MAX-PLANCK-INSTITUT
FÜR PLASMAPHYSIK

Towards data-driven non-intrusive reduced-order modeling for plasma turbulence via Operator Inference

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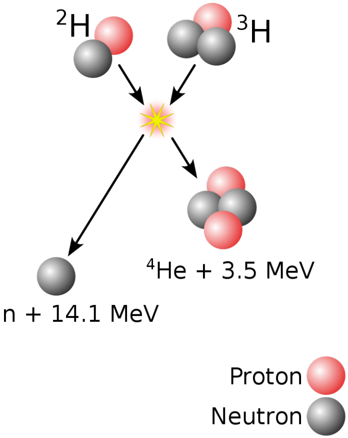
Max-Planck-Institute for Plasma Physics (Munich)

Takeaway

I'm not from MPI Magdeburg

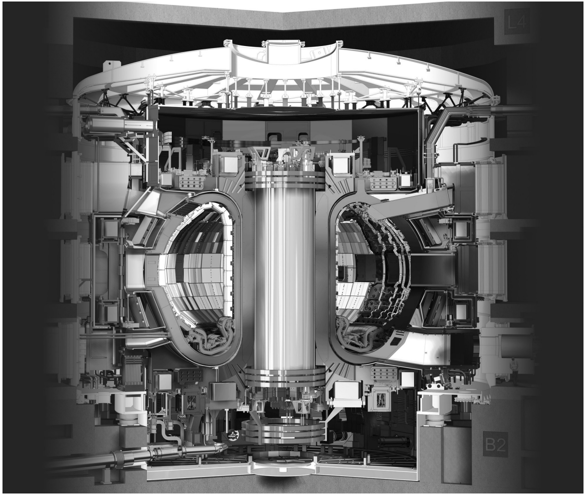
Thermonuclear fusion: Deuterium-Tritium fusion



Source: Wikipedia, Deuterium-Tritium Fusion

- uses magnetic fields to contain particles
- needs temperatures $10\times$ hotter then the sun
- planed to be achieved in ITER

International Thermonuclear Experiment Reactor (ITER)



Source: `https://www.iter.org/mach`

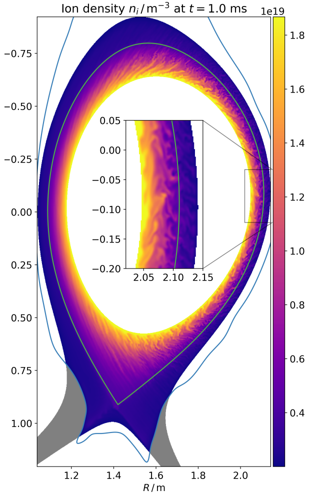
Numerical models and plasma turbulence

Challenge:

- measuring QoIs in a plasma is hard
- build a new new reactor/experiment is very expensive
- numerical simulations are (slightly) less hard and less expensive:
  - needs to resolve several magnitude of spatial and temporal scales
  - highly non-linear

Solution:

reduced order model for plasma turbulence



Source: Dominik Michels

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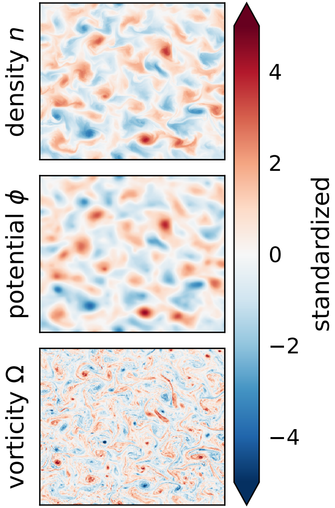
Hasegawa-Wakatani (HW) equation

Hasegawa-Wakatani equation

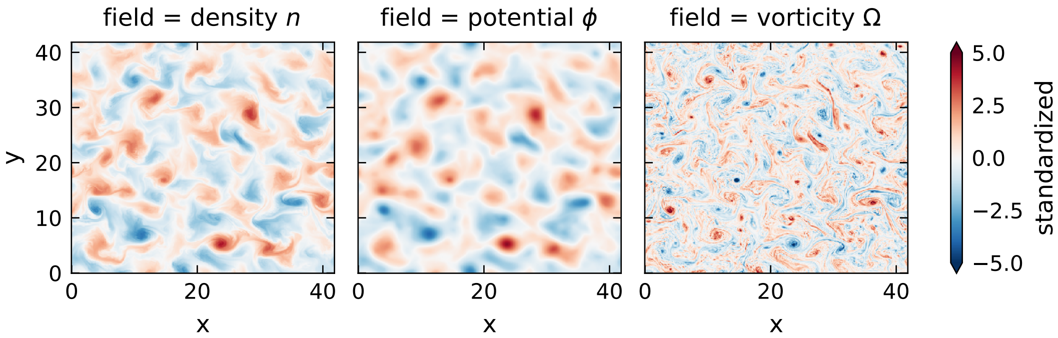
$$\partial_t n = c_1(\phi - n) - \{\phi, n\} + \kappa \partial_y \phi + \nu \nabla^6 n$$
$$\partial_t \nabla^2 \phi = c_1(\phi - n) - \{\phi, \nabla^2 \phi\} + \nu \nabla^8 \phi$$

where

- n is the density,
- ϕ the potential,
- $\nabla^2 \phi$ the vorticity,
- and $\{\cdot, \cdot\}$ the Poisson brackets
  $\{f, g\} := \partial_x f \partial_y g - \partial_y f \partial_x g$



Hasegawa-Wakatani (HW) equation (cont.)



- 40 001 snaphots per simulation
- 512×512 grid points
- 120*GB* per simulation
- periodic in x and y direction
- solved using RK4 method
- 1 simulation takes $\sim$ 24 hours

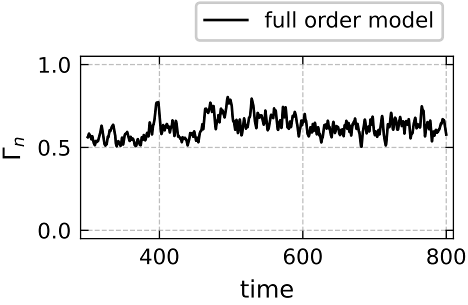
Quantity of interest: particle flux Γ_n

- accurate state predictions not important
- “Is the system statistically correct?”

Particle Flux Γ_n:

$$\Gamma_n(t) = \iint n\, \partial_y \phi \,\mathrm{d}x\mathrm{d}y$$

- rate at which free energy is extracted from the background gradient[1]
- characterizes turbulent behavior



[1] Camargo et al., *“Resistive drift-wave turbulence”* (1995)

Wasserstein metric W_1

Given

- distributions μ_1 and μ_2
- with cumulative distribution functions P_1 and P_2,

define

$$W_1(\mu_1, \mu_2) = \int_{-\infty}^{\infty} |P_1(x) - P_2(x)| \,\mathrm{d}x$$

Loss function

compare the distributions of Γ_n using Wasserstein distance

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Reduced order model - setup

1. State:

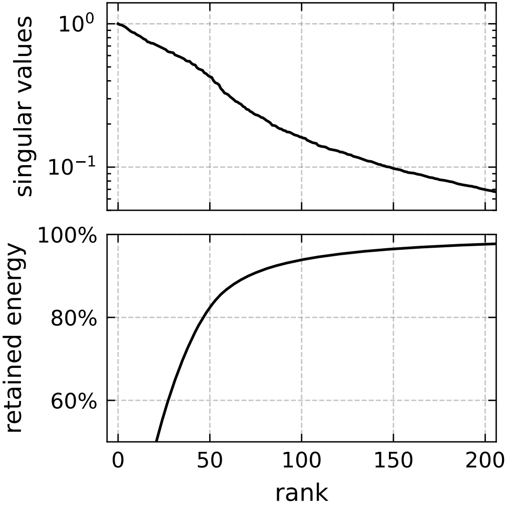
$$\mathbf{q}_t = \begin{pmatrix} n_t \\ \phi_t \end{pmatrix} \in \mathbb{R}^N$$

2. Snapshot matrix $\mathbf{Q}$:

$$\mathbf{Q} = \begin{pmatrix} | & & | \\ \mathbf{q}_1 & \dots & \mathbf{q}_m \\ | & & | \end{pmatrix} \in \mathbb{R}^{N \times m}$$

3. rank r POD basis from the thin SVD of $\mathbf{Q}$:

$$\mathbf{Q} \approx \mathbf{V}_r \mathbf{\Sigma}_r \mathbf{U}_r^T, \qquad \mathbf{V}_r \in \mathbb{R}^{N \times r}$$



Operator Inference (OpInf)

Reduced state:

$$\hat{\mathbf{q}}_t = \mathbf{V}_r^T \mathbf{q}_t \in \mathbb{R}^r, \qquad \mathbf{q}_t \approx \mathbf{V}_r \hat{\mathbf{q}}_t$$

Full order model:

$$\dot{\mathbf{q}}_t = \mathbf{A}\mathbf{q}_t + \mathbf{H}\mathbf{q}_t \otimes \mathbf{q}_t$$

Reduced order model:

$$\dot{\hat{\mathbf{q}}}_t = \mathbf{V}_r^T \mathbf{A}\mathbf{V}_r \hat{\mathbf{q}}_t + \mathbf{V}_r^T \mathbf{H}(\mathbf{V}_r \hat{\mathbf{q}}_t) \otimes (\mathbf{V}_r \hat{\mathbf{q}}_t)$$

Instead of computing $\hat{\mathbf{A}}, \hat{\mathbf{H}}$ intrusively, solve

$$\underset{\hat{\mathbf{A}},\hat{\mathbf{H}}}{argmin} \sum_t \|\dot{\hat{\mathbf{q}}}_t - \hat{\mathbf{A}}\hat{\mathbf{q}}_t - \hat{\mathbf{H}}(\hat{\mathbf{q}}_t \otimes \hat{\mathbf{q}}_t)\|_2^2 + \alpha_1 \|\hat{\mathbf{A}}\|_F^2 + \alpha_2 \|\hat{\mathbf{H}}\|_F^2.$$

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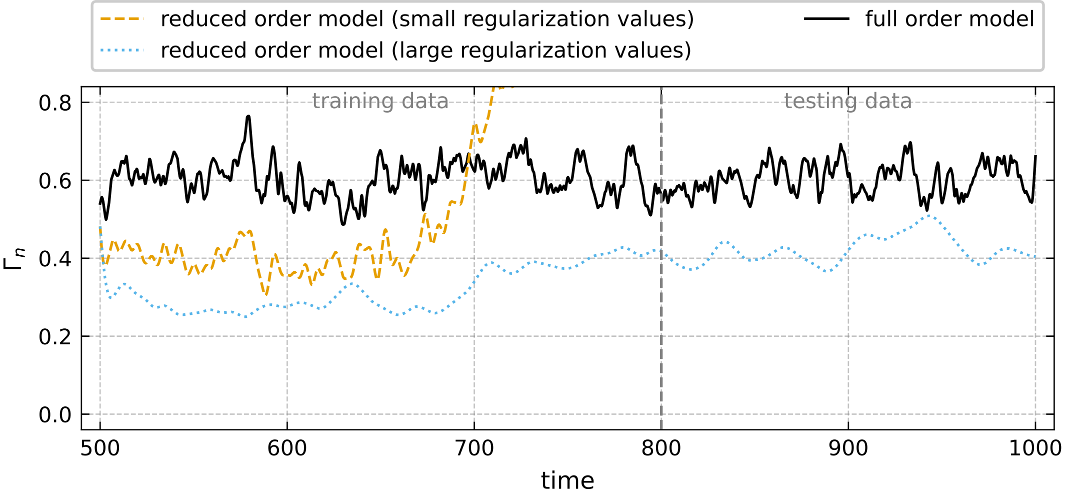
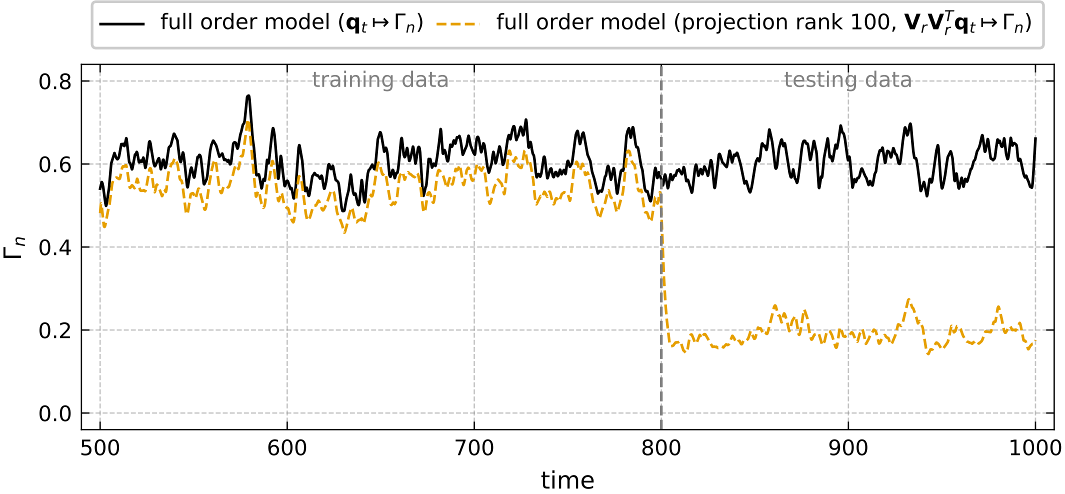
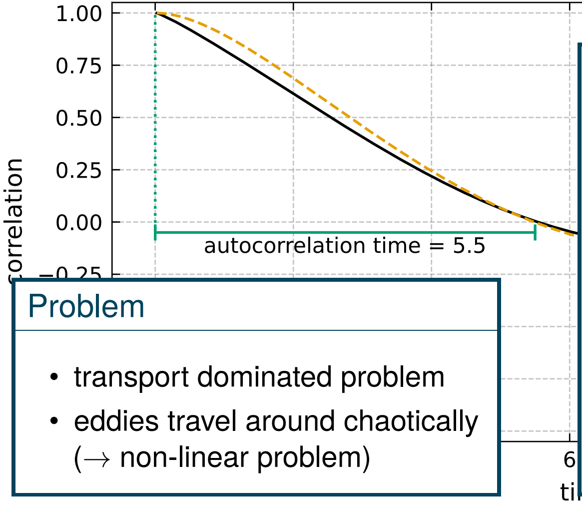
Predicting Γ_n using Operator Inference ROM (rank $r = 100$)

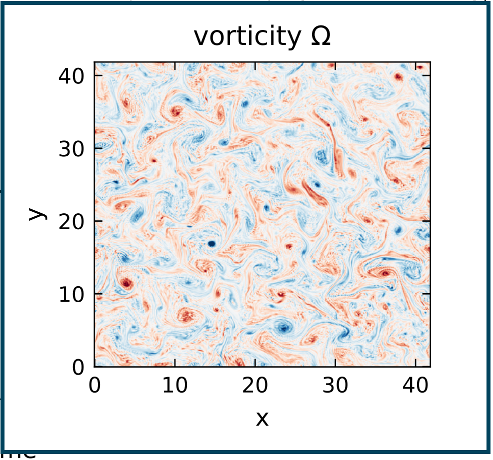


POD basis doesn't generalize beyond training data



Snapshots are not correlated over time

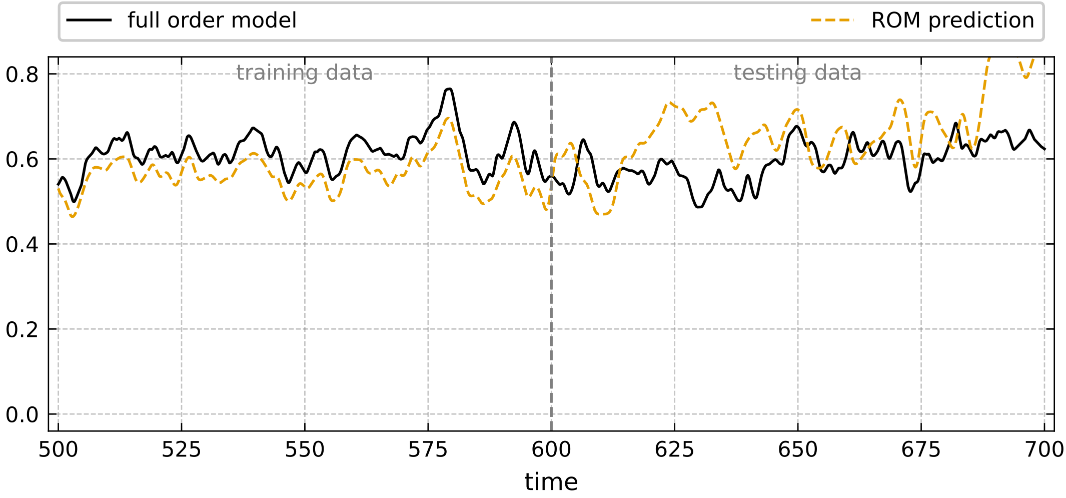




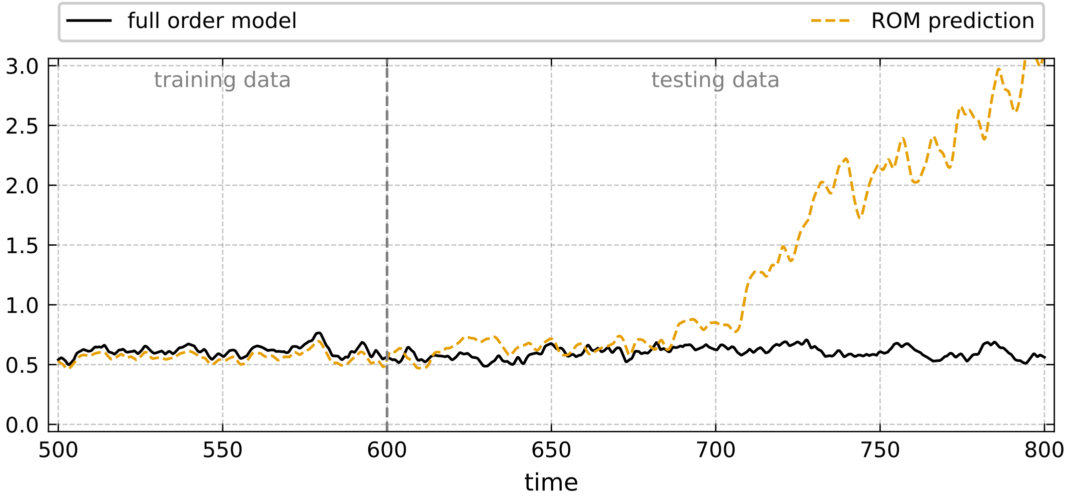
Problem

- transport dominated problem
- eddies travel around chaotically ($\rightarrow$ non-linear problem)

How good can I possibly be? (rank = 80)



Well...


full order model
ROM prediction
training data
testing data
3.0
2.5
2.0
1.5
1.0
0.5
0.0
500
550
600
650
700
750
800
time

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Idea I: shifted POD / operator inference[1]

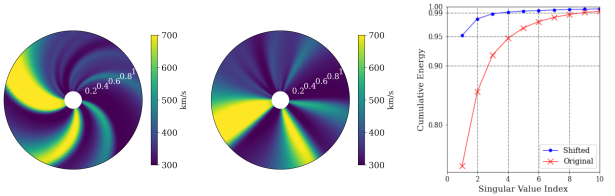
Predicting Solar Wind Streams from the Inner-Heliosphere to Earth via Shifted Operator Inference

Opal Issan[a,*], Boris Kramer[a]

[a] *Department of Mechanical and Aerospace Engineering, University of California San Diego, CA, United States*

March 28, 2022

(a) Original Coordinates (b) Shifted Coordinates (c) Cumulative Energy obtained by Eq. (18)



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[1] Issan et al., *"Predicting Solar Wind Streams from the Inner-Heliosphere to Earth via Shifted Operator Inference"* (2022)

Idea I: shifted POD / operator inference (cont.)

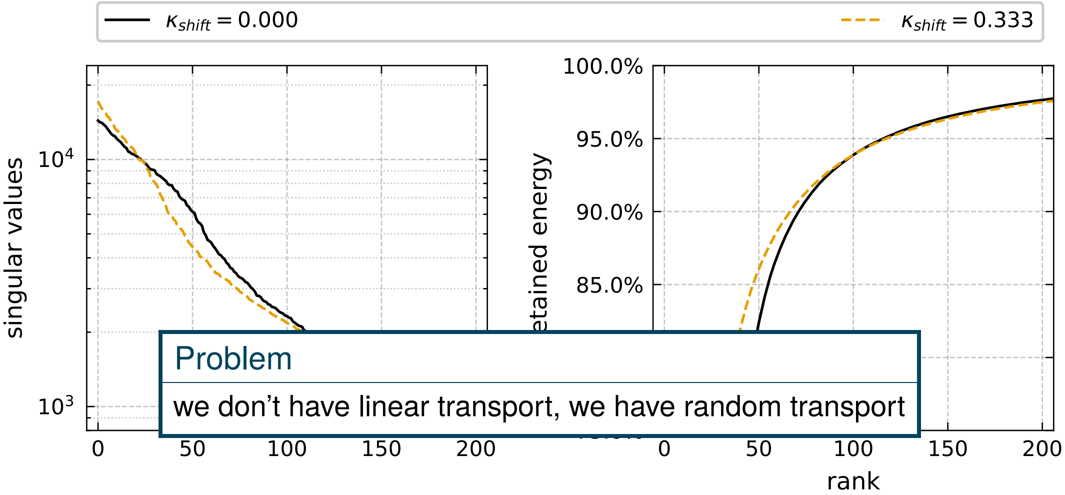
$$\partial_t n = c_1(\phi - n) - \{\phi, n\} + \nu\nabla^6 n + \boxed{\kappa\partial_y\phi}$$
$$\partial_t \nabla^2\phi = c_1(\phi - n) - \{\phi, \nabla^2\phi\} + \nu\nabla^8\phi$$

$\rightarrow$ constant velocity driftwaves in y-direction

Steps:

1. determine velocity κ
2. remove drift from data
3. better decay of singular values

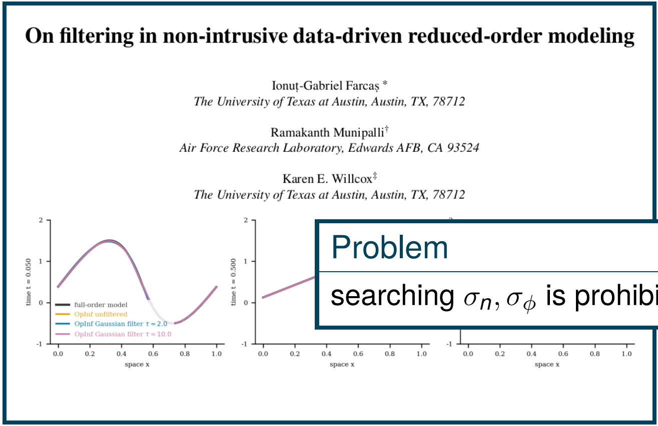
Idea I: shifted POD / OpInf - singular values and retained energy



Problem

we don’t have linear transport, we have random transport

Idea II: filtering & OpInf[2]



On filtering in non-intrusive data-driven reduced-order modeling

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Karen E. Willcox‡
The University of Texas at Austin, Austin, TX, 78712

Problem

searching σ_n, σ_ϕ is prohibitively slow

Procedure:

1. choose filter values σ_n, σ_ϕ
2. filter data ($\sim 4h$)
3. compute SVD ($\sim 3h$)
4. …npute ROM (negligable)
5. …rch regularization values ($10 - 20h$)

[2] Farcas et al., *"On Filtering in Non-Intrusive Data-Driven Reduced-Order Modeling"* (2022)

Idea III: quadratic manifolds[3]

Operator inference for non-intrusive model reduction with nonlinear manifolds

Rudy Geelen* Stephen Wright† Karen Willcox*

Idea:

1. normal SVD to project:

$$\hat{\mathbf{q}}_t = \mathbf{V}_r^T \mathbf{q}_t$$

2. quadratic map to reconstruct:

$$\mathbf{q}_t = \Xi(\hat{\mathbf{q}}_t) = \mathbf{V}_r \hat{\mathbf{q}}_t + \mathbf{W} \hat{\mathbf{q}}_t \otimes \hat{\mathbf{q}}_t$$

Challenge:

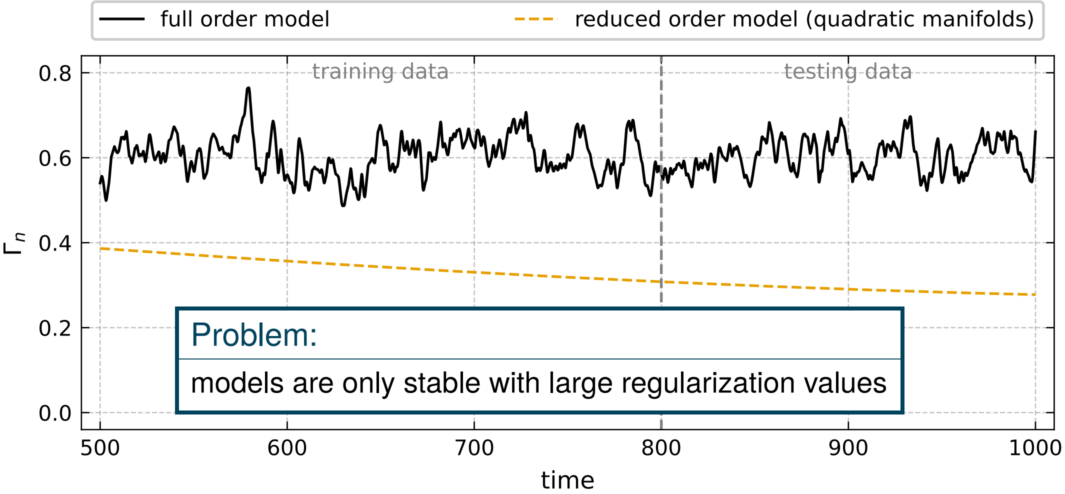
ROM becomes

$$\begin{aligned}\dot{\hat{\mathbf{q}}}_t = \hat{\mathbf{A}}\hat{\mathbf{q}}_t &+ \hat{\mathbf{H}}_2(\hat{\mathbf{q}}_t \otimes \hat{\mathbf{q}}_t) \\ &+ \hat{\mathbf{H}}_3(\hat{\mathbf{q}}_t \otimes \hat{\mathbf{q}}_t \otimes \hat{\mathbf{q}}_t) \\ &+ \hat{\mathbf{H}}_4(\hat{\mathbf{q}}_t \otimes \hat{\mathbf{q}}_t \otimes \hat{\mathbf{q}}_t \otimes \hat{\mathbf{q}}_t)\end{aligned}$$

$\Longrightarrow$ we are restricted to rank ~ 20

[3] Jain et al., *"A Quadratic Manifold for Model Order Reduction of Nonlinear Structural Dynamics"* (2017)

Idea III: quadratic manifolds (cont.)

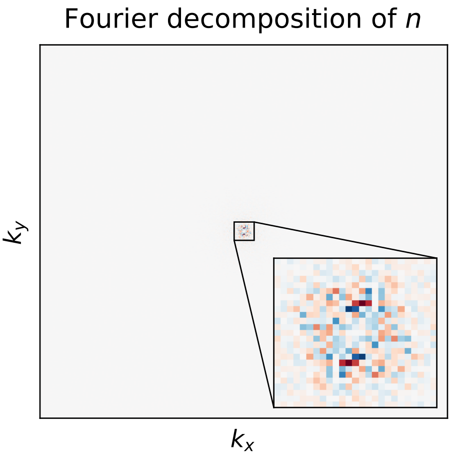


Problem:

models are only stable with large regularization values

Idea IV (WIP): using a different basis

- Fourier basis
- wavelet basis[4]



[4] Farge, *“Wavelet transforms and their applications to turbulence”* (1992)

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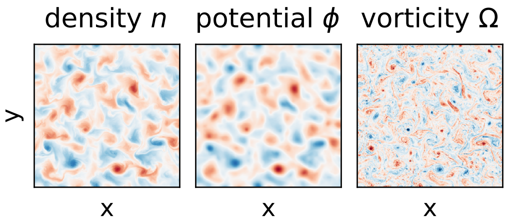
Constructing predictive and accurate ROMs for plasma turbulence models is very challenging

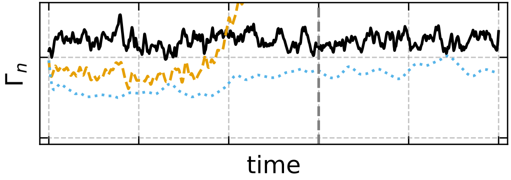
- Hasegawa-Wakatani equations:



- OpInf: learn reduced operators via

$$\underset{\hat{\mathbf{A}},\hat{\mathbf{H}}}{argmin} \sum_t \|\dot{\hat{\mathbf{q}}}_t - \hat{\mathbf{A}}\hat{\mathbf{q}}_t - \hat{\mathbf{H}}(\hat{\mathbf{q}}_t \otimes \hat{\mathbf{q}}_t)\|_2^2 + \alpha_1 \|\hat{\mathbf{A}}\|_F^2 + \alpha_2 \|\hat{\mathbf{H}}\|_F^2.$$

- OpInf reduced order model:



- possible improvements:
  - shifted POD
  - filtering
  - quadratic manifolds
  - different basis

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