Stability and Transient Dynamics for Linearized Reduced Order Models

Mark Embree · Virginia Tech

Nonlinear Model Reduction for Control
Blacksburg, Virginia

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Prelude

On Monday, Boris Kramer mentioned the simple model cf. [Kawano & Scherpen, 2017]

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & \mathbf{1} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -x_2(t)^2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t).$$

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Move the nonlinearity to the second component, and adjust the off-diagonal, and drop the input:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & \gamma \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ x_1(t)^2 \end{bmatrix}.$$

Note that $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a fixed point.

Is it stable?

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Is it stable?

Linear stability analysis: linearize about $\mathbf{x} = \mathbf{0}$ to get $\dot{\boldsymbol{\xi}} = \mathbf{A}\boldsymbol{\xi}$,

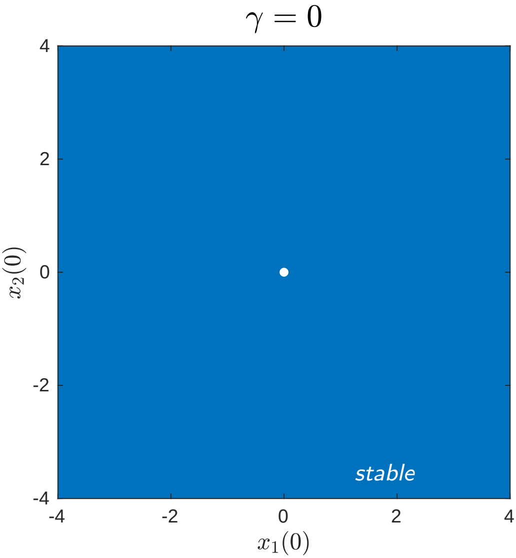
$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & \gamma \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix},$$

and note that $\mathbf{A}$ has negative eigenvalues: therefore, $\mathbf{x} = \mathbf{0}$ is stable.

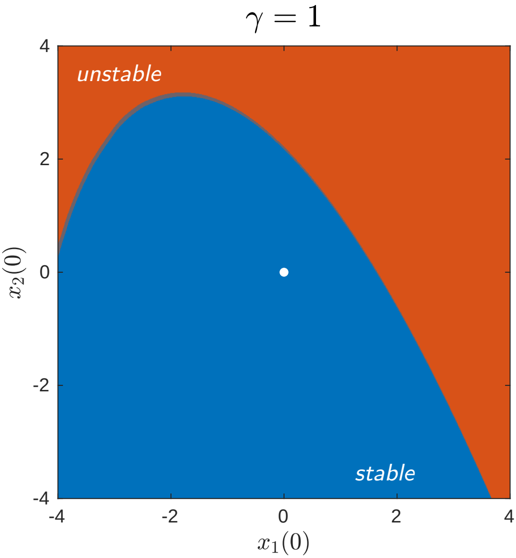
What is the basin of attraction? Does it depend on γ?

Let's use numerical simulations to assess the stability. . . .

Prelude



blue: basin of attraction of stable fixed point $\mathbf{x} = \mathbf{0}$



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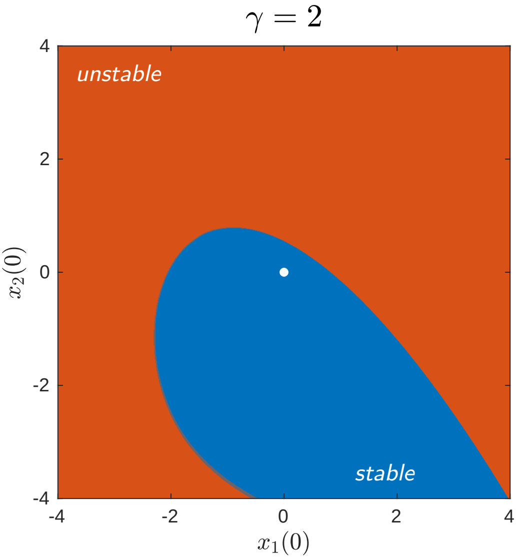

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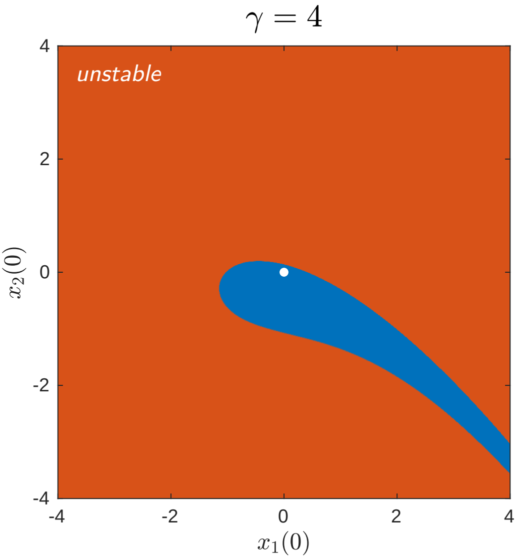

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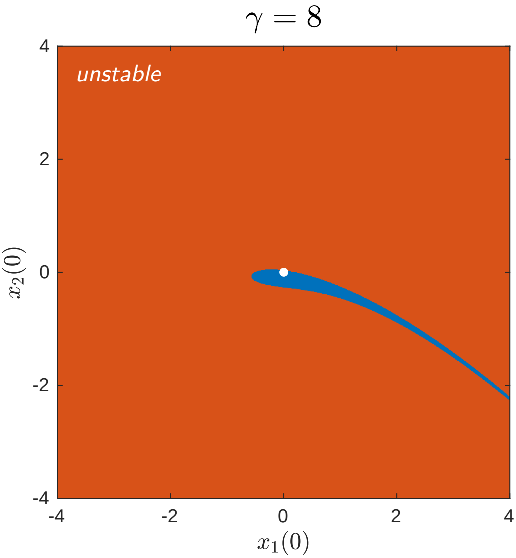

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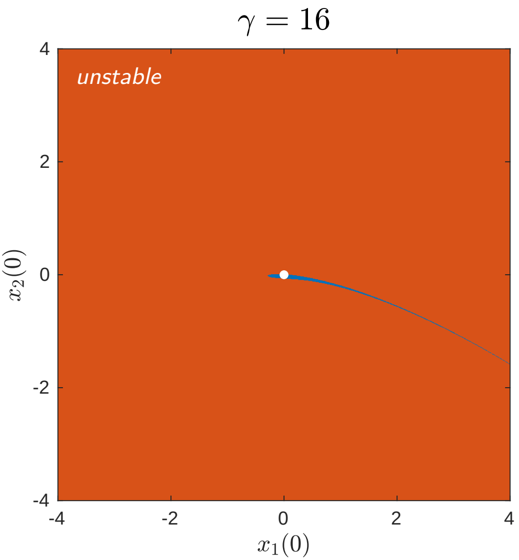

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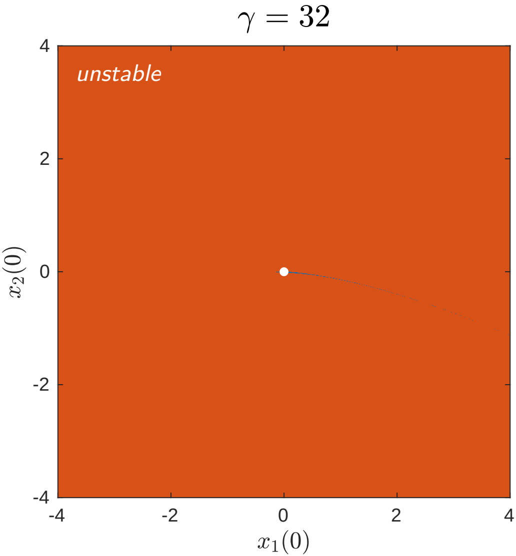

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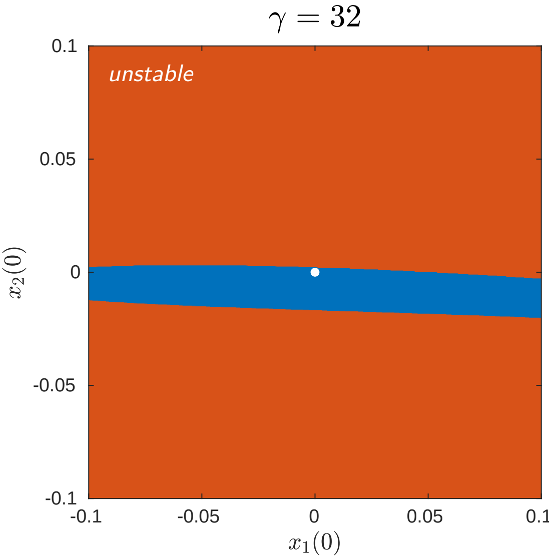

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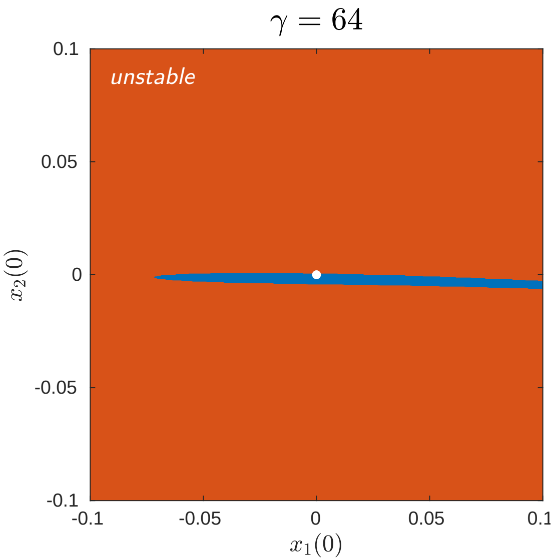

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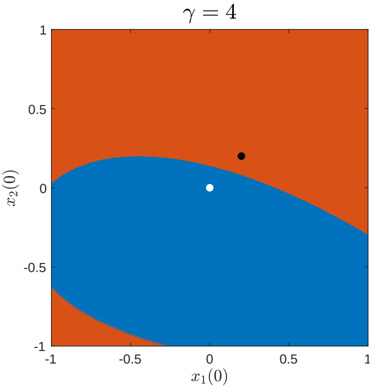
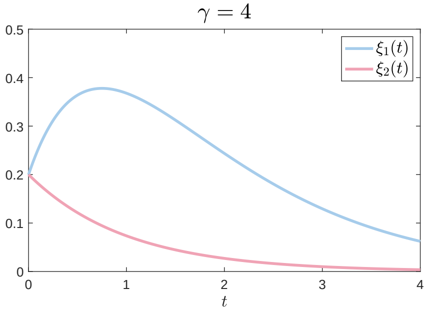
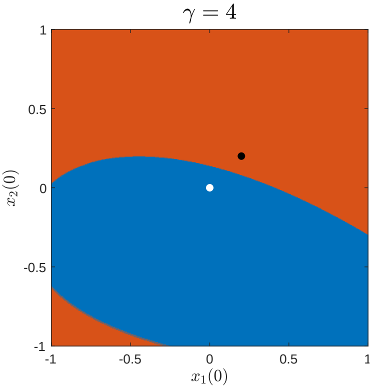


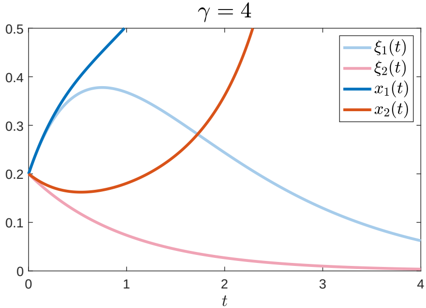
- initial condition $\mathbf{x}(0) = [0.2, 0.2]^T$



transient growth of a linearized system

Prelude





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linear transient growth feeds nonlinearity

This transient linear + nonlinear coupling has been proposed as model for transition to turbulence in fluid mechanics.

See [Butler & Farrell 1992], [Trefethen, Trefethen, Reddy, Driscoll 1993]; [Baggett, Driscoll, Trefethen 1995]; . . . , [Singler 2017, 2022].

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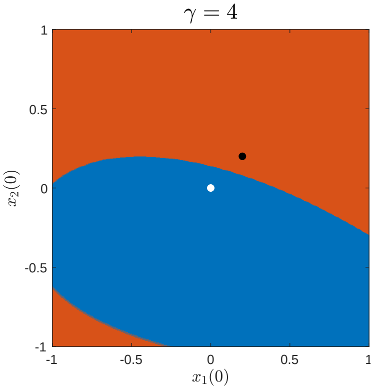

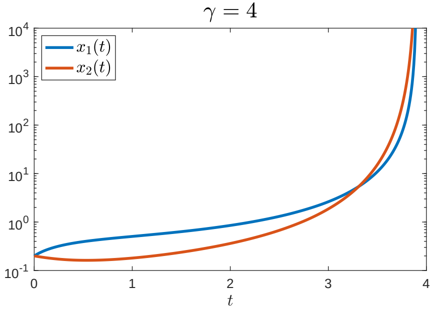

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The Mechanism Behind Transient Growth

Consider the (diagonalizable) example

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 100 & -2 \end{bmatrix}$$

with eigenvalues and (*nearly aligned*) eigenvectors

$$\lambda_1 = -1, \quad \mathbf{v}_1 = \begin{bmatrix} 1/100 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -2, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Expand the initial condition in this basis (*much cancellation*):

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 100 \begin{bmatrix} 1/100 \\ 1 \end{bmatrix} - 99 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now evolve the system in time:

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0) = 100e^{-t} \begin{bmatrix} 1/100 \\ 1 \end{bmatrix} - 99e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

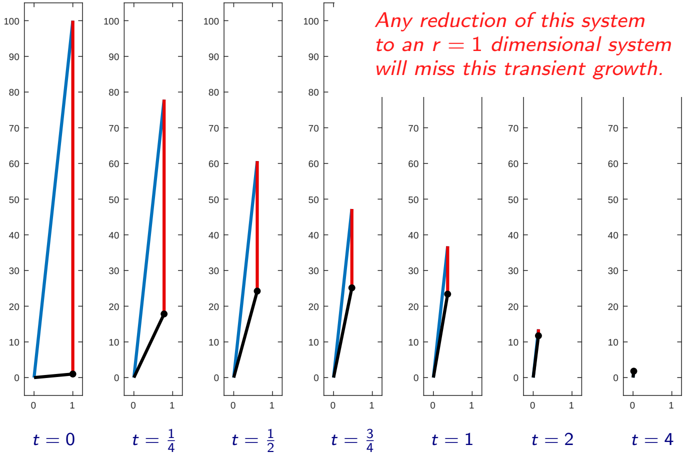
The exponentials decay at different rates, breaking the cancellation.

The Mechanism Behind Transient Growth

Seven snapshots of the state vector

$$\mathbf{x}(t) = 100e^{-t}\begin{bmatrix} 1/100 \\ 1 \end{bmatrix} - 99e^{-2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Any reduction of this system to an $r = 1$ dimensional system will miss this transient growth.



The Mechanism Behind Transient Growth

For $\mathbf{A} \in \mathbb{C}^{n\times n}$, the *numerical range* is the set

$$W(\mathbf{A}) = \left\{ \frac{\mathbf{x}^*\mathbf{A}\mathbf{x}}{\mathbf{x}^*\mathbf{x}} : \mathbf{x} \in \mathbb{C}^n \right\}.$$

- $W(\mathbf{A})$ is a closed, bounded, convex subset of $\mathbb{C}$ that contains the origin.
- If $\mathbf{A}$ is normal, $W(\mathbf{A})$ is the *convex hull of the spectrum*.
- If $\mathbf{A}$ is Hermitian, $W(\mathbf{A}) = [\lambda_{\min}, \lambda_{\max}] \subset \mathbb{R}$.

The *numerical abscissa* is the rightmost point in $W(\mathbf{A})$:

$$\omega(\mathbf{A}) = \max_{z\in W(\mathbf{A})} \mathrm{Re}(z) = \lambda_{\max}\left(\frac{\mathbf{A}+\mathbf{A}^*}{2}\right).$$

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Classical results from semigroup theory...

Theorem (see, e.g., Trefethen & E. 2005, Part IV)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{e}^{t\mathbf{A}}\|\bigg|_{t=0} = \omega(\mathbf{A}), \qquad \|\mathrm{e}^{t\mathbf{A}}\| \le \mathrm{e}^{t\omega(\mathbf{A})}$$

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- Solutions $\mathrm{e}^{t\mathbf{A}}\mathbf{x}(0)$ to $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ can transiently grow only if $\omega(\mathbf{A}) > 0$.
- Potentially $\omega(\mathbf{A}) > 0$ even if all eigenvalues of $\mathbf{A}$ are in the left-half plane.

Projection Methods for Model Reduction

Let $\mathbf{V} \in \mathbb{C}^{n\times r}$ have orthonormal columns, $\mathbf{V}^*\mathbf{V} = \mathbf{I}$.

To compute eigenvalues and to reduce models, we can restrict

$$\mathbf{A} \in \mathbb{C}^{n\times n} \quad \text{down to} \quad \mathbf{V}^*\mathbf{A}\mathbf{V} \in \mathbb{C}^{r\times r}.$$

For the bulk of this talk we focus on *Galerkin* projection of a SISO system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) &= \mathbf{c}^*\mathbf{x}(t), \end{aligned}$$

e.g., as generated by the *Arnoldi process* applied to $(\mathbf{A}, \mathbf{b})$, or POD:

$$\begin{aligned} \dot{\mathbf{x}}_r(t) &= (\mathbf{V}^*\mathbf{A}\mathbf{V})\mathbf{x}_r(t) + (\mathbf{V}^*\mathbf{b})u(t) \\ y_r(t) &= (\mathbf{c}^*\mathbf{V})\mathbf{x}_r(t). \end{aligned}$$

- The eigenvalues of $\mathbf{V}^*\mathbf{A}\mathbf{V}$ are in the numerical range $W(\mathbf{A})$:

$$(\mathbf{V}^*\mathbf{A}\mathbf{V})\boldsymbol{\xi} = \theta\boldsymbol{\xi} \qquad \Longrightarrow \qquad \frac{(\mathbf{V}\boldsymbol{\xi})^*\mathbf{A}(\mathbf{V}\boldsymbol{\xi})}{(\mathbf{V}\boldsymbol{\xi})^*(\mathbf{V}\boldsymbol{\xi})} = \theta.$$

- When $\mathbf{A} = \mathbf{A}^*$, the *Cauchy Interlacing Theorem* describes precisely how the eigenvalues of $\mathbf{V}^*\mathbf{A}\mathbf{V}$ distribute amongst the eigenvalues of $\mathbf{A}$.
- For nonnormal $\mathbf{A}$, very little is understood about the eigenvalues of $\mathbf{V}^*\mathbf{A}\mathbf{V}$.

Eigenvalues of Galerkin Projections for Non-Hermitian Matrices

Does there exist some notion of "interlacing" for non-Hermitian matrices?

Consider an extreme example:

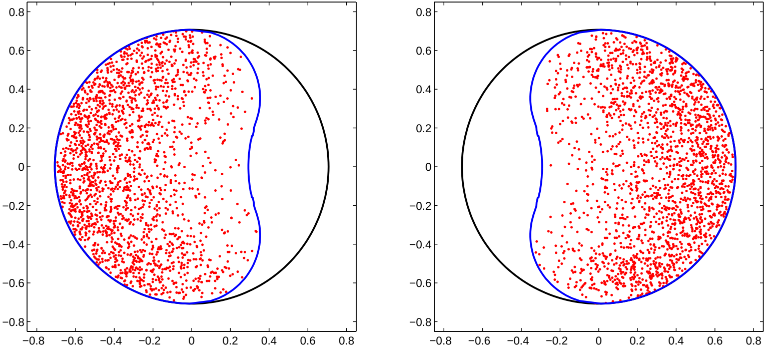
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Repeat the following experiment many times:

- Generate random *two* dimensional subspaces, $\mathcal{V} = \operatorname{Ran} \mathbf{V}$, where $\mathbf{V}^*\mathbf{V} = \mathbf{I}$.
- Form $\mathbf{V}^*\mathbf{A}\mathbf{V} \in \mathbb{C}^{2\times 2}$ and compute its eigenvalues: θ_1, θ_2.
- Sort by real part: $\operatorname{Re}\theta_1 \geq \operatorname{Re}\theta_2$.
- Since $\mathbf{A}$ has eigenvalues $\lambda_1 = \lambda_2 = 0$, "interlacing" is meaningless here. . . .

Two Dimensional Reduction of a Three-Dimensional Jordan Block

Eigenvalues of $\mathbf{V}^*\mathbf{AV}$



leftmost eigenvalue

rightmost eigenvalue

Eigenvalues of $\mathbf{V}^*\mathbf{AV}$ for random (complex) two dimensional subspaces

Black circle shows boundary of $W(\mathbf{A}) = \{z \in \mathbb{C} : |z| \leq \sqrt{2}/2\}$

Eigenvalues of Galerkin Projections (Sorted by Real Part)

Denote the eigenvalues of the Hermitian part $\frac{1}{2}(\mathbf{A} + \mathbf{A}^*)$, labeled

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n.$$

Theorem (Carden)

Let $\theta_1, \ldots, \theta_r$ denote the eigenvalues of $\mathbf{V}^*\mathbf{A}\mathbf{V} \in \mathbb{C}^{r \times r}$ for an $r < n$ dimensional subspace $\mathrm{Range}(\mathbf{V})$, labeled by decreasing real part: $\mathrm{Re}\,\theta_1 \geq \cdots \geq \mathrm{Re}\,\theta_r$. Then for $k = 1, \ldots, r$,

$$\frac{\mu_{n-r+k} + \cdots + \mu_n}{r-k+1} \leq \mathrm{Re}\,\theta_k \leq \frac{\mu_1 + \cdots + \mu_k}{k}.$$

- Ky Fan similarly bounded the real parts of the eigenvalues of $\mathbf{A}$ [Fan 1950].
- The fact that $\theta_j \in W(\mathbf{A})$ gives the well-known bound

  $$\mu_n \leq \mathrm{Re}\,\theta_j \leq \mu_1, \qquad j = 1, \ldots, r.$$

  The theorem provides sharper bounds for interior eigenvalues of $\mathbf{V}^*\mathbf{A}\mathbf{V}$.

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$$\frac{\mu_{n-r+k} + \cdots + \mu_n}{r - k + 1} \leq \mathrm{Re}\,\theta_k \leq \frac{\mu_1 + \cdots + \mu_k}{k}.$$

Corollary (for Galerkin Model Reduction)

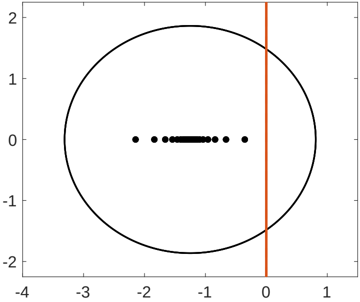
If for some $1 \leq k \leq r$,

$$\mu_1 + \cdots + \mu_k < 0,$$

then $\mathbf{V}^*\mathbf{A}\mathbf{V}$ has no more than $k - 1$ eigenvalues in the right-half plane.

Bounds on the Number of Unstable Modes: Example

$$\mathbf{A} = \frac{1}{8}\begin{bmatrix} -10 & 32\varrho & & & \\ 1 & -10 & 32\varrho^2 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & -10 & 32\varrho^{n-1} \\ & & & 1 & -10 \end{bmatrix}, \qquad \begin{array}{l} \varrho = 3/4 \\ n = 128 \end{array}$$

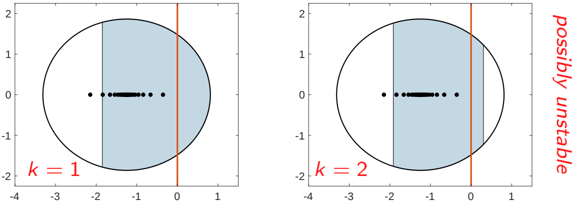


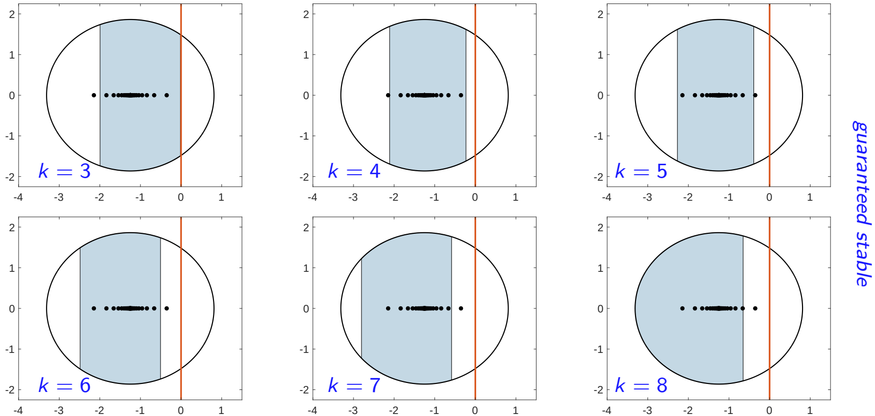
A is stable, but $W(\mathbf{A})$ extends into the RHP.

How many unstable modes can $\mathbf{V}^*\mathbf{A}\mathbf{V}$ *have?*

Bounds on the Number of Unstable Modes: Example

The containment regions for θ_k for $r = 8$ guarantee that $\mathbf{V}_r^* \mathbf{A} \mathbf{V}_r$ has at most two unstable modes.



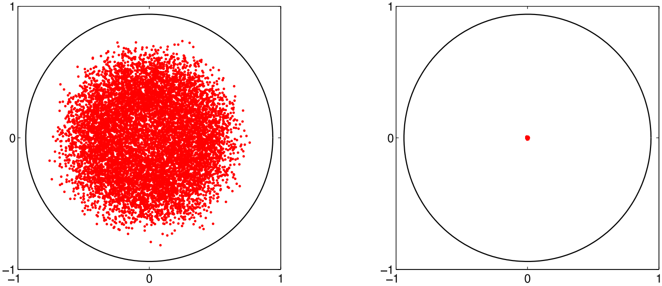


Two Matrices with Identical $W(\mathbf{A})$

Compute $r = 4$ eigenvalues of $\mathbf{V}^*\mathbf{A}\mathbf{V}$ for these 8×8 matrices $\mathbf{A}$:

$$\begin{bmatrix} 0 & 1 & & & & & & \\ & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & & 0 & & & & \\ & & & & 0 & 1 & & \\ & & & & & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & & 0 \end{bmatrix} \qquad \gamma \begin{bmatrix} 0 & \varrho^1 & & & & & & \\ & 0 & \varrho^2 & & & & & \\ & & 0 & \varrho^3 & & & & \\ & & & 0 & \varrho^4 & & & \\ & & & & 0 & \varrho^5 & & \\ & & & & & 0 & \varrho^6 & \\ & & & & & & 0 & \varrho^7 \\ & & & & & & & 0 \end{bmatrix}.$$

(Choose γ to give the same $W(\mathbf{A})$ for both examples; $\varrho = 1/8$.)



Smallest *magnitude* eigenvalue of $\mathbf{V}^*\mathbf{A}\mathbf{V}$, 10,000 random complex subspaces.

Eigenvalues of Galerkin Projections (Sorted by Magnitude)

Now sort the eigenvalues of $\mathbf{V}^*\mathbf{A}\mathbf{V}$ by magnitude: $|\theta_1| \geq |\theta_2| \geq \cdots \geq |\theta_r|$.

- For any $\mathbf{A} \in \mathbb{C}^{n\times n}$, the product of eigenvalues is *log-majorized* by the product of singular values; see, e.g., [Marshall, Olkin, Arnold 2011]. Sort the eigenvalues and singular values of $\mathbf{A}$ by magnitude, $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Then

$$\prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} \sigma_j.$$

Theorem (Carden)

Let $\theta_1, \ldots, \theta_r$ denote the eigenvalues of $\mathbf{V}^*\mathbf{A}\mathbf{V} \in \mathbb{C}^{r\times r}$ for an $r < n$ dimensional subspace Range($\mathbf{V}$)*, labeled by decreasing magnitude: $|\theta_1| \geq \cdots \geq |\theta_r|$. Then for $k = 1, \ldots, r$,*

$$|\theta_k| \leq \left(\sigma_1 \cdots \sigma_k\right)^{1/k},$$

where $\sigma_1 \geq \cdots \geq \sigma_n$ are the singular values of $\mathbf{A}$.

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Corollary (for Galerkin Model Reduction)

If for some $1 \leq k \leq r$,

$$\sigma_1 \cdots \sigma_k < 1,$$

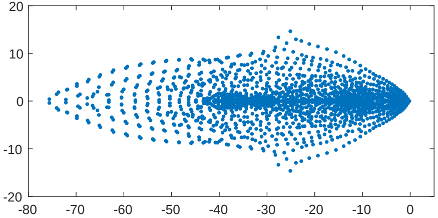
then $\mathbf{V}^*\mathbf{A}\mathbf{V}$ has no more than $k-1$ eigenvalues outside the unit disk.

Illustration for a Fluid Dynamics Problem

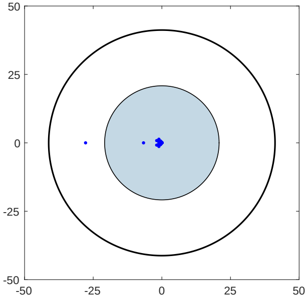
Lid driven cavity fluid stability problem from IFISS [Elman, Ramage Silvester].
Q2-Q1 elements, 32×32 mesh, viscosity $\nu = 0.01$, dimension $n = 2178$.

We seek the rightmost eigenvalue of a generalized eigenvalue problem.

Compute eigenvalues via shift-invert Arnoldi: $\mathbf{A}_\gamma := (\mathbf{A} - \gamma\mathbf{B})^{-1}\mathbf{B}$.
We now seek the largest magnitude eigenvalue of $\mathbf{A}_\gamma$.



finite eigenvalues of $\mathbf{A} - \lambda\mathbf{B}$



eigenvalues of $\mathbf{A}_\gamma$

By the theorem, at least $r - 1$ eigenvalues of $\mathbf{V}^*\mathbf{A}\mathbf{V}$ are located in the blue disk

How many unstable modes can $\mathbf{V}^*\mathbf{A}\mathbf{V}$ have when $\mathbf{A}$ is stable?

Theorem (Duintjer Tebbens & Meurant 2012)

Specify the following complex scalars:

- $\lambda_1, \ldots, \lambda_n$;
- $\theta_1^{(1)}$;
- $\theta_1^{(2)}, \theta_2^{(2)}$;
- $\vdots$
- $\theta_1^{(n-1)}, \theta_2^{(n-1)}, \ldots, \theta_{n-1}^{(n-1)}$.

IMPORTANT NOTE:
This construction allows you to specify the eigenvalues of $\mathbf{A}$, *but you cannot specify* $W(\mathbf{A})$.

There exists $\mathbf{A} \in \mathbb{C}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{C}^n$ *such that*

- $\mathbf{A}$ *has the specified eigenvalues:* $\lambda_1, \ldots, \lambda_n$;
- $\mathbf{V}_r^*\mathbf{A}\mathbf{V}_r$ *has the specified eigenvalues:* *for* $r = 1, \ldots, n-1$,

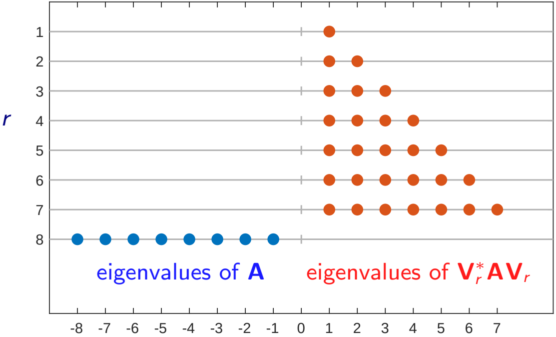
$$\text{eigenvalues of } \mathbf{V}_r^*\mathbf{A}\mathbf{V}_r = \{\theta_1^{(r)}, \ldots, \theta_r^{(r)}\}$$

when the columns of $\mathbf{V}_r$ *are an orthonormal basis for the Krylov subspace*

$$\mathcal{K}_r(\mathbf{A}, \mathbf{b}) = \text{span}\{\mathbf{b}, \mathbf{A}\mathbf{b}, \ldots, \mathbf{A}^{r-1}\mathbf{b}\}.$$

Adversarial Construction for Galerkin Reduction

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -362880 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & -1451520 \\ & 1 & 3 & 0 & 0 & 0 & 0 & -1693440 \\ & & 1 & 4 & 0 & 0 & 0 & -846720 \\ & & & 1 & 5 & 0 & 0 & -211680 \\ & & & & 1 & 6 & 0 & -28224 \\ & & & & & 1 & 7 & -2016 \\ & & & & & & 1 & -64 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



All modes of $\mathbf{V}_r^*\mathbf{A}\mathbf{V}_r$ *are unstable for* $1 \leq r < n$.

Petrov–Galerkin Projection

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t)\\ y(t) &= \mathbf{c}^*\mathbf{x}(t).\end{aligned}$$

Thus far we have focused on *Galerkin* projection, $\mathbf{V}_r^*\mathbf{A}\mathbf{V}_r$ with $\mathbf{V}_r^*\mathbf{V}_r = \mathbf{I}$, e.g., as generated by the *Arnoldi process* applied to $(\mathbf{A}, \mathbf{b})$.
The resulting model will match r moments of the transfer function at $z = \infty$:

$$\begin{aligned}\dot{\mathbf{x}}_r(t) &= (\mathbf{V}_r^*\mathbf{A}\mathbf{V}_r)\mathbf{x}_r(t) + (\mathbf{V}_r^*\mathbf{b})\,u(t)\\ y_r(t) &= (\mathbf{c}^*\mathbf{V}_r)\mathbf{x}_r(t).\end{aligned}$$

We briefly consider *Petrov–Galerkin* projection, $\mathbf{W}_r^*\mathbf{A}\mathbf{V}_r$ with $\mathbf{W}_r^*\mathbf{V}_r = \mathbf{I}$, e.g., as generated by the *bi-Lanczos process* applied to $(\mathbf{A}, \mathbf{b}, \mathbf{c})$.
The resulting model will match $2r$ moments of the transfer function at $z = \infty$:

$$\begin{aligned}\dot{\mathbf{x}}_r(t) &= (\mathbf{W}_r^*\mathbf{A}\mathbf{V}_r)\mathbf{x}_r(t) + (\mathbf{W}_r^*\mathbf{b})\,u(t)\\ y_r(t) &= (\mathbf{c}^*\mathbf{V}_r)\mathbf{x}_r(t).\end{aligned}$$

What are the stability properties of this Petrov–Galerkin reduced order model?

Can $\mathbf{W}^*\mathbf{A}\mathbf{V}$ have unstable modes when $\mathbf{A}$ is stable?

Theorem (Greenbaum 1998)

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$, *and suppose* $1 \le r \le n/2$. *Specify:*

- $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$ *and* $\beta_1, \ldots, \beta_{r-1} \in \mathbb{C}$;
- *nonzero starting vector,* $\mathbf{b} \in \mathbb{C}^n$ *with* $\mathbf{v}_1 := \mathbf{b}/\|\mathbf{b}\|$;
- *vectors* $\mathbf{v}_2, \ldots, \mathbf{v}_{r+1}$ *and scalars* $\gamma_1, \ldots, \gamma_{r-1} \in \mathbb{C}$ *generated by:*

$$\begin{aligned} \widehat{\mathbf{v}}_{j+1} &:= \mathbf{A}\mathbf{v}_j - \alpha_j\mathbf{v}_j - \beta_{j-1}\mathbf{v}_{j-1} \\ \gamma_j &:= \|\widehat{\mathbf{v}}_{j+1}\| \\ \mathbf{v}_{j+1} &;= \widehat{\mathbf{v}}_{j+1}/\gamma_j \end{aligned}$$

- *vector* $\mathbf{c} \perp \mathrm{span}\{\mathbf{v}_2, \ldots, \mathbf{v}_{r+1}, \mathbf{A}\mathbf{v}_{r+1}, \ldots, \mathbf{A}^{r-1}\mathbf{v}_{r+1}\}$.

Then r steps of the bi-Lanczos process applied to $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ either breaks down, or generates

$$\mathbf{W}_r^*\mathbf{A}\mathbf{V}_r = \begin{bmatrix} \alpha_1 & \beta_1 & & \\ \gamma_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_{r-1} \\ & & \gamma_{r-1} & \alpha_r \end{bmatrix}.$$

Adversarial Construction for Petrov–Galerkin Reduction

Consider the *Hermitian matrix* **A** and the Greenbaum construction:

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & \ddots & & & \\ & & & \ddots & \ddots & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & -2 \end{bmatrix} \in \mathbb{C}^{16\times 16}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{C}^{16}.$$

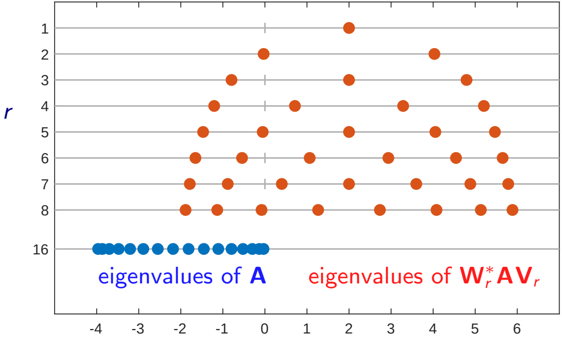
- $\mathbf{A} = \mathbf{A}^*$ has eigenvalues $\lambda_j = -2 + \cos(j\pi/17)$ in the left-half plane.
- $W(\mathbf{A}) = [\lambda_n, \lambda_1]$ is also contained in the left-half plane.
- *Any Galerkin projection of* **A** *will produce a stable reduced order model.*

Use Greenbaum's Theorem to construct

$$\mathbf{W}_r^* \mathbf{A} \mathbf{V}_r = \begin{bmatrix} +2 & 1 & & \\ \gamma_1 & +2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & \gamma_{r-1} & +2 \end{bmatrix} \in \mathbb{C}^{r\times r}.$$

Adversarial Construction for Petrov–Galerkin Reduction

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} \in \mathbb{C}^{16\times 16} \qquad \mathbf{W}_r^*\mathbf{A}\mathbf{V}_r = \begin{bmatrix} +2 & 1 & & \\ \gamma_1 & +2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & \gamma_{r-1} & +2 \end{bmatrix} \in \mathbb{C}^{r\times r}$$



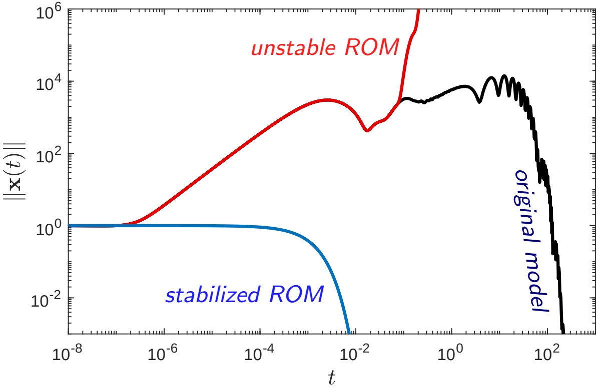
All modes of $\mathbf{W}_r^*\mathbf{A}\mathbf{V}_r$ are unstable for $1 \le r \le n/2$ here, despite the fact that $\mathbf{A}$ is a stable Hermitian matrix.

What Can We Learn From an Unstable ROM?

Unstable ROMs for stable systems are distasteful. One might go to lengths to *suppress the instability*; see, e.g., [Grimme, Sorensen, van Dooren 1995].

However, an unstable ROM might better capture transient dynamics than a stabilized version.

Boeing 767 example: stable linear system, $n = 55$; reduce to dimension $r = 20$ [Anderson, Ly, Liu 1990; Burke, Lewis, Overton 2003]



What Can We Learn From an Unstable ROM?

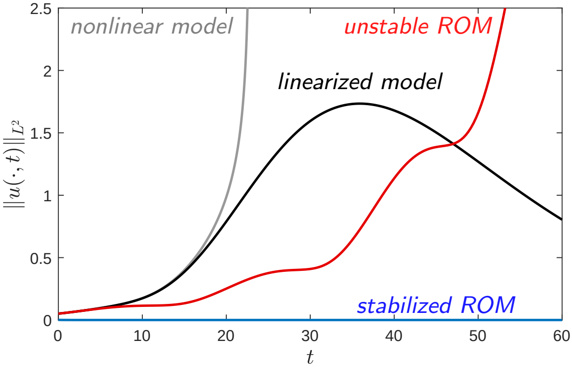
On the domain $x \in (0, \ell)$, $t > 0$, consider the *nonlinear heat equation*

$$u_t(x,t) = u_{xx}(x,t) + u_x(x,t) + \frac{1}{8}u(x,t) + u(x,t)^3,$$

with Dirichlet boundary conditions: $u(0,t) = u(\ell,t) = 0$.

[Sandsted & Scheel, 2005], [Galkowski, 2012] consider stability of this equation with small initial data, as a function of ℓ.

We take $\ell = 30$ and $u_0(x) = 10^{-5}x(x-\ell)(x-\ell/2)$ and reduce to $r = 40$.



Concluding Thoughts

- The interplay of linear transient growth and nonlinearity requires care.
- Reduction methods that preserve structure, nonlinearity, energy provide a major step in the right direction.
- Use a physically relevant inner product / norm.

  Eigenvalues (and the transfer function) are independent of the state-space representation, but $W(\mathbf{A})$ depends highly only the choice of coordinates. It is possible that $W(\mathbf{A})$ extends into the right-half plane in the Euclidean (vector) inner product, but not in the "energy inner product" motivated by the application.
- We still have much to learn about the eigenvalues of $\mathbf{V}^*\mathbf{A}\mathbf{V}$.

  Insight about these eigenvalues informs both model reduction and algorithms for solving large-scale eigenvalue problems.

References

- Russell Carden. "Ritz Values and Arnoldi Convergence for Non-Hermitian Matrices." PhD thesis, Rice University, 2011.
- Russell L. Carden and Mark Embree. "Ritz value localization for non-Hermitian matrices," *SIAM J. Matrix Anal. Appl.* 33 (2012) 1320–1338.
- Mark Embree. "Unstable modes in projection-based reduced-order models: How many can there be, and what do they tell you?" *Systems Control Lett.* 124 (2019) 49–59.
- Jurjen Duintjer Tebbens and Gérard Meurant. "Any Ritz value behavior is possible for Arnoldi and for GMRES," *SIAM J. Matrix Anal. Appl.* 33 (2012) 958–978.
- Anne Greenbaum. "On the role of the left starting vector in the two-sided Lanczos algorithm and nonsymmetric linear system solvers," *Numerical Analysis 1997*, D.F. Griffiths, D.J. Higham, and G.A. Watson, eds. Addison Wesley Longman, Harlow, UK, 1998.