Extended (differential) balancing based model reduction for structure preservation

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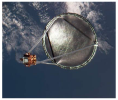
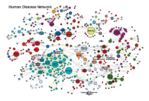
Background

Large scale systems

- Models stemming from discretization procedures, such as
  - Continuum mechanics models such as flexible beams
  - Multi-phase systems
  - Maxwell equations
  - Coupling of the above domains
  - ....



- Biochemical processes



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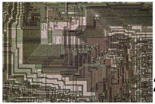
- Chip design, electrical circuit simulators



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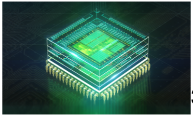

[3]

[1]The proteine interaction Network of Treponema pallidum, Wikimedia commons
[2]Wired.com
[3]Siemens blogs

Problem description

Problem

Given system

$$\begin{cases} \dot{x} = f(x) + g(x)u, & x \in \mathbb{R}^n \\ y = h(x) \end{cases}$$

Find a lower dimensional system ($r < n$)

$$\begin{cases} \dot{x}_r = f_r(x_r) + g_r(x_r)u, & x_r \in \mathbb{R}^r \\ y_r = h_r(x_r) \end{cases}$$

such that $y \approx y_r$ for the same u

- How to decide r?
- How to find a lower dimensional system?

Background

Important to realize: purpose of model reduction!

- Control design based on system model often results in a controller of the same size as the model, potentially hampering implementation
- Interaction with environment through inputs and outputs is important to take into account

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Balancing methods for other structure preservation (on top of input-output structure) topic of this presentation

- port-Hamiltonian structure
- RLC structure
- Network structure
- ......

Outline

1. Generalized and extended Gramians, MOR and an error bound
2. Linear port Hamiltonian systems structure preservation
3. Differential extended balancing for nonlinear systems
4. Conclusion and Outlook

Generalized balancing

Consider (A, B, C) as. stable, but not necessarily minimal. Using Lyapunov inequalities for balancing, the error bound can be improved, i.e., using the so called generalized controllability and observability Gramians (*Hinrichsen/Pritchard 1990*):

$$A\tilde{P} + \tilde{P}A^T + BB^T \leq 0$$

$$A^T Q + QA + C^T C \leq 0$$

$Q, \tilde{P} \geq 0$. Assume $Q, \tilde{P} > 0$.

Generalized balancing: simultaneously diagonalise Q and $\tilde{P}$ with on the diagonal $\lambda_i(Q\tilde{P})$.

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Freedom can potentially be used for preserving more structure.

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Generalized balancing: simultaneously diagonalise Q and $\tilde{P}$ with on the diagonal $\lambda_i(Q\tilde{P})$.

Freedom can potentially be used for preserving more structure.

We may need to consider the inverse of $\tilde{P}$, i.e., $P = \tilde{P}^{-1} > 0$ is a stabilizing solution of

$$PA + A^T P + PBB^T P \leq 0$$

Extended Gramians

- Sometimes even more structure needs to be preserved, e.g., network structure or physical structure.
- Discrete time version of extended balancing provided in *Sandberg 2010*.
- Continuous time version is not straightforwardly obtained, i.e., in discrete time there is robust control theory with LMI's (*Oliveira et al. 1999, 2002*) that provide the basis for *Sandberg 2010*.

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- Continuous time version is not straightforwardly obtained, i.e., in discrete time there is robust control theory with LMI's (*Oliveira et al. 1999, 2002*) that provide the basis for *Sandberg 2010*.
- Extension by *Scherpen/ Fujimoto, ECC18* based on dissipativity thinking that underlies discrete version, i.e., aim at using storage function

$$(x-x_r)^T Q(x-x_r) + (x+x_r)^T P(x+x_r)$$

where $P = \tilde{P}^{-1}$.

Extended Gramians

Consider the LMI's

$$\begin{pmatrix} -QA - A^TQ - C^TC & Q - \alpha S - A^TS \\ Q - \alpha S^T - S^TA & S + S^T \end{pmatrix} \geq 0$$

and

$$\begin{pmatrix} -PA - A^TP & -P + \beta T + A^TT & -2PB \\ -P + \beta T^T + T^TA & T + T^T & 2T^TB \\ -2B^TP & 2B^TT & 4I \end{pmatrix} \geq 0$$

with Q and P the generalized observability and inverse controllability Gramian, resp., $S, T \in \mathbb{R}^{n \times n}$, and $\alpha, \beta > 0$ large enough scalars.

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with Q and P the generalized observability and inverse controllability Gramian, resp., $S, T \in \mathbb{R}^{n\times n}$, and $\alpha, \beta > 0$ large enough scalars.

Solutions define the extended observability Gramian (Q, S, α) and extended inverse controllability Gramian (P, T, β).

Extended controllability Gramian

Another characterization for the extended controllability Gramian. $\tilde{P} = P^{-1}$ is given by

$$\begin{pmatrix} -A\tilde{P} - \tilde{P}A^\top & -T^{-1} + \tilde{P}(\beta I_n + A)^\top \\ -T^{-\top} + (\beta I_n + A)\tilde{P} & T^{-\top} + T^{-1} \end{pmatrix} - \begin{pmatrix} BB^\top & -BB^\top \\ -BB^\top & BB^\top \end{pmatrix} \geq 0.$$

with $\tilde{P}$ the generalized controllability Gramian. Solutions define the extended controllability Gramian $(\tilde{P}, T^{-1}, \beta)$.

Extended Gramians

Theorem *extension of (Scherpen/Fujimoto 2018)*

The generalized observability Lyapunov inequality has a solution $Q > 0$ if and only if the extended observability LMI has a solution (Q, S, α) with $Q = Q^T > 0$, and $\alpha \in \mathbb{R}$, $\alpha > 0$ large enough, i.e., such that $2\alpha Q \geq C^T C$.

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The "dual" version:

Theorem *extension of (Scherpen/Fujimoto 2018)*

The generalized inverse controllability Riccati inequality has a solution $P > 0$ (and thus the controllability Lyapunov inequality has a solution $\tilde{P} > 0$) if and only if the extended inverse controllability LMI has a solution (P, T, β) with $P > 0$, for any $\beta \in \mathbb{R}$, $\beta > 0$.

Model comparison

Now compare the following two models:

$$\Sigma : \begin{cases} \dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ y &= Cx + Du \end{cases}$$

$$\Sigma_r : \begin{cases} \dot{x}_r &= Ax_r + Bu + v, \quad x_r(0) = x_{r0} \\ y_r &= Cx_r + Du. \end{cases}$$

Idea: by picking a specific v, the error after balanced truncation based on the extended Gramians can be obtained.

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First important to compare these two systems.

Model comparison

Write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix},$$

Lemma

Consider the reduced order system $\hat{\Sigma}$ given by the triple (A_{11}, B_1, C_1) with state $\hat{x}$, and output $\hat{y}$. Now partition x_r of Σ_r accordingly in $x_r = (x_{r1}^T, x_{r2}^T)^T$ Choose

$$v(t) = \begin{pmatrix} 0 \\ -A_{21}x_{r1}(t) - B_2 u(t) \end{pmatrix}$$

and $x_r(0) = 0$, $\hat{x}(0) = 0$, then $y_r(t) = \hat{y}(t)$, and $x_{r2}(t) = 0$ for $t \geq 0$.

Comparison and the extended LMI's

$$\begin{pmatrix} x - x_r \\ v \end{pmatrix}^T \begin{pmatrix} -QA - A^TQ - C^TC & Q - \alpha S - A^TS \\ Q - \alpha S^T - S^TA & S + S^T \end{pmatrix} \begin{pmatrix} x - x_r \\ v \end{pmatrix} \geq 0$$

$$\Leftrightarrow \frac{d}{dt}\left((x - x_r)^T Q (x - x_r)^T\right) \leq -|y - y_r|^2 - 2\alpha (x - x_r)^T S v - 2(\dot{x} - \dot{x}_r)^T S v$$

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To be combined with the expression for the extended inverse controllability

Assume S and T symmetric. Eigenvalues of ST^{-1} similarity invariants.

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & \sigma_n \end{pmatrix}, \quad T = \begin{pmatrix} T_1 & 0 \\ 0 & \sigma_n^{-1} \end{pmatrix}$$

Pick $\alpha = \beta$ large enough, then using the previous expressions:

$$\frac{d}{dt}\left((x - x_r)^TQ(x - x_r) + \sigma_n^2(x + x_r)^TP(x + x_r)^T\right) = -|y - y_r|^2 + 4\sigma_n^2|u|^2$$

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Useful for error bound! Inspired by a presentation of Jan Willems 2002.

Error bound

Iteratively taking all steps the following is obtained:

Theorem *S., Fujimoto ECC'18, Borja, S., Fujimoto TAC '22*

Suppose that Σ has balanced extended Gramians in the sense that T and S are balanced, i.e.,

$$S = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix}, \quad T = \begin{pmatrix} \sigma_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n^{-1} \end{pmatrix}.$$

Assume $\sigma_k > \sigma_{k+1}$, and write $S = diag(S_k, S_{n-k})$, $T = diag(T_k, T_{n-k})$ Then the truncated k^{th} order system $\hat{\Sigma}$ has balanced extended Gramians (P_{11}, T_k, α) and (Q_{11}, S_k, α). Furthermore

$$\|\Sigma - \hat{\Sigma}\|_\infty \leq 2 \sum_{i=k+1}^{n} \sigma_i$$

Error bound

- This often gives a better error bound than standard balancing, and may also give a better error bound then generalized balancing!

Linear port Hamiltonian systems

- Introduced in 1992 *(Maschke/Van der Schaft)*, useful for multi-physics systems, passive interconnections, and control design.
- Important to preserve the structure when reduction is performed.

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Linear PH system:

$$\begin{aligned} \dot{x} &= (J-R)\Gamma x + Bu \\ y &= B^T \Gamma x \end{aligned}$$

wtih $J = -J^T$, $R = R^T \geq 0$, and Hamiltonian $H = x^T \Gamma x$, $\Gamma > 0$.

$$\dot{H}(x) = -x^T \Gamma R \Gamma x + u^T y \leq u^T y \quad \text{PH systems are passive!}$$

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Note

For structure preserving reduction it is necessary that in Γ is "block" diagonal.

Generalized balancing for PH systems

Another possibility *(Borja, S. , Fujimoto, MTNS'18, TAC'22)*:

Proposition

Consider factorized matrices

$$\begin{aligned} P^{-1} &= \tilde{P} = \phi_P^\top \phi_P, \quad \phi_P \Gamma \phi_P^\top = U_\Gamma \Sigma_\Gamma U_\Gamma^\top \\ \hat{F} &= U_\Gamma^\top \phi_P^{-\top} (J-R) \phi_P^{-1} U_\Gamma, \quad \hat{B} = U_\Gamma^\top \phi_P^{-\top} B, \end{aligned}$$

and assume that the following inequality holds

$$-\Sigma_{QP}^2 \Sigma_\Gamma^{-1} \hat{F} - \hat{F}^\top \Sigma_\Gamma^{-1} \Sigma_{QP}^2 - \hat{B}\hat{B}^\top \geq 0$$

for a diagonal matrix Σ_{QP}^2. Then, a generalized observability Gramian is given by $Q = \phi_P^{-1} U_\Gamma \Sigma_{QP}^2 U_\Gamma^\top \phi_P^{-\top}$.

The transformation $W = \phi_P^\top U_\Gamma \Sigma_{QP}^{-\frac{1}{2}}$ balances the system and diagonalizes Γ.

Generalized and extended balancing for PH systems

A dual version starting from factorizing Q, and then obtaining the generalized controllability Gramian $\tilde{P}$ is also available.

- Solvability of the inequality in the propositions is not straightforward. Nevertheless, often computable.

Extended controllability Gramian $(\tilde{P}, T^{-1}, \beta)$ and observability Gramian (Q, S, α). Similar construction, now on T and S. PH structure preserved as well, somewhat more freedom to do balancing.

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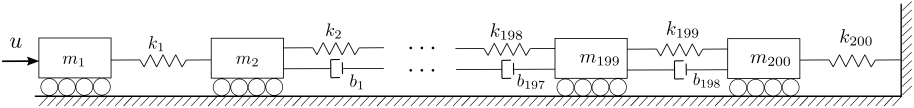
- Observation from examples: when the actuated coordinates are damped, the generalized solutions provide the best error bound, otherwise the extended procedure provides the best error bound!

Example

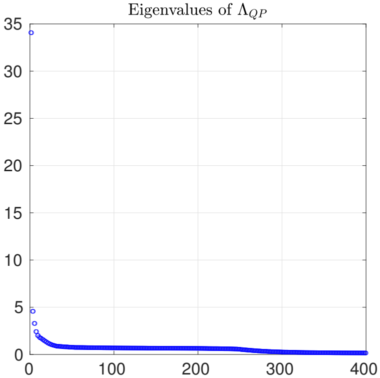
Consider a mechanical system consisting of 200 mass-spring-damper systems



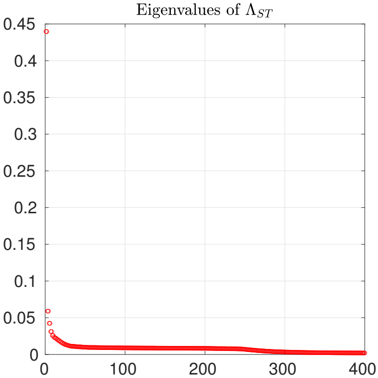
$$\begin{bmatrix}\dot q\\ \dot p\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{0}_{200\times 200} & I_{200}\\ -I_{200} & R_2\end{bmatrix}}_{F}\underbrace{\begin{bmatrix}K & \mathbf{0}\\ \mathbf{0} & M^{-1}\end{bmatrix}}_{H}\begin{bmatrix}q\\ p\end{bmatrix} + \underbrace{\begin{bmatrix}\mathbf{0}\\ G\end{bmatrix}}_{B}u,$$

Masses vary between $0.4[kg]$ and $0.6[kg]$
Spring constants between $0.9[kg/s^2]$ and $1.1[kg/s^2]$
Damping coefficients between $1.8[kg/s]$ and $2.2[kg/s]$.

Example, singular values



Generalized balancing.



Extended balancing.

Example, errors

		GB	GB	EB	EB
k	ω	$\mathcal{H}_\infty$	$2\sum_{j=k+1}^{n}\sigma_j$	$\mathcal{H}_\infty$	$2\sum_{j=k+1}^{n}\sigma_j$
300	1.8078	0.0120	38.3047	0.0120	0.4943
200	1.7864	0.3801	140.0848	0.2492	1.7635
100	1.8081	0.6898	276.8439	0.4701	3.4875

ω corresponds to the input frequency, i.e., $u = 2sin(\omega t)$
ω chosen at peak frequency of error system

Nonlinear prolonged system

Consider the nonlinear system

$$\Sigma : \begin{cases} \dot{x} = f(x) + Bu, \\ y = Cx, \end{cases}$$

$x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is at least class C^1, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$.
$\phi(t, x_0, u)$ denotes the solution at time t starting from x_0 for input u.

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Associated variational system

$$d\Sigma : \begin{cases} \delta\dot{x} = \dfrac{\partial f(x)}{\partial x}\delta x + B\delta u, \\ \delta y = C\delta x, \end{cases}$$

Differential Gramian extensions

Differential Controllability function

$$L_{\mathcal{C}}(x_0, \delta x_0) := \inf_{\delta u \in L_2^m(-\infty, t_0]} \frac{1}{2} \int_{-\infty}^{t_0} ||\delta u(t)||^2 dt$$

where $x(t_0) = x_0 \in \mathbb{R}^n$, $\delta x(t_0) = \delta x_0 \in \mathbb{R}^n$, $u \in L_2^m(-\infty, t_0]$ and $\delta x(-\infty) = 0$.

Differential Observability function

$$L_{\mathcal{O}}(x_0, \delta x_0) := \frac{1}{2} \int_{t_0}^{\infty} ||\delta y(t)||^2 dt$$

where $x(t_0) = x_0 \in \mathbb{R}^n$, $\delta x(t_0) = \delta x_0 \in \mathbb{R}^n$, and $\delta x(\infty) = 0$.

Generalized Differential Gramians

Generalized differential controllability and observability Gramians are defined as the solutions $P \succ 0$, $Q \succ 0$ of

$$\frac{\partial f(x)}{\partial x} P + P \frac{\partial^\top f(x)}{\partial x} + BB^\top \preceq -\epsilon P,$$

$$Q \frac{\partial f(x)}{\partial x} + \frac{\partial^\top f(x)}{\partial x} Q + C^\top C \preceq -\epsilon Q,$$

for all $x \in \mathbb{R}^n$, $\epsilon > 0$.

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for all $x \in \mathbb{R}^n$, $\epsilon > 0$. If solutions exist then

$\bar{L}_{\mathcal{C}}(\delta x_0) := \frac{1}{2}\delta x_0^\top P^{-1}\delta x_0$ generalized differential controllabilty function
$\bar{L}_{\mathcal{O}}(\delta x_0) := \frac{1}{2}\delta x_0^\top Q\delta x_0$ generalized differential observability function.

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$$\bar{L}_{\mathcal{C}}(\delta x_0) \leq L_{\mathcal{C}}(\delta x_0, x_0),$$

$$\bar{L}_{\mathcal{O}}(\delta x_0) \geq L_{\mathcal{O}}(\delta x_0, x_0),$$

where $L_{\mathcal{C}}$ and $L_{\mathcal{O}}$ differential controllability and observability functions.

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Extended differential gramians

Consider the LMIs

$$\begin{bmatrix} X_o & Q - (\alpha I_n + \frac{\partial^\top f}{\partial x})S \\ Q - S^\top(\alpha I_n + \frac{\partial f}{\partial x}) & (S + S^\top) \end{bmatrix} \succeq 0$$

and

$$\begin{bmatrix} -\check{P}\frac{\partial f}{\partial x} - \frac{\partial^\top f}{\partial x}\check{P} - \epsilon\check{P} & -\check{P} + (\beta I_n + \frac{\partial^\top f}{\partial x})T & -2\check{P}B \\ \check{P} - T^\top(\beta I_n + \frac{\partial f}{\partial x}) & T + T^\top & 2T^\top B \\ -2B^\top\check{P} & 2B^\top T & 4I_m \end{bmatrix} \succeq 0$$

with $\check{P} := P^{-1}$ and $X_o(x) := -Q\frac{\partial f(x)}{\partial x} - \frac{\partial^\top f(x)}{\partial x}Q - C^\top C - \epsilon Q$.

(Q, S, α) and $(\check{P}, T, \beta)$ extended differential observability and controllability Gramians.

Extended differential gramians

Consider the LMIs

$$\begin{bmatrix} X_o & Q - (\alpha I_n + \frac{\partial^\top f}{\partial x})S \\ Q - S^\top(\alpha I_n + \frac{\partial f}{\partial x}) & (S + S^\top) \end{bmatrix} \succeq 0$$

and

$$\begin{bmatrix} -\check{P}\frac{\partial f}{\partial x} - \frac{\partial^\top f}{\partial x}\check{P} - \epsilon\check{P} & -\check{P} + (\beta I_n + \frac{\partial^\top f}{\partial x})T & -2\check{P}B \\ \check{P} - T^\top(\beta I_n + \frac{\partial f}{\partial x}) & T + T^\top & 2T^\top B \\ -2B^\top\check{P} & 2B^\top T & 4I_m \end{bmatrix} \succeq 0$$

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System transformed into extended differentially balanced form by $x = W_e z$,

$$W_e^\top T^{-1} W_e = W_e^{-1} S W_e^{-\top} = \Lambda_{ST} = \text{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_n\}, \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$$

Truncation Error bound

Assumption

The drift vector field of the nonlinear system is an odd function of the state vector, i.e. $f(x) = -f(-x)$.

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Theorem

Suppose the system is differentially extended balanced (Q, S, α) and $(\check{P}, T, \beta)$. With

$$S = T^{-1} = \Lambda_{ST} = \text{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_n\},$$

and reduction performed when we pick $\alpha = \beta$, $\sigma_r >> \sigma_{r+1}$, then

$$||y - \hat{y}||_2 \leq 2 \sum_{j=r+1}^{n} \sigma_j ||u||_2$$

where $\hat{y}$ is the output of the reduced order system.

Nonlinear mass-spring-damper system

Consider a mass-spring-damping system with 40 masses and Coulomb friction.

The variational system is

$$\begin{aligned}\begin{bmatrix}\delta\dot{q}\\ \delta\dot{p}\end{bmatrix} &= \begin{bmatrix}0 & I_n\\ -I_n & -(D+\bar{R}(p))\end{bmatrix}\begin{bmatrix}K & 0\\ 0 & M^{-1}\end{bmatrix}\begin{bmatrix}\delta q\\ \delta p\end{bmatrix} + \begin{bmatrix}0\\ G\end{bmatrix}\delta u\\ y &= \begin{bmatrix}0 & G^\top\end{bmatrix}\begin{bmatrix}K & 0\\ 0 & M^{-1}\end{bmatrix}\begin{bmatrix}\delta q\\ \delta p\end{bmatrix}\end{aligned}$$

with a smooth approximation of Coulomb friction as follows

$$\bar{R}(p) = \mathrm{diag}\left\{\frac{\delta_i\gamma_i m_i^2}{(\gamma_i m_i^2 + p_i^2)^{3/2}}\right\} \geq 0$$

for $i = 1, 2, \cdots, n$.

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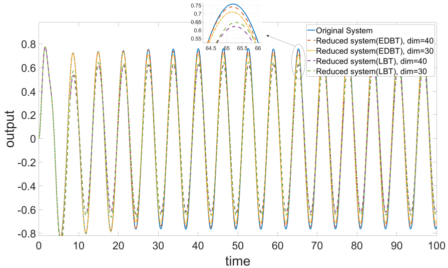
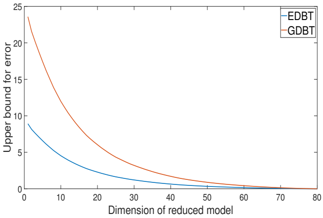
Simulations



Comparison of outputs of actual system and reduced order system of different orders



Upper bounds for the error versus dimension of the reduced model

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- Useful for PH systems to preserve structure, more "symmetric" than existing methods (e.g., Polyuga et al.)
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Outlook

- General RLC structure preservation.
- Nonlinear port-Hamiltonian system structure preservation (A. Sarkar, S., SCL'23, presentation on Thursday).

Publications

- J.M.A. Scherpen, K. Fujimoto, Extended Balanced Truncation for Continuous Time LTI Systems, *Proc. European Control Conference*, Limassol, Cyprus (2018).
- L.P. Borja Rosales, J.M.A. Scherpen, Model Reduction of Linear Port-Hamiltonian Systems: A Structure Preserving Approach, extended abstract *Proc. 23rd International Symposium on Mathematical Theory of Networks and Systems (MTNS 2018)*, Hong Kong (2018).
- L.P. Borja Rosales, J.M.A. Scherpen, K. Fujimoto, Extended balancing of continuous LTI systems: a structure-preserving approach, *IEEE Trans. Automatic Control*, vol. 68, no. 1 (2023) 257-271.
- A. Sarkar, J.M.A. Scherpen, Extended differential balancing for nonlinear dynamical systems, *IEEE Control Systems Letters*, Vol. 6, (2022) 3170-3175.
- A. Sarkar, J.M.A. Scherpen, Structure-preserving generalized balanced truncation for nonlinear port-Hamiltonian systems, *Systems & Control Letters*, Special issue 80th birthday Arthur Krener, Vol. 174 (2023) 105501.

Questions?