Approximation of nonlinear optimal control problems in infinite dimension

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Joint work with

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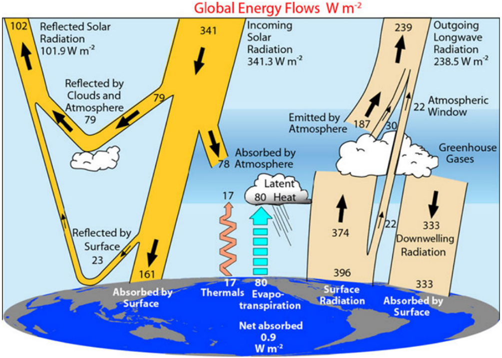
Outline

A motivating example: Optimal control of climate?

"Probably intervention in atmospheric and climate matters will come in a few decades, and will unfold on a scale difficult to imagine at present."
(John von Neumann, "Can we survive technology?", Fortune (June, 1955).)



The global annual mean Earth's energy budget for the Mar 2000 to May 2004 period. (K. E. Trenberth, J. T. Fasullo, and J. Kiehl, *Earth's Global Energy Budget*, **AMS**, March 2009.) [See also "Heat stored in the Earth system 1960–2020: where does the energy go?", ESSD article, 2023]

The consideration of climate engineering (a.k.a. geoengineering) is raising in the scientific community. (See e.g. National Research Council, Climate intervention: Carbon dioxide removal and reliable sequestration, National Academies Press, 2015.)

Functional Framework

Class of Equations considered and Functional Framework

We consider the following class of controlled **nonlinear evolution equations**:

$$\frac{\mathrm{d}y}{\mathrm{d}s} = Ly + F(y) + \mathfrak{C}(u(s)), \quad s \in (t, T],\ u \in \mathcal{U}_{ad}[t, T],$$

- $(\mathcal{H}, \|\cdot\|)$ a separable Hilbert space;
- $L : D(L) \subset \mathcal{H} \to \mathcal{H}$ a linear operator generating a C_0-semigroup, $\{T(s)\}_{s\geq 0}$;
- $F : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz;
- $\mathfrak{C} : V \to \mathcal{H}$ is allowed to be nonlinear and V is a separable Hilbert space;
- The set of admissible controls is taken to be:

  $$\mathcal{U}_{ad} := \{f \in L^q(0, T; V) \ : \ f(s) \in U \text{ for a.e. } s \in [0, T]\}, \ \ q \geq 1,$$

  with U a bounded set in V.
- $u \in \mathcal{U}_{ad}[t, T] := \{v|_{[t,T]} \ : \ v \in \mathcal{U}_{ad}\}$.

Note: This framework covers a range of equations, including **semilinear parabolic PDEs** and **delay differential equations**; and allows for **a broad class of nonlinear control laws**.

The optimal control problem

Cost functional: For each (t,x) in $[0,T) \times \mathcal{H}$, we consider the following type of cost functional $J_{t,x} : \mathcal{U}_{ad}[t,T] \to \mathbb{R}^+$:

$$J_{t,x}(u) := \int_t^T \left[\mathcal{G}(y_{t,x}(s,u(s))) + \mathcal{E}(u(s))\right] \mathrm{d}s.$$

- $\mathcal{G} \colon \mathcal{H} \to \mathbb{R}^+$ is locally Lipschitz;
- $\mathcal{E} \colon V \to \mathbb{R}^+$ is continuous;
- $y_{t,x}(\cdot,u)$ denotes the mild solution of the state equation with $y(t) = x$.

We consider the following family of **optimal control problems**:

$$\min\ J_{t,x}(u) \ \text{ subject to } \ (y,u) \in L^2(t,T;\mathcal{H}) \times \mathcal{U}_{ad}[t,T]) \ \text{ solves } \begin{cases} \frac{\mathrm{d}y}{\mathrm{d}s} = L_\lambda y + F(y) + \mathfrak{C}(u(s)), & s \in (t,T], \\ y(t) = x \in \mathcal{H}. \end{cases}$$

Galerkin Approximation: Convergence of Value Functions

Galerkin approximation

- Let $\{\mathcal{H}_N : N \in \mathbb{N}^*\}$ be a sequence of finite-dimensional subspaces of $\mathcal{H}$ associated with *orthogonal projectors* $\Pi_N : \mathcal{H} \to \mathcal{H}_N$, such that

$$\|(\Pi_N - \mathrm{Id})x\| \underset{N\to\infty}{\longrightarrow} 0, \quad \forall x \in \mathcal{H}, \quad \mathcal{H}_N \subset D(L), \text{ for all } N.$$

- The corresponding Galerkin approximation of the state equation reads:

$$\begin{aligned} &\frac{\mathrm{d}y_N}{\mathrm{d}s} = L_N y_N + \Pi_n F(y_N) + \Pi_N \mathfrak{C}(u(s)), \quad s \in (t,T],\ u \in \mathcal{U}_{ad}[t,T], \\ &y_N(t) = \Pi_N x,\ x \in \mathcal{H}, \end{aligned} \tag{4.1}$$

where $L_N := \Pi_N L \Pi_N : \mathcal{H} \to \mathcal{H}_N$.

- The associated cost functional is:

$$J^N_{t,x_N}(u) := \int_t^T \left[\mathcal{G}(y^N_{t,x_N}(s;u)) + \mathcal{E}(u(s))\right] \mathrm{d}s,$$

where $y^N_{t,x_N}(\cdot, u)$ denotes the solution of the Galerkin approximation (4.1).

Value functions and the main convergence result

The value functions corresponding to the above optimal control problems and their Galerkin approximations are defined by:

$$v(t,x) := \inf_{u \in \mathcal{U}_{ad}[t,T]} J_{t,x}(u), \ \forall\, (t,x) \in [0,T) \times \mathcal{H} \qquad \text{and} \quad v(T,x) := 0,$$

$$v_N(t,x_N) := \inf_{u \in \mathcal{U}_{ad}[t,T]} J^N_{t,x_N}(u), \ \forall\, (t,x_N) \in [0,T) \times \mathcal{H}_N \qquad \text{and} \quad v_N(T,x_N) := 0.$$

We identify **checkable conditions** that guarantee the following convergence result:

$$\boxed{\lim_{N\to\infty} \sup_{t\in[0,T]} |v_N(t,\Pi_N x) - v(t,x)| = 0, \quad \forall\, x \in \mathcal{H}.}$$

Reference:

[CKL17] M. D. Chekroun, A. Kröner and H. Liu, *Galerkin approximations of nonlinear optimal control problems in Hilbert spaces*, EJDE, 1–40, 2017. [arXiv link]

Sufficient conditions for the convergence of value functions

The conditions identified in [CKL17] to ensure the above convergence result can be put into three groups.

Group I: Conditions on the linear operator L and its Galerkin approximation L_N.

- $L : D(L) \subset \mathcal{H} \to \mathcal{H}$ generates a C_0-semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ on $\mathcal{H}$. In particular, there are constants $M > 0$ and $\omega \in \mathbb{R}$, such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

- The linear flow $e^{L_N t} : \mathcal{H}_N \to \mathcal{H}_N$ extends to a C_0-semigroup $T_N(t)$ on $\mathcal{H}$ for each $N \geq 1$. The following **stability condition** is satisfied by the family $\{T_N(t)\}_{N\geq 1, t\geq 0}$

$$\|T_N(t)\| \leq Me^{\omega t}, \quad N \geq 1, \quad t \geq 0.$$

- The following **consistency condition** holds:

$$\lim_{N\to\infty} \|L_N\phi - L\phi\|_{\mathcal{H}} = 0, \qquad \forall\, \phi \in D(L).$$

Sufficient conditions (cont'd)

Group II: Local Lipschitz conditions on F and $\mathcal{G}$, **continuity** of $\mathfrak{C}$ and $\mathcal{E}$, as well as **compactness** of U in V are required.

Group III: A uniform in u **a priori bound** as well as a condition on the **residual energy** are required:

(**III-1**) For each $x \in \mathcal{H}$ and $T > 0$, there exists a constant $\mathcal{C} := \mathcal{C}(T, x)$ such that

$$\|y(s; x, u)\|_{\mathcal{H}} \leq \mathcal{C}, \qquad \forall\, s \in [0, T],\ u \in \mathcal{U}_{ad},$$
$$\|y_N(s; \Pi_N x, u)\|_{\mathcal{H}} \leq \mathcal{C}, \qquad \forall\, s \in [0, T],\ u \in \mathcal{U}_{ad},\ N \in \mathbb{N}^*,$$

where $y(\cdot; x, u) := y_{0,x}(\cdot, u)$ and $y_N(\cdot; x, u) := y^N_{0,x}(\cdot, u)$.

(**III-2**) It is required that the residual energy of the solution $y(\cdot; x, u)$ satisfies

$$\lim_{N \to \infty} \sup_{u \in \mathcal{U}_{\rm ad}} \sup_{s \in [0,T]} \|\Pi_N^{\perp} y(s; x, u)\|_{\mathcal{H}} = 0,$$

where $\Pi_N^{\perp} := \mathrm{Id}_{\mathcal{H}} - \Pi_N$.

Note: Sufficient conditions to ensure (**III-2**) are also identified in [CKL17] for Galerkin approx. based on eigenbasis. It requires essential L to be self-adjoint with compact resolvent, and (**III-1**) is satisfied.

Main result and sketch of the proof

Theorem (Chekroun, Kröner, L., 2017)

Assume that

- *the conditions given in Groups I,II, III above hold;*
- *there exists for each pair* (t, x) *a minimizer* $u^*_{t,x}$ *(resp.* $u^{N,*}_{t,x}$*) in* $\mathcal{U}_{ad}[t, T]$ *for the the value function* $v(t, x)$ *(resp.* $v_N(t, \Pi_N x)$*).*

Then, it holds that

$$\lim_{N\to\infty} \sup_{t\in[0,T]} |v_N(t, \Pi_N x) - v(t, x)| = 0, \qquad \forall\, x \in \mathcal{H}.$$

Sketch of the proof: Recall that
$v(t, x) := \inf_{u\in\mathcal{U}_{ad}[t,T]} J_{t,x}(u),\ v_N(t, x_N) := \inf_{u\in\mathcal{U}_{ad}[t,T]} J^N_{t,x_N}(u)$, with

$$J_{t,x}(u) = \int_t^T [\mathcal{G}(y_{t,x}(s, u)) + \mathcal{E}(u(s))]\, \mathrm{d}s,\ J^N_{t,x_N}(u) = \int_t^T [\mathcal{G}(y^N_{t,x_N}(s; u)) + \mathcal{E}(u(s))]\, \mathrm{d}s.$$

We only need to estimate $\|y_{t,x}(s, u)) - y^N_{t,x_N}(s; u)\|_{\mathcal{H}}$. But, since L and F are time-independent, it suffices to estimate $\|y(s; x, u)) - y_N(s; x_N, u)\|_{\mathcal{H}}$.

Preparatory Lemma I: Assume that

- $F : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz;
- Conditions in Group III (i.e., (**III-1**) and (**III-2**)) hold.

Then,

$$\lim_{N\to\infty} \sup_{u\in\mathcal{U}_{\mathrm{ad}}} \sup_{t\in[0,T]} \|\Pi_N^{\perp} F(y(t;x,u))\|_{\mathcal{H}} = 0.$$

Proof: Note that

$$\|\Pi_N^{\perp} F(y(t;x,u))\|_{\mathcal{H}} \le \underbrace{\|\Pi_N^{\perp}\big(F(y(t;x,u)) - F(\Pi_{N_0} y(t;x,u))\big)\|_{\mathcal{H}}}_{=:I_1(N,N_0;u)} + \underbrace{\|\Pi_N^{\perp} F(\Pi_{N_0} y(t;x,u))\|_{\mathcal{H}}}_{=:I_2(N,N_0;u)}.$$

Denoting $\mathfrak{B} := B(0,\mathcal{C}) \subset \mathcal{H}$, for any $N \in \mathbb{N}^*$, we have

$$I_1(N,N_0;u) \le \mathrm{Lip}(F|_{\mathfrak{B}}) \|\Pi_{N_0}^{\perp} y(t;x,u)\|_{\mathcal{H}} \underset{N_0\to\infty}{\longrightarrow} 0, \qquad \forall\, t \in [0,T],\ u \in \mathcal{U}_{ad}.$$

Note also for each fixed $N_0 \in \mathbb{N}^*$, by compactness of $\Pi_{N_0}\mathfrak{B}$, we have

$$I_2(N,N_0;u) \underset{N\to\infty}{\longrightarrow} 0, \qquad \forall\, t \in [0,T],\ u \in \mathcal{U}_{ad}.$$

□

Preparatory Lemma II: Assume all the conditions in Groups I–III hold. Then, for any (x, u) in $\mathcal{H} \times \mathcal{U}_{ad}$, the following uniform convergence result holds:

$$\lim_{N\to\infty} \sup_{u\in\mathcal{U}_{ad}} \sup_{t\in[0,T]} \|y_N(t;\Pi_N x, u) - y(t;x,u)\|_{\mathcal{H}} = 0.$$

Proof: Let $w_N(t;u) := y(t;u) - y_N(t;u)$. We have

$$\begin{aligned} w_N(t;u) = & \underbrace{T(t)x - e^{L_N t}\Pi_N x}_{J_1(t)} + \int_0^t \underbrace{\big(T(t-s) - e^{L_N(t-s)}\Pi_N\big)F(y(s;u))}_{J_2(t,s;u)} \,\mathrm{d}s \\ & + \int_0^t \underbrace{e^{L_N(t-s)}\Pi_N\big(F(y(s;u)) - F(y_N(s;u))\big)}_{J_3(t,s;u)} \,\mathrm{d}s \\ & + \int_0^t \underbrace{\Big(T(t-s) - e^{L_N(t-s)}\Pi_N\Big)\mathfrak{C}(u(s))}_{J_4(t,s;u)} \,\mathrm{d}s. \end{aligned}$$

For $J_3(t,s;u)$, we have

$$\begin{aligned}\|J_3(t,s;u)\|_{\mathcal{H}} &= \|e^{L_N(t-s)}\Pi_N\big(F(y(s;u)) - F(y_N(s;u))\|_{\mathcal{H}} \\ &\leq M\mathsf{Lip}(F|_{\mathfrak{B}})e^{\omega(t-s)}\|w_N(s;u)\|_{\mathcal{H}}\end{aligned}$$

By Gronwall's inequality, we get for all t in $[0,T]$:

$$\begin{aligned}\|w_N(t;u)\|_{\mathcal{H}} \leq \Big(\|J_1(t)\|_{\mathcal{H}} &+ \int_0^T \sup_{t\in[s,T]} \|J_2(t,s;u)\|_{\mathcal{H}}\,\mathrm{d}s \\ &+ \int_0^T \sup_{t\in[s,T]} \|J_4(t,s;u)\|_{\mathcal{H}}\,\mathrm{d}s\Big)\exp\big(M\mathsf{Lip}(F|_{\mathfrak{B}})e^{\omega T}T\big).\end{aligned}$$

Conditions in Group I ensures (Trotter-Kato theorem [Pazy83,Thm. 4.5, p.88]):

$$\|J_1(t)\|_{\mathcal{H}} = \|T(t)x - e^{L_N t}\Pi_N x\|_{\mathcal{H}} \to 0, \quad \forall\, t \in [0,T].$$

For $J_2(t,s;u)$, we have

$$\|J_2(t,s;u)\|_{\mathcal{H}} \leq \underbrace{\|\big(T(t-s)-e^{L_N(t-s)}\Pi_N\big)\Pi_{N_0}F(y(s;x,u))\|_{\mathcal{H}}}_{K_1(N,N_0;u)} + \underbrace{\|\big(T(t-s)-e^{L_N(t-s)}\Pi_N\big)\Pi_{N_0}^{\perp}F(y(s;x,u))\|_{\mathcal{H}}}_{K_2(N,N_0;u)}.$$

Note that by **Lemma I**,

$$\sup_{t\in[s,T]} K_2(N,N_0;u) \leq 2Me^{\omega T}\|\Pi_{N_0}^{\perp}F(y(s;x,u))\|_{\mathcal{H}} \underset{N_0\to\infty}{\longrightarrow} 0,\ \forall s\in[0,T],\ u\in\mathcal{U}_{ad}.$$

By again the compactness of $\Pi_{N_0}\mathfrak{B}$ and the Trotter-Kato theorem, we have

$$\sup_{t\in[s,T]} K_1(N,N_0;u) \underset{N\to\infty}{\longrightarrow} 0,\ \forall s\in[0,T],\ u\in\mathcal{U}_{ad}.$$

It follows that $\sup_{t\in[s,T]}\|J_2(t,s;u)\|_{\mathcal{H}} \underset{N\to\infty}{\longrightarrow} 0, \forall s\in[0,T],\ u\in\mathcal{U}_{ad}$.
The $\sup_{t\in[s,T]}\|J_4(t,s;u)\|_{\mathcal{H}}$ term can be dealt with in the same way. □

Application to Optimal Control of Delay Differential Equations

Reference:

[CKL18] M. D. Chekroun, A. Kröner and H. Liu, *Galerkin approximations for optimal control of nonlinear delay differential equations*, Chapter 4 in *Hamilton-Jacobi-Bellman Equations: Numerical Methods and Applications in Optimal Control*, Edited by D. Kalise, K. Kunisch, and Z. Rao, De Gruyter, 2018. [arXiv link]

Application to the Wright equation

For any u in $L^2(0,T;\mathbb{R})$ and $T>0$, we consider

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -m(t-\tau)(1+m(t)) + u(t), \qquad t \in (0,T), \tag{5.1}$$

supplemented with

$$\begin{aligned} &m(t) = \phi(t),\ t \in [-\tau, 0), \quad \text{where } \phi \in L^2(-\tau,0;\mathbb{R}), \\ &m(0) = m_0 \in \mathbb{R}. \end{aligned} \tag{5.2}$$

Cost functional:

$$J(m,u) := \int_0^T \left[\frac{1}{2}m(t)^2 + \frac{\mu}{2}u(t)^2\right] \mathrm{d}t, \quad \mu > 0$$

Optimal control problem:

$\min J(m,u)$ subject to $(m,u) \in L^2(0,T;\mathbb{R}) \times L^2(0,T;\mathbb{R})$ solves the problem (5.1)–(5.2).

Recasting into an evolution equation in a Hilbert space

The reformulation is classical. Denote by m_t the time evolution of the history segments of a solution

$$m_t(\theta) := m(t+\theta), \qquad t \geq 0, \qquad \theta \in [-\tau, 0],$$

we introduce then a new variable

$$y(t,\theta) := (m_t(\theta), m_t(0)), \qquad t \geq 0,$$

and take the state space to be:

$$\mathcal{H} := L^2([-\tau, 0); \mathbb{R}) \times \mathbb{R}.$$

The problem (5.1)–(5.2) can be rewritten as:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mathcal{A}y + \mathcal{F}(y) + \mathfrak{C}u(t), \quad y(0) = \Phi := (\phi, m_0).$$

The linear operator $\mathcal{A}\colon D(\mathcal{A}) \to \mathcal{H}$ is defined by

$$[\mathcal{A}\Psi](\theta) := \begin{cases} \frac{\mathrm{d}^+\Psi^D}{\mathrm{d}\theta}, & \theta \in [-\tau, 0), \\ -\Psi^D(-\tau), & \theta = 0, \end{cases}$$

with domain

$$D(\mathcal{A}) = \Big\{(\Psi^D, \Psi^S) \in \mathcal{H} : \Psi^D \in H^1([-\tau, 0); \mathbb{R}), \lim_{\theta \to 0^-} \Psi^D(\theta) = \Psi^S\Big\}.$$

The nonlinearity $\mathcal{F}\colon \mathcal{H} \to \mathcal{H}$ is defined by

$$[\mathcal{F}(\Psi)](\theta) := \begin{cases} 0, & \theta \in [-\tau, 0), \\ -\Psi^D(-\tau)\Psi^S, & \theta = 0, \end{cases} \quad \text{for all } \Psi = (\Psi^D, \Psi^S) \in \mathcal{H}.$$

The control operator $\mathfrak{C}\colon V \to \mathcal{H}$ is taken here to be linear and given by

$$[\mathfrak{C}v](\theta) := \begin{cases} 0, & \theta \in [-\tau, 0) \\ v, & \theta = 0 \end{cases}, \quad v \in V,$$

where $V = \mathbb{R}$.

Galerkin approximation

We take $\mathcal{H}_N$ to be spanned by the Koornwinder polynomials $\{K_j^\tau : [-\tau, 0] \to \mathbb{R}\}$:

$$\mathcal{H}_N := \operatorname{span}\{\mathcal{K}_j^\tau := (K_j^\tau, K_j^\tau(0)) \ : \ j = 0, \cdots, N-1\}.$$

Using such polynomials to build Galerkin approx. of DDEs is first introduced in [CGLW16], to which we refer for its **theoretical advantages** and **numerical efficiencies**.

The corresponding Galerkin approximation is an ODE system given by:

$$\boxed{\frac{\mathrm{d}\boldsymbol{\xi}_N}{\mathrm{d}t} = M\boldsymbol{\xi}_N + G(\boldsymbol{\xi}_N) + \mathfrak{C}_N u(t), \qquad \boldsymbol{\xi}_N(0) = \boldsymbol{\zeta}_N \in \mathbb{R}^N, \quad \text{with}}$$

$$(M)_{j,n} = \frac{1}{\|\mathcal{K}_j\|_{\mathcal{E}}^2}\Big(- K_n(-1) + \frac{2}{\tau}\sum_{k=0}^{n-1} a_{n,k}\left(\delta_{j,k}\|\mathcal{K}_j\|_{\mathcal{E}}^2 - 1\right)\Big), \ 0 \le j, n \le N-1,$$

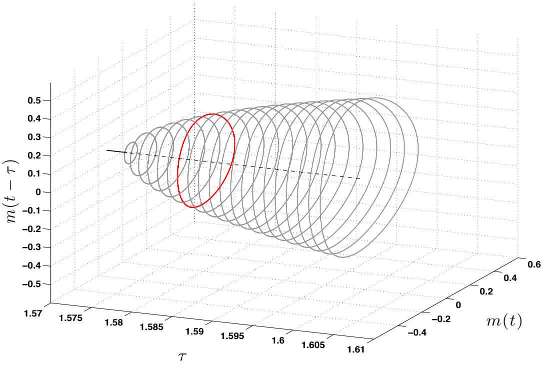
$$G_j(\boldsymbol{\xi}_N) = -\frac{1}{\|\mathcal{K}_j\|_{\mathcal{E}}^2}\left[\sum_{n=0}^{N-1} \xi_n^N(t)\right]\left[\sum_{n=0}^{N-1} \xi_n^N(t) K_n(-1)\right], \ 0 \le j \le N-1,$$

$$\mathfrak{C}_N v = \Big(\frac{1}{\|\mathcal{K}_0^\tau\|_{\mathcal{H}}^2}, \cdots, \frac{1}{\|\mathcal{K}_{N-1}^\tau\|_{\mathcal{H}}^2}\Big)^{\mathrm{tr}} v.$$

[CGLW16] M. D. Chekroun, M. Ghil, H. Liu & S. Wang, *Low–dimensional Galerkin approximations of nonlinear DDEs*. DCDS-A, Vol. 36, pp 4133–4177, 2016. [arXiv link]

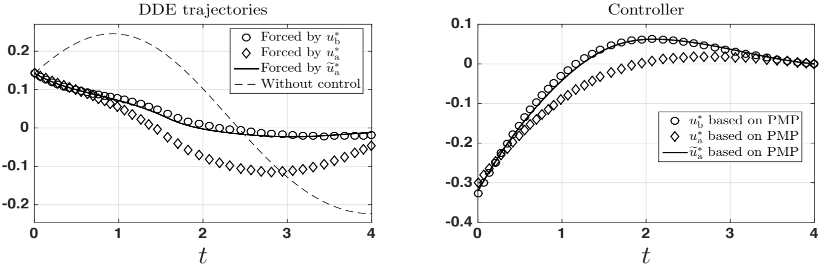
Numerical setup: Amplitude oscillation reduction

The uncontrolled equation experiences a supercritical Hopf bifurcation when τ crosses the critical delay $\tau_c = \frac{\pi}{2}$ from below:



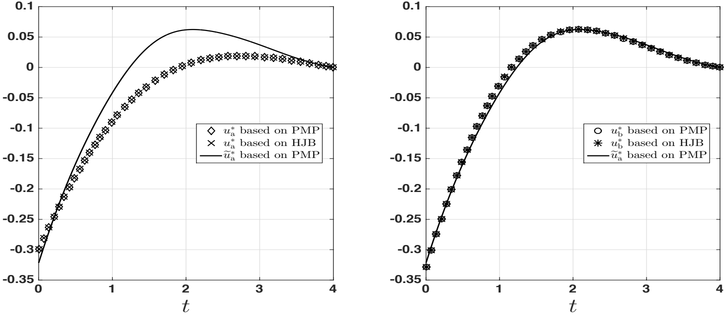
Our goal is to show that close to the criticality (i.e. $\tau \approx \tau_c$), these amplitudes can be reduced at a nearly optimal cost, by solving efficiently a low-dimensional optimal control problem.

Numerical results via Pontryagin Maximum Principle



- $\widetilde{u}_{\mathrm{a}}^*$: from a 6-dim Galerkin-Koornwinder approximation (as a benchmark);
- u_{a}^*: from a 2-dim Galerkin-Koornwinder approximation;
- u_{b}^*: from another 2-dim ODE system obtained by projecting the 6-dim Galerkin-Koornwinder approximation onto the eigen-subspace spanned by the first two eigenvectors of the matrix M.

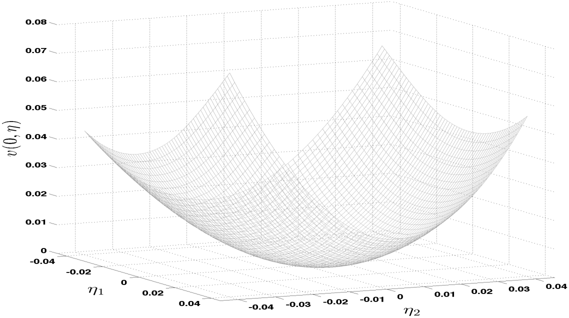
Numerical results via Dynamic Programming



- **Left panel**: Control obtained without basis transformation (HJB vs PMP).
- **Right panel**: Control obtained after basis transformation (HJB vs PMP).

Numerical results via Dynamic Programming (cont'd)

- **Value function:**



Handling non-polynomial nonlinearities

There are applications for which non-polynomial nonlinearities can arise, and the conditions in the developed framework can still be checked. Performing **Taylor expansion** of such nonlinearity **may require high degree polynomials**.

Example [CL20]: Optimal management of harvested population

$$\min_{u \in L^2(t,T;L^2(\Omega))} \left(\frac{1}{2} \int_t^T \left| y(s) - p_{\delta'} \right|^2_{L^2(\Omega)} \mathrm{d}s + \frac{\kappa}{2} \int_t^T \left| u(s) \right|^2_{L^2(\Omega)} \mathrm{d}s \right)$$

where $y(s,x)$ solves the Kolmogorov-Petrovsky-Piskunov Eqn

$$\frac{\partial y}{\partial s} = D\nabla^2 y + \mu(x)y - \nu(x)y^2 - \delta\rho_\epsilon(y) + u(s,x),\ (s,x) \in [t,T] \times \Omega,$$

$$\frac{\partial y}{\partial \boldsymbol{n}} = 0,\ (s,x) \in [t,T] \times \partial\Omega, \quad \text{with } y(t,x) = y_0(x).$$

The harvest function ρ_ϵ takes the form

$$\rho_\epsilon(y) = \begin{cases} 1, & \text{if } y \geq \epsilon, \\ 0.5\sin(\pi(y - 0.5\epsilon)/\epsilon) + 0.5, & \text{if } 0 < y < \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Summary

- **Convergence results** for Galerkin approximations of optimal control of nonlinear evolution equations are obtained.
- **Checkable conditions** are delineated. Error estimates for the value function and the optimal control can also be obtained under additional conditions [CKL17].
- The framework is **general and flexible**, which covers not only **semilinear parabolic PDEs** but also **delay differential equations**; and allows for **a broad class of nonlinear control laws**.

References

[CKL17] M. D. Chekroun, A. Kröner and H. Liu, *Galerkin approximations of nonlinear optimal control problems in Hilbert spaces*, EJDE, 1–40, 2017. [arXiv link]

[CKL18] M. D. Chekroun, A. Kröner and H. Liu, *Galerkin approximations for optimal control of nonlinear delay differential equations*, Chapter 4 in *Hamilton-Jacobi-Bellman Equations: Numerical Methods and Applications in Optimal Control*, Edited by D. Kalise, K. Kunisch, and Z. Rao, De Gruyter, 2018. [arXiv link]

[CL20] M. D. Chekroun and H. Liu, *Optimal management of harvested population at the edge of extinction*, Chapter 2 in *Advances in Nonlinear Biological Systems: Modeling and Optimal Control*, Edited by J. Kotas, AIMS, 2020. [arXiv link]

Some related references

- Galerkin approximations for the case of linear evolution equations
    - Ferretti (1997).
- Feedback control using POD and dynamic programming
    - Kunisch, Volkwein, Xie (2004), Kunisch, Xie (2005), Alla, Falcone (2013), Alla, Falcone, Volkwein (2015), Alla, Falcone, Kalise (2016)