n) z^n$$

to the system equations (2.25) yields

$$\widehat{y}(z) = T_{\mathbf{U}}(z) \cdot \widehat{u}(z) \tag{2.29}$$

where

$$T_{\mathbf{U}}(z) = D + zC(I - zA)^{-1}B \tag{2.30}$$

is the *transfer function* of the system $\Sigma(\mathbf{U})$, or, more simply, of the colligation $\mathbf{U}$. A consequence of the the summed form of the energy balance relation (2.26) is that

$$\|\widehat{y}\|_{L^2_{\mathcal{E}_*}(\mathbb{T})} = \|T_{\mathbf{U}} \cdot \widehat{u}\|_{L^2_{\mathcal{E}_*}(\mathbb{T})} \le \|\widehat{u}\|^2_{L^2_{\mathcal{E}}(\mathbb{T})}$$

so the multiplication operator $M_{T_{\mathbf{U}}}\colon f(z) \to T_{\mathbf{U}} \cdot f(z)$ extends to define a bounded contraction operator from $L^2_{\mathcal{E}}(\mathbb{T})$ into $L^2_{\mathcal{E}_*}(\mathbb{T})$. Moreover, since the system equations (2.25) are causal, it is easily seen that $M_{T_{\mathbf{U}}}$ takes $H^2_{\mathcal{E}}(\mathbb{D})$ into $H^2_{\mathcal{E}_*}(\mathbb{D})$, i.e., $T_{\mathbf{U}} \in \mathcal{S}(\mathcal{E}, \mathcal{E}_*)$.

Now let us suppose that $\mathbf{U}$ is the unitary colligation embedded in a Lax-Phillips scattering system $\mathfrak{S}$ as given by (2.21). Then we have the scattering function $S = S_{\mathfrak{S}} \in \mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ associated with the scattering system as in (2.17) and we also have the transfer function $T_{\mathbf{U}}$ given by (2.30) in the same Schur-class $\mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ associated with the unitary colligation $\mathbf{U} = \mathbf{U}(\mathfrak{S})$ embedded in the scattering system via (2.21). The remarkable fact is that these are the same.

Theorem 2.2. *Suppose that* $\mathbf{U} = \mathbf{U}(\mathfrak{S})$ *is the unitary colligation associated with the Lax-Phillips scattering system as in* (2.21). *Then the scattering function of* $\mathfrak{S}$ *is the same as the transfer function of* $\mathbf{U}$, *i.e.,*

$$S_{\mathfrak{S}}(z) = T_{\mathbf{U}}(z).$$

Among the many proofs of this existing in the literature (e.g. [16, 6]), we sketch here the proof from [5] specialized to our 1-D setting.

For this purpose we define the set $\mathcal{T}$ of *admissible trajectories* of $\Sigma(\mathbf{U})$ to consist of all solutions $(u, x, y) = (u(n), x(n), y(n))_{n\in\mathbb{Z}}$ of the system equations over all of $\mathbb{Z}$ such that

$$u|_{\mathbb{Z}_+} \in \ell^2_{\mathcal{E}}(\mathbb{Z}_+) \text{ and } y|_{\mathbb{Z}_-} \in \ell^2_{\mathcal{E}_*}(\mathbb{Z}_-) \tag{2.31}$$

with trajectory norm

$$\|(u, x, y)\|^2_{\mathcal{T}} = \|y|_{\mathbb{Z}_-}\|^2_{\ell^2_{\mathcal{E}_*}(\mathbb{Z}_-)} + \|x(0)\|^2_{\mathcal{H}} + \|u|_{\mathbb{Z}_+}\|^2_{\ell^2_{\mathcal{E}}(\mathbb{Z}_+)}. \tag{2.32}$$

The key points are summarized in the following Lemma.

Lemma 2.3. (1) *The map*

$$\mathcal{I}_{\mathcal{T}\to\boldsymbol{\mathcal{K}}} \colon (u, x, y) \mapsto y|_{\mathbb{Z}_-} \oplus x(0) \oplus u|_{\mathbb{Z}_+}$$

maps the space of admissible trajectories $\mathcal{T}$ *isometrically onto the space* $\boldsymbol{\mathcal{K}} := \ell^2_{\mathcal{E}_*}(\mathbb{Z}_-) \oplus \mathcal{H} \oplus \ell^2_{\mathcal{E}}(\mathbb{Z}_+)$.

(2) *Let* $\mathbf{U} = \mathbf{U}(\mathfrak{S})$ *be the unitary colligation associated with the scattering system* $\mathfrak{S}$ *as in* (2.21). *The map*

$$\mathcal{I}_{\mathfrak{S}\to\mathcal{T}} \colon k \mapsto \{(i^*_{\mathcal{E}_*}\mathcal{U}^{*n}k,\, i^*_{\mathcal{H}}k,\, i^*_{\mathcal{E}}\mathcal{U}^{*n})\}_{n\in\mathbb{Z}} = \{(\Phi_* k)(n),\, i^*_{\mathcal{H}}\mathcal{U}^{*n}k,\, (\Phi k)(n))\}_{n\in\mathbb{Z}}$$

maps the ambient space $\mathcal{K}$ *of the scattering system onto the space* $\mathcal{T}$ *of admissible trajectories for the colligation* $\mathbf{U}(\mathfrak{S})$. *Moreover, if* $\mathcal{I}_{\mathfrak{S}\to\boldsymbol{\mathcal{K}}}$ *is the map given by* (2.19), *then*

$$\mathcal{I}_{\mathcal{K}\to\boldsymbol{\mathcal{K}}} = \mathcal{I}_{\mathcal{T}\to\boldsymbol{\mathcal{K}}} \circ \mathcal{I}_{\mathcal{K}\to\mathcal{T}}. \tag{2.33}$$

Proof. To prove (1), note that $\mathcal{I}_{\mathcal{T}\to\boldsymbol{\mathcal{K}}}$ maps $\mathcal{T}$ into $\boldsymbol{\mathcal{K}}$ by the definition (2.31) of admissible trajectories. Suppose now that

$$y_0 \oplus x_0 \oplus u_0 \in \ell^2_{\mathcal{E}_*}(\mathbb{Z}_-) \oplus \mathcal{H} \oplus \ell^2_{\mathcal{E}}(\mathbb{Z}_+) =: \boldsymbol{\mathcal{K}}.$$

Use the forward system equations (2.25) with $x(0) = x_0$ and $u(n) = u_0(n)$ for $n \geq 0$ to generate $y(n) \in \mathcal{E}_*$ for $n \geq 0$ and $x(n) \in \mathcal{H}$ for $n > 0$. Similarly use the backward system equations (2.28) with $x(0) = x_0$ and $u_*(n) = y_0(n)$ for $n < 0$ to generate $y_*(n) =: u(n)$ and $x(n)$ for $n < 0$. By using the unitary property of the colligation $\mathbf{U}$, one can then check that the resulting globally defined sequence $\{(u(n), x(n), y(n))\}_{n\in\mathbb{Z}}$ is a system trajectory such that $u|_{\mathbb{Z}_+} = u_0$, $x(0) = x_0$ and $y|_{\mathbb{Z}_-} = y_0$, i.e., such that $\mathcal{I}_{\mathcal{T}}(u, x, y) = y_0 \oplus x_0 \oplus u_0$. We conclude that $\mathcal{I}_{\mathcal{T}\to\boldsymbol{\mathcal{K}}}$ is onto. Moreover, $\mathcal{I}_{\mathcal{T}\to\boldsymbol{\mathcal{K}}}$ is isometric by the definition of the trajectory norm (2.32) and part (1) of Lemma 2.3 follows.

To verify part (2) of Lemma 2.3 we must verify

$$\begin{aligned} i_{\mathcal{H}}^* \mathcal{U}^{*n+1} k &= A i_{\mathcal{H}}^* \mathcal{U}^{*n} k + B i_{\mathcal{E}}^* \mathcal{U}^{*n} k, \\ i_{\mathcal{E}_*}^* \mathcal{U}^{*n} k &= C i_{\mathcal{H}}^* \mathcal{U}^{*n} k + D i_{\mathcal{E}}^* \mathcal{U}^{*n} k \end{aligned} \tag{2.34}$$

for each $k \in \mathcal{K}$. Plugging in the definition (2.21) of $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, multiplying the first equation by $\mathcal{U} i_{\mathcal{H}}$ and the second by $i_{\mathcal{E}_*}$, and using the identities

$$P_{\mathcal{H}} = i_{\mathcal{H}} i_{\mathcal{H}}^*, \qquad P_{\mathcal{E}_8} = i_{\mathcal{E}_*} i_{\mathcal{E}_*}^*,$$

we see that (2.34) is equivalent to

$$\begin{aligned} \mathcal{U} P_{\mathcal{H}} \mathcal{U}^{*n+1} k &= \mathcal{U} P_{\mathcal{H}} \mathcal{U}^* P_{\mathcal{H}} \mathcal{U}^{*n} k + \mathcal{U} P_{\mathcal{H}} \mathcal{U}^* P_{\mathcal{E}} \mathcal{U}^{*n} k \\ P_{\mathcal{E}_*} \mathcal{U}^{*n} k &= P_{\mathcal{E}_*} P_{\mathcal{H}} \mathcal{U}^{*n} k + P_{\mathcal{E}_*} P_{\mathcal{E}} \mathcal{U}^{*n} k. \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \mathcal{U} P_{\mathcal{H}} \mathcal{U}^{*n+1} k &= \mathcal{U} P_{\mathcal{H}} \mathcal{U}^* (P_{\mathcal{H}} + P_{\mathcal{E}}) \mathcal{U}^{*n} k \\ P_{\mathcal{E}_*} \mathcal{U}^{*n} k &= P_{\mathcal{E}_*} (P_{\mathcal{H}} + P_{\mathcal{E}}) \mathcal{U}^{*n} k. \end{aligned} \tag{2.35}$$

Note that $\mathcal{U} P_{\mathcal{H}} \mathcal{U}^* = P_{\mathcal{U}\mathcal{H}}$. Since $\mathcal{E}_* = \mathcal{U}\mathcal{G}_* \ominus \mathcal{G}_* \subset \mathcal{U}\mathcal{G}_*$ and $\mathcal{G}_* \perp \mathcal{H}$, we see that $\mathcal{E}_* \perp \mathcal{U}\mathcal{H}$. Therefore (2.35) can be rewritten as a single equation

$$P_{\mathcal{E}_* \oplus \mathcal{U}\mathcal{H}} \mathcal{U}^{*n} k = P_{\mathcal{E}_* \oplus \mathcal{U}\mathcal{H}} P_{\mathcal{H} \oplus \mathcal{E}} \mathcal{U}^{*n} k. \tag{2.36}$$

This identity (2.36) finally follows from the general identity (2.7). It follows that $\mathcal{I}_{\mathcal{K} \to \mathcal{T}}$ indeed maps $\mathcal{K}$ into trajectories. As we have already noted that Φ_* maps $\mathcal{K}$ into $\ell^2_{\mathcal{E}_*}(\mathbb{Z})$ and that Φ maps $\mathcal{K}$ into $\ell^2_{\mathcal{E}}(\mathbb{Z})$, it follows that $\mathcal{I}_{\mathcal{K} \to \mathcal{T}}$ in fact maps $\mathcal{K}$ into admissible trajectories. Finally the property (2.33) follows from the definitions. This completes the proof of Lemma 2.3. □

Proof of Theorem 2.2. It suffices to show that

$$M_{S_{\mathfrak{S}}}|_{H^2_{\mathcal{E}}(\mathbb{D})} = M_{T_{\mathbf{U}}}|_{H^2_{\mathcal{E}}(\mathbf{D})}. \tag{2.37}$$

Choose $u \in \ell^2_{\mathcal{E}}(\mathbb{Z}_+)$, so $\widehat{u}$ is a general element of $H^2_{\mathcal{E}}(\mathbb{D})$.

By our convention identifying $\ell^2_{\mathcal{E}}(\mathbb{Z}_+)$ as a subspace of $\ell^2_{\mathcal{E}}(\mathbb{Z})$, we also view u as an element of $\ell^2_{\mathcal{E}}(\mathbb{Z})$. Set $k = \Phi^* u \in \mathcal{G}$. Then

$$\Phi_* k = \Phi_* \Phi^* u = \mathbf{S} u \text{ and } \widehat{\Phi}_* k = M_S \widehat{u}. \tag{2.38}$$

Since Φ is a coisometry, we know that $\Phi k = \Phi\Phi^* u = u$ and $\mathcal{I}_{\mathcal{K} \to \mathcal{T}}$ has the form

$$\mathcal{I}_{\mathcal{K} \to \mathcal{T}} k = (u, \{i_{\mathcal{H}}^* \mathcal{U}^{*n} k\}_{n \in \mathbb{Z}}, \mathbf{S} u) =: \{(u(n), x(n), y(n))\}_{n \in \mathbb{Z}}. \tag{2.39}$$

Since $k \in \mathcal{G}$, $\mathcal{H} \perp \mathcal{G}$ and $\mathcal{G}$ is invariant under $\mathcal{U}$, it follows that

$$i_{\mathcal{H}} x(n) = P_{\mathcal{H}} \mathcal{U}^{*n} k = 0 \text{ for } n = 0, -1, -2, \dots.$$

We have already observed as a consequence of the causality assumption (DT-A3) that $\mathbf{S}$ maps $\ell^2_{\mathcal{E}}(\mathbb{Z}_+)$ into $\ell^2_{\mathcal{E}_*}(\mathbb{Z}_+)$. Thus (u, x, y) is a system trajectory for $\Sigma(\mathbf{U})$ with u and y supported on $\mathbb{Z}_+$ and with $x(0) = 0$. The defining property (2.29) for the transfer function $T_{\mathbf{U}}$ now implies that

$$\widehat{y} = T_{\mathbf{U}} \cdot \widehat{u}. \tag{2.40}$$

Combining (2.38), (2.39) and (2.40) now gives (2.37) as wanted.

□

Remark 2.4. One can also prove Theorem 2.2 by computing directly that $S_{\mathfrak{S}}(z)$ and $T_{\mathbf{U}}(z)$ have the same Fourier coefficients:

$$S_{\mathfrak{S}}(z) = D + \sum_{n=1}^{\infty} CA^{n-1}Bz^n = T_{\mathbf{U}}(z).$$

This calculation can be found e.g. in [16, Proposition VI.2.2].

2.4. **Unitary colligation to scattering system: the coordinate-free approach.** Given a unitary colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H} \oplus \mathcal{E}_*$, we have already seen that U can be embedded in a scattering system $\mathfrak{S} = \mathfrak{S}(\mathbf{U})$ by using (2.19), (2.22) and (2.23) to define a Schäffer-matrix model scattering system $\boldsymbol{\mathfrak{S}} = \boldsymbol{\mathfrak{S}}(\mathbf{U})$. Here we turn the map $\mathcal{I}_{\mathcal{K}\to\mathcal{T}}$ around to produce an alternative more coordinate-free version of a Lax-Phillips scattering system $\mathfrak{S} = \mathfrak{S}(\mathbf{U})$ with preassigned unitary colligation $\mathbf{U}$ embedded within it. The added flexibility of this coordinate-free version of $\mathfrak{S}(\mathbf{U})$ is often more convenient for analysis in more general situations (see [5] and [8]). For this purpose, suppose that we are given a unitary colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H} \oplus \mathcal{E}_*$ with associated conservative system $\Sigma(\mathbf{U})$ (2.25) or, equivalently, $\Sigma(\mathbf{U}^*)_{\text{backward}}$ (2.28). Then we may define the set $\mathcal{T}$ of all admissible trajectories $(u(\cdot), x(\cdot), y(\cdot))$ as in (2.31) with norm given by (2.32). Part (1) of Lemma 2.3 applies to any unitary colligation (independent of $\mathbf{U}$ being of the form (2.21) coming from some scattering system $\mathfrak{S}$): hence the map $\mathcal{I}_{\mathcal{T}\to\boldsymbol{\mathcal{K}}}$ is unitary from $\mathcal{T}$ onto $\boldsymbol{\mathcal{K}} := \ell^2_{\mathcal{E}_*}(\mathbb{Z}_-) \oplus \mathcal{H} \oplus \ell^2_{\mathcal{E}}(\mathbb{Z}_+)$ and $\mathcal{T}$ is a Hilbert space in the $\mathcal{T}$ norm (2.32). The added observation here is that the map

$$\mathcal{U}_{\mathcal{T}} \colon (u(\cdot), x(\cdot), y(\cdot)) \mapsto (u(\cdot - 1), x(\cdot - 1), y(\cdot - 1)) \tag{2.41}$$

is unitary on $\mathcal{T}$. The fact that $\mathcal{U}$ takes a system trajectory to a system trajectory is a simple consequence of the time-invariance of the system equations: the colligation operators A, B, C, D in (2.25) are independent of the time n. The isometry property of $\mathcal{U}_{\mathcal{T}}$ follows from the assumed isometry property of $\mathbf{U}$: given a system trajectory $(u(\cdot), x(\cdot), y(\cdot))$ we must check that $\|\mathcal{U}_{\mathcal{T}}(u,x,y)\|^2_{\mathcal{T}} = \|(u,x,y)\|^2_{\mathcal{T}}$, i.e., that

$$\begin{aligned} &\sum_{n=-\infty}^{0} \|y(n)\|^2_{\mathcal{E}_*} + \|x(1)\|^2_{\mathcal{H}} + \sum_{n=1}^{\infty} \|u(n)\|^2_{\mathcal{E}} \\ &\quad = \sum_{n=-\infty}^{-1} \|y(n)\|^2_{\mathcal{E}_*} + \|x(0)\|^2_{\mathcal{H}} + \sum_{n=0}^{\infty} \|u(n)\|^2. \end{aligned}$$

After performing the obvious cancellations we are left to check that

$$\|y(0)\|^2_{\mathcal{E}_*} + \|x(1)\|^2_{\mathcal{H}} = \|x(0)\|^2_{\mathcal{H}} + \|u(0)\|^2_{\mathcal{E}}$$

which follows immediately from the fact that $\mathbf{U}$ is isometric. Similarly, the fact that

$$\mathcal{U}_{\mathcal{T}}^{-1} \colon (u(\cdot), x(\cdot), y(\cdot)) \to (u(\cdot + 1), x(\cdot + 1), y(\cdot + 1)$$

is also isometric follows from the fact that $\mathbf{U}$ is coisometric. Thus $\mathcal{U}_{\mathcal{T}}$ given by (2.41) defines a unitary operator on $\mathcal{T}$. Let us define incoming and outgoing subspaces $\mathcal{G}_{\mathcal{T},*} \subset \mathcal{T}$ and $\mathcal{G}_{\mathcal{T}} \subset \mathcal{T}$ by

$$\begin{aligned} \mathcal{G}_{*\mathcal{T}} &= \{(u,x,y) \in \mathcal{T} \colon x(0) = 0 \text{ and } y|_{\mathbb{Z}_-} = 0\}, \\ \mathcal{G}_{\mathcal{T}} &= \{(u,x,y) \in \mathcal{T} \colon u|_{\mathbb{Z}_-} = 0 \text{ and } x(0) = 0\} \end{aligned} \tag{2.42}$$

with associated scattering subspace $\mathcal{H}_{\mathcal{T}} := \mathcal{T} \ominus [\mathcal{G}_{*\mathcal{T}} \oplus \mathcal{G}_{\mathcal{T}}]$ given by

$$\mathcal{H}_{\mathcal{T}} = \{(u, x, y) \in \mathcal{T} \colon u|_{\mathbb{Z}_+} = 0 \text{ and } y|_{\mathbb{Z}_-} = 0\}. \tag{2.43}$$

Then we arrive at the following proposition. Given the preceding discussion, the details of the proof are routine and left to the reader.

Proposition 2.5. *Suppose that we are given a unitary colligation* $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H} \oplus \mathcal{E}_*$. *Define Hilbert spaces* $\mathcal{T}$, $\mathcal{G}_{*\mathcal{T}}$ *and* $\mathcal{G}_{\mathcal{T}}$ *as in* (2.31), (2.42) *and the operator* $\mathcal{U}$ *on* $\mathcal{T}$ *by* (2.41). *Then*

$$\mathfrak{S} = \mathfrak{S}(\mathbf{U}) = (\mathcal{U}_{\mathcal{T}}; \mathcal{T}, \mathcal{G}_{\mathcal{T}}, \mathcal{G}_{*\mathcal{T}})$$

is a (discrete-time) Lax-Phillips scattering system (i.e., axioms (DT-A1), (DT-A2) and (DT-A3) are satisfied). Moreover, we recover $\mathbf{U}$ *as the associated unitary colligation* $\mathbf{U} = \mathbf{U}(\mathfrak{S}(\mathbf{U}))$ *embedded in* $\mathfrak{S}(\mathbf{U})$ *according to* (2.21).

2.5. **Spectral theory of the scattering function/transfer function: the discrete-time case.** Given a pair of operators (A, B) with $A \colon \mathcal{H} \to \mathcal{H}$ and $B \colon \mathcal{U} \to \mathcal{H}$, we say that (A, B) is *approximately controllable* if

$$\overline{\operatorname{span}}\{\operatorname{Ran} A^n B \colon n = 0, 1, 2, \dots\} = \mathcal{X}.$$

Similarly, given a pair of operators (C, A) with $A \colon \mathcal{X} \to \mathcal{X}$ and $C \colon \mathcal{X} \to \mathcal{Y}$, we say that (C, A) is *approximately observable* if

$$\bigcap_{n=0}^{\infty} \operatorname{Ker} C A^n = \{0\},$$

or, equivalently, if the pair (A^*, C^*) is approximately controllable. Given a contraction operator A on a Hilbert space $\mathcal{H}$, it is known that $\mathcal{X}$ has an orthogonal decomposition as

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{cnu} \tag{2.44}$$

such that

(1) both $\mathcal{H}_u$ and $\mathcal{H}_{cnu}$ are reducing for A, and
(2) $A|_{\mathcal{H}_u}$ is unitary while there is no nonzero reducing subspace $\mathcal{H}'_{cnu} \subset \mathcal{H}_{cnu}$ such that $A|_{\mathcal{H}'_{cnu}}$ is unitary

(see e.g. [16, Theorem I.3.2]). We say that the contraction operator A is *completely nonunitary* if the subspace $\mathcal{X}_u$ in the above decomposition is zero, or equivalently, there is no nonzero reducing subspace for A on which A is unitary. The following theorem gives basic connections between scattering geometry and system geometry when the unitary colligation $\mathbf{U}$ is embedded in the Lax-Phillips scattering system $\mathfrak{S}$ as in (2.21). This result is probably well known but we sketch the ideas of the proof for completeness.

Theorem 2.6. *Suppose that* $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *is the unitary colligation embedded in the Lax-Phillips scattering system* $\mathfrak{S} = (\mathcal{U}; \mathcal{K}, \mathcal{G}, \mathcal{G}_*)$ *as in* (2.21). *Define* $\widetilde{\mathcal{G}}$ *and* $\widetilde{\mathcal{G}}_*$ *as in* (2.2). *Then:*

(1) $\widetilde{\mathcal{G}} = \mathcal{K}$ *if and only if* (A, B) *is approximately controllable.*
(2) $\widetilde{\mathcal{G}}_* = \mathcal{K}$ *if and only if* (C, A) *is approximately observable.*
(3) $\mathfrak{S}$ *is minimal, i.e.,* $\widetilde{\mathcal{G}}_* + \widetilde{\mathcal{G}}$ *is dense in* $\mathcal{K}$, *if and only if* A *is completely nonunitary.*

Proof. For the purposes of this proof we drop the inclusion operators $i_{\mathcal{E}}$, $i_{\mathcal{H}}$ and $i_{\mathcal{E}_*}$ and consider these spaces as already sitting in the ambient space $\mathcal{K}$. The adjoint operators $i_{\mathcal{E}}^*$, $i_{\mathcal{H}}^*$ and $i_{\mathcal{E}_*}^*$ are then replaced by the orthogonal projections $P_{\mathcal{E}}$, $P_{\mathcal{H}}$ and $P_{\mathcal{E}_*}$. From the orthogonal decomposition $\mathcal{K} = \mathcal{G}_* \oplus \mathcal{H} \oplus \mathcal{H}$ and the fact that $\widetilde{\mathcal{G}} \supset \mathcal{G}$, we see that

$$\widetilde{\mathcal{G}}^{\perp} \subset \mathcal{G}_* \oplus \mathcal{H}.$$

Moreover, as the orthogonal complement of the $\mathcal{U}$-reducing subspace $\widetilde{\mathcal{G}}$, the space $\widetilde{\mathcal{G}}^{\perp}$ is reducing for $\mathcal{U}$ and hence in particular invariant for $\mathcal{U}$. Hence if $k \in \widetilde{\mathcal{G}}^{\perp}$, then $\mathcal{U}^n k \in \widetilde{\mathcal{G}}^{\perp}$ for all $n = 0, 1, 2, \dots$ and

$$\|k\|^2 = \|\mathcal{U}^n k\|^2 = \|P_{\mathcal{G}_*}\mathcal{U}^n k\|^2 + \|P_{\mathcal{H}}\mathcal{U}^n k\|^2$$

where

$$\lim_{n\to\infty} \|P_{\mathcal{G}_*}\mathcal{U}^n k\|^2 = 0.$$

We conclude that

$$\widetilde{\mathcal{G}}^{\perp} \neq \{0\} \Longleftrightarrow P_{\mathcal{H}}\widetilde{\mathcal{G}}^{\perp} \neq \{0\}. \tag{2.45}$$

Next note that

$$\begin{aligned} h \in P_{\mathcal{H}}\widetilde{\mathcal{G}}^{\perp} &\Longleftrightarrow h \in \mathcal{H} \text{ and } h \perp \widetilde{\mathcal{G}} = \bigoplus_{n=-\infty}^{\infty} \mathcal{U}^n \mathcal{E} \\ &\Longleftrightarrow h \in \mathcal{H} \text{ and } h \perp \bigoplus_{n=-\infty}^{-1} \mathcal{U}^n \mathcal{E} \text{ (since } \mathcal{H} \perp \mathcal{G} = \bigoplus_{n=0}^{\infty} \mathcal{U}^n \mathcal{E}) \\ &\Longleftrightarrow h \in \mathcal{H} \text{ and } h \perp P_{\mathcal{H}}\mathcal{U}^{*n}\mathcal{E} \text{ for } n = 1, 2, 3, \dots \\ &\Longleftrightarrow h \in \mathcal{H} \text{ and } h \perp A^n B \text{ for } n = 0, 1, 2, \dots \end{aligned}$$

where the last step is by the definition of A and B in (2.21). We conclude that

$$P_{\mathcal{H}}\widetilde{\mathcal{G}}^{\perp} \neq \{0\} \Longleftrightarrow (A, B) \text{ is not approximately controllable.} \tag{2.46}$$

Combining (2.45) and (2.46) gives us the first assertion of Theorem 2.6. The second assertion is verified by an analogous argument using that $\lim_{n\to\infty} P_{\mathcal{G}}\mathcal{U}^{*n}k = 0$ for all $k \in \mathcal{K}$.

To verify the third assertion, from the orthogonal decomposition $\mathcal{K} = \mathcal{G}_* \oplus \mathcal{H} \oplus \mathcal{G}$ and the fact that $\widetilde{\mathcal{G}}_* + \widetilde{\mathcal{G}} \supset \mathcal{G}_* \oplus \mathcal{G}$, we see that $[\widetilde{\mathcal{G}}_* + \widetilde{\mathcal{G}}]^{\perp} \subset \mathcal{H}$. Further careful analysis using that $[\widetilde{\mathcal{G}}_* + \widetilde{\mathcal{G}}]^{\perp}$ is reducing for $\mathcal{U}$ and that $A = P_{\mathcal{H}}\mathcal{U}^*|_{\mathcal{H}}$ (from (2.21)) shows that one can make the identification

$$[\widetilde{\mathcal{G}}_* + \widetilde{\mathcal{G}}]^{\perp} = \mathcal{H}_u \tag{2.47}$$

where $\mathcal{H}_u$ is the unitary subspace for A in the decomposition (2.44) for A. The third assertion of Theorem 2.6 is an immediate consequence of (2.47). □

The scattering system $\mathfrak{S} = (\mathcal{U}; \mathcal{K}, \mathcal{G}, \mathcal{G}_*)$ is minimal in the sense of Lax-Phillips when $\widetilde{G}_* + \widetilde{\mathcal{G}}$ is dense in $\mathcal{K}$. By Theorem 2.6, this is equivalent to the embedded unitary colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ being not only unitary but also approximately controllable and approximately observable. Note that the condition $\widetilde{\mathcal{G}} = \mathcal{K}$ corresponds to the coisometric Fourier representation operator $\widehat{\Phi}$ being unitary while the condition $\widetilde{\mathcal{G}}_* = \mathcal{K}$ corresponds to the coisometry $\widehat{\Phi}_* \colon \mathcal{K} \to L^2_{\mathcal{E}_*}(\mathbb{T})$ being unitary. In the case of Lax-Phillips minimality, we therefore have that the scattering operator $M_S = \widehat{\mathbf{S}} = \widehat{\Phi}_* \widehat{\Phi}^*$ is unitary from which we get that the associated scattering function $S(z)$ is a *two-sided inner function*, i.e., $S \in H^\infty_{\mathcal{L}(\mathcal{E},\mathcal{E}_*)}(\mathbb{T})$ with boundary-value function $S(\zeta)$ on the unit circle having unitary values for almost all $\zeta \in \mathbb{T}$. Theorem VI.4.1 in [16] identifies the spectrum of the operator A with the points λ in the unit disk $\mathbb{D}$ where $S(\lambda)$ is not invertible together with the points ζ on the unit circle $\mathbb{T}$ where S fails to have analytic continuation to a neighborhood of ζ extending into the exterior of the unit disk. In particular, isolated eigenvalues λ for A are identified as points λ where $S(\lambda)$ has nontrivial kernel. As the formula $S(z) = S(1/\overline{z})^{*-1}$ gives a pseudomeromorphic continuation of S to the exterior of the unit disk, we see that point spectrum of A can also be identified with poles of $S(z)$ in the exterior of the unit disk. The following refinement of these spectral results is based on connections with system theory.

Theorem 2.7. *Suppose that $\mathfrak{S} = (\mathcal{U}; \mathcal{K}, \mathcal{G}, \mathcal{G}_*)$ is a Lax-Phillips scattering system which is minimal in the sense of Lax-Phillips, i.e., $\widetilde{\mathcal{G}}_* + \widetilde{\mathcal{G}}$ is dense in $\mathcal{K}$, and that $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is the associated embedded unitary colligation given by* (2.21)*. Then the point $\lambda \in \mathbb{D}$ is a pole of order m for $(zI - A)^{-1}$ if and only if $1/\lambda$ is a pole of order m for $S(z)$.*

Proof. By Theorem 2.2 we know that $S(z) = T_{\mathbf{U}}(z)$ and hence has the representation

$$S(z) = D + zC(I - zA)^{-1}B = D + C(z^{-1}I - A)^{-1}B \tag{2.48}$$

By the discussion preceding the theorem, in the Lax-Phillips-minimal case we know that (A, B) is approximately controllable and that (C, A) is approximately observable. In the rational case where A, B, C, D are finite matrices, the approximate controllability is exact controllability and the approximate observability is exact observability and hence the formula (2.48) gives rise to the following *minimal realization* for $S(z^{-1})$:

$$S(z^{-1}) = D + C(zI - A)^{-1}B. \tag{2.49}$$

It is a standard fact in finite-dimensional system theory that, in the minimal (i.e., controllable and observable) case, there is no pole-zero cancellation and poles of A show up (to the same order) as poles of $S(z^{-1})$ in (2.49). These results (for isolated poles) have been extended to the infinite-dimensional setting, under the assumption that the realization is both approximately controllable and approximately observable, in [27]. With this result, Theorem 2.7 now follows. □

Remark 2.8. Theorem VI.4.1 in the book of Sz.-Nagy-Foias [16] actually includes the general case where $\mathfrak{S}$ is only minimal in the Adamjan-Arov sense (i.e., $\mathcal{G}_* + \mathcal{G}$ is dense in $\mathcal{K}$), or, equivalently, where the main operator A of $\mathbf{U}$ is a general completely nonunitary operator. In case either or both of the conditions $\widetilde{\mathcal{G}}_* = \mathcal{K}$ and $\widetilde{\mathcal{G}} = \mathcal{K}$ fails, then the spectrum of A is the whole closed unit disk $\overline{\mathbb{D}}$.

3. Lax-Phillips scattering and conservative linear systems: the continuous-time case

3.1. **Lax-Phillips scattering system: axiomatic form.**

By a *continuous-time Lax-Phillips scattering system* we mean a collection

$$\mathfrak{S} = (\mathcal{U}(t); \mathcal{K}, \mathcal{G}, \mathcal{G}_*) \tag{3.1}$$

where $\mathcal{K}$ is a Hilbert space, $t \mapsto \mathcal{U}(t)$ is a strongly continuous group of unitary linear operators on $\mathcal{K}$, $\mathcal{G}$ (the *outgoing subspace*) and $\mathcal{G}_*$ (the *incoming subspace*) are subspaces of $\mathcal{K}$ subject to the following axioms:

(CT-A1) $\mathcal{U}(t)\mathcal{G} \subset \mathcal{G}$ for $t \geq 0$ and $\bigcap_{t\colon t\geq 0} \mathcal{U}(t)\mathcal{G} = \{0\}$,
(CT-A2) $\mathcal{U}(t)\mathcal{G}_* \subset \mathcal{G}_*$ for $t \leq 0$ and $\bigcap_{t\colon t\leq 0} \mathcal{U}(t)\mathcal{G}_* = \{0\}$, and
(CT-A3) the *causality condition* $\mathcal{G}_* \perp \mathcal{G}$ holds.

We say that the scattering system is *minimal* if

$$\widetilde{\mathcal{G}}_* + \widetilde{\mathcal{G}} \text{ is dense in } \mathcal{K}$$

where we have set

$$\widetilde{\mathcal{G}} = \text{ closure of } \bigcup_{t\colon t\leq 0} \mathcal{U}(t)\mathcal{G}, \qquad \widetilde{\mathcal{G}}_* = \text{ closure of } \bigcup_{t\colon t\geq 0} \mathcal{U}(t)\mathcal{G}_*. \tag{3.2}$$

Given this setup, there are translation representation operators Φ_* and Φ analogous to (2.8) for the discrete-time case; it is however more technical to define and derive their properties.

We let $L^2(\mathbb{R})$ denote the space of measurable, modulus-squared integrable complex-valued functions on the real line. For a coefficient Hilbert space $\mathcal{X}$, we will use the notation $L^2_{\mathcal{X}}(\mathbb{R})$ for the space $L^2(\mathbb{R}) \otimes \mathcal{X}$ of $L^2(\mathbb{R})$-functions with values in $\mathcal{X}$. We let $L^2_{\mathcal{X}}(\mathbb{R}_+)$ (respectively $L^2_{\mathcal{X}}(\mathbb{R}_-)$) denote the subspace of $L^2_{\mathcal{X}}(\mathbb{R})$ consisting of functions with support inside $\mathbb{R}_+ = \{t \in \mathbb{R}\colon t \geq 0\}$ (respectively, inside $\mathbb{R}_- := \{t \in \mathbb{R}\colon t \leq 0\}$). Finally, we let $t \mapsto \tau^t \otimes I_{\mathcal{X}}$ denote the left translation semigroup on $L^2_{\mathcal{X}}(\mathbb{R})$

$$\tau^t \otimes I_{\mathcal{X}} \colon f(\cdot) \mapsto f(\cdot + t)$$

while $\tau^t_- \otimes I_{\mathcal{X}}$ and $\tau^t_+ \otimes I_{\mathcal{X}}$ denotes its respective compressions to $L^2_{\mathcal{X}}(\mathbb{R}_-)$ and $L^2_{\mathcal{X}}(\mathbb{R}_+)$. On occasion we abbreviate $\tau^t \otimes I_{\mathcal{X}}$ to τ^t; the meaning should be clear from the context.

Proposition 3.1. *Suppose that $\mathfrak{S}$ as in* (3.1) *is a continuous-time Lax-Phillips scattering system. Then there are coefficient Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$ together with coisometries*

$$\Phi_* \colon \mathcal{K} \to L^2_{\mathcal{E}_*}(\mathbb{R}), \qquad \Phi \colon \mathcal{K} \to L^2_{\mathcal{E}}(\mathbb{R})$$

so that:

(1) *The initial space of Φ is $\widetilde{\mathcal{G}}$, Φ maps $\mathcal{G}$ isometrically onto $L^2_{\mathcal{E}}(\mathbb{R}_+)$ and we have the intertwining*

$$(\tau^{-t} \otimes I_{\mathcal{E}})\Phi = \Phi\mathcal{U}(t).$$

(2) *The initial space of $\Phi_*$is $\widetilde{\mathcal{G}}_*$, Φ_* maps $\mathcal{G}_*$ isometrically onto $L^2_{\mathcal{E}_*}(\mathbb{R}_-)$ and we have the intertwining*

$$(\tau^{-t} \otimes I_{\mathcal{E}_*})\Phi_* = \Phi_*\mathcal{U}(t).$$

Proof. It suffices to verify part (1) as part (2) is completely symmetric (after replacing t by $-t$).

One approach is to consider the *cogenerator* of the semigroup $\mathcal{V}(t)$ defined below (see (3.3)), i.e., the Cayley transform $V = (I + A)(I - A)^{-1}$ of the infinitesimal generator A of $\mathcal{V}$. Then V^* turns out to be a shift and one can construct the discrete-time representation operator Φ_{discrete} as is done in Section 2.1 above. This approach is carried out in [14, Section II.3]. A slick proof working with translation-invariant spectral measures can be found in [9, Section 12.1.3]. Here we prefer a direct approach which amounts to a more detailed version of the approach sketched in [15, Section 35.5] (see also [14, page 67]).

We assume that we are given a unitary group $t \mapsto \mathcal{U}(t)$ on the Hilbert space $\mathcal{K}$ together with a subspace $\mathcal{G}$ satisfying axiom (CT-A1). We define a contractive semigroup $t \mapsto \mathcal{V}(t)$ on $\mathcal{G}$ by

$$\mathcal{V}(t) = P_{\mathcal{G}} \mathcal{U}(t)^*|_{\mathcal{G}}. \tag{3.3}$$

The group property of $\mathcal{U}(t)$ combined with the fact that $\mathcal{G}$ is invariant for $\mathcal{U}(t)$ (and hence $\mathcal{G}^\perp$ is invariant for $\mathcal{U}(t)^*$) for $t \geq 0$ implies that $t \mapsto \mathcal{V}(t)$ has the semigroup property: for $s, t > 0$ we have

$$\begin{aligned} \mathcal{V}(s+t)x &= P_{\mathcal{G}} \mathcal{U}(s+t)^* x \\ &= P_{\mathcal{G}} \mathcal{U}(s)^* \mathcal{U}(t)^* x \\ &= P_{\mathcal{G}} \mathcal{U}(s)^* (P_{\mathcal{G}} + P_{\mathcal{G}^\perp}) \mathcal{U}(t)^* x \\ &= P_{\mathcal{G}} \mathcal{U}(s)^* P_{\mathcal{G}} \mathcal{U}(t)^* x \\ &= \mathcal{V}(s)\mathcal{V}(t)x. \end{aligned}$$

The semigroup $\mathcal{V}(t)$ also has the property that, for all $x \in \mathcal{G}$, $\mathcal{V}(t)x \to 0$ as $t \to \infty$. To see this, note first that the set

$$\{\mathcal{U}(t)y : y \in \mathcal{G}^\perp,\ t > 0\}$$

is dense in $\mathcal{K}$. If not, there must exist a nonzero $z \in \mathcal{K}$ such that $\langle z, \mathcal{U}(t)y \rangle = 0$ for all $y \in \mathcal{G}^\perp$ and $t > 0$. Since the unitary operators $\mathcal{U}(t)$ preserve orthogonal complements, this forces $z \in \mathcal{U}(t)\mathcal{G}$ for all $t > 0$, which in turn forces $z = 0$ by axiom (CT-A1). Hence, for $x \in \mathcal{G}$ and $\varepsilon > 0$, there exists $y \in \mathcal{G}^\perp$ and $t > 0$ which satisfy $\|x - \mathcal{U}(t)y\| < \varepsilon$. Then

$$\|\mathcal{U}(t)^* x - y\| = \|\mathcal{U}(t)^*(x - \mathcal{U}(t)y)\| = \|x - \mathcal{U}(t)y\| < \epsilon$$

as well, and applying the projection operator $P_{\mathcal{G}}$,

$$\|\mathcal{V}(t)x\| = \|P_{\mathcal{G}} \mathcal{U}(t)^* x\| = \|P_{\mathcal{G}}(\mathcal{U}(t)^* x - y)\| \leq \|\mathcal{U}(t)^* x - y\| < \varepsilon.$$

So for $s > t$, it follows that

$$\|\mathcal{V}(s)x\| = \|\mathcal{V}(s-t)\mathcal{V}(t)x\| \leq \|\mathcal{V}(t)x\| < \varepsilon,$$

and hence $\mathcal{V}(s)x \to 0$ as $s \to \infty$ for every $x \in \mathcal{G}$.

Let A be the infinitesimal generator of the semigroup $\mathcal{V}(t)$ with domain $\mathcal{D}(A)$. Define a new norm on $\mathcal{D}(A)$ by

$$\|x\|_{\mathcal{E}}^2 = -\left.\frac{d}{dt}\right|_{t=0} \|\mathcal{V}(t)x\|_{\mathcal{G}}^2 = -\langle Ax, x \rangle_{\mathcal{G}} - \langle x, Ax \rangle_{\mathcal{G}}.$$

Since the operators $\mathcal{V}(t)$ are contractions, for $t > t'$,

$$\begin{aligned} \|\mathcal{V}(t)x\|_{\mathcal{G}}^2 &= \|\mathcal{V}(t - t' + t')x\|_{\mathcal{G}}^2 = \|\mathcal{V}(t-t')\mathcal{V}(t')x\|_{\mathcal{G}}^2 \\ &\leq \|\mathcal{V}(t-t')\|^2 \|\mathcal{V}(t')x\|_{\mathcal{G}}^2 \leq \|\mathcal{V}(t')x\|_{\mathcal{G}}^2, \end{aligned}$$

so the function $\|\mathcal{V}(t)x\|_{\mathcal{G}}^2$ is nonincreasing and $\|x\|_{\mathcal{E}}^2$ is indeed nonnegative. The auxiliary space $\mathcal{E}$ is defined to be the completion of $\mathcal{D}(A)$ under $\|\cdot\|_{\mathcal{E}}$, modulo the vectors of norm zero. Note that this identification of equivalence classes creates a bounded mapping $\Pi : \mathcal{D}(A) \to \mathcal{E}$ if $\mathcal{D}(A)$ is given the graph norm

$$\|y\|_{\mathcal{D}}^2 = \|y\|_{\mathcal{G}}^2 + \|Ay\|_{\mathcal{G}}^2.$$

We define an operator Φ_0 from $\mathcal{D}(A)$ into a space of $\mathcal{E}$-valued functions by

$$(\Phi_0 x)(s) = \Pi[\mathcal{V}(s)x] \text{ for } x \in \mathcal{D}(A). \tag{3.4}$$

(Here we use that $\mathcal{V}(s)$ maps $\mathcal{D}(A)$ into itself for $s > 0$.) We note that, if we view Π as the continuous-time analogue of the projection operator $i^*_{\mathcal{E}}$, then formula (3.4) can be viewed as the continuous-time analogue of the formula (2.8) for the discrete-time case. We next check that Φ_0 is an isometry from $\mathcal{D}(A)$ (with the $\mathcal{G}$-norm) into $L^2_{\mathcal{E}}(\mathbb{R}_+)$ as follows: if the $\mathcal{E}$-valued function f is defined by the right-hand side of (3.4), then

$$\begin{aligned}
\|f\|^2_{L^2_{\mathcal{E}}(\mathbb{R}_+)} &= \int_0^\infty \|\mathcal{V}(s)x\|^2_{\mathcal{E}}\, ds \\
&= \int_0^\infty (-\langle A\mathcal{V}(s)x, \mathcal{V}(s)x\rangle_{\mathcal{G}} - \langle \mathcal{V}(s)x, A\mathcal{V}(s)x\rangle_{\mathcal{G}})\, ds \\
&= \int_0^\infty -\frac{d}{ds}\|\mathcal{V}(s)x\|^2_{\mathcal{G}}\, ds \\
&= \|\mathcal{V}(0)x\|^2_{\mathcal{G}} - \lim_{s\to\infty}\|\mathcal{V}(s)x\|^2_{\mathcal{G}} \\
&= \|x\|^2_{\mathcal{G}}.
\end{aligned}$$

Since $\mathcal{D}(A)$ is dense in $\mathcal{G}$, we can extend Φ_0 by continuity to an isometry (still denoted by Φ_0) from all of $\mathcal{G}$ into $L^2_{\mathcal{E}}(\mathbb{R}_+)$.

To verify the intertwining properties of Φ_0 with respect to $\mathcal{V}(t)$, observe first that for arbitrary $x \in \mathcal{D}(A)$ and $t, s > 0$ we have

$$(\Phi_0\mathcal{V}(t)x)(s) = \Pi\mathcal{V}(s)\mathcal{V}(t)x = \Pi\mathcal{V}(s+t)x = (\tau^t_+\Phi_0 x)(s),$$

and thus $\Phi_0\mathcal{V}(t) = \tau^t_+\Phi_0$ for $t > 0$. Similarly, the case for $\mathcal{U}(t)$, for $x \in \mathcal{D}(A)$ and $s > t$,

$$(\Phi_0\mathcal{V}(t)^*x)(s) = \Pi P_{\mathcal{G}}\mathcal{U}(s)^*\mathcal{U}(t)x = \Pi P_{\mathcal{G}}\mathcal{U}(s-t)^*x = \Pi\mathcal{V}(s-t)x = (\tau^{-t}_+\Phi_0 x)(s).$$

Now suppose $s < t$; then for $r < s - t$, $\mathcal{V}(r)\mathcal{U}(t-s)x = \mathcal{U}(t-s-r)x$. However,

$$\|\mathcal{U}(t-s-r)x\|^2_{\mathcal{G}} = \|x\|^2_{\mathcal{G}} = \|\mathcal{U}(t-s)x\|^2_{\mathcal{G}},$$

implying that $\|\mathcal{V}(r)\mathcal{U}(t-s)x\|^2_{\mathcal{G}}$ is constant for $r < t - s$. Since the norm of $\mathcal{E}$ is defined by the derivative of this quantity at $r = 0$, it follows that $\Pi P_{\mathcal{G}}\mathcal{U}(t-s)x = 0$ for $s < t$. Hence $\Phi_0\mathcal{V}(t)^*|_{\mathcal{G}} = \tau^{-t}_+\Phi_0 = (\tau^t_+)^*\Phi_0$, and thus $\operatorname{Ran}(\Phi_0)$ is a reducing subspace for the operator τ^t_+ on $L^2(\mathbb{R}_+, \mathcal{E})$ for each $t > 0$.

Closed reducing subspaces $\mathcal{R}$ of $L^2_{\mathcal{R}}(\mathbb{R}_+)$ are known to have the form $L^2_{\mathcal{E}_0}(\mathbb{R}_+)$ where $\mathcal{E}_0$ is a closed subspace of $\mathcal{E}$. To see this, we note that a subspace $\mathcal{R}$ is reducing for τ^t_+ for all $t > 0$ if and only if $\mathcal{R}$ is reducing for $\mathcal{S} \otimes I_{\mathcal{E}} = (I - \mathcal{A}^*)(I + \mathcal{A}^*)^{-1}$ where $\mathcal{A} = \frac{\mathsf{d}}{\mathsf{d}t}$ is the infinitesimal generator of the semigroup τ^t_+. The operator $\mathcal{S}$ in turn is the shift operator

$$\mathcal{S} \colon \phi_n \mapsto \phi_{n+1}$$

on the orthonormal basis for $L^2(\mathbb{R})$ given by the Laguerre polynomials

$$\phi_n(t) = (-1)^n\sqrt{2}e^{-t}\sum_{k=0}^{n}\binom{n}{k}\frac{(-2t)^k}{k!} \text{ for } t \geq 0 \text{ and } n = 0, 1, 2, \ldots$$

(see [26, Theorem 12.3.3] with $\alpha = 1$). The asserted form for a reducing subspace for τ^t_+ (or equivalently, for $\mathcal{S} \otimes I_{\mathcal{E}}$) now follows from the known form of reducing subspaces for a shift operator (see e.g. [18]).

We conclude that the subspace $\mathcal{R} := \operatorname{Ran}\Phi_0$ has the form $L^2_{\mathcal{E}_0}(\mathbb{R}_+)$ for a closed subspace $\mathcal{E}_0$ of $\mathcal{E}$. Hence, for any $x \in \mathcal{D}(A)$, $\Pi[V(t)x] \in \mathcal{E}_0$. Since $\Pi[\mathcal{D}(A)]$ is dense in $\mathcal{E}$ by construction, $\mathcal{E}_0$ must also be dense in $\mathcal{E}$. As $\mathcal{E}_0$ is closed, necessarily $\mathcal{E}_0 = \mathcal{E}$. We conclude that Φ_0 is actually unitary from $\mathcal{G}$ onto $L^2_{\mathcal{E}}(\mathbb{R}_+)$.

The next step is to extend the representation operator Φ_0 from $\mathcal{G}$ to $\widetilde{\mathcal{G}} :=$ clos. $\cup_{t<0}\mathcal{U}(t)\mathcal{G}$. Let $\tilde{x} = \mathcal{U}(t)x$ for some $t < 0$ and $x \in \mathcal{G}$; then define $\Phi\tilde{x} = \tau_+^{-t}\Phi_0 x$. It must be verified that this definition is independent of the representation of $\tilde{x}$. Suppose $U(t_1)x_1 = U(t_2)x_2$ for $t_1 \le t_2 < 0$ and $x_1, x_2 \in \mathcal{G}$. Then $x_1 = U(t_2 - t_1)x_2$; the right-hand term remains in $\mathcal{G}$ since $t_2 - t_1 \ge 0$. Now

$$\begin{aligned} \Phi_0 x_1 &= \Phi_0 U(t_2 - t_1)x_2 \\ &= \tau_+^{t_1 - t_2}\Phi_0 x_2, \end{aligned}$$

and thus $\tau_+^{-t_1}\Phi_0 x_1 = \tau_+^{-t_2}\Phi_0 x_2$. Hence

$$\Phi : \bigcup_{t<0} U(t)\mathcal{G} \to \bigcup_{t>0} \tau^t L^2(\mathbb{R}_+, \mathcal{E})$$

is a surjection, by the unitarity of Φ_0. In addition,

$$\|\tau_+^{-t}\Phi_0 x\|_{L^2([t,\infty),\mathcal{E})} = \|\Phi_0 x\|_{L^2(\mathbb{R}_+,\mathcal{E})} = \|x\|_{\mathcal{G}} = \|U(t)x\|_{\widetilde{\mathcal{G}}},$$

proving that Φ is an isometry. The domain space of Φ is dense in $\widetilde{\mathcal{G}}$, and its range space is dense in $L^2(\mathbb{R}, \mathcal{E})$. Therefore, Φ extends to a unitary operator from $\widetilde{\mathcal{G}}$ onto $L^2(\mathbb{R}, \mathcal{E})$. Setting $\Phi \equiv 0$ on $\mathcal{K} \ominus \widetilde{\mathcal{G}}$ yields a coisometry, which completes the proof. □

Remark 3.2. It is possible to obtain an explicit formula for the adjoint of the translation representation operator Φ_0. Let $\mathcal{D}'$ be the completion of $\mathcal{G}$ in the linear functional norm

$$\|x\|_{\mathcal{D}'} = \sup\{|\langle y, x\rangle| : y \in \mathcal{D}(A),\ \|y\|_{\mathcal{D}} \le 1\},$$

where $\mathcal{D}$ is $\mathcal{D}(A)$ imbued with the graph norm. The pairing $\langle y, x\rangle_{\mathcal{G}}$ now extends to a pairing $\langle y, y'\rangle_{\mathcal{D}\times\mathcal{D}'}$ which satisfies

$$\langle y, y'\rangle_{\mathcal{D}\times\mathcal{D}'} \le \|y\|_{\mathcal{D}}\, \|y'\|_{\mathcal{D}'}.$$

In fact, $\|y'\|_{\mathcal{D}'} = \sup\{|\langle y, y'\rangle| : \|y\|_{\mathcal{D}} \le 1\}$, and thus $\mathcal{D}'$ can be identified with the space of bounded linear functionals on $\mathcal{D}$ via the pairing $\ell_{y'}(y) = \langle y, y'\rangle_{\mathcal{D}\times\mathcal{D}'}$. Since the operator $\Pi : \mathcal{D} \to \mathcal{E}$ is bounded, there exists an adjoint operator $\Pi^* : \mathcal{E} \to \mathcal{D}'$. For any $\alpha \in \rho(A)$, the formula

$$\mathcal{V}(t)|_{\mathcal{D}'} = (\alpha I - A)|_{\mathcal{D}}\mathcal{V}(t)(\alpha I - A)^{-1}$$

extends the semigroup $\mathcal{V}(t)$ to a semigroup on the new space $\mathcal{D}'$. For any $f \in L^2_{\mathcal{E}}(\mathbb{R}_+)$ we define

$$\Phi_0^* f = \int_0^\infty \mathcal{V}(s)^*|_{\mathcal{D}'}\Pi^* f(s)\, ds.$$

Superficially, this is an element of $\mathcal{D}'$ and not $\mathcal{G}$; however, note that for $x \in \mathcal{G}$,

$$\begin{aligned} |\langle x, \Phi_0^* f\rangle_{\mathcal{D}\times\mathcal{D}'}| &= \left| \left\langle x, \int_0^\infty \mathcal{V}(s)^*|_{\mathcal{D}'} \Pi^* f(s)\, ds \right\rangle_{\mathcal{D}\times\mathcal{D}'} \right| \\ &= \left| \int_0^\infty \langle \Pi \mathcal{V}(s) x, f(s)\rangle_{\mathcal{E}}\, ds \right| \\ &= \left| \int_0^\infty \langle (\Phi x)(s), f(s)\rangle_{\mathcal{E}}\, ds \right| \\ &\leq \|\Phi x\|_{L^2_{\mathcal{E}}(\mathbb{R}_+)} \, \|f\|_{L^2_{\mathcal{E}}(\mathbb{R}_+)} \\ &= \|x\|_{\mathcal{G}} \, \|f\|_{L^2_{\mathcal{E}}(\mathbb{R}_+)}. \end{aligned}$$

Thus $\Phi_0^* f$ defines a bounded linear functional on $\mathcal{G}$ and hence is an element of $\mathcal{G}$. This is the continuous-time analogue of the formula (2.10) for the discrete-time case.

Once the translation representation operators Φ and Φ_* are understood, we can define the *scattering operator* $\mathbf{S}$ by

$$\mathbf{S} = \Phi_* \Phi^* \colon L^2_{\mathcal{E}}(\mathbb{R}) \to L^2_{\mathcal{E}_*}(\mathbb{R}).$$

Just as in the discrete-time case we see that $\mathbf{S}$ intertwines the translation semigroups

$$(\tau^t \otimes I_{\mathcal{E}_*})\mathbf{S} = \mathbf{S}(\tau^t \otimes I_{\mathcal{E}}) \text{ for all } t \in \mathbb{R}. \tag{3.5}$$

To arrive at the scattering function, we need to introduce the bilateral Laplace transform (the continuous-time analogue of the bilateral Z-transform)

$$f \in L^2_{\mathcal{X}}(\mathbb{R}) \mapsto \mathcal{L}f(s) = \widehat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-st} f(s)\ dt \text{ for } s \in i\mathbb{R}.$$

which maps (after proper interpretation) $L^2_{\mathcal{X}}(\mathbb{R})$ unitarily onto $L^2_{\mathcal{X}}(i\mathbb{R})$ and $L^2_{\mathcal{X}}(\mathbb{R}_+)$ unitarily onto $H^2_{\mathcal{X}}(\mathbb{C}_+)$ and $L^2_{\mathcal{X}}(\mathbb{R}_-)$ onto $H^2_{\mathcal{X}}(\mathbb{C}_-)$, where $H^2_{\mathcal{X}}(\mathbb{C}_+)$ and $H^2_{\mathcal{X}}(\mathbb{C}_-)$ are the Hardy spaces for the right half plane $\mathbb{C}_+$ (respectively, for the left half plane $\mathbb{C}_-$). Then the Laplace-transformed translation-representation operators (or *Fourier representation operators*) are given by

$$\widehat{\Phi} = \mathcal{L} \circ \Phi \colon \mathcal{K} \to L^2_{\mathcal{E}}(i\mathbb{R}),$$
$$\widehat{\Phi}_* = \mathcal{L} \circ \Phi_* \colon \mathcal{K} \to L^2_{\mathcal{E}_*}(i\mathbb{R})$$

and the Laplace-transformed scattering operator $\widehat{\mathbf{S}}$ is given by

$$\widehat{\mathbf{S}} = \widehat{\Phi}_* \widehat{\Phi}^* \colon L^2_{\mathcal{E}}(i\mathbb{R}) \to L^2_{\mathcal{E}_*}(i\mathbb{R}).$$

The intertwining relation (3.5) means that $\widehat{\mathbf{S}}$ satisfies

$$(M_f \otimes I_{\mathcal{E}_*})\widehat{\mathbf{S}} = \widehat{\mathbf{S}}(M_f \otimes I_{\mathcal{E}}) \text{ for all } f \in L^\infty(i\mathbb{R}) \tag{3.6}$$

whence it follows that $\widehat{\mathbf{S}}$ itself must be a multiplication operator

$$\widehat{\mathbf{S}} = M_S \colon f(s) \to S(s) \cdot f(s) \text{ for } s \in i\mathbb{R}$$

for some $S \in L^\infty_{\mathcal{L}(\mathcal{E},\mathcal{E}_*)}(i\mathbb{R})$. The causality assumption (CT-A3) then implies that in fact $S \in H^\infty_{\mathcal{L}(\mathcal{E},\mathcal{E}_*)}(\mathbb{C}_+)$. We shall call this S the *scattering function* for the continuous-time scattering system $\mathfrak{S}$ (often called *scattering matrix* in the literature). The scattering function is a complete unitary invariant for minimal continuous-time Lax-Phillips scattering systems just as in the discrete-time case.

3.2. **Lax-Phillips Schäffer-matrix model scattering systems: the continuous-time case.** One can use the translation representation operators Φ and $\Phi_*$to obtain a model for the abstract Lax-Phillips scattering system as follows. For any coefficient space $\mathcal{X}$ we let $\pi_{t_0,t_1} \colon L^2_{\mathcal{X}}(\mathbb{R}) \to L^2_{\mathcal{X}}([t_0,t_1))$ be the restriction operator

$$\pi_{[t_0,t_1)} \colon f \mapsto f|_{[t_0,t_1)}.$$

As we have noted in the statement of Proposition 3.1, Φ_* maps $\mathcal{G}_*$ unitarily onto $L^2_{\mathcal{E}_*}(\mathbb{R}_-)$ and Φ maps $\mathcal{G}$ unitarily onto $L^2_{\mathcal{E}}(\mathbb{R}_+)$. Hence the continuous-time analogue of the map $\mathcal{I}_{\mathcal{K}\to\boldsymbol{\mathcal{K}}}$ given by (2.19), which we also denote as $\mathcal{I}_{\mathcal{K}\to\boldsymbol{\mathcal{K}}}$ is given by

$$\mathcal{I}_{\mathcal{K}\to\boldsymbol{\mathcal{K}}} = \begin{bmatrix} \Phi_* P_{\mathcal{G}_*} \\ i^*_{\mathcal{H}} \\ \Phi P_{\mathcal{G}} \end{bmatrix} \colon \mathcal{K} \to \begin{bmatrix} L^2_{\mathcal{E}_*}(\mathbb{Z}_-) \\ \mathcal{H} \\ L^2_{\mathcal{E}}(\mathbb{R}_+) \end{bmatrix} =: \boldsymbol{\mathcal{K}} \tag{3.7}$$

and is unitary. The unitary evolution $\boldsymbol{\mathcal{U}}(t) := \mathcal{I}_{\mathcal{K}\to\boldsymbol{\mathcal{K}}} \mathcal{U}^* (\mathcal{I}_{\mathcal{K}\to\boldsymbol{\mathcal{K}}})^*$ has adjoint given by, for $t \geq 0$,

$$\boldsymbol{\mathcal{U}}(t)^* = \begin{bmatrix} \tau^t_- & 0 & 0 \\ 0 & I_{\mathcal{H}} & 0 \\ 0 & 0 & \tau^t_+ \end{bmatrix} \begin{bmatrix} I & \mathfrak{C}^t_0 & \mathfrak{D}^t_0 \pi_{[0,t)} \\ 0 & \mathfrak{A}(t) & \mathfrak{B}^t_0 \pi_{[0,t)} \\ 0 & 0 & I \end{bmatrix} \tag{3.8}$$

where the entries in the upper right 2×2 corner are given by

$$\begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B}^t_0 \\ \mathfrak{C}^t_0 & \mathfrak{D}^t_0 \end{bmatrix} = \begin{bmatrix} i^*_{\mathcal{H}} \mathcal{U}(t)^* \\ \pi_{[0,t)} \Phi \end{bmatrix} \begin{bmatrix} i_{\mathcal{H}} & \Phi^*|_{L^2_{\mathcal{E}}([0,t))} \end{bmatrix} \colon \begin{bmatrix} \mathcal{H} \\ L^2_{\mathcal{E}}([0,t)) \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ L^2_{\mathcal{E}_*}([0,t)) \end{bmatrix}. \tag{3.9}$$

If we write $\boldsymbol{\mathcal{U}}(t)^*$ in (3.8) in the form

$$\boldsymbol{\mathcal{U}}(t)^* = \begin{bmatrix} \tau^t_- & \mathbb{C}^t & \mathbb{D}^t \\ 0 & \mathbb{A}^t & \mathbb{B}^t \\ 0 & 0 & \tau^t_+ \end{bmatrix} \colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \tag{3.10}$$

then we see that $t \mapsto \mathbb{A}^t$ (for $t \geq 0$) is a strongly continuous semigroup and that $\mathbb{B}^t$, $\mathbb{C}^t$ and $\mathbb{D}^t$ have the causality properties

$$\mathbb{C}^t = \pi_{[-t,0)} \mathbb{C}^t, \qquad \mathbb{D}^t = \pi_{[-t,0)} \mathbb{D}^t,$$
$$\mathbb{D}^t = \mathbb{D}^t \pi_{[0,t)}, \qquad \mathbb{B}^t = \mathbb{B}^t \pi_{[0,t)}$$

and hence $t \mapsto \boldsymbol{\mathcal{U}}(t)^*$ is a *Lax-Phillips model semigroup of type* L^2_0 on $\boldsymbol{\mathcal{K}} = L^2_{\mathcal{E}_*}(\mathbb{R}_-) \oplus \mathcal{H} \oplus L^2_{\mathcal{E}}(\mathbb{R}_+)$ in the sense of Staffans (see [23]). If we introduce the model incoming and outgoing subspaces

$$\boldsymbol{\mathcal{G}} = \begin{bmatrix} 0 \\ 0 \\ L^2_{\mathcal{E}}(\mathbb{R}) \end{bmatrix} \subset \boldsymbol{\mathcal{K}}, \qquad \boldsymbol{\mathcal{G}}_* = \begin{bmatrix} L^2_{\mathcal{E}_*}(\mathbb{R}_-) \\ 0 \\ 0 \end{bmatrix} \subset \boldsymbol{\mathcal{K}} \tag{3.11}$$

then we see that $\boldsymbol{\mathfrak{S}} = (\boldsymbol{\mathcal{U}}(t); \boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{G}}_*)$ is a Lax-Phillips scattering system unitarily equivalent to the minimal part of the original Lax-Phillips scattering system $(\mathcal{U}(t); \mathcal{K}, \mathcal{G}, \mathcal{G}_*)$ via the coisometric identification map $\mathcal{I}_{\mathcal{K}\to\boldsymbol{\mathcal{K}}}$.

Remark 3.3. Given a continuous-time Lax-Phillips scattering system and a fixed time $t_0 > 0$, then it is easily seen that

$$\mathfrak{S}_{\text{dis},t_0} = (\mathcal{U}(t_0); \mathcal{K}, \mathcal{G}, \mathcal{G}_*) \tag{3.12}$$

is a discrete-time Lax-Phillips scattering system in the sense introduced above in Section 2.1 with associated wandering subspaces $\mathcal{E}_{\text{dis},t_0} := \mathcal{G} \ominus \mathcal{U}(t_0)\mathcal{G}$ and $\mathcal{E}_{*\text{dis},t_0} := \mathcal{U}(t_0)\mathcal{G}_* \ominus \mathcal{G}_*$. The associated embedded unitary colligation

$$U_{\text{dist},t_0} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix}$$

is then given by

$$A_{\text{dis},t_0} = i_{\mathcal{H}}^* \mathcal{U}(t_0)^* i_{\mathcal{H}}, \qquad B_{\text{dis},t_0} = i_{\mathcal{H}}^* \mathcal{U}(t_0)^* i_{\mathcal{E}_{\text{dis},t_0}}$$
$$C_{\text{dis},t_0} = i_{\mathcal{E}}^* i_{\mathcal{H}}, \qquad D_{\text{dis},t_0} = i_{\mathcal{E}_{*\text{dis},t_0}}^* i_{\mathcal{E}_{\text{dis},t_0}}$$

If we assume that the original continuous-time Lax-Phillips scattering system is in the Schäffer model $(\boldsymbol{\mathcal{U}}(t); \boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{G}}_*)$ given by (3.8), (3.7) and (3.11), then the associated wandering subspaces $\boldsymbol{\mathcal{E}}_{\text{dis},t_0}$ and $\boldsymbol{\mathcal{E}}_{*\text{dis},t_0}$ can be identified explicitly as

$$\boldsymbol{\mathcal{E}}_{\text{dis},t_0} = L^2_{\mathcal{E}}([0,t)), \qquad \boldsymbol{\mathcal{E}}_{*\text{dis},t_0} = L^2_{\mathcal{E}}([0,t))$$

with identification maps

$$i_{\boldsymbol{\mathcal{E}}_{\text{dis},t_0}} : f \mapsto \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} \in \boldsymbol{\mathcal{K}}, \qquad i_{\boldsymbol{\mathcal{E}}_{*\text{dis},t_0}} : g \mapsto \boldsymbol{\mathcal{U}}(t_0) \begin{bmatrix} \tau^{t_0} g \\ 0 \\ 0 \end{bmatrix} \in \boldsymbol{\mathcal{K}}.$$

and with associated embedded unitary colligation

$$\mathbf{U}_{\text{dis},t_0} = \begin{bmatrix} \mathbf{A}_{\text{dis},t_0} & \mathbf{B}_{\text{dis},t_0} \\ \mathbf{C}_{\text{dis},t_0} & \mathbf{D}_{\text{dis},t_0} \end{bmatrix} = \begin{bmatrix} \mathfrak{A}(t_0) & \mathfrak{B}_0^{t_0} \\ \mathfrak{C}_0^{t_0} & \mathfrak{D}_0^{t_0} \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ L^2_{\mathcal{E}}([0,t_0)) \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ L^2_{\mathcal{E}_*}([0,t_0)) \end{bmatrix}$$

given by (3.9) with $t = t_0$.

3.3. **Well-posed linear systems.** Our goal is to make a connection between Lax-Phillips scattering systems and conservative linear systems for the continuous-time setting. For the continuous-time setting, the definition of what is a conservative linear system (and more generally, what is a well-posed linear system) takes considerably more explanation than in the discrete-time case if one is to handle the most general case involving rigged spaces and unbounded operators. We recall what we need concerning these topics in this subsection; for a comprehensive treatment, we refer to [26].

Many infinite-dimensional linear systems are described in input/state/output form as

$$\begin{aligned} \frac{\mathtt{d}}{\mathtt{d}t} x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{3.13}$$

where the *system node*

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

consists of bounded linear operators

$$A \colon \mathcal{X} \to \mathcal{X}, \qquad B \colon \mathcal{U} \to \mathcal{X}, \qquad C \colon \mathcal{X} \to \mathcal{Y}, \qquad D \colon \mathcal{U} \to \mathcal{Y}$$

where we take $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$ to be Hilbert spaces (although more generality is possible) which have interpretations as the *state space*, *input space* and *output space* respectively. However this formalism excludes many natural examples unless we allow unbounded operators and introduce various auxiliary rigged spaces. The idea of *well-posed systems* of Staffans and Weiss (see [26]) is to start with the integral form of the system equations

$$x(t) = \mathfrak{A}_s^t x(s) + \mathfrak{B}_s^t u|_{[s,t)}$$
$$y|_{[s,t)} = \mathfrak{C}_s^t x(s) + \mathfrak{D}_s^t u|_{[s,t)}$$

where we have set

$$\mathfrak{A}_s^t = e^{(t-s)A}, \qquad \mathfrak{B}_s^t \colon u|_{[s,t)} \mapsto \int_s^t e^{(t-s')A} Bu(s')\, ds',$$
$$\mathfrak{C}_s^t \colon x(s) = Ce^{(t'-s)A} x(s) \text{ for } s \le t' < t,$$
$$\mathfrak{D}_s^t \colon u|_{[s,t)} \mapsto \int_s^{t'} Ce^{(t'-s')A} Bu(s')\, ds' + Du(t') \text{ for } s \le t' < t. \tag{3.14}$$

If we assume that the state $x(t)$ is initialized to be 0 in the distant past and if we then feed in an input signal $u|_{\mathbb{R}_-}$ compactly supported in the past, then the state at time 0 is given by $x(0) = \mathfrak{B}(u|_{\mathbb{R}_-})$ where the *control operator* $\mathfrak{B}$ is given by

$$\mathfrak{B} \colon u|_{\mathbb{R}_-} \mapsto \int_{-\infty}^0 e^{s'A} Bu(s')\, ds'. \tag{3.15}$$

Similarly, if $x(0)$ is the state at time 0 and if no input signal $u(t)$ is fed into the system in the future $t > 0$, then the future output signal $y|_{\mathbb{R}_+}$ is given by $y|_{\mathbb{R}_+} = \mathfrak{C}x(0)$ where the *observation operator* $\mathfrak{C}$ is given by

$$\mathfrak{C}x(0) = Ce^{tA} x(0) \text{ for } t \ge 0. \tag{3.16}$$

Finally, if the state is initialized to be 0 in the distant past and if an input signal u of compact support in the past is fed into the system, then the resulting output signal $y(t)$ $(t \in \mathbb{R})$ is given by $y = \mathfrak{D}u$ where we define the *input-output operator* $\mathfrak{D}$ by

$$\mathfrak{D} \colon u(t) \mapsto y(t) := \int_{-\infty}^t Ce^{(t-s')A} Bu(s')\, ds' + Du(t). \tag{3.17}$$

Furthermore, if A is exponentially stable, then the control operator $\mathfrak{B}$ extends to a bounded operator from $L^2_{\mathbb{R}_-}$ to $\mathcal{X}$, the observation operator $\mathfrak{C}$ can be viewed as a bounded operator from $\mathcal{X}$ into $L^2_{\mathcal{Y}}(\mathbb{R}_+)$, the input-output operator $\mathfrak{D}$ makes sense as a bounded operator from $L^2_{\mathcal{U}}(\mathbb{R})$ into $L^2_{\mathcal{Y}}(\mathbb{R})$, and the system is what we shall call *L^2-well-posed*. The following definition (see [26, Definition 2.2.1]) catches the structural properties of the collection of operators $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ given by (3.15), (3.16), (3.17) and the interplay with the group (or more generally, semigroup) of operators $\mathfrak{A}(t) = e^{tA}$. Here we find it more convenient to use the formulation from [26] in terms of the analogues of the operators $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ given by (3.15), (3.16) and (3.17) rather than the original formulation of Weiss [28, 25] in terms of the analogues of the operators $\mathfrak{B}_s^t$, $\mathfrak{C}_s^t$ and $\mathfrak{D}_s^t$ as in (3.14).

Definition 3.4. *Let $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$ be Hilbert spaces. A (causal, time-invariant) L^2-*well-posed linear system Σ *on $(\mathcal{Y}, \mathcal{X}, \mathcal{U})$ consists of a quadruple* $\Sigma = \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$ *so that*

1. $t \mapsto \mathfrak{A}(t)$ *is a strongly continuous semigroup on* $\mathcal{X}$,
2. $\mathfrak{B} \colon L^2_{\mathcal{U}}(\mathbb{R}_-) \to \mathcal{X}$ *satisfies* $\mathfrak{A}(t)\mathfrak{B}u = \mathfrak{B}\tau_-^t u$ *for all* $u \in L^2_{\mathcal{U}}(\mathbb{R}_-)$,
3. $\mathfrak{C} \colon \mathcal{X} \to L^2_{\mathcal{Y}}(\mathbb{R}_+)$ *satisfies* $\mathfrak{C}\mathfrak{A}(t)x = \tau_+^t \mathfrak{C}x$ *for all* $x \in \mathcal{X}$ *and* $t \ge 0$, *and*

(4) $\mathfrak{D}\colon L^2_{\mathcal{U}}(\mathbb{R}) \to L^2_{\mathcal{Y}}(\mathbb{R})$ *satisfies* $\tau^t \mathfrak{D} u = \mathfrak{D}\tau^t u$, $\pi_- \mathfrak{D}\pi_+ u = 0$ *and* $\pi_+\mathfrak{D}\pi_- u = \mathfrak{C}\mathfrak{B}u$ *for all* $u \in L^2_{\mathcal{U}}(\mathbb{R})$ *for all* $t \in \mathbb{R}$.

By the well-understood theory of C_0-semigroups (see e.g. [26, Chapter 3]), the semigroup $t \mapsto \mathfrak{A}(t)$ is determined by its infinitesimal generator A, a closed operator satisfying the conditions of the Hille-Yosida theorem with dense domain $\mathcal{D}(A) \subset \mathcal{X}$. If we introduce auxiliary spaces $\mathcal{X}_1 = \mathcal{D}(A)$ with the A-graph norm and its dual $\mathcal{X}_{-1}$ (with respect to the $\mathcal{X}$-inner product), we have the nesting of spaces $\mathcal{X}_{-1} \subset \mathcal{X} \subset \mathcal{X}_{-1}$. If $\mathfrak{B}$ and $\mathfrak{C}$ satisfy the conditions of Definition 3.4, then it is possible to define operators $B\colon \mathcal{U} \to \mathcal{X}_{-1}$ and $C\colon \mathcal{X}_1 \to \mathcal{Y}$ so that $\mathfrak{B}$ and $\mathfrak{C}$ can be expressed in the form familiar from the classical case

$$\begin{aligned}\mathfrak{B}u &= \int_{-\infty}^{0} \mathfrak{A}(-s)Bu(s)\, ds \text{ for } u \in L^2_{\mathcal{U}}(\mathbb{R}_-)\\ \mathfrak{C}x &= C\mathfrak{A}(t)x \text{ for } x \in \mathcal{X}_1\end{aligned}$$

(see [26, Chapter 4]).

Given an L^2-well-posed linear system Σ, for any real numbers s, t with $s < t$ associate the *state transition map* $\mathfrak{A}^t_s$, the *input map* $\mathfrak{B}^t_s$, the *output map* $\mathfrak{C}^t_s$ and the *input/output map* $\mathfrak{D}^t_s$ by

$$\begin{bmatrix} \mathfrak{A}^t_s & \mathfrak{B}^t_s \\ \mathfrak{C}^t_s & \mathfrak{D}^t_s \end{bmatrix} = \begin{bmatrix} \mathfrak{A}(t-s) & \mathfrak{B}\tau^t \pi_{[s,t)} \\ \pi_{[s,t)}\tau^{-s}\mathfrak{C} & \pi_{[s,t)}\mathfrak{D}\pi_{[s,t)} \end{bmatrix} \colon \begin{bmatrix} \mathcal{X} \\ L^2_{\mathcal{U}}([s,t)) \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ L^2_{\mathcal{Y}}([s,t)) \end{bmatrix}.$$

Then system trajectories for the system Σ can be defined as functions $(u(\cdot)$, $x(\cdot)$, $y(\cdot))$ from the real line $\mathbb{R}$ to $\mathcal{U} \times \mathcal{X} \times \mathcal{Y}$ so that, for each $s < t$ we have the equality

$$\begin{bmatrix} x(t) \\ y|_{[s,t)} \end{bmatrix} = \begin{bmatrix} \mathfrak{A}^t_s & \mathfrak{B}^t_s \\ \mathfrak{C}^t_s & \mathfrak{D}^t_s \end{bmatrix} \begin{bmatrix} x(s) \\ u|_{[s,t)} \end{bmatrix}. \tag{3.18}$$

(It is easily checked that these formulas recover the system-update operators given by (3.14) from the collection $\mathfrak{A}(t) = e^{tA}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ given by (3.15), (3.16) and (3.17) for the classical case described above.) The infinitesimal form of a L^2-well-posed linear system is a system node

$$S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \colon \mathcal{D}(S) \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

and for smooth trajectories it is possible to make sense of the infinitesimal form of the system equations

$$\begin{bmatrix} \frac{d}{dt}x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{3.19}$$

In case the operator S is bounded and we can write S in the split form

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix},$$

then the infinitesimal form of the system equations (3.19) have the familiar form as in (3.13).

If we impose an initial condition $x(0) = x_0$ and feed in an input signal $u \in L^2_{\mathcal{U}}(\mathbb{R}_+)$ into the system equations (3.18) or (3.19), upon taking Laplace transform we arrive at the following frequency-domain relation between the Laplace-transformed input signal $\widehat{u}(z)$ and the Laplace-transformed output signal $\widehat{y}(z)$:

$$\widehat{y}(z) = C(zI - A)^{-1}x_0 + \widehat{\mathfrak{D}}(z) \cdot \widehat{u}(z) \tag{3.20}$$

where $\widehat{\mathfrak{D}}(z)$ is the *transfer function* of Σ (a $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function analytic on the right-half plane $\mathbb{C}_+$ for the L^2-well-posed case which we are considering here) given by

$$\widehat{\mathfrak{D}}(z) = C\&D \begin{bmatrix} (zI - A|_{\mathcal{X}})^{-1} B \\ I_{\mathcal{U}} \end{bmatrix}. \tag{3.21}$$

The L^2-well-posed system Σ is in addition (scattering) *energy-preserving* (see [26, Definition 11.2.1]) if all system trajectories $(u(\cdot), x(\cdot), y(\cdot))$ in addition satisfy

$$\|x(t)\|^2_{\mathcal{X}} + \int_s^t \|y(r)\|^2_{\mathcal{Y}}\, dr = \|x(s)\|^2_{\mathcal{X}} + \int_s^t \|u(r)\|^2_{\mathcal{U}}\, dr \tag{3.22}$$

for all s, t with $s < t$.

There is an (anticausal) *dual system* $\Sigma^\dagger$

$$\Sigma^\dagger = \begin{bmatrix} \mathfrak{A}^\dagger & \mathfrak{B}^\dagger \\ \mathfrak{C}^\dagger & \mathfrak{D}^\dagger \end{bmatrix}$$

where

$$\mathfrak{A}^\dagger(t) = \mathfrak{A}(-t)^* \text{ on } \mathcal{X} \text{ for } t < 0, \qquad \mathfrak{B}^\dagger = \mathfrak{C}^* \colon L^2_{\mathcal{Y}}(\mathbb{R}_+) \to \mathcal{X},$$
$$\mathfrak{C}^\dagger = \mathfrak{B}^* \colon \mathcal{X} \to L^2_{\mathcal{U}}(\mathbb{R}_-), \qquad \mathfrak{D}^\dagger = \mathfrak{D}^* \colon L^2_{\mathcal{Y}}(\mathbb{R}) \to L^2_{\mathcal{U}}(\mathbb{R}).$$

Trajectories of the dual system are defined (in local integral form) via

$$\Sigma^\dagger \colon \left\{ \begin{bmatrix} x_*(t) \\ y_*|_{[t,s)} \end{bmatrix} = \begin{bmatrix} (\mathfrak{A}^\dagger)^s_t & (\mathfrak{B})^s_t \\ (\mathfrak{C}^\dagger)^s_t & (\mathfrak{D})^s_t \end{bmatrix} \begin{bmatrix} x_*(s) \\ u_*|_{[t,s)} \end{bmatrix} := \begin{bmatrix} (\mathfrak{A}^s_t)^* & (\mathfrak{C}^s_t)^* \\ (\mathfrak{B}^s_t)^* & (\mathfrak{D}^s_t)^* \end{bmatrix} \begin{bmatrix} x_*(s) \\ u_*|_{[t,s)} \end{bmatrix} \right. \text{ for } t < s$$

and satisfy a dual-pairing with respect to any system trajectory $(u(\cdot), x(\cdot), y(\cdot))$ of the original system:

$$\langle x(t), x_*(t) \rangle_{\mathcal{X}} + \int_s^t \langle y(r), u_*(r) \rangle_{\mathcal{Y}} = \langle x(s), x_*(s) \rangle_{\mathcal{X}} + \int_s^t \langle u(r), y_*(r) \rangle_{\mathcal{U}}. \tag{3.23}$$

In fact, trajectories of the adjoint system (u_*, x_*, y_*) can be characterized as those functions $t \mapsto (u_*(t), x_*(t), y_*(t))$ such that the dual pairing (3.23) holds with respect to all trajectories (u, x, y) of the original system Σ. If we assume that $x(0) = 0$ and then take $s = 0$ while letting $t \to \infty$, we arrive at

$$\|y\|^2_{L^2_{\mathcal{Y}}(\mathbb{R}_+)} \le \|u\|^2_{L^2_{\mathcal{Y}}(\mathbb{R}_+)}.$$

An application of the Plancherel theorem combined with the relation (3.20) then tells us that $\|\widehat{\mathfrak{D}}(z)\| \le 1$ for all $z \in \mathbb{C}_+$, i.e., *the transfer function* $\widehat{\mathfrak{D}}$ *is in the Schur-class* $\mathcal{S}_{\mathbb{C}_+}(\mathcal{U}, \mathcal{Y})$ *of analytic contractive-valued functions over the right half plane* $\mathbb{C}_+$.

Finally, we say that the L^2-well-posed system is (scattering) *conservative* if it is (scattering) energy-preserving (i.e., system trajectories $(u(\cdot), x(\cdot), y(\cdot))$ satisfy the energy-balance condition (3.22)) and trajectories $(u_*(\cdot), x_*(\cdot), y_*(\cdot))$ of the dual system also satisfy an energy-balance condition:

$$\|x_*(s)\|^2_{\mathcal{X}} + \int_t^s \|u_*(r)\|^2_{\mathcal{Y}}\, dr = \|x_*(t)\|^2 + \int_t^s \|y_*(r)\|^2_{\mathcal{U}}\, dr \text{ for } t < s. \tag{3.24}$$

Using the characterization of the dual-system trajectories as those satisfying the dual-pairing relation (3.23) with respect to all trajectories of the original system, the effect is that *the* L^2*-well-posed system is conservative if and only if a given* $(\mathcal{U} \times \mathcal{X} \times \mathcal{Y})$*-valued function* $(u(\cdot), x(\cdot), y(\cdot))$ *is a trajectory of* Σ *exactly when* $(u_*(\cdot), x_*(\cdot), y_*(\cdot)) = (y(\cdot), x(\cdot), u(\cdot))$ *is a trajectory of the dual system* $\Sigma^\dagger$.

3.4. **The conservative linear system embedded in a Lax-Phillips scattering system.** Let us now suppose that we are given a (continuous-time) Lax-Phillips scattering system $\mathfrak{S} = (\mathcal{U}(t); \mathcal{K}, \mathcal{G}, \mathcal{G}_*)$ as in (3.1) with outgoing translation representation $\Phi\colon \mathcal{K} \to L^2_{\mathcal{E}}(\mathbb{R})$ and incoming translation representation $\Phi_*\colon \mathcal{K} \to L^2_{\mathcal{E}_*}(\mathbb{R})$. We define the scattering subspace $\mathcal{H}$ to be $\mathcal{H} := \mathcal{K} \ominus [\mathcal{G} \oplus \mathcal{G}_*]$. In addition to the operators $\mathfrak{A}(t)\colon \mathcal{H} \to \mathcal{H}$ defined via (3.9) as

$$\mathfrak{A}(t) = i^*_{\mathcal{H}} \mathcal{U}(t)^* i_{\mathcal{H}} \tag{3.25}$$

we define operators $\mathfrak{B} : L^2_{\mathcal{E}}(\mathbb{R}_-) \to \mathcal{H}$, $\mathfrak{C} : \mathcal{H} \to L^2_{\mathcal{E}_*}(\mathbb{R}_+)$, and $\mathfrak{D} : L^2_{\mathcal{E}}(\mathbb{R}) \to L^2_{\mathcal{E}_*}(\mathbb{R})$ by

$$\mathfrak{B} = i^*_{\mathcal{H}} \Phi^*|_{L^2_{\mathcal{E}}(\mathbb{R}_-)}, \tag{3.26}$$

$$\mathfrak{C} = \Phi_* i_{\mathcal{H}}, \tag{3.27}$$

$$\mathfrak{D} = \Phi_* \Phi^*. \tag{3.28}$$

Note that these operators are connected with the operators in (3.9) via

$$\mathfrak{B}^t_0 = \mathfrak{B}\tau^t|_{L^2_{\mathcal{E}}([0,t))},$$
$$\mathfrak{C}^t_0 = \pi_{[0,t)}\mathfrak{C},$$
$$\mathfrak{D}^t_0 = \pi_{[0,t)}\mathfrak{D}|_{L^2_{\mathcal{E}}([0,t))}$$

Then we have the following result.

Theorem 3.5. *Let $\mathfrak{S} = (\mathcal{U}(t); \mathcal{K}, \mathcal{G}, \mathcal{G}_*)$ be a continuous-time Lax-Phillips scattering system and define the operators $\mathfrak{A}(t)$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ as in* (3.25)-(3.28). *Then the collection $\Sigma = \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$ is a scattering-conservative (and in particular L^2-well-posed) linear system on the triple of spaces $(\mathcal{E}_*, \mathcal{H}, \mathcal{E})$.*

Proof. To check that $\Sigma = \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$ is a well-posed linear system, we must verify conditions (1), (2), (3), (4) in Definition 3.4 and then verify the energy-balance relations (3.22) and (3.24).

Verification of (1): As $\mathcal{H}$ is the difference of invariant subspaces for $\mathcal{U}(t)$, it follows that $\mathfrak{A}(t)$ forms a strongly continuous semigroup of operators on $\mathcal{H}$ by the Sarason lemma (see [21, Lemma 0]) and (1) follows.

Verification of (2): Note first that, for $t \geq 0$,

(1) $i^*_{\mathcal{H}} \mathcal{U}(t)^* P_{\mathcal{G}_*} = 0$ since $\mathcal{G}_*$ is invariant under $\mathcal{U}(t)^*$ and $\mathcal{H} \perp \mathcal{G}_*$, and
(2) $P_{\mathcal{G}} \Phi^* u = 0$ for $u \in L^2_{\mathcal{E}}(\mathbb{R}_-)$ since then $\Phi^* u \in \widetilde{\mathcal{G}} \ominus \mathcal{G}$.

Hence

$$\begin{aligned} \mathfrak{A}(t)\mathfrak{B}u &= i^*_{\mathcal{H}} \mathcal{U}(t)^* P_{\mathcal{H}} \Phi^* u = i^*_{\mathcal{H}} \mathcal{U}(t)^* (P_{\mathcal{G}_*} + P_{\mathcal{H}} + P_{\mathcal{G}}) \Phi^* u \\ &= i^*_{\mathcal{H}} \mathcal{U}(t)^* \Phi^* u = i^*_{\mathcal{H}} \Phi^* (\tau^{-t})^* u = i^*_{\mathcal{H}} \Phi^* \tau^t_- u \\ &= \mathfrak{B}\tau^t_- u \end{aligned}$$

and (2) follows.

Verification of (3): Similarly for $\mathfrak{C}$, note that

(1) $\Phi_* P_{\mathcal{H}} = P_{L^2_{\mathcal{E}_*}(\mathbb{R}_+)} \Phi_* P_{\mathcal{H}}$ since $\mathcal{H} \perp \mathcal{G}_*$,
(2) $P_{L^2_{\mathcal{E}_*}(\mathbb{R}_+)} \Phi_* P_{\mathcal{G}_*} = 0$ since Φ_* maps $\mathcal{G}_*$ into $L^2_{\mathcal{E}_*}(\mathbb{R}_-)$, and
(3) $P_{\mathcal{G}} \mathcal{U}(t)^* i_{\mathcal{H}} = 0$ since $\mathcal{U}(t)^* \mathcal{H} \subset \mathcal{G}_* \oplus \mathcal{H}$ which is orthogonal to $\mathcal{G}$.

Hence

$$\begin{aligned}\mathfrak{C}\mathfrak{A}(t)x &= \Phi_* P_{\mathcal{H}} \mathcal{U}(t)^* i_{\mathcal{H}} x = P_{L^2_{\mathcal{E}_*}(\mathbb{R}_+)} \Phi_* P_{\mathcal{H}} \mathcal{U}(t)^* i_{\mathcal{H}} x \\ &= P_{L^2_{\mathcal{E}_*}(\mathbb{R}_+)} \Phi_* (P_{\mathcal{G}_*} + P_{\mathcal{H}} + P_{\mathcal{G}}) \mathcal{U}(t)^* i_{\mathcal{H}} x \\ &= P_{L^2_{\mathcal{E}_*}(\mathbb{R}_+)} \Phi_* \mathcal{U}(t)^* i_{\mathcal{H}} x = P_{L^2_{\mathcal{E}_*}(\mathbb{R}_+)} \tau^t \Phi_* i_{\mathcal{H}} x = \tau^t_+ \Phi_* i_{\mathcal{H}} x \\ &= \tau^t_+ \mathfrak{C} x\end{aligned}$$

and (3) follows.

Verification of (4): We first note that

$$\begin{aligned}\tau^t \mathfrak{D} u &= \tau^t \Phi_* \Phi^* u = \Phi_* \mathcal{U}(t)^* \Phi^* u = \Phi_* \Phi^* \tau^t u \\ &= \mathfrak{D} \tau^t u\end{aligned}$$

and the first of conditions (4) follows.

For $u \in L^2_{\mathcal{E}}(\mathbb{R}_+)$, note that $\pi_+ u \in L^2_{\mathcal{E}}(\mathbb{R}_+)$ and hence $\Phi^* \pi_+ u \in \mathcal{G}$. From the intertwining relation $\pi_- \Phi_* = \Phi_* P_{\mathcal{G}_*}$, it then follows that

$$\pi_- \mathfrak{D} \pi_+ u = \pi_- \Phi_* \Phi^* \pi_+ u = \Phi_* P_{\mathcal{G}_*} \Phi^* \pi_+ u = 0$$

and the second of relations (4) follows.

Finally note that

$$\begin{aligned}\pi_+ \Phi_* &= \pi_+ \Phi_* P_{\mathcal{G}_*^\perp}, \\ \Phi^* \pi_- &= P_{\mathcal{G}^\perp} \Phi^* \pi_-.\end{aligned}$$

Hence, for any $u \in L^2_{\mathcal{E}}(\mathbb{R})$, we have

$$\begin{aligned}\pi_+ \mathfrak{D} \pi_- u &= \pi_+ \Phi_* \Phi^* \pi_- u \\ &= \pi_+ \Phi_* P_{\mathcal{G}_*{}^\perp} P_{\mathcal{G}^\perp} \Phi^* \pi_- u \\ &= \pi_+ \Phi_* P_{\mathcal{G}_*{}^\perp \cap \mathcal{G}^\perp} \Phi^* \pi_- u \\ &= \pi_+ \Phi_* P_{\mathcal{H}} \Phi^* \pi_- u \\ &= \pi_+ \mathfrak{C} \mathfrak{B} \pi_- u\end{aligned}$$

and the last of conditions (4) follows as well. We conclude that $\Sigma = \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$ is a well-posed linear system.

Verification of energy-balance relations: It remains to check that the energy balance relation (3.22) is satisfied by system trajectories and that the dual energy balance relation (3.24) is satisfied by dual system trajectories. By Lemma 11.2.3 in [26], we need only check that the system update operator

$$\Sigma^t_s = \begin{bmatrix} \mathfrak{A}^t_s & \mathfrak{B}^t_s \\ \mathfrak{C}^t_s & \mathfrak{D}^t_s \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ L^2_{\mathcal{U}}([s,t)) \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ L^2_{\mathcal{Y}}([s,t)) \end{bmatrix}$$

is unitary for each $s < t$. By time-invariance, we may assume that $s = 0$. The operator Σ^t_0 in turn is exactly the operator given by (3.9). As explained in Remark 3.3, this operator is exactly the unitary colligation $\mathbf{U}_{\mathrm{dis},t}$ embedded in the discrete-time Lax-Phillips scattering system $(\mathcal{U}(t); \mathcal{H}, \mathcal{G}, \mathcal{G}_*)$ (where here t is considered as fixed), and hence $\Sigma^t_0 = \mathbf{U}_{\mathrm{dis},t}$ is unitary as wanted. □

It is now a straightforward matter to verify that the scattering function $S(z)$ of a (continuous-time) Lax-Phillips scattering system is identical to the transfer function $\widehat{\mathfrak{D}}(z)$ of the embedded (scattering) conservative linear system.

Theorem 3.6. *Suppose that $\Sigma = \Sigma(\mathfrak{S})$ is the conservative well-posed linear system embedded in the (continuous-time) Lax-Phillips scattering system*

$$\mathfrak{S} = (\mathcal{U}(t); \mathcal{K}, \mathcal{G}, \mathcal{G}_*).$$

Then the scattering function of $\mathfrak{S}$ is the same as the transfer function of Σ:

$$S_{\mathfrak{S}}(z) = \widehat{\mathfrak{D}}_\Sigma(z) \text{ for } z \in \mathbb{C}_+.$$

The proof of Theorem 3.6 proceeds in parallel to the analysis for the discrete-time case in Section 2. We first prove the continuous-time analogue of Lemma 2.3. Given any (scattering) conservative (L^2-well-posed) linear system, we let $\mathcal{T}$ denote the set of *admissible trajectories*, i.e., trajectories $(u(\cdot), x(\cdot), y(\cdot))$ as in (3.18) or (3.19) (for the smooth case) such that

$$u|_{\mathbb{R}_+} \in L^2_{\mathcal{U}}(\mathbb{R}_+), \qquad y|_{\mathbb{R}_-} \in L^2_{\mathcal{Y}}(\mathbb{R}_-) \tag{3.29}$$

with trajectory norm

$$\|(u,x,y)\|^2_{\mathcal{T}} = \|y\|^2_{L^2_{\mathcal{Y}}(\mathbb{R}_-)} + \|x(0)\|^2_{\mathcal{X}} + \|u\|^2_{L^2_{\mathcal{U}}(\mathbb{R}_+)}. \tag{3.30}$$

For the case where Σ is the conservative linear system embedded in a Lax-Phillips scattering system, the input space $\mathcal{U}$ is equal to the coefficient space $\mathcal{E}$ and the output space $\mathcal{Y}$ is equal to the coefficient space $\mathcal{E}_*$.

Lemma 3.7. *Let $\Sigma = \left[\begin{smallmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ be a scattering-conservative linear system. Then:*

(1) *The map*

$$\mathcal{I}_{\mathcal{T}\to\boldsymbol{\mathcal{K}}} \colon (u,x,y) \mapsto y|_{\mathbb{R}_-} \oplus x(0) \oplus u|_{\mathbb{R}_+}$$

maps the space of admissible trajectories isometrically onto the space $\boldsymbol{\mathcal{K}} := L^2_{\mathcal{E}_*}(\mathbb{R}_-) \oplus \mathcal{H} \oplus L^2_{\mathcal{E}}(\mathbb{R}_+)$.

(2) *Suppose that $\Sigma(\mathfrak{S}) = \left[\begin{smallmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ is the scattering-conservative linear system embedded in the (continuous-time) Lax-Phillips scattering system $\mathfrak{S} = (\mathcal{U}(t); \mathcal{K}, \mathcal{G}, \mathcal{G}_*)$ as in* (3.25)*,* (3.26)*,* (3.27) *and* (3.28)*. Then the map*

$$\mathcal{I}_{\mathfrak{S}\to\mathcal{T}} \colon k \mapsto [t \mapsto ((\Phi_* k)(t), P_{\mathcal{H}}\mathcal{U}(t)^* k, (\Phi k)(t))]$$

maps the ambient space $\mathcal{K}$ of the scattering system $\mathfrak{S}$ onto the space $\mathcal{T}$ of admissible trajectories for the conservative linear system $\Sigma(\mathfrak{S})$. Moreover

$$\mathcal{I}_{\mathcal{K}\to\boldsymbol{\mathcal{K}}} = \mathcal{I}_{\mathcal{T}\to\boldsymbol{\mathcal{K}}} \circ \mathcal{I}_{\mathcal{K}\to\mathcal{T}}.$$

Proof. To prove (1), we first note that $\mathcal{I}_{\mathcal{T}\to\boldsymbol{\mathcal{K}}}$ maps $\mathcal{K}$ into $\boldsymbol{\mathcal{K}}$ by definition. To verify surjectivity, note that any element (y_-, x_0, y_+) of $\boldsymbol{\mathcal{K}}$ generates a system trajectory $(u(\cdot), x(\cdot), y(\cdot))$ with the property that

$$y|_{\mathbb{R}_-} = y_-, \qquad x(0) = x_0, \qquad u|_{\mathbb{R}_+} = u_+$$

by feeding u_+ as input signal over $\mathbb{R}_+$ with initial condition $x(0) = x_0$ into the forward system equations

$$\begin{aligned} x(t) &= \mathfrak{A}(t)x(0) + \mathfrak{B}_0^t \pi_{[0,t)} u \text{ for } t \geq 0, \\ y|_{\mathbb{R}_+} &= \mathfrak{C}x(0) + \mathfrak{D}\pi_{\mathbb{R}_+} u \end{aligned}$$

and by feeding in y_- as input signal over $\mathbb{R}_-$ with final condition $x(0) = x_0$ into the backward system equations

$$x(t) = \mathfrak{A}(t)^* x(0) + (\mathfrak{C}_t^0)^* y|_{[t,0)} \text{ for } t \le 0,$$
$$u(t) = \mathfrak{B}^* x(0) + \mathfrak{D}^* \pi_{\mathbb{R}_-} y.$$

Note that here we use the characterization of conservative systems that (u, x, y) being a system trajectory is equivalent to (y, x, u) being a dual system trajectory. Part (1) of Lemma 3.7 now follows.

To prove part (2) of the lemma, we must verify that

$$\begin{aligned} x(t) &= \mathfrak{A}_0^t x(0) + \mathfrak{B}_0^t \pi_{[0,t)} u \\ \pi_{[0,t)} y &= \mathfrak{C}_0^t x(0) + \mathfrak{D}_0^t \pi_{[0,t)} u \end{aligned} \tag{3.31}$$

where

$$\mathfrak{A}(t) = i_{\mathcal{H}}^* \mathcal{U}(t)^* i_{\mathcal{H}}, \qquad \mathfrak{B}_0^t = \mathfrak{B} \tau^t \pi_{[0,t)} = i_{\mathcal{H}}^* \Phi^* \tau^t \pi_{[0,t)}$$
$$\mathfrak{C}_0^t = \pi_{[0,t)} \mathfrak{C} = \pi_{[0,t)} \Phi_* i_{\mathcal{H}}, \qquad \mathfrak{D}_0^t = \pi_{[0,t)} \mathfrak{D} \pi_{[0,t)} = \pi_{[0,t)} \Phi_* \Phi^* \pi_{[0,t)}$$

and where

$$x(t) = i_{\mathcal{H}}^* \mathcal{U}(t)^* k, \qquad u(t) = (\Phi k)(t), \qquad y(t) = (\Phi_* k)(t).$$

The first of equations (3.31), upon multiplication on the left by $i_{\mathcal{H}}$, becomes

$$\begin{aligned} P_{\mathcal{H}} \mathcal{U}(t)^* k &= P_{\mathcal{H}} \mathcal{U}(t)^* P_{\mathcal{H}} k + P_{\mathcal{H}} \Phi^* \tau^t \pi_{[0,t)} \Phi k \\ &= P_{\mathcal{H}} \mathcal{U}(t)^* P_{\mathcal{H}} k + P_{\mathcal{H}} \mathcal{U}(t)^* \Phi^* \pi_{[0,t)} \Phi k \\ &= P_{\mathcal{H}} \mathcal{U}(t)^* \left[P_{\mathcal{H}} k + P_{\Phi^* L^2_{\mathcal{E}}([0,t))} k \right] \end{aligned}$$

which in turn, upon multiplication on the left by $\mathcal{U}(t)$, becomes

$$P_{\mathcal{U}(t)\mathcal{H}} k = P_{\mathcal{U}(t)\mathcal{H}} \left[P_{\mathcal{H}} k + P_{\Phi^* L^2_{\mathcal{E}}([0,t))} k \right]. \tag{3.32}$$

The second of equations (3.31) becomes

$$\pi_{[0,t)} \Phi_* k = \pi_{[0,t)} \Phi_* P_{\mathcal{H}} k + \pi_{[0,t)} \Phi_* \Phi^* \pi_{[0,t)} \Phi k$$

which, upon multiplication on the left by Φ_*^*, becomes

$$P_{\Phi_*^* L^2_{\mathcal{E}_*}([0,t))} k = P_{\Phi_*^* L^2_{\mathcal{E}_*}([0,t))} (P_{\mathcal{H}} + P_{\Phi^* L^2_{\mathcal{E}}([0,t))}) k. \tag{3.33}$$

Since $\Phi_*^* L^2_{\mathcal{E}_*}([0,t))$ and $\mathcal{U}(t)\mathcal{H}$ are orthogonal, (3.32) and (3.33) can be combined into the single equation

$$(P_{\Phi_*^* L^2_{\mathcal{E}_*}([0,t))} + P_{\mathcal{U}(t)\mathcal{H}}) k = (P_{\Phi_*^* L^2_{\mathcal{E}_*}([0,t))} + P_{\mathcal{U}(t)\mathcal{H}})(P_{\mathcal{H}} + P_{\Phi^* L^2_{\mathcal{E}}([0,t))}) k. \tag{3.34}$$

This last equation (3.34) in turn is a direct consequence of the identity of subspaces

$$\mathcal{H} \oplus \Phi^* L^2_{\mathcal{E}}([0,t)) = \Phi_*^* L^2_{\mathcal{E}_*}([0,t)) \oplus \mathcal{U}(t)\mathcal{H}$$

which follows as a consequence of the general identity (2.7) for a discrete-time Lax-Phillips scattering system applied to the discrete-time approximation $\mathfrak{S}_{\mathrm{dis},t}$ defined as in (3.12). Part (2) of Lemma 3.7 now follows as wanted. □

Proof of Proposition 3.6. Theorem 3.6 can now be proved in exactly the same way as was done for the proof of the discrete-time case (Theorem 2.2). □

3.5. **Conservative linear system to scattering system: the coordinate-free approach in the continuous-time case.** Given a scattering-conservative linear system $\Sigma = \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$, one can always embed Σ into a (continuous-time) Lax-Phillips scattering system $\mathfrak{S}(\Sigma)$ by using the formula (3.10) to define the unitary evolution group on a Schäffer-matrix model space. In this section we present a coordinate-free approach to the construction of a Lax-Phillips scattering system with embedded conservative linear system $\Sigma = \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$ equal to a preassigned conservative system Σ. This section parallels the work of Section 2.4 for the discrete-time setting.

Suppose therefore that a conservative linear system $\Sigma = \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$ on a triple of spaces $(\mathcal{Y}, \mathcal{H}, \mathcal{U})$ is given. The energy balance (3.22) specialized to the case where $s = 0 < t$ can be written as

$$\|x(t)\|^2_{\mathcal{H}} + \int_0^t \|y(s)\|^2_{\mathcal{Y}}\, ds = \|x(0)\|^2_{\mathcal{H}} + \int_0^t \|u(s)\|^2_{\mathcal{U}}\, ds. \tag{3.35}$$

Specialization of the dual energy balance condition (3.24) applied to the dual system trajectory $(u_*, x_*, y_*) = (y, x, u)$ and specialized to the case where $t < s = 0$ gives

$$\|x(0)\|^2_{\mathcal{H}} + \int_t^0 \|y(s)\|^2\, ds = \|x(t)\|^2 + \int_t^0 \|u(s)\|^2_{\mathcal{U}}\, ds. \tag{3.36}$$

A trajectory $(u(t), x(t), y(t))$ of the system Σ is said to be *admissible* if its input and output signals satisfy $\pi_+ u \in L^2_{\mathcal{U}}(\mathbb{R}_+)$ and $\pi_- y \in L^2_{\mathcal{Y}}(\mathbb{R}_-)$. Note that, as explained in Lemma 3.7, $y(t)$ for $t > 0$ is obtained via the system equations for Σ with input signal $u(\cdot)$; similarly, $u(t)$ for $t < 0$ is the output of the dual system $\Sigma^\dagger$ with input signal $y(\cdot)$. From the energy-balance relation (3.35) we see that $x(0) \in \mathcal{H}$ and $u|_{\mathbb{R}_+} \in L^2_{\mathcal{U}}(\mathbb{R}_+)$ implies that then $y|_{\mathbb{R}_+} \in L^2_{\mathcal{Y}}(\mathbb{R}_+)$. Similarly, from the energy-balance relation (3.36) we see that $x(0) \in \mathcal{H}$ and $y|_{\mathbb{R}_-} \in L^2_{\mathcal{Y}}(\mathbb{R}_-)$ implies that $u|_{\mathbb{R}_-} \in L^2_{\mathcal{U}}(\mathbb{R}_-)$. Thus the set of admissible trajectories $\mathcal{T}$ could have equivalently been defined as

$$\mathcal{T} = \{(u, x, y) \colon (u, x, y) \text{ is a trajectory of } \Sigma \text{ with } u \in L^2_{\mathcal{U}}(\mathbb{R}) \text{ and } y \in L^2_{\mathcal{Y}}(\mathbb{R})\}. \tag{3.37}$$

We take the ambient space for the Lax-Phillips scattering system $\mathfrak{S}(\Sigma)$ associated with a given scattering-conservative linear system Σ to be the set $\mathcal{T}$ of admissible trajectories (u, x, y) satisfying (3.29) with norm given by (3.30). The unitary group $\mathcal{U}(t)$ is the group of translations acting on $\mathcal{T}$

$$\mathcal{U}(t) \colon (u, x, y) \mapsto (u(\cdot - t), x(\cdot - t), y(\cdot - t)). \tag{3.38}$$

while the incoming subspace is defined to be

$$\mathcal{G} = \{(u, x, y) \in \mathcal{T} : x(0) = 0,\ \pi_- y = 0\} \tag{3.39}$$

and the outgoing subspace is taken to be

$$\mathcal{G}_* = \{(u, x, y) \in \mathcal{T} : x(0) = 0,\ \pi_+ u = 0\}. \tag{3.40}$$

It is now a matter of checking that these objects form a Lax-Phillips scattering system with the desired properties.

Theorem 3.8. *Let* $\Sigma = \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$ *be a scattering-conservative linear system and define the spaces* $\mathcal{K}, \mathcal{G}, \mathcal{G}_*$ *and operator* $\mathcal{U}(t)$ *as in* (3.29), (3.38), (3.39) *and* (3.40). *Then the collection* $\mathfrak{S} = \mathfrak{S}(\Sigma) := (\mathcal{T}; \mathcal{U}(t), \mathcal{G}, \mathcal{G}_*)$ *forms a Lax-Phillips scattering* *system. Moreover, the scattering-conservative linear system* $\Sigma = \Sigma_{\mathfrak{S}}$ *associated with* $\mathfrak{S} = \mathfrak{S}(\Sigma)$ *as in* (3.25), (3.26), (3.27) *and* (3.28) *is equal to the originally given scattering-conservative linear system* $\Sigma = \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$.

Proof. As Σ is time-invariant, it is clear that $\mathcal{U}(t)$ takes trajectories into trajectories for all $t \in \mathbb{R}$. By the second characterization (3.37) of admissible system trajectories, we see that $\mathcal{U}(t)$ takes admissible system trajectories into admissible system trajectories for all t. As $\mathcal{U}(-t) = \mathcal{U}(t)^{-1}$, we see that in fact $\mathcal{U}(t)$ takes $\mathcal{T}$ onto $\mathcal{T}$. If we add the quantity

$$\int_{-\infty}^{0} \|y(s)\|^2_{\mathcal{Y}}\, ds + \int_{t}^{\infty} \|u(s)\|^2_{\mathcal{Y}}\, ds$$

to each side of the energy balance equation (3.35) for any $t > 0$, we get

$$\|(u, x, y)\|^2_{\mathcal{T}} = \int_{t}^{\infty} \|u(s)\|^2_{\mathcal{U}}\, ds + \|x(t)\|^2_{\mathcal{H}} + \int_{-\infty}^{t} \|y(s)\|^2_{\mathcal{Y}}\, ds \tag{3.41}$$

which shows that $\mathcal{U}(t)$ is isometric for $t > 0$. Similarly, for any $t < 0$, adding the quantity

$$\int_{-\infty}^{t} \|y(s)\|^2_{\mathcal{Y}}\, ds + \int_{0}^{\infty} \|u(s)\|^2\, ds$$

to both sides of (3.36) yields (3.41) for $t < 0$ and hence $\mathcal{U}(t)$ is isometric for any $t < 0$. We conclude that $\mathcal{U}(t)$ defines a unitary group on $\mathcal{K}$.

Confirmation that $\mathcal{G} \perp \mathcal{G}_*$ is a straightforward calculation using the definition (3.30) of the $\mathcal{K}$-inner product. By Lemma 3.7, $\mathcal{G}$ can be identified with $L^2_{\mathcal{U}}(\mathbb{R}_+)$ and $\mathcal{U}(t)|_{\mathcal{G}}$ with $\tau^{-t}|_{L^2_{\mathcal{U}}(\mathbb{R}_+)}$ and $\mathcal{G}_*$ can be identified with $L^2_{\mathcal{Y}}(\mathbb{R}_-)$ and $\mathcal{U}(t)|_{L^2_{\mathcal{Y}}(\mathbb{R}_-)}$ with $\tau^{-t}|_{L^2_{\mathcal{Y}}(\mathbb{R}_-)}$ for $t < 0$. With these identifications the simple invariance properties of $\mathcal{G}$ with respect to $\mathcal{U}(t)$ for $t > 0$ and of $\mathcal{G}_*$ for $\mathcal{U}(t)$ for $t < 0$ follow easily.

From the construction we see that the outgoing and incoming translation representation operators for this scattering system $\mathfrak{S} = \mathfrak{S}(\Sigma)$ are given by

$$\Phi \colon (u, x, y) \mapsto u, \qquad \Phi_* \colon (u, x, y) \mapsto y.$$

Moreover, there is an isometric identification of the state-space $\mathcal{H}$ with $\mathcal{T} \ominus [\mathcal{G} \oplus \mathcal{G}_*]$ given by

$i_{\mathcal{H}} \colon h \mapsto (u, x, y)$ where (u, x, y) is the unique system trajectory satisfying
$x(0) = h,\ u|_{\mathbb{R}_+} = 0,\ y|_{\mathbb{R}_-} = 0.$

It is then clear that the adjoint $i^*_{\mathcal{H}}$ of this inclusion map has the action $i^*_{\mathcal{H}} \colon (u, , y) \mapsto x(0)$ and hence

$$i^*_{\mathcal{H}} \mathcal{U}(t)^* \colon (u, x, y) \mapsto x(t).$$

It follows that the map $\mathcal{I}_{\mathfrak{S}(\Sigma) \to \mathcal{T}(\mathfrak{S}(\Sigma))}$ appearing in part (2) of Lemma 3.7 is the identity, and that the operators $\mathfrak{A}_{\mathfrak{S}}(t)$, $\mathfrak{B}_{\mathfrak{S}}$, $\mathfrak{C}_{\mathfrak{S}}$ and $\mathfrak{D}_{\mathfrak{S}}$ given by (3.25), (3.26), (3.27) and (3.28) (associated with the scattering system $\mathfrak{S}(\Sigma)$) coincide with the components $\mathfrak{A}(t)$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ of our original scattering-conservative linear system Σ, as required. □

3.6. **Spectral theory of the scattering function/transfer function: the continuous-time case.** For convenience we formulate the following ideas for L^2-well-posed systems although they make sense equally well for more general well-posed systems as developed in [26].

Given a well-posed linear system $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$, we say that

(1) Σ is *approximately controllable* if the control operator $\mathfrak{B}\colon L^2_{\mathcal{U}}(\mathbb{R}_-) \to \mathcal{X}$ has dense range:
$$\overline{\operatorname{Ran}}\,\mathfrak{B} = \mathcal{X}.$$
(2) Σ is *approximately observable* if the observation operator $\mathfrak{C}\colon \mathcal{X} \to L^2_{\mathcal{Y}}(\mathbb{R}_+)$ has trivial kernel
$$\operatorname{Ker}\,\mathfrak{C} = \{0\},$$
or, equivalently, if the backward-time control operator $\mathfrak{C}^*\colon L^2_{\mathcal{Y}}(\mathbb{R}_+) \to \mathcal{X}$ has dense range:
$$\overline{\operatorname{Ran}}\,\mathfrak{C}^* = \mathcal{X}.$$

We also note that, given a contraction semigroup $t \mapsto \mathfrak{A}(t)$ on a Hilbert space $\mathcal{X}$, there is a direct sum decomposition of $\mathcal{X}$
$$\mathcal{X} = \mathcal{X}_u \oplus \mathcal{X}_{cnu}$$
so that both $\mathcal{X}_u$ and $\mathcal{X}_{cnu}$ are invariant for $\mathfrak{A}(t)$ and $t \mapsto \mathfrak{A}(t)|\mathcal{X}_u$ is a unitary group while $t \mapsto \mathfrak{A}(t)|_{\mathcal{X}_{cnu}}$ is *completely nonunitary* in the sense there is no nonzero invariant subspace $\mathcal{X}'_{cnu}$ invariant for $\mathfrak{A}(t)$ so that $\mathfrak{A}(t)|_{\mathcal{X}'_{cnu}}$ is a unitary group (see [26, Section 11.1]).

We have the following analogue of Theorem 2.6 for the continuous-time case.

Theorem 3.9. *Suppose that* $\Sigma = \left[\begin{smallmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ *is the scattering-conservative linear system embedded in the continuous-time Lax-Phillips scattering system* $\mathfrak{S} = (\mathcal{U}(t); \mathcal{K}, \mathcal{G}, \mathcal{G}_*)$ *as in Theorem 3.5. Define* $\widetilde{\mathcal{G}}$ *and* $\widetilde{\mathcal{G}}_*$ *as in* (3.2)*. Then:*

(1) $\widetilde{\mathcal{G}} = \mathcal{K}$ *if and only if* Σ *is approximately controllable.*
(2) $\widetilde{\mathcal{G}}_* = \mathcal{K}$ *if and only if* Σ *is approximately observable.*
(3) $\mathfrak{S}$ *is minimal, i.e.,* $\widetilde{\mathcal{G}}_* + \widetilde{\mathcal{G}}$ *is dense in* $\mathcal{K}$*, if and only if* $\mathfrak{A}(t)$ *is completely nonunitary.*

Proof. This can be proved in much the same way as the discrete-time case (Theorem 2.6) by letting the continuous-time parameter t play the role of the discrete-time parameter n appearing there. □

Just as in the discrete-time case, we see that the scattering function for a Lax-Phillips continuous-time scattering system which is Lax-Phillips minimal is necessarily two-sided inner with respect to the right half-plane, i.e., $S(z)$ is analytic and contraction-valued on the right half plane with boundary-value function $S(s)$ on the imaginary line having unitary values for almost all $s \in i\mathbb{R}$.

One of the themes in the book of Lax-Phillips (see [14, Section III.3]) is the connection between the spectrum of the infinitesimal generator of the contraction semigroup $t \to \mathfrak{A}(t)$ obtained by compressing the unitary group $\mathcal{U}(t)^*$ to the scattering subspace $\mathcal{H} = \mathcal{K} \ominus [\mathcal{G}_* \oplus \mathcal{G}]$ in a Lax-Phillips continuous-time scattering system $\mathfrak{S} = (\mathcal{U}(t); \mathcal{K}, \mathcal{G}, \mathcal{G}_*)$ and the poles of the associated scattering function $S(z)$; for a discussion of the more general case including the more general continuous-time analogue of the setting in Remark 2.8, see [26, Section 11.9]. Here we present the continuous-time analogue of Theorem 2.7.

Theorem 3.10. *Suppose that $\mathfrak{S} = (\mathcal{U}(t); \mathcal{K}, \mathcal{G}, \mathcal{G}_*)$ is a continuous-time Lax-Phillips scattering system which is minimal in the sense of Lax-Phillips, i.e., $\widetilde{\mathcal{G}}_* + \widetilde{\mathcal{G}}$ is dense in $\mathcal{K}$, and that $\Sigma = \begin{bmatrix} \mathfrak{A}(t) & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$ is the associated embedded scattering-conservative L^2-well-posed linear system given as in Theorem 3.5. Then the point λ in the left half plane is an isolated pole of order m for the resolvent $(zI - A)^{-1}$ if and only if λ is an isolated pole of the same order m for the associated scattering function $S(z)$.*

Proof. A cheap way to get the result is to apply the Cayley transform to the discrete-time result (Theorem 2.7 above). Alternatively, by Theorem 2.2 we know that $S(z)$ has the generalized realization form

$$S(z) = C\&D \begin{bmatrix} (zI - A)|_{\mathcal{X}})^{-1} B \\ I_{\mathcal{U}} \end{bmatrix} \tag{3.42}$$

where $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is the (infinitesimal) system node associated with the conservative well-posed system $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$. If the system is *weakly regular* (see Definition 5.6.1 in [26]), then the operator $C \colon \mathcal{X}_1 \to \mathcal{Y}$ can be split in a certain precise sense and the operator C has a certain extension C_{L_w} (the *weak Lebesgue extension*) which enables us to rewrite (3.42) in the more conventional realization form

$$S(z) = C_{L_w}(zI - A)^{-1}B + D \tag{3.43}$$

for a bounded operator $D \colon \mathcal{U} \to \mathcal{Y}$. As we are assuming that $\mathfrak{S}$ is Lax-Phillips minimal, Theorem 3.10 tells us that the well-posed linear system $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ is approximately controllable and approximately observable. Now the results from [27] can be applied directly to give the result (see also [10, Theorem XV.2.1]). The general case can be handled in the same way contingent upon carrying out the technical exercise of extending the results from [27] to the case of a general well-posed linear system. □

References

[1] V.M. Adamjan and D.Z. Arov, On unitary coupling of semiunitary operators, *Dokl. Akad. Nauk. Arm. SSR* XLIII, 5 (1966), 257-263 [in Russian] = *Amer. Math. Soc. Transl.* 95 (1970), 75-129.

[2] D.Z. Arov and M.A. Nudelman, Passive linear stationary dynamical scattering systems with continuous time, *Integral Equations and Operator Theory* **24** (1996), 1-45.

[3] J.A. Ball, Linear systems, operator model theory and scattering: multivariable generalizations, in *Operator Theory and Its Applications (Winnipeg, MB, 1998)* (Ed. A.G. Ramm, P.N. Shivakumar and A.V. Strauss), Fields Institute Communications Vol. 25, Amer. Math. Soc., Providence, 2000, pp. 151-178.

[4] J.A. Ball, C. Sadosky and V. Vinnikov, Conservative linear systems, unitary colligations and Lax-Phillips scattering: multidimensional generalizations, *International J. Control* **77** (2004) 802-811.

[5] J.A. Ball, C. Sadosky and V. Vinnikov, Scattering systems with several evolutions and multidimensional input/state/output systems, *Integral Equations and Operator Theory* **52** (2005), 323-393.

[6] J.A. Ball and T.T. Trent, The abstract interpolation problem and commutant lifting: A coordinate-free approach, in *Operator Theory and Interpolation: International Workshop on Operator Theory and Applications, IWOTA96* (Ed. H. Bercovici and C. Foias), **OT 115** Birkhäuser-Verlag, Basel-Boston, 2000, pp. 55-83.

[7] J.A. Ball and V. Vinnikov, Overdetermined multidimensional systems: state space and frequency domain methods, in *Mathematical systems Theory in Biology, Communications, Computation, and Finance* (Ed. J. Rosenthal and D.S. Gilliam), IMA Volumes in Mathematics and its Applications **134**, Springer-Verlag, Berlin-New York, 2003, pp. 63-119.

[8] J.A. Ball and V. Vinnikov, Lax-Phillips scattering and conservative linear systems: A Cuntz-algebra multidimensional setting, *Memoirs of the American Mathematical Society*, Volume 178 Number 837, American Mathematical Society, Providence, 2005.
[9] H. Baumgärtel and M. Wollenberg, *Mathematical Scattering Theory*, **OT 9** Birkhäuser-Verlag, Basel-Boston, 1983.
[10] I. Gohberg, S. Goldberg and M.A. Kaashoek, *Classes of Linear Operators, Vol. I*, **OT 49**, Birkhäuser-Verlag, Basel-Boston, 1990.
[11] J.W. Helton, The characteristic functions of operator theory and electrical network realization, *Ind. U. Math. J.* **22** (1972/3), 403-414.
[12] J.W. Helton, Discrete time systems, operator models and scattering theory, *J. Funct. Anal.* **16** (1974), 15-28.
[13] J.W. Helton, Systems with infinite-dimensional state space: the Hilbert space approach, *Proceedings of the IEEE* **64** (1976), 145-160.
[14] P.D. Lax and R.S. Phillips, *Scattering Theory*, Academic Press, 1967.
[15] P.D. Lax, *Functional Analysis*, Wiley-Interscience, 2002.
[16] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam-London, 1970.
[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Math. Sciences 44, Springer-Verlag, New York, 1983.
[18] M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford Mathematical Monographs, Oxford University Press, New York, 1988.
[19] D. Salamon, Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach, *Trans. Amer. Math. Soc.* **300** (1987), 383-431.
[20] D. Salamon, *Realization theory in Hilbert space*, Math. systems Theory **21** (1989), 147-164.
[21] D. Sarason, On spectral sets having connected complement, *Acta Sci. Math. (Szeged)* **26** (1965), 289-299.
[22] J.J. Schäffer, On unitary dilations of contractions, *Proc. Amer. Math. Soc.* **6** 91955), 322.
[23] O.J. Staffans, Lax-Phillips scattering and well-posed linear systems, in *Proceedings of the 7th IEEE Mediterranean Conference on Control and Systems*, CD-ROM, Haifa, Israel, July 28-30, 1999.
[24] O.J. Staffans, Well-posed linear systems, Lax-Phillips scattering, and L^p-multipliers, in *Systems, Approximation, Singular Integral Operators, and Related Topics* (Ed. A.A. Borichev and N.K. Nikolskii), **OT 129** Birkhäuser-Verlag, Basel-Boston, 2001, pp. 445-464.
[25] O.J. Staffans and G. Weiss, Transfer functions of regular linear systems. Part II: the system operator and the Lax-Phillips semigroup, *Trans. Amer. Math. Soc.* **354** (2004), 517-558.
[26] O.J. Staffans, *Well-Posed Linear Systems*, Cambridge University Press, Cambridge-New York, 2005.
[27] Y. Uetake, Some local properties of spectrum of linear dynamical systems in Hilbert space, *Integral Equations and Operator Theory* **51** (2005), 283-302.
[28] G. Weiss, Transfer functions of regular linear systems. Part I: Characterizations of regularity, *Trans. Amer. Math. Soc.* **342** (1994), 827-854.

Department of Mathematics, Virginia Tech, Blacksburg, Virginia 24061
E-mail address: `ball@math.vt.edu`

Department of Mathematics, Virginia Tech, Blacksburg, Virginia 24061
E-mail address: `phcarrol@vt.edu`

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87, 61-614 Poznań, Poland
E-mail address: `uetake@amu.edu.pl`