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(Nonlinear) balanced truncation

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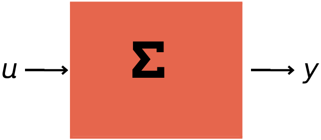
Blacksburg, 22 May 2023

Problem setting

Consider an input-output system Σ:



and a time-invariant asymptotically stable state space realization

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \qquad (*)$$

Assume that $(*)$ is a valid state space realization of Σ about x_0.

Problem setting

Questions treated here:

- How to reduce the order of $(*)$ such that it is useful for control purposes?
- When is $(*)$ of minimal order?
- When is $(*)$ in balanced form?
- Is there a relation with the Hankel operator?
- What if $(*)$ is unstable?
- Can we use some duality notion for observability and controllability?
- Can we preserve certain structures: energy, network,....?

Problem setting

Many large scale modeling examples such as

> Electrical circuit simulators (VLSI design, etc.)

> Any model stemming from spatial discretization procedures such as

- transmission line, Maxwell equations
- piezoelectric material
- Ovens, (bio-)chemical processes, etc.
- Flexible beams, plates, etc..

> .....

Model order reduction methods

Ad hoc model order reduction methods are often applied, i.e., modeler decides based upon physical and engineering intuition.

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For us, roughly a division in two branches (see book Antoulas)

> Moment matching approaches (Krylov, Lanczos, Arnoldi, etc.) based upon series expansion of the transfer matrix, and then matching the moments for a certain frequency up to a certain order.

> Singular value decomposition based methods: balancing procedures.

Model reduction for control

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- Controller design is based upon model of system. In general: size of controller is same as size of model, potentially hampering implementation of the controller.
- Part of system that is not controllable or observable does not show in input-output characteristics of equations.

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- Part of system that is not controllable or observable does not show in input-output characteristics of equations.

Not necessarily the same consideration as model reduction for simulation purposes.

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> Balancing is based upon the singular values of the Hankel operator. Computationally more tedious, and thus type of systems that can be handled are smaller. However, a priori error bound exists, and direct relation to minimality.

Focus of this course: **Balancing**

Overview

> **Linear systems:**
- Balancing for asymptotically stable linear systems
- Model order reduction by truncation

> Nonlinear systems:
- Energy functions and minimality
- State space balanced realizations
- Model reduction
- Hankel operator considerations

Stable linear systems

Continuous-time, causal linear input-output system $\Sigma : u \to y$ with impulse response $H(t)$.

If Σ is also BIBO stable then the system **Hankel operator**:

$$
\begin{aligned}
\mathcal{H} \quad &: \quad L_2^m[0,+\infty) \to L_2^p[0,+\infty) \\
&: \quad \hat{u} \to \hat{y}(t) = \int_0^\infty H(t+\tau)\hat{u}(\tau)\, d\tau.
\end{aligned}
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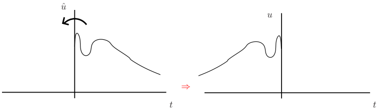
Time flipping operator $\mathcal{F} : L_2^m[0,+\infty) \to L_2^m(-\infty, 0]$



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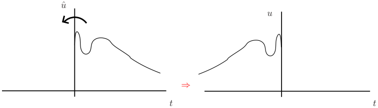

$$\mathcal{H}(\hat{u}) = \Sigma \circ \mathcal{F}(\hat{u})$$

The Hankel operator

When $\mathcal{H}$ is compact, then its (Hilbert) adjoint operator, $\mathcal{H}^*$, is also compact, and $\mathcal{H}^*\mathcal{H}$, is a self-adjoint compact operator :

$$\mathcal{H}^*\mathcal{H} = \sum_{i=1}^{\infty} \sigma_i^2 \, \langle \cdot, \psi_i \rangle_{L_2} \, \psi_i, \quad \sigma_i \geq 0,$$

$$\langle \psi_i, \psi_j \rangle_{L_2} = \delta_{ij}, \quad \langle \psi_i, (\mathcal{H}^*\mathcal{H})(\psi_i) \rangle_{L_2} = \sigma_i^2.$$

where σ_i^2 is an eigenvalue of $\mathcal{H}^*\mathcal{H}$ with corresponding eigenvector ψ_i, ordered as $\sigma_1 \geq \cdots \geq \sigma_n > 0$.

$\sigma_1, \ldots, , \sigma_n$ are the **Hankel singular values** of Σ.

State space formulation

Let (A, B, C) be a state space realization of Σ with dimension n,

$$\dot{x} = Ax + Bu,$$
$$y = Cx$$

where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. Assume that (A,B,C) is **as. stable and minimal** (controllable and observable).

Then $\mathcal{H} = \mathcal{O}\mathcal{C}$, where the controllability and observability operators are

$$\mathcal{C} \quad : \quad L_2^m[0, +\infty) \to \mathbb{R}^n : \hat{u} \to \int_0^\infty e^{At} B\hat{u}(t)\, dt$$
$$\mathcal{O} \quad : \quad \mathbb{R}^n \to L_2^p[0, +\infty) : x \to \hat{y}(t) = Ce^{At}x.$$

Energy functions

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2} \int_{-\infty}^{0} \| u(t) \|^2 \, dt$$

Minimum amount of control energy necessary to reach state x_0. L_c is the so-called **controllability function**.

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$$L_o(x_0) = \frac{1}{2} \int_{0}^{\infty} \| y(t) \|^2 \, dt, \qquad \substack{x(0) = x_0 \\ u(\tau) = 0, \quad 0 \le \tau < \infty}$$

Output energy generated by state x_0.
L_o is the so-called **observability function**.

Gramians

Theorem: Consider (A, B, C). Then $L_c(x_0) = \frac{1}{2}x_0^T P^{-1} x_0$ and $L_o(x_0) = \frac{1}{2}x_0^T Q x_0$, where $P = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$ is the controllability Gramian and $Q = \int_0^\infty e^{A^Tt}C^TCe^{At}dt$ is the observability Gramian.

Furthermore $P = P^T > 0$ and $Q = Q^T > 0$ are unique solutions of the Lyapunov equations

$$AP + PA^T = -BB^T$$

and

$$A^TQ + QA = -C^TC.$$

Gramians and operators

For any $x_1, x_2 \in \mathbb{R}^n$:

$$
\begin{aligned}
< x_1, \mathcal{C}\mathcal{C}^* x_2 > &= x_1^T \int_0^\infty e^{At} B B^T e^{A^T t} dt\, x_2 = x_1^T P x_2 \\
< x_1, \mathcal{O}^* \mathcal{O} x_2 > &= x_1^T \int_0^\infty e^{A^T t} C^T C e^{At} dt\, x_2 = x_1^T Q x_2.
\end{aligned}
$$

and the relation with the energy functions is given as

$$
\begin{aligned}
L_c(x) &= \frac{1}{2} x^T P^{-1} x = \frac{1}{2} \langle x, (\mathcal{C}\mathcal{C}^*)^{-1} x \rangle \\
L_o(x) &= \frac{1}{2} x^T Q x = \frac{1}{2} \langle x, (\mathcal{O}^* \mathcal{O}) x \rangle.
\end{aligned}
$$

Balanced realization

Theorem: (*Moore 1981)* The eigenvalues of QP are similarity invariants, i.e., they do not depend on the choice of the state space coordinates. There exists a state space representation where

$$\Sigma := Q = P = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ the square roots of the eigenvalues of QP.

The system is in **balanced form**. The σ_i's, i=1,..,n, equal the Hankel singular values.

Hankel norm

The Hankel norm of the system:

$$\| G \|_H^2 = \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{L_o(x)}{L_c(x)} = \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T Q x}{x^T P^{-1} x} = \max_{\substack{\bar{x} \in \mathbb{R}^n \\ \bar{x} \neq 0}} \frac{\bar{x}^T \Sigma^2 \bar{x}}{\bar{x}^T \bar{x}} = \sigma_1^2,$$

where $G = C(sI - A)^{-1}B$. The other Hankel singular values may be characterized inductively in a similar way.

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where $G = C(sI - A)^{-1}B$. The other Hankel singular values may be characterized inductively in a similar way.

Theorem: If (A, B, C) is asymptotically stable, then the realization is minimal if and only if $P > 0$ and $Q > 0$.

Question

What is the meaning of a small Hankel singular value:

a). Well controllable and observable

b). Well controllable, badly observable

c). Badly controllable and observable

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Minimality $\Leftrightarrow$ controllable and observable. No pole-zero cancellation. Useful for model reduction.

Model reduction by truncation

$\bar{L}_c(\bar{x}_0) = \frac{1}{2}\bar{x}_0^T\Sigma^{-1}\bar{x}_0$ and $\bar{L}_o(\bar{x}_0) = \frac{1}{2}\bar{x}_0^T\Sigma\bar{x}_0$.
Small σ_i, then amount of control energy required to reach $\tilde{x} = (0, \ldots, 0, x_i, 0, \ldots, 0)$ is large, and generated output energy is small → **badly controllable and observable, almost non-minimal, almost pole-zero cancellation**.

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Hence, if $\sigma_k \gg \sigma_{k+1}$, x_{k+1} to x_n can be removed. Partition system accordingly

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix},$$

$$x^1 = (x_1, \dots, x_k)^T, \quad x^2 = (x_{k+1}, \dots, x_n)^T, \quad \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix},$$

where $\Sigma_1 = \mathrm{diag}(\sigma_1, \dots, \sigma_k)$ and $\Sigma_2 = \mathrm{diag}(\sigma_{k+1}, \dots, \sigma_n)$.

Properties reduced order models

Theorem: Both subsystems (A_{ii}, B_i, C_i), $i = 1, 2$, are again in balanced form, and their controllability and observability Gramians are equal to Σ_i, $i = 1, 2$.

Theorem: Assume that $\sigma_k > \sigma_{k+1}$. Then both subsystems (A_{ii}, B_i, C_i), $i = 1, 2$, are asymptotically stable.

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$\mathcal{H}_\infty$-norm of $G(s) = C(sI - A)^{-1}B$ is

$$\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \lambda_{max}^{\frac{1}{2}}(G(-j\omega)^T G(j\omega)).$$

Denote $\tilde{G}(s) = C_1(sI - A_{11})^{-1}B_1$, then

Theorem: Error bound *(Glover 1984)*
$\|G - \tilde{G}\|_H \;\leq\; \|G - \tilde{G}\|_\infty \;\leq\; 2(\sigma_{k+1} + \cdots + \sigma_n)$.

Singular perturbation reduction

Set $\dot{x}^2 = 0$ (thus interpreting x^2 as a very fast stable state, which may be approximated by a constant function of x^1 and u), then.

$$x^2 = -A_{22}^{-1}\left(A_{21}x^1 + B_2 u\right).$$

Substitution leads to a reduced order model $(\hat{A}, \hat{B}, \hat{C})$

$$\begin{aligned}\hat{A} &:= A_{11} - A_{12}A_{22}^{-1}A_{21}\\ \hat{B} &:= B_1 - A_{12}A_{22}^{-1}B_2\\ \hat{C} &:= C_1 - C_2A_{22}^{-1}A_{21}\end{aligned}$$

For $(\hat{A}, \hat{B}, \hat{C})$ same properties hold as for truncated reduced order model. Intermediate forms by *(Heuberger 1990)*.

Balancing for stable linear systems

- Non-minimal state space realizations: zero Hankel singular values → tool in e.g. system identification.
- Balancing "equally" weights frequencies. Frequency weighting is used to deal with this → next week.
- Standard tool in matlab, "balreal", etc.

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Other types of balancing

> LQG balancing

> Coprime balancing

> H_∞ balancing

> Positive and bounded real balancing

> Structure preserving balanced reduction techniques such as

- port-Hamiltonian system structure
- consensus network structure
- gradient system structure

Generalized and extended balancing

Some literature

> Many of the linear methods up to 2005: A.C. Antoulas, Approximation of Large-Scale Dynamical Systems, SIAM, Philadelphia, 2005.

> Overview of reduction methods for networks: X. Cheng, J.M.A. Scherpen, Model Reduction Methods for Complex Network Systems. In NE Leonard (ed.), Annual Review of Control Robotics and Autonomous Systems, Vol. 4, Annual Reviews, 425-453 (2021) https://doi.org/10.1146/annurev-control-061820-083817

> port-Hamiltonian structure preservation, e.g.,

- R. Polyuga and A. J. van der Schaft. Structure preserving model reduction of port-Hamiltonian systems. In Proc. 18th Int. Symposium on Mathematical Theory of Networks and Systems, 2008.
- P. Borja, J.M.A. Scherpen, K. Fujimoto, Extended Balancing of Continuous LTI Systems: a Structure-preserving Approach, IEEE Trans. Aut. Contr., 2022.
- ....

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> Linear systems:
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- Maybe most frequently used is model reduction based on singular perturbation,(e.g., Jardon/Scherpen 2017).
- Proper Orthogonal Decomposition, Karhoenen-Loève expansions, empirical balancing (e.g., Lall et. al., 2002)
  $\Rightarrow$ data-based linear projection methods (based on SVD), while taking nonlinearities into account.
- Moment matching approach extended to nonlinear systems by Astolfi et al. (since 2010), with a signal generator and the internal model approach.

Today

- "Analytical" methods based on nonlinear extension of balancing theory $\Rightarrow$ Topic of today, with focus on balancing around a stable equilibrium.

Not treated: work of Besselink et al. (incremental balancing), Kawano et al. (differential balancing), Ionescu et al. (dissipativity based), etc.

The controllability and observability function

Smooth system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

Assumptions:

> $f(0) = 0$, 0 as. stable eq. point for $u = 0$, $x \in W$.

> $h(0) = 0$.

> Controllability function L_c and observability function L_o smooth and exist.

Gramian extensions

Controllability function:

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty) = 0,\, x(0) = x_0}} \frac{1}{2} \int_{-\infty}^{0} \| u(t) \|^2 \, dt$$

Observability function:

$$L_o(x_0) = \frac{1}{2} \int_{0}^{\infty} \| y(t) \|^2 \, dt, \qquad \begin{matrix} x(0) = x_0 \\ u(\tau) = 0, \quad 0 \le \tau < \infty \end{matrix}$$

Characterization of these functions

Are you familiar with the characterization of the observability function?

Characterization of these functions

Are you familiar with the characterization of the observability function?

And of the controllability function? Do you recognize this as an optimal control problem?

Observability function

- $L_o(x)$ is unique smooth solution of **Lyapunov** type of equation

$$\frac{\partial L_o}{\partial x}(x)f(x) + \frac{1}{2}h^T(x)h(x) = 0, \qquad L_o(0) = 0.$$

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How to obtain?

Assume $\check{L}_o(x)$ is solution on W.

$$\begin{aligned} L_o(x_0) &= \frac{1}{2}\int_0^\infty \|y(t)\|^2 dt = \frac{1}{2}\int_0^\infty h^T(x(t))h(x(t))dt \\ &= -\int_0^\infty \frac{\partial \check{L}_o}{\partial x}(x(t))f(x(t))dt = -\int_0^\infty \frac{d}{dt}\check{L}_o(x(t))dt \\ &= -\check{L}_o(x(\infty)) + \check{L}_o(x(0)) = \check{L}_o(x_0), \qquad \forall x_0 \in W, \end{aligned}$$

since $x(0) = x_0$ and $x(\infty) = 0$ by the asymptotic stability of $f(x)$.

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How to obtain? Vice versa:

$$L_o(x(t)) = \frac{1}{2}\int_t^{\infty} h(x(\tau))^T h(x(\tau))d\tau, \qquad u(\tau) \equiv 0, \qquad t \leq \tau < \infty.$$

Differentiating with respect to the time t gives us:

$$\frac{dL_o}{dt}(x(t)) = -\frac{1}{2}h(x(t))^T h(x(t)) \Longrightarrow \frac{\partial L_o}{\partial x}(x)f(x) + \frac{1}{2}h^T(x)h(x) = 0.$$

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Linear case: $x^T Q A x + \frac{1}{2} x^T C^T C x = 0$

Controllability function

$L_c(x)$ is unique smooth solution of **Hamilton-Jacobi** equation

$$\frac{\partial L_c}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial L_c}{\partial x}(x)g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x) = 0, \qquad L_c(0) = 0$$

with $-\left(f(x) + g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x)\right)$ as. stable on W.

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with $-\left(f(x) + g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x)\right)$ as. stable on W.

Assume $\check{L}_c(x)$ smooth solution on W. Then

$$\begin{aligned} \frac{d}{dt}\check{L}_c(x) &= \frac{\partial \check{L}_c}{\partial x}(x)\dot{x} = \frac{\partial \check{L}_c}{\partial x}(x) f(x) + \frac{\partial \check{L}_c}{\partial x}(x) g(x) u \\ &= -\frac{1}{2}\frac{\partial \check{L}_c}{\partial x}(x) g(x) g^T(x) \frac{\partial^T \check{L}_c}{\partial x}(x) + \frac{\partial \check{L}_c}{\partial x}(x) g(x) u \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\|u - g^T(x)\frac{\partial^T \check{L}_c}{\partial x}(x)\|^2, \end{aligned}$$

Thus

$$\check{L}_c(x_0) \quad \leq \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 dt, \qquad \forall x_0 \in W.$$

Hence with $-\left(f(x) + g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x)\right)$ as. stable on W, $\check{L}_c(x_0) = L_c(x_o)$.

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with $-\left(f(x) + g(x) g^T(x) \frac{\partial^T L_c}{\partial x}(x)\right)$ as. stable on W.

Linear: $x^T P^{-1} A x + x^T P^{-1} B B^T P^{-1} x = 0, -(A + BB^T P^{-1})$ as. stab. Verify!

Contr. and obs. function

- Converse statements (about existence) also possible.
- Role of observability and controllability for linear systems is replaced by **zero-state observability** and **asymptotic reachability** (or anti-stabilizability).

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- Σ is **zero-state observable** on $W \iff L_o(x_0) > 0$, $\forall x_0 \in W$, $x_0 \neq 0$.

Contr. and obs. function

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- Σ is **zero-state observable** on $W \iff L_o(x_0) > 0$, $\forall x_0 \in W$, $x_0 \neq 0$.

- Σ is **asymptotically reachable** (i.e., $-\left(f(x) + g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x)\right)$ is as. stable on W) $\iff$ $L_c(x) > 0$ for $x \in W$, $x \neq 0$.

Observability and reachability

Are you familiar with nonlinear notions?

Observability

Observability and zero-state observability are **different** for nonlinear systems. Recall that state is of influence in the input vector field, i.e., the input also plays a role with observability!

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Definition: The system is **locally observable at** x_0 if there exists a neighborhood W of x_0 such that for every neighborhood $V \subset W$ of x_0 the relation $x_0 I^V x_1$

(i.e., x_0 and x_1 are indistinguishable on V, i.e., starting from x_0 or x_1, apply same input, results in equal outputs)

implies that $x_1 = x_0$.

Observability

Definition: If the system is locally observable at each x_0 then it is called locally observable.

The system is called **observable** if $x_1 I^M x_2$ implies that $x_1 = x_2$.

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Definition: The **observation space** $\mathcal{O}$ is the linear space of functions on M containing $h_1, \ldots, h_p$, and all repeated Lie derivatives $L_{X_1} L_{X_2} \cdots L_{X_k} h_j$, for $j = 1, \ldots, p$ and with X_i, $i = 1, 2, \ldots$ in the set $\{f, g_1, \ldots, g_m\}$.
The observation space $\mathcal{O}$ defines the **observability codistribution** $d\mathcal{O}$, by setting $d\mathcal{O}(x) = \text{span}\{dH(x) | H \in \mathcal{O}\}$, $x \in M$.

Observability

If $\dim\ d\mathcal{O}(x_0) = n$, then the system is **locally observable at** x_0. If $\dim d\mathcal{O}(x) = n$ **for all** $x \in M$ then the system is **locally observable**.

Converse results also available!

Observability

If $\dim\ d\mathcal{O}(x_0) = n$, then the system is **locally observable at** x_0. If $\dim d\mathcal{O}(x) = n$ **for all** $x \in M$ then the system is **locally observable**.

Converse results also available!

Notice role of input vector field $g(x)$! If $u \equiv 0$, then no $g(x) \Rightarrow$ **zero-state observability** (i.e., $u \equiv 0, y \equiv 0 \Rightarrow x \equiv 0$).

Zero-state observation space $\mathcal{O}_0$ as $\mathcal{O}$, but now with $X_i = f$, $i = 1, 2, \ldots$.

Note that zero-state observability implies observability!

Observability

> So far, observability function L_o defined for $u \equiv 0$, i.e., can be related to zero-state observability.

Observability

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> Extension possible, i.e., **general** observability function is defined as (Gray and Mesko)

$$L_o^G(x_0) = \max_{\substack{u \in L_2(0,\infty),\ \|u\|_{L_2} \leq \alpha \\ x(0)=x_0,\ x(\infty)=0}} \frac{1}{2} \int_0^\infty \|y(t)\|^2 \, dt,$$

where $\alpha \geq 0$ is fixed. Assume existence and smoothness. The **natural** observability function is defined as

$$L_o^N(x_0) = L_o^G(x_0) - L_o^G(0).$$

(Strong) accessibility

No controllability characterization for nonlinear systems, only accessibility and strong accessibility. Complete state space is often not reachable, only a part.

$R^V(x_0, T)$: **reachable set** from x_0 at time $T > 0$, following the trajectories which remain in the neighborhood V of x_0 for $t \leq T$.

(Strong) accessibility

No controllability characterization for nonlinear systems, only accessibility and strong accessibility. Complete state space is often not reachable, only a part.

$R^V(x_0, T)$: **reachable set** from x_0 at time $T > 0$, following the trajectories which remain in the neighborhood V of x_0 for $t \leq T$.

Definition: System is **locally strongly accessible** from x_0 if for any neighborhood V of x_0 the set $R^V(x_0, T)$ contains a non-empty open set for any $T > 0$ sufficiently small. If this holds for every $x_0 \in M$ then the system is locally strongly accessible.

(Strong) accessibility

Accessibility is less strong, i.e.,

System is **locally accessible** from x_0 if $R_T^V(x_0) = \bigcup_{\tau \leq T} R^V(x_0, \tau)$ contains a non-empty open set of M for all neighborhoods V of x_0 and all $T > 0$. If this holds for every $x_0 \in M$ then the system is called locally accessible.

(Strong) accessibility

> Every element of $\mathcal{C}$ is linear combination of repeated Lie brackets as

$$[X_k, [X_{k-1}, [\cdots, [X_2, X_1]\cdots]]]$$

where $X_i \in \{f, g_1, \ldots, g_m\}$, $i = 1, 2, \ldots$

> Every element of $\mathcal{C}_0$ is linear combination of repeated Lie brackets as

$$[X_k, [X_{k-1}, [\cdots, [X_2, g_j]\cdots]]] \qquad j = 1, \ldots, m$$

where $X_i \in \{f, g_1, \ldots, g_m\}$, $i = 2, 3, \ldots$.

(Strong) accessibility

> If $\dim C(x_0) = n$, then system is locally accessible from x_0. If $\dim C(x) = n$ for all $x \in M$ then the system is locally accessible.

(Strong) accessibility

- If $\dim C(x_0) = n$, then system is locally accessible from x_0. If $\dim C(x) = n$ for all $x \in M$ then the system is locally accessible.
- If $\dim C_0(x_0) = n$, then system is locally strongly accessible from x_0.
- If $\dim C_0(x) = n$ for all $x \in M$ then the system is locally strongly accessible.

Converse results available, see Nijmeijer/van der Schaft 1990/2016.

Are (strong) accessibility and asymptotic reachability the same?

Minimality

Why are observability and accessibility important?

- Still certain minimality characterization, i.e., (see e.g., Isidori, 1995). An analytic realization (f, g, h) about x_0 of a formal power series c is **minimal if and only if** $\dim C(x_0) = n$ **and** $\dim d\mathcal{O}(x_0) = n$.

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> Recall that for linear case **positivity** of Gramians and **minimality** of system are equivalent. Model reduction due to **small** Hankel singular values corresponds to **“almost non-minimality”** of system!

Minimality

Under the assumption of constant dimension of $\mathcal{C}_0$ and of $d\mathcal{O}$, ($d\mathcal{O}_0$), and under the assumption that the analytic system (f, g, h) is a realization of the formal power series c, and that $f(x)$ is asymptotically stable, then we can proof that

- If $0 < L_c(x) < \infty$ and $0 < L_o^N(x) < \infty$ ($0 < L_o(x) < \infty$) for $x \in W$, $x \neq 0$, then (f, g, h) is **minimal** realization of c.

Minimality

Application of previous theory to nonlinear extension of **Kalman decomposition** is possible.

Under appropriate constant dimensionality assumptions for distributions, there exist local coordinates $x = (x^1, x^2, x^3, x^4)$ such that

> x_1 is strongly accessible **and** zero-state observable.

> x_2 is strongly accessible **but** not zero-state observable.

> x_3 is zero-state observable **but** not strongly accessible.

> x_4 is **not** zero-state observable **nor** strongly accessible.

Minimality

In these coordinates:

$$
\begin{aligned}
\dot{x}^1 &= f^1(x^1, x^3) + \sum_{j=1}^{m} g_j^1(x^1, x^2, x^3, x^4) u_j \\
\dot{x}^2 &= f^2(x^1, x^2, x^3, x^4) + \sum_{j=1}^{m} g_j^2(x^1, x^2, x^3, x^4) u_j \\
\dot{x}^3 &= f^3(x^3) \\
\dot{x}^4 &= f^4(x^3, x^4) \\
y &= h(x^1, x^3).
\end{aligned}
$$

Minimality

Assume that **as. reachable from 0** part equals **strongly accessible** part. Then, under appropriate assumptions on existence of L_o and L_c

- $L_o(x^1, x^2, x^3, x^4) > 0$ whenever $(x^1, x^3) \neq (0, 0)$.
- $L_o(0, x^2, 0, x^4) = 0$ for $(0, x^2, 0, x^4)$.
- $L_c(x^1, x^2, x^3, x^4)$ is infinite whenever $(x^3, x^4) \neq (0, 0)$.
- $0 < L_c(x^1, x^2, 0, 0) < \infty$, for $(x^1, x^2, 0, 0), (x^1, x^2) \neq (0, 0)$.

Introduction balanced realizations

Example:

$$f(x) = -\frac{1}{625}\begin{pmatrix} 625x_1 + 112x_1^3 + 552x_1^2x_2 + 639x_1x_2^2 + 216x_2^3 \\ 384x_1^3 + 625x_2 + 464x_1^2x_2 + 48x_1x_2^2 - 63x_2^3 \end{pmatrix},$$

$$g(x) = \begin{pmatrix} \frac{3\sqrt{2}}{5} & \frac{4\sqrt{2}}{25}\sqrt{25 + 7x_1^2 + 48x_1x_2 - 7x_2^2} \\ \frac{-4\sqrt{2}}{5} & \frac{3\sqrt{2}}{25}\sqrt{25 + 7x_1^2 + 48x_1x_2 - 7x_2^2} \end{pmatrix},$$

$$h(x) = \begin{pmatrix} \frac{2}{5}(3x_1 - 4x_2) \\ \frac{\sqrt{2}}{25}(4x_1 + 3x_2)^2 \end{pmatrix}.$$

Introduction balanced realizations

This system is zero-state observable, and f is asymptotically stable.

Solving the Hamilton-Jacobi equations we obtain:

$$L_c(x) = \frac{1}{2}x^T x, \quad L_o(x) = \frac{1}{2}x^T \begin{pmatrix} m_{11}(x) & m_{12}(x) \\ m_{21}(x) & m_{22}(x) \end{pmatrix} x,$$

$$\begin{aligned} m_{11}(x) = & \quad \tfrac{2}{625}(425 + 72x_1^2 - 192x_1x_2 + 128x_2^2) \\ m_{12}(x) = m_{21}(x) = & \quad \tfrac{12}{625}(-25 + 9x_1^2 - 24x_1x_2 + 16x_2^2) \\ m_{22}(x) = & \quad \tfrac{1}{625}(1025 + 81x_1^2 - 216x_1x_2 + 144x_2^2). \end{aligned}$$

Introduction balanced realizations

System has the **input-normal form**.

The eigenvalues of $M(x)$ are:

$$\lambda_1(x) = \frac{1}{25}(25 + 9x_1^2 - 24x_1x_2 + 16x_2^2) = 1 + (\frac{1}{5}(3x_1 - 4x_2))^2$$
$$\lambda_2(x) = 2.$$

The neighborhood V of 0 where the **number of distinct eigenvalues** is constant, is

$$V = \{x | (3x_1 - 4x_2)^2 < 25\},$$

i.e., $\lambda_1(x) < 2$ for $x \in V$, thus $\lambda_2(x) > \lambda_1(x)$.

Introduction balanced realizations

The unitary matrix of eigenvectors is

$$T(x) = T = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}.$$

Thus, **coordinate transformation** to diagonalize $M(x)$ is

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \nu(x) = T^T x = \frac{1}{5}\begin{pmatrix} 3x_1 - 4x_2 \\ 4x_1 + 3x_2 \end{pmatrix}, \quad x \in V.$$

Consider coordinates $\psi(z) = \nu^{-1}(z)$ for $z \in W$, where

$$W = \psi^{-1}(V) = \nu(V) = \{z | z_1^2 < 1\}.$$

Introduction balanced realizations

In new coordinates controllability and observability functions become for $z \in W$

$$\tilde{L}_c(z) = \frac{1}{2} z^T z, \qquad \tilde{L}_o(z) = \frac{1}{2} z^T \begin{pmatrix} 2 & 0 \\ 0 & 1 + z_1^2 \end{pmatrix} z.$$

The so-called **singular value functions** are $\tau_1(z) = 2$ and $\tau_2(z) = 1 + z_1^2$.

Coordinates $z \in W$ transform system into input-normal/output-diagonal form.

Introduction balanced realizations

Input-normal/output-diagonal form:

$$\begin{cases} \dot{z}_1 = -z_1 + z_1 z_2^2 + u_1\sqrt{2} \\ \dot{z}_2 = -z_2 - z_2^3 + u_2\sqrt{2 - 2z_1^2 + 2z_2^2} \end{cases}$$

$$\begin{cases} y_1 = 2z_1 \\ y_2 = \sqrt{2}z_2 \end{cases}$$

Additional coordinate transformation to bring system in **balanced** form! **Only** on the axes!

Introduction balanced realizations

Additional coordinate transformation $\bar{z} = \eta(z)$ as follows:

$$\begin{cases} \bar{z}_1 = 2^{\frac{1}{4}} z_1 \\ \bar{z}_2 = z_2 \end{cases}, \qquad \bar{z} \in \bar{W} = \nu(W) = \{\bar{z} | \bar{z}_1^2 < 2^{\frac{1}{2}}\}.$$

Now controllability and observability functions are in **balanced form**, i.e.,

$$\check{L}_c(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} 2^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} \bar{z}, \ \ \check{L}_o(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} 2^{\frac{1}{2}} & 0 \\ 0 & 1 + 2^{-\frac{1}{2}}\bar{z}_1^2 \end{pmatrix} \bar{z}.$$

Introduction balanced realizations

Then at coordinate axes:

$$\check{L}_c(\bar{z}_1, 0) = \frac{1}{2\sqrt{2}}\bar{z}_1^2, \qquad \check{L}_c(0, \bar{z}_2) = \frac{1}{2}\bar{z}_2^2,$$
$$\check{L}_o(\bar{z}_1, 0) = \frac{\sqrt{2}}{2}\bar{z}_1^2, \qquad \check{L}_o(0, \bar{z}_2) = \frac{1}{2}\bar{z}_2^2.$$

i.e., **less input energy** is needed to reach states $(\bar{z}_1, 0)$ than to reach the states $(0, \bar{z}_2)$, and states $(\bar{z}_1, 0)$ generate a **larger output energy** than the states $(0, \bar{z}_2)$.

Hence, on neighborhood $\bar{W}$ $\bar{z}_1$ is a more important state component than $\bar{z}_2$.

Introduction balanced realizations

For **model reduction** set $\bar{z}_2 = 0$.

Reduced order system:

$$\begin{cases} \dot{\tilde{z}} = -\tilde{z} + 2^{\frac{3}{4}} u_1 \\ \tilde{y} = 2^{\frac{3}{4}} \tilde{z} \end{cases}, \qquad \tilde{z} \in \tilde{W} = \{\tilde{z} | \tilde{z} < 2^{\frac{1}{2}}\}.$$

Controllability and observability function

$$\tilde{L}_c(\tilde{z}) = \frac{1}{2\sqrt{2}} \tilde{z}^2 = 2^{-\frac{3}{2}} \tilde{z}^2, \quad \text{and} \quad \tilde{L}_o(\tilde{z}) = \frac{\sqrt{2}}{2} \tilde{z}^2 = 2^{-\frac{1}{2}} \tilde{z}^2, \qquad \tilde{z} \in \tilde{W},$$

Reduced system is asymptotically stable on $\tilde{W}$.

Semi-quadratic forms

Smooth nonlinear system:

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}$$

Standing assumptions:

- $f(x)$ is as. stab. on some neighborhood Y of 0.
- System is zero-state observable on Y.
- L_o and L_c exist and are smooth on Y.
- $\frac{\partial^2 L_c}{\partial x^2}(0) > 0$ and $\frac{\partial^2 L_o}{\partial x^2}(0) > 0$.

Semi-quadratic forms

Version of the Fundamental Theorem of Integral Calculus:
Let L be smooth function in convex neighborhood V of 0 in $\mathbb{R}^n$, with $L(0) = 0$. Then

$$L(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i a_i(x_1, \ldots, x_n)$$

for some suitable smooth functions a_i defined on V, with $a_i(0) = \frac{\partial L}{\partial x_i}(0)$.

Semi-quadratic forms

Construction given by proof, i.e.,

$$
\begin{aligned}
L(x_1,\ldots,x_n) &= \int_0^1 \frac{\partial L(tx_1,\ldots,tx_n)}{\partial t} dt \\
&= \int_0^1 \sum_{i=1}^{n} \frac{\partial L}{\partial x_i}(tx_1,\ldots,tx_n) x_i dt
\end{aligned}
$$

Thus, $a_i(x_1,\ldots,x_n) = \int_0^1 \frac{\partial L}{\partial x_i}(tx_1,\ldots,tx_n)dt$.

Semi-quadratic forms

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&= \int_0^1 \sum_{i=1}^{n} \frac{\partial L}{\partial x_i}(tx_1,\dots,tx_n) x_i dt
\end{aligned}
$$

Thus, $a_i(x_1,\dots,x_n) = \int_0^1 \frac{\partial L}{\partial x_i}(tx_1,\dots,tx_n)dt$.

Apply twice, then **“semi-quadratic”** form!

Next result uses Morse’s lemma for **input-normal** form.

Semi-quadratic form

Lemma: There exists coordinate transf. $x = \phi(\bar{x})$, $\phi(0) = 0$ (defined on neighborhood of 0), such that in $\bar{x} = \phi^{-1}(x)$ function $L_c(x)$ is of the form

$$L_c(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T\bar{x}.$$

In $\bar{x} = \phi^{-1}(x)$ we can write $L_o(x)$ in the form

$$L_o(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T M(\bar{x})\bar{x} \quad \text{where} \quad M(0) = \frac{\partial^2 L_o}{\partial x^2}(0),$$

with $M(\bar{x})$ an $n \times n$ symmetric matrix such that its entries are smooth functions of $\bar{x}$.

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with $M(\bar{x})$ an $n \times n$ symmetric matrix such that its entries are smooth functions of $\bar{x}$.

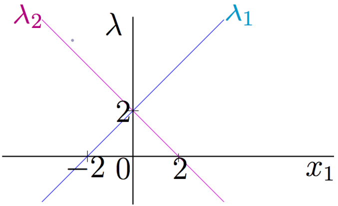
Is $M(\bar{x})$ unique?

Kato's result

Lemma: If there exists a neighborhood V of 0 where the number of distinct eigenvalues of $M(\bar{x})$ is constant for $\bar{x} \in V$, then on V the eigenvalues $\lambda_i(\bar{x})$, $i = 1, \ldots, n$, are smooth functions of $\bar{x}$, as well as the associated normalized eigenvectors.

Ex.: # distinct eig. val. is not constant:

$$M(\bar{x}) = \begin{pmatrix} 2 + x_1 & 0 \\ 0 & 2 - x_1 \end{pmatrix}$$



Input-normal/output-diagonal form

Theorem: Assume condition of Kato's result is fulfilled. On neighborhood U of 0 there exists ψ, $x = \psi(z)$, $\psi(0) = 0$, s.t. for $z \in W := \psi^{-1}(U)$ L_c and L_o are of form

$$\tilde{L}_c(z) := L_c(\psi(z)) = \frac{1}{2} z^T z,$$

$$\tilde{L}_o(z) := L_o(\psi(z)) = \frac{1}{2} z^T \begin{pmatrix} \tau_1(z) & & 0 \\ & \ddots & \\ 0 & & \tau_n(z) \end{pmatrix} z.$$

Here $\tau_1(z) \geq \cdots \geq \tau_n(z)$ are smooth functions of z, called the **singular value functions** of system.

Balanced form

- Recall for linear systems in balanced form: $\Sigma = P = Q$.
- Input-normal/output-diagonal means $P = I$, $Q = \Sigma^2$.

Hence, nonlinear system is not really in balanced form yet!

Balanced form

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Hence, nonlinear system is not really in balanced form yet!

To make model reduction stage easier, **no** aim for full balance, but only for **balance on the coordinate axes**, i.e., aim for extension of

$$L_c(0, \dots, x_i, \dots, 0) = \frac{1}{2} x_i^2 \sigma_i^{-1}$$

$$L_o(0, \dots, x_i, \dots, 0) = \frac{1}{2} x_i^2 \sigma_i.$$

Balanced form

- Take $\bar{z}_i = \eta_i(z_i) := \tau_i(0, \ldots, 0, z_i, 0, \ldots 0)^{\frac{1}{4}} z_i$, $i = 1, \ldots, n$. Since $\tilde{L}_o(z) > 0$, $\tau_i(0, \ldots, 0, z_i, 0, \ldots, 0) > 0$, for $z_i \neq 0$.

Balanced form

- Take $\bar{z}_i = \eta_i(z_i) := \tau_i(0,\ldots,0,z_i,0,\ldots0)^{\frac{1}{4}} z_i$, $i = 1,\ldots,n$. Since $\tilde{L}_o(z) > 0$, $\tau_i(0,\ldots,0,z_i,0,\ldots,0) > 0$, for $z_i \neq 0$.
- Define $\check{L}_c(\bar{z}) := \tilde{L}_c(\eta^{-1}(\bar{z}))$, $\check{L}_o(\bar{z}) := \tilde{L}_o(\eta^{-1}(\bar{z}))$. Then

$$\check{L}_c(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} \sigma_1(\bar{z}_1)^{-1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n(\bar{z}_n)^{-1} \end{pmatrix} \bar{z},$$

$$\check{L}_o(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} \sigma_1(\bar{z}_1)^{-1}\tau_1(\eta^{-1}(\bar{z})) & & 0 \\ & \ddots & \\ 0 & & \sigma_n(\bar{z}_n)^{-1}\tau_n(\eta^{-1}(\bar{z})) \end{pmatrix} \bar{z},$$

with $\sigma_i(\bar{z}_i) = \tau_i(0,\ldots,0,\eta_i^{-1}(\bar{z}_i),0,\ldots,0)^{\frac{1}{2}}$, $i = 1,\ldots,n$.

Balanced form

In $\bar{z}$ coordinates

$$\begin{aligned}\check{L}_c(0,\ldots,0,\bar{z}_i,0,\ldots,0) &= \frac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i)^{-1}\\ \check{L}_o(0,\ldots,0,\bar{z}_i,0,\ldots,0) &= \frac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i)\end{aligned}$$

Balanced form

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Linearization of the complete nonlinear procedure results in the linear balancing procedure.

Linearized system:

$$\dot{\hat{z}} = \bar{A}\hat{z} + \bar{B}u$$

$$y = \bar{C}\hat{z}$$

where $\bar{A} = \frac{\partial \bar{f}}{\partial \bar{z}}(0)$, $\bar{B} = \bar{g}(0)$ and $\bar{C} = \frac{\partial \bar{h}}{\partial \bar{z}}(0)$.

Balanced form

Based on the linearization, we have the following result:

Theorem: Assume linearized system is **minimal** and A is **asymptotically stable**. If Hankel singular values, σ_i, $i = 1, \ldots, n$, of linear system satisfy $\sigma_i \neq \sigma_j$ for $i \neq j$, $i, j = 1, \ldots, n$, then locally the nonlinear system may be brought into balanced form.

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Uses the fact that $\sigma_i \neq \sigma_j$ implies that there exists neighborhood V of 0 such that # of distinct eigenvalues of the matrix $M(\bar{x})$ is constant and equal to n (Kato's result).

Model reduction

Assume that

$$\tau_k(z) > \tau_{k+1}(z)$$
$$\left(\Rightarrow \sigma(\bar{z}_k)^{-1}\tau_k(\eta^{-1}(\bar{z})) > \sigma(\bar{z}_{k+1})^{-1}\tau_{k+1}(\eta^{-1}(\bar{z}))\right).$$

and that nonlinear system is in balanced form.

Partition system as follows:

$$\bar{f}(\bar{z}) = \begin{pmatrix} \bar{f}_a(\bar{z}^a, \bar{z}^b) \\ \bar{f}_b(\bar{z}^a, \bar{z}^b) \end{pmatrix}, \;\; \bar{g}(\bar{z}) = \begin{pmatrix} \bar{g}_a(\bar{z}^a, \bar{z}^b) \\ \bar{g}_b(\bar{z}^a, \bar{z}^b) \end{pmatrix}, \;\; \bar{h}(\bar{z}) = \bar{h}(\bar{z}^a, \bar{z}^b)$$

where $\bar{z}^a = (\bar{z}_1, \ldots, \bar{z}_k)$ and $\bar{z}^b = (\bar{z}_{k+1}, \ldots, \bar{z}_n)$.

Model reduction

Truncation, then $\bar{z}^b = 0$.

$$\frac{\partial \check{L}_o}{\partial \bar{z}^a}(\bar{z}^a, 0)\bar{f}_a(\bar{z}^a, 0) + \frac{\partial \check{L}_o}{\partial \bar{z}^b}(\bar{z}^a, 0)\bar{f}_b(\bar{z}^a, 0) + \frac{1}{2}\bar{h}^T(\bar{z}^a, 0)\bar{h}(\bar{z}^a, 0) = 0$$

$$\frac{\partial \check{L}_c}{\partial \bar{z}^a}(\bar{z}^a, 0)\bar{f}_a(\bar{z}^a, 0) + \frac{1}{2}\frac{\partial \check{L}_c}{\partial \bar{z}^a}(\bar{z}^a, 0)\bar{g}_a(\bar{z}^a, 0)\bar{g}_a^T(\bar{z}^a, 0)\frac{\partial^T \check{L}_c}{\partial \bar{z}^a}(\bar{z}^a, 0) = 0$$

From the Hamilton-Jacobi equations it follows that the **controllability** function of the reduced order system is equal to $\check{L}_c(\bar{z}^a, 0)$. For observability new condition necessary.

Model reduction

Theorem: If $\frac{\partial \check{L}_o}{\partial \bar{z}^b}(\bar{z}^a, 0)\bar{f}_b(\bar{z}^a, 0) = 0$ for $(\bar{z}^a, 0) \in \bar{W}$ then the **observability** function of the reduced order system is given by $\check{L}_o(\bar{z}^a, 0)$. Furthermore, the reduced order system is in balanced form having singular value functions $\tau_1(z^a, 0) \geq \cdots \geq \tau_k(z^a, 0)$, for $(z^a, 0) = \eta^{-1}(\bar{z}^a, 0)$.

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Theorem: The subsystems $\left(\bar{f}_a(\bar{z}^a, 0), \bar{g}_a(\bar{z}^a, 0), \bar{h}(\bar{z}^a, 0)\right)$ and $(\bar{f}_b(0, \bar{z}^b), \bar{g}_b(0, \bar{z}^b), \bar{h}(0, \bar{z}^b))$, respectively, are both **locally asymptotically stable**.

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Theorem: If $\bar{f}_b(\bar{z}^a, 0) = 0$ and $\check{L}_o$ is proper (for each $c > 0$ the set $\{x \in M | 0 \leq \check{L}_o(x) \leq c\}$ is compact) on $\bar{W}$, then the reduced system is asymptotically stable on $(\bar{z}^a, 0) \in \bar{W}$.

Relation with minimality

$L_o(x^1,0,0,0)$ and $L_c(x^1,0,0,0)$ can be transformed to input-normal/output-diagonal form, i.e., there exists $x^1=\psi(z)$, $\psi(0)=0$, $(\psi^{-1}(x^1),0,0,0)\in Y$, such that

$$L_c(\psi(z),0,0,0)=\frac{1}{2}z^Tz$$

$$L_o(\psi(z),0,0,0)=\frac{1}{2}z^T\begin{pmatrix}\tau_1(z) & & 0\\ & \ddots & \\ 0 & & \tau_{n_1}(z)\end{pmatrix}z.$$

Thus x^1-part can be **balanced**, with singular value functions $\tau_1(z)\geq\cdots\geq\tau_{n_1}(z)$.

Relation with minimality

Also consider x^2, then there exists $(z^1, z^2) = \phi^{-1}(x^1, x^2)$ s.t.

$$L_c(\phi(z^1, z^2), 0, 0) = \frac{1}{2}{z^1}^T z^1 + \frac{1}{2}{z^2}^T z^2$$

$$L_o(\phi(z^1, z^2), 0, 0) = \frac{1}{2}\begin{pmatrix} {z^1}^T & {z^2}^T \end{pmatrix} M(z^1, z^2) \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}.$$

If Kato's condition is fulfilled, we may diagonalize $M(z^1, z^2)$.

Then functions on the diagonal are
$\bar{\tau}_1(z^1, z^2) \geq \cdots \geq \bar{\tau}_{n_1+n_2}(z^1, z^2)$, where $\bar{\tau}_i(z^1, 0) = \tau_i(z)$,
$i = 1, \ldots, n_1$, and $\bar{\tau}_j(0, x^2) = 0$, $j = n_1 + 1, \ldots, n_1 + n_2$.

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In accordance with the linear case, where **unobservable** part yields **zero** ‘Hankel singular values’.

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Hence the part of the system that is not strongly accessible yields an ‘**inverse of the controllability Gramian**’ that is **infinite**, and thus a ‘**controllability Gramian**’ that is **zero**.

Hankel considerations

- Note that this notion of balanced form for nonlinear systems is not unique, due to non-uniqueness in semi-quadratic form!
- Restricted form of balancing, more related to Hankel operator and similarity invariants are obtained.
- In case semi-quadratic form is given, then unique up to orthogonal transformation related to multiplicity of singular value function.
- Other possibility is “block” diagonalization based on ordering of eigenvalues of matrix $M(\bar{x})$.

Hankel considerations

- Hankel norm for **linear** systems

$$\begin{aligned}\|\Sigma\|_H^2 &= \max_{u\in L_{2+}} \frac{\|\mathcal{H}(u)\|^2}{\|u\|^2} = \max_{u\in L_{2+}} \frac{\langle u, \mathcal{H}^*\mathcal{H}(u)\rangle}{\langle u, u\rangle} \\ &= \max_x \frac{x^T Q x}{x^T P^{-1} x} = \lambda_{\max}(\mathcal{H}^*\mathcal{H}) = \lambda_{\max}(PQ) = \sigma_1^2\end{aligned}$$

- Hankel norm for **nonlinear** systems

$$\begin{aligned}\|\Sigma\|_H^2 &= \max_{u\in L_{2+}} \frac{\|\mathcal{H}(u)\|^2}{\|u\|^2} = \max_{u\in L_{2+}} \frac{\langle u, \mathcal{H}^*(\mathcal{H}(u), u)\rangle}{\langle u, u\rangle} \\ &= \max_x \frac{L_o(x)}{L_c(x)} = \textbf{???}\end{aligned}$$

Hankel considerations

How to determine **???**

- For relation with Hankel operator and Hankel norm, balanced state-space form does not suffice.

- By considering both eigenstructure of

⋆ **differential adjoint** $(d\mathcal{H}(\cdot))^*\,(\mathcal{H}(\cdot))$ and

⋆ **full nonlinear Hilbert adjoint** $\mathcal{H}^*(\mathcal{H}(u), u)$,

characterization based on sort of **parametrization** that is related to the input value yields form that fill in the **???**, i.e., give explicit expression for Hankel norm.

Hankel considerations

- Appropriate assumptions, then there exists $x = \Phi(z)$ s.t.

$$\begin{aligned} L_c(\Phi(z)) &= \frac{1}{2} z^T z \\ L_o(\Phi(z)) &= \frac{1}{2} z^T \mathrm{diag}(\tau_1(z), \ldots, \tau_n(z)) z \end{aligned}$$

$$\frac{\partial L_c(\Phi(z))}{\partial z_i} = 0 \iff \frac{\partial L_o(\Phi(z))}{\partial z_i} = 0$$

$$\tau_i(0, \ldots, 0, z_i, 0, \ldots 0) = \rho_i^2(z_i), \qquad \|\Sigma\|_H = \max_{z_1}\{\rho_1(z_1)\}$$

Now **unique** and balanced structure preserving model reduction tool on coordinate axes!

Concluding remarks

> Balancing for stable systems treated

> Unstable, structure preserving, etc., exist, also for nonlinear systems

> Variations: incremental (Besselink et al. 2014), differential (Kawano/Scherpen 2017), flow balancing (Verriest/Gray 00’s), etc.

> Computations have always been an issue, Boris is up next to enlighten us on this!

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