Nonlinear Balancing for Quadratic-Polynomial Systems

Nicholas A. Corbin

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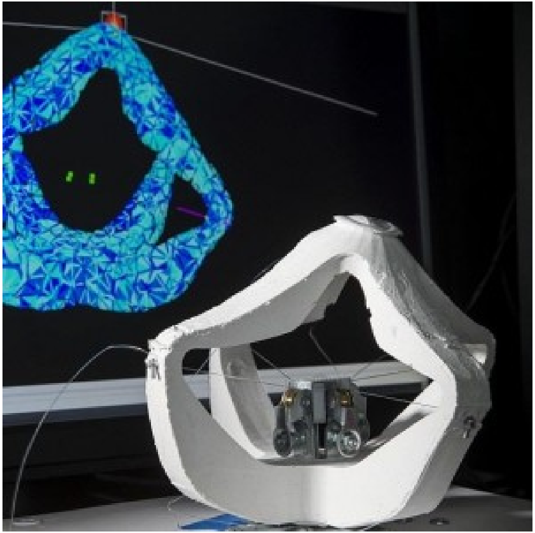
UC San Diego
JACOBS SCHOOL OF ENGINEERING
Mechanical and Aerospace Engineering

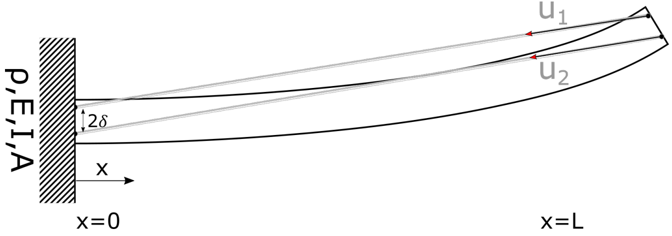
Motivating application: soft robotics

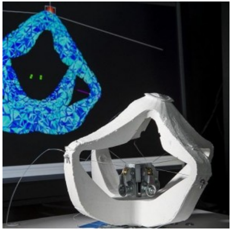


[Zhang et al., 2016]

"Even if different results exist in the literature for linear model reduction, like balanced truncation or iterative tangential interpolation, the only method suitable for non-linear system is the Proper Orthogonal Decomposition (POD)." - [Thieffry et al., 2018]

Challenges with soft robot control





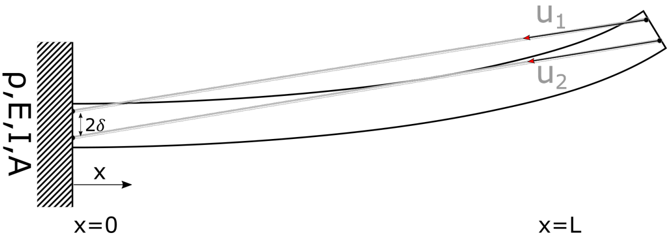
Control-affine dynamical system

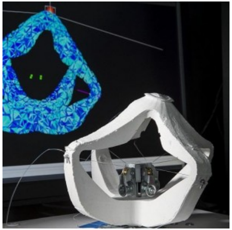
Material and geometric nonlinearities

Large state dimension due to PDE discretization

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$
$$\mathbf{y} = \mathbf{h}(\mathbf{x})$$

Challenges with soft robot control





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Material and geometric nonlinearities

Large state dimension due to PDE discretization

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$
$$\mathbf{y} = \mathbf{h}(\mathbf{x})$$

Geometric nonlinearities, e.g. cable angles

Background on nonlinear balancing and computation

$\mathcal{H}_\infty$ energy function definitions

Definition ([Scherpen, 1996])

Let γ be a positive constant $\gamma > 0, \gamma \neq 1$, and define $\eta := 1 - \gamma^{-2}$. The $\mathcal{H}_\infty$ past energy of the nonlinear system is defined as

$$\mathcal{E}_\gamma^-(\mathbf{x}_0) := \min_{\substack{\mathbf{u} \in L_2(-\infty, 0] \\ \mathbf{x}(-\infty) = \mathbf{0},\, \mathbf{x}(0) = \mathbf{x}_0}} \frac{1}{2} \int_{-\infty}^{0} \eta \|\mathbf{y}(t)\|^2 + \|\mathbf{u}(t)\|^2 \mathrm{d}t. \tag{1}$$

If $\gamma < 1$, the $\mathcal{H}_\infty$ future energy of the nonlinear system is defined as

$$\mathcal{E}_\gamma^+(\mathbf{x}_0) := \max_{\substack{\mathbf{u} \in L_2[0, \infty) \\ \mathbf{x}(0) = \mathbf{x}_0,\, \mathbf{x}(\infty) = \mathbf{0}}} \frac{1}{2} \int_{0}^{\infty} \|\mathbf{y}(t)\|^2 + \frac{\|\mathbf{u}(t)\|^2}{\eta} \mathrm{d}t, \tag{2}$$

whereas if $\gamma > 1$, the $\mathcal{H}_\infty$ future energy is defined as

$$\mathcal{E}_\gamma^+(\mathbf{x}_0) := \min_{\substack{\mathbf{u} \in L_2[0, \infty) \\ \mathbf{x}(0) = \mathbf{x}_0,\, \mathbf{x}(\infty) = \mathbf{0}}} \frac{1}{2} \int_{0}^{\infty} \|\mathbf{y}(t)\|^2 + \frac{\|\mathbf{u}(t)\|^2}{\eta} \mathrm{d}t. \tag{3}$$

Energy functions solve Hamilton-Jacobi-Bellman PDEs

Theorem ([Scherpen, 1996])

Assume that the HJB equation

$$0 = \frac{\partial \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + \frac{1}{2} \frac{\partial \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} \mathbf{g}(\mathbf{x}) \mathbf{g}(\mathbf{x})^\top \frac{\partial^\top \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} - \frac{\eta}{2} \mathbf{h}(\mathbf{x})^\top \mathbf{h}(\mathbf{x}) \tag{4}$$

has a solution with $\mathcal{E}_\gamma^-(\mathbf{0}) = 0$ *such that* $-\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^\top \partial^\top \mathcal{E}_\gamma^-(\mathbf{x})/\partial \mathbf{x}$ *is asymptotically stable. Then this solution is the past energy function* $\mathcal{E}_\gamma^-(\mathbf{x})$. *Furthermore, assume that the HJB equation*

$$0 = \frac{\partial \mathcal{E}_\gamma^+(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) - \frac{\eta}{2} \frac{\partial \mathcal{E}_\gamma^+(\mathbf{x})}{\partial \mathbf{x}} \mathbf{g}(\mathbf{x}) \mathbf{g}(\mathbf{x})^\top \frac{\partial^\top \mathcal{E}_\gamma^+(\mathbf{x})}{\partial \mathbf{x}} + \frac{1}{2} \mathbf{h}(\mathbf{x})^\top \mathbf{h}(\mathbf{x}) \tag{5}$$

has a solution with $\mathcal{E}_\gamma^+(\mathbf{0}) = 0$ *such that* $\mathbf{f}(\mathbf{x}) - \eta \mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^\top \partial^\top \mathcal{E}_\gamma^+(\mathbf{x})/\partial \mathbf{x}$ *is asymptotically stable. Then this solution is the future energy function* $\mathcal{E}_\gamma^+(\mathbf{x})$.

Analytic dynamics → analytic energy function

If the dynamics are analytic (polynomial):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \quad \rightarrow \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum_{\xi=2}^{\ell} \mathbf{F}_\xi \mathbf{x}^{\textcircled{\xi}} + \sum_{\xi=1}^{\ell} \mathbf{G}_\xi \left(\mathbf{x}^{\textcircled{\xi}} \otimes \mathbf{u}\right) + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) \quad \rightarrow \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \sum_{\xi=2}^{\ell} \mathbf{H}_\xi \mathbf{x}^{\textcircled{\xi}}$$

then the energy functions are also analytic (polynomial) [Al'brekht, 1961, Lukes, 1969]:

$$\mathcal{E}_\gamma^-(\mathbf{x}) = ??? \quad \rightarrow \quad \mathcal{E}_\gamma^-(\mathbf{x}) = \frac{1}{2}\left(\mathbf{v}_2^\top \mathbf{x}^{\textcircled{2}} + \mathbf{v}_3^\top \mathbf{x}^{\textcircled{3}} + \dots\right)$$

$$\mathcal{E}_\gamma^+(\mathbf{x}) = ??? \quad \rightarrow \quad \mathcal{E}_\gamma^+(\mathbf{x}) = \frac{1}{2}\left(\mathbf{w}_2^\top \mathbf{x}^{\textcircled{2}} + \mathbf{w}_3^\top \mathbf{x}^{\textcircled{3}} + \dots\right)$$

HJB PDEs become a set of algebraic equations for the coefficients $\mathbf{v}_i, \mathbf{w}_i$.

Some notes on the Kronecker product[1]

The Kronecker product of $\mathbf{A} \in \mathbb{R}^{p\times q}$ and $\mathbf{B} \in \mathbb{R}^{s\times t}$ is the $ps \times qt$ block matrix:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1q}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{p1}\mathbf{B} & \cdots & a_{pq}\mathbf{B} \end{bmatrix}$$

where a_{ij} denotes the (i,j)th entry of $\mathbf{A}$. We write repeated Kronecker products as

$$\mathbf{x}^{\textcircled{k}} := \underbrace{\mathbf{x} \otimes \cdots \otimes \mathbf{x}}_{k \text{ times}} \in \mathbb{R}^{n^k}.$$

We adopt the following notation to define the *k-way Lyapunov matrix*

$$\mathcal{L}_k(\mathbf{A}) := \underbrace{\mathbf{A} \otimes \cdots \otimes \mathbf{I}}_{k \text{ times}} + \cdots + \underbrace{\mathbf{I} \otimes \cdots \otimes \mathbf{A}}_{k \text{ times}}$$

ID. 1. $(\mathbf{A} \otimes \mathbf{B})(\mathbf{D} \otimes \mathbf{G}) = \mathbf{AD} \otimes \mathbf{BG}$

ID. 2. $\mathbf{A} \otimes \mathbf{B} = \mathbf{S}_{s\times p}(\mathbf{B} \otimes \mathbf{A})\mathbf{S}_{q\times t}$

ID. 3. $(\mathbf{I}_p \otimes \mathbf{x})\mathbf{A} = (\mathbf{A} \otimes \mathbf{x})$

ID. 4. $(\mathbf{x} \otimes \mathbf{I}_p)\mathbf{A} = (\mathbf{x} \otimes \mathbf{A})$

ID. 5. $\mathrm{vec}(\mathbf{ADB}) = (\mathbf{B}^\top \otimes \mathbf{A})\mathrm{vec}(\mathbf{D})$

ID. 6.
$$\begin{aligned} \mathrm{vec}(\mathbf{AD}) &= (\mathbf{I}_s \otimes \mathbf{A})\mathrm{vec}(\mathbf{D}) \\ &= (\mathbf{D}^\top \otimes \mathbf{I}_p)\mathrm{vec}(\mathbf{A}) \\ &= (\mathbf{D}^\top \otimes \mathbf{A})\mathrm{vec}(\mathbf{I}_q) \end{aligned}$$

ID. 7. $\mathbf{u}^\top \mathbf{Z}\mathbf{x} = \mathrm{vec}(\mathbf{Z})^\top(\mathbf{x} \otimes \mathbf{u})$

ID. 8. $\mathrm{vec}\left(\mathbf{x}^\top \otimes \mathbf{I}_m\right) = (\mathbf{x} \otimes \mathrm{vec}(\mathbf{I}_m))$

ID. 9. $\mathrm{vec}(\mathbf{A} \otimes \mathbf{B}) = (\mathbf{I}_q \otimes \mathbf{S}_{p\times t} \otimes \mathbf{I}_s)\left(\mathrm{vec}\left(\mathbf{A}\right) \otimes \mathrm{vec}\left(\mathbf{B}\right)\right)$

[1] IDs from various sources, including [Brewer, 1978, Magnus and Neudecker, 2019], etc.

Nonlinear balancing for quadratic systems [Kramer et al., 2022]

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F}\mathbf{x}^{②} + \mathbf{B}\mathbf{u},\qquad \mathcal{E}_\gamma^-(\mathbf{x}) \approx \frac{1}{2}\left(\mathbf{v}_2^\top \mathbf{x}^{②} + \mathbf{v}_3^\top \mathbf{x}^{③} + \cdots + \mathbf{v}_d^\top \mathbf{x}^{ⓓ}\right)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x},$$

Theorem

$\mathbf{v}_2 = \mathrm{vec}\left(\mathbf{V}_2\right)$ *is the symmetric positive definite solution to the $\mathcal{H}_\infty$ Riccati equation*

$$\mathbf{0} = \mathbf{A}^\top \mathbf{V}_2 + \mathbf{V}_2\mathbf{A} - \eta \mathbf{C}^\top \mathbf{C} + \mathbf{V}_2\mathbf{B}\mathbf{B}^\top \mathbf{V}_2. \tag{6}$$

For $3 \leq k \leq d$, let $\tilde{\mathbf{v}}_k \in \mathbb{R}^{n^k}$ solve the linear system

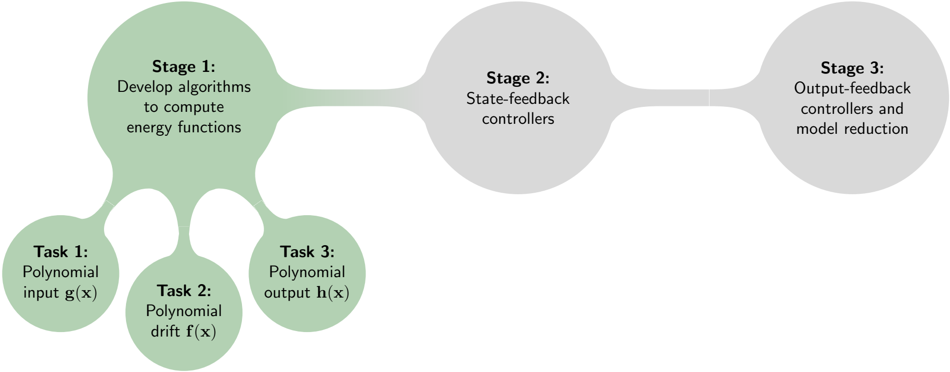
$$\mathcal{L}_k(\mathbf{A}^\top + \mathbf{V}_2\mathbf{B}\mathbf{B}^\top)\tilde{\mathbf{v}}_k = -\mathcal{L}_{k-1}(\mathbf{F})^\top \mathbf{v}_{k-1} - \frac{1}{4}\sum_{\substack{i,j>2 \\ i+j=k+2}} ij\ \mathrm{vec}(\mathbf{V}_i^\top \mathbf{B}\mathbf{B}^\top \mathbf{V}_j).$$

Then the coefficient vector $\mathbf{v}_k = \mathrm{vec}\left(\mathbf{V}_k\right)$ is obtained by the symmetrization of $\tilde{\mathbf{v}}_k$.

Roadmap

Control-affine dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$
$$\mathbf{y} = \mathbf{h}(\mathbf{x})$$



Extension to polynomial inputs

Quadratic-Polynomial System

Consider the quadratic-polynomial system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F}\mathbf{x}^{\circled{2}} + \sum_{\xi=1}^{\ell} \mathbf{G}_\xi \left(\mathbf{x}^{\circled{\xi}} \otimes \mathbf{u}\right) + \mathbf{B}\mathbf{u}, \tag{7}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}. \tag{8}$$

Now we wish to solve the HJB PDE

$$0 = \frac{\partial \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) + \frac{1}{2}\frac{\partial \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^\top \frac{\partial^\top \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} - \frac{\eta}{2}\mathbf{h}(\mathbf{x})^\top \mathbf{h}(\mathbf{x}) \tag{9}$$

with the quadratic-polynomial dynamics given by

$$\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{F}\mathbf{x}^{\circled{2}}, \qquad \mathbf{g}(\mathbf{x}) = \sum_{\xi=1}^{\ell} \mathbf{G}_\xi \left(\mathbf{x}^{\circled{\xi}} \otimes \mathbf{I}_m\right) + \mathbf{B}, \qquad \mathbf{h}(\mathbf{x}) = \mathbf{C}\mathbf{x}. \tag{10}$$

What do the additional $\mathbf{G}_\xi$ terms change?

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F}\mathbf{x}^{②} + \sum_{\xi=1}^{\ell} \mathbf{G}_\xi \left(\mathbf{x}^{ⓧ} \otimes \mathbf{u}\right) + \mathbf{B}\mathbf{u}, \qquad \mathcal{E}_\gamma^-(\mathbf{x}) \approx \frac{1}{2}\left(\mathbf{v}_2^\top \mathbf{x}^{②} + \mathbf{v}_3^\top \mathbf{x}^{③} + \cdots + \mathbf{v}_d^\top \mathbf{x}^{ⓓ}\right)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

Lemma

The additional $\mathbf{G}_\xi$ terms only add terms to the right-hand side of the linear systems for $\mathbf{v}_k$ for $3 \leq k \leq d$.

So $\mathbf{v}_2 = \mathrm{vec}\left(\mathbf{V}_2\right)$ still solves the $\mathcal{H}_\infty$ Riccati equation

$$\mathbf{0} = \mathbf{A}^\top \mathbf{V}_2 + \mathbf{V}_2 \mathbf{A} - \eta \mathbf{C}^\top \mathbf{C} + \mathbf{V}_2 \mathbf{B}\mathbf{B}^\top \mathbf{V}_2.$$

For $3 \leq k \leq d$, $\tilde{\mathbf{v}}_k \in \mathbb{R}^{n^k}$ solves the linear system

$$\mathcal{L}_k(\mathbf{A}^\top + \mathbf{V}_2\mathbf{B}\mathbf{B}^\top)\tilde{\mathbf{v}}_k = -\mathcal{L}_{k-1}(\mathbf{F})^\top \mathbf{v}_{k-1} - \frac{1}{4}\sum_{\substack{i,j>2 \\ i+j=k+2}} ij \ \mathrm{vec}(\mathbf{V}_i^\top \mathbf{B}\mathbf{B}^\top \mathbf{V}_j) \quad + \ ???$$

Main result: additional terms in the linear system for $\mathbf{v}_k$

Theorem

For $3 \leq k \leq d$, let $\tilde{\mathbf{v}}_k \in \mathbb{R}^{n^k}$ solve the linear system

$$\mathcal{L}_k(\mathbf{A}^\top + \mathbf{V}_2\mathbf{B}\mathbf{B}^\top)\tilde{\mathbf{v}}_k = -\mathcal{L}_{k-1}(\mathbf{F})^\top \mathbf{v}_{k-1} - \frac{1}{4}\sum_{\substack{i,j>2 \\ i+j=k+2}} ij \ \mathrm{vec}(\mathbf{V}_i^\top \mathbf{B}\mathbf{B}^\top \mathbf{V}_j)$$

$$-\frac{1}{4}\sum_{o=1}^{2\ell}\left(\sum_{\substack{p,q\geq 0 \\ p+q=o}}\left(\sum_{\substack{i,j\geq 2 \\ i+j=k-o+2}} ij \ \mathrm{vec}\left[\left(\mathbf{I}_{n^p} \otimes \mathrm{vec}\left[\mathbf{I}_m\right]^\top\right)\left(\mathrm{vec}\left[\mathbf{G}_q^\top \mathbf{V}_j\right]^\top \otimes \left(\mathbf{G}_p^\top \mathbf{V}_i \otimes \mathbf{I}_m\right)\right) \times \left(\mathbf{I}_{n^{j-1}} \otimes \mathbf{S}_{n^{i-1}\times n^q m} \otimes \mathbf{I}_m\right)\left(\mathbf{I}_{n^{k-p}} \otimes \mathrm{vec}\left[\mathbf{I}_m\right]\right)\right]\right)\right) \tag{11}$$

Then the coefficient vector $\mathbf{v}_k = \mathrm{vec}\left(\mathbf{V}_k\right)$ *is obtained by the symmetrization of* $\tilde{\mathbf{v}}_k$.

Main result proof sketch

The gradient of the energy function $\mathcal{E}_\gamma^-(\mathbf{x}) \approx \frac{1}{2}\left(\mathbf{v}_2^\top \mathbf{x}^{\textcircled{2}} + \mathbf{v}_3^\top \mathbf{x}^{\textcircled{3}} + \cdots + \mathbf{v}_d^\top \mathbf{x}^{\textcircled{d}}\right)$ is

$$\begin{aligned}\frac{\partial \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} = \frac{1}{2}\Big(&\mathbf{v}_2^\top(\mathbf{I}_n \otimes \mathbf{x}) + \mathbf{v}_2^\top(\mathbf{x} \otimes \mathbf{I}_n) \\ &+ \mathbf{v}_3^\top(\mathbf{I}_n \otimes \mathbf{x} \otimes \mathbf{x}) + \mathbf{v}_3^\top(\mathbf{x} \otimes \mathbf{I}_n \otimes \mathbf{x}) + \mathbf{v}_3^\top(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{I}_n) \\ &+ \mathbf{v}_4^\top(\mathbf{I}_n \otimes \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x}) + \mathbf{v}_4^\top(\mathbf{x} \otimes \mathbf{I}_n \otimes \mathbf{x} \otimes \mathbf{x}) \\ &+ \mathbf{v}_4^\top(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{I}_n \otimes \mathbf{x}) + \mathbf{v}_4^\top(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{I}_n) + \cdots\Big). \end{aligned} \tag{12}$$

According to the Lemma, the terms we consider only appear on the right-hand side, so we may assume symmetry of the already computed $\mathbf{v}_i$ and write

$$\begin{aligned}\frac{\partial \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} &= \frac{1}{2}\left(2\mathbf{v}_2^\top(\mathbf{I}_n \otimes \mathbf{x}) + 3\mathbf{v}_3^\top(\mathbf{I}_n \otimes \mathbf{x} \otimes \mathbf{x}) + 4\mathbf{v}_4^\top(\mathbf{I}_n \otimes \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x}) + \cdots\right) \\ &= \frac{1}{2}\sum_{i=2}^{k-1} i\mathbf{v}_i^\top(\mathbf{I}_n \otimes \mathbf{x}^{\textcircled{i-1}}). \end{aligned} \tag{13}$$

Main result proof sketch

HJB PDE: $$0 = \frac{\partial \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + \frac{1}{2} \frac{\partial \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} \mathbf{g}(\mathbf{x}) \mathbf{g}(\mathbf{x})^\top \frac{\partial^\top \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} - \frac{\eta}{2} \mathbf{h}(\mathbf{x})^\top \mathbf{h}(\mathbf{x}).$$

Gradient of the energy function: $$\frac{\partial \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} = \frac{1}{2} \sum_{i=2}^{k-1} i \mathbf{v}_i^\top (\mathbf{I}_n \otimes \mathbf{x}^{\textcircled{\scriptsize i-1}}).$$

Input vector field: $$\mathbf{g}(\mathbf{x}) = \sum_{\xi=1}^{\ell} \mathbf{G}_\xi \left(\mathbf{x}^{\textcircled{\scriptsize \xi}} \otimes \mathbf{I}_m\right) + \mathbf{B} := \sum_{\xi=0}^{\ell} \mathbf{G}_\xi \left(\mathbf{x}^{\textcircled{\scriptsize \xi}} \otimes \mathbf{I}_m\right).$$

Then we can write an arbitrary kth-order HJB term containing $\mathbf{G}_\xi$ generally as

$$\frac{1}{8} i \mathbf{v}_i^\top (\mathbf{I}_n \otimes \mathbf{x}^{\textcircled{\scriptsize i-1}}) \mathbf{G}_p (\mathbf{x}^{\textcircled{\scriptsize p}} \otimes \mathbf{I}_m)(\mathbf{x}^{\textcircled{\scriptsize q}\,\top} \otimes \mathbf{I}_m) \mathbf{G}_q^\top (\mathbf{I}_n \otimes \mathbf{x}^{\textcircled{\scriptsize j-1}\,\top}) \mathbf{v}_j j \tag{14}$$

$$\text{with} \quad \begin{cases} p \in [0, o], & o \in [1, 2\ell], \\ q = o - p, & i + j + o = k + 2. \end{cases} \tag{15}$$

Algebraic manipulations using Kronecker product identities

$$\frac{1}{8} i \mathbf{v}_i^\top (\mathbf{I}_n \otimes \mathbf{x}^{(i-1)}) \mathbf{G}_p (\mathbf{x}^{(p)} \otimes \mathbf{I}_m)(\mathbf{x}^{(q)^\top} \otimes \mathbf{I}_m) \mathbf{G}_q^\top (\mathbf{I}_n \otimes \mathbf{x}^{(j-1)^\top}) \mathbf{v}_j j,$$

Table: Dimensions of matrices used in identities

$\mathbf{A}(p \times q)$	$\mathbf{D}(q \times s)$	$\mathbf{u}(s \times 1)$
$\mathbf{B}(s \times t)$	$\mathbf{G}(t \times u)$	$\mathbf{w}(t \times 1)$

ID. 1. $(\mathbf{A} \otimes \mathbf{B})(\mathbf{D} \otimes \mathbf{G}) = \mathbf{AD} \otimes \mathbf{BG}$

ID. 2. $\mathbf{A} \otimes \mathbf{B} = \mathbf{S}_{s \times p}(\mathbf{B} \otimes \mathbf{A})\mathbf{S}_{q \times t}$

ID. 3. $(\mathbf{I}_p \otimes \mathbf{w})\mathbf{A} = (\mathbf{A} \otimes \mathbf{w})$

ID. 4. $(\mathbf{w} \otimes \mathbf{I}_p)\mathbf{A} = (\mathbf{w} \otimes \mathbf{A})$

ID. 5. $\mathrm{vec}(\mathbf{ADB}) = (\mathbf{B}^\top \otimes \mathbf{A})\mathrm{vec}(\mathbf{D})$

ID. 6. $\mathbf{u}^\top \mathbf{B} \mathbf{w} = \mathrm{vec}(\mathbf{B})^\top (\mathbf{w} \otimes \mathbf{u})$

ID. 7. $\mathrm{vec}\left(\mathbf{w}^\top \otimes \mathbf{I}_m\right) = (\mathbf{w} \otimes \mathrm{vec}(\mathbf{I}_m))$

ID. 8. $\mathrm{vec}(\mathbf{A} \otimes \mathbf{B}) = (\mathbf{I}_q \otimes \mathbf{S}_{p \times t} \otimes \mathbf{I}_s)\left(\mathrm{vec}\left(\mathbf{A}\right) \otimes \mathrm{vec}\left(\mathbf{B}\right)\right)$

Algebraic manipulations using Kronecker product identities

$$\frac{1}{8}i \; \underbrace{\mathbf{v}_i^\top(\mathbf{I}_n \otimes \mathbf{x}^{\textcircled{\scriptsize i-1}})}_{=\mathbf{x}^{\textcircled{\scriptsize i-1}^\top}\mathbf{V}_i^\top \text{ by ID. 5}} \; \mathbf{G}_p(\mathbf{x}^{\textcircled{\scriptsize p}} \otimes \mathbf{I}_m)(\mathbf{x}^{\textcircled{\scriptsize q}^\top} \otimes \mathbf{I}_m)\mathbf{G}_q^\top \underbrace{(\mathbf{I}_n \otimes \mathbf{x}^{\textcircled{\scriptsize j-1}^\top})\mathbf{v}_j}_{=\mathbf{V}_j\mathbf{x}^{\textcircled{\scriptsize j-1}} \text{ by ID. 5}} j,$$

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ID. 6. $\mathbf{u}^\top\mathbf{Bw} = \text{vec}(\mathbf{B})^\top(\mathbf{w} \otimes \mathbf{u})$

ID. 7. $\text{vec}\left(\mathbf{w}^\top \otimes \mathbf{I}_m\right) = (\mathbf{w} \otimes \text{vec}(\mathbf{I}_m))$

ID. 8. $\text{vec}(\mathbf{A} \otimes \mathbf{B}) = (\mathbf{I}_q \otimes \mathbf{S}_{p\times t} \otimes \mathbf{I}_s)\left(\text{vec}\left(\mathbf{A}\right) \otimes \text{vec}\left(\mathbf{B}\right)\right)$

Algebraic manipulations using Kronecker product identities

$$\dots \frac{1}{8} ij \ \mathrm{vec}\left[\left(\mathbf{I}_{n^p} \otimes \mathrm{vec}\left[\mathbf{I}_m\right]^\top\right)\left(\mathrm{vec}\left[\mathbf{G}_q^\top \mathbf{V}_j\right]^\top \otimes \left(\mathbf{G}_p^\top \mathbf{V}_i \otimes \mathbf{I}_m\right)\right) \times \right.$$

$$\left. \left(\mathbf{I}_{n^{j-1}} \otimes \mathbf{S}_{n^{i-1} \times n^q m} \otimes \mathbf{I}_m\right)\left(\mathbf{I}_{n^{k-p}} \otimes \mathrm{vec}\left[\mathbf{I}_m\right]^\top\right)\right] \mathbf{x}^{\textcircled{k}}$$

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ID. 4. $(\mathbf{w} \otimes \mathbf{I}_p)\mathbf{A} = (\mathbf{w} \otimes \mathbf{A})$

ID. 5. $\mathrm{vec}(\mathbf{ADB}) = (\mathbf{B}^\top \otimes \mathbf{A})\mathrm{vec}(\mathbf{D})$

ID. 6. $\mathbf{u}^\top \mathbf{B} \mathbf{w} = \mathrm{vec}(\mathbf{B})^\top (\mathbf{w} \otimes \mathbf{u})$

ID. 7. $\mathrm{vec}\left(\mathbf{w}^\top \otimes \mathbf{I}_m\right) = (\mathbf{w} \otimes \mathrm{vec}(\mathbf{I}_m))$

ID. 8. $\mathrm{vec}(\mathbf{A} \otimes \mathbf{B}) = \left(\mathbf{I}_q \otimes \mathbf{S}_{p \times t} \otimes \mathbf{I}_s\right)\left(\mathrm{vec}\left(\mathbf{A}\right) \otimes \mathrm{vec}\left(\mathbf{B}\right)\right)$

Recap before numerical results

- Interested in soft robot control with control-affine nonlinear models
- Use Taylor expansions to get polynomial models
- Polynomial model yields polynomial energy functions
- **Contribution: Kronecker product form of the algebraic equations for the energy coefficients when you have polynomial $\mathbf{g}(\mathbf{x})$**
- Solve Riccati equation and then a series of linear systems instead of a PDE
- Resulting energy function can be used for
  - control,
  - model reduction,
  - etc.

Numerical results

1D Example: exact solution

Consider the 1D quadratic-polynomial model

$$\dot{x} = ax + nx^2 + bu + g_1 xu + g_2 x^2 u,$$
$$y = cx.$$

The HJB PDE is actually an ODE for this 1D model:

$$0 = \frac{\mathrm{d}\mathcal{E}_\gamma^-(x)}{\mathrm{d}x} f(x) + \frac{1}{2}\left(\frac{\mathrm{d}\mathcal{E}_\gamma^-(x)}{\mathrm{d}x}\right)^2 g(x)^2 - \frac{\eta}{2} h(x)^2.$$

Let $q(x) = \mathrm{d}\mathcal{E}_\gamma^-(x)/\mathrm{d}x$, $\mathrm{a}(x) = g(x)^2/2$, $\mathrm{b}(x) = f(x)$, and $\mathrm{c}(x) = -\eta h(x)^2/2$. The HJB PDE takes the form of a standard quadratic equation in $q(x)$

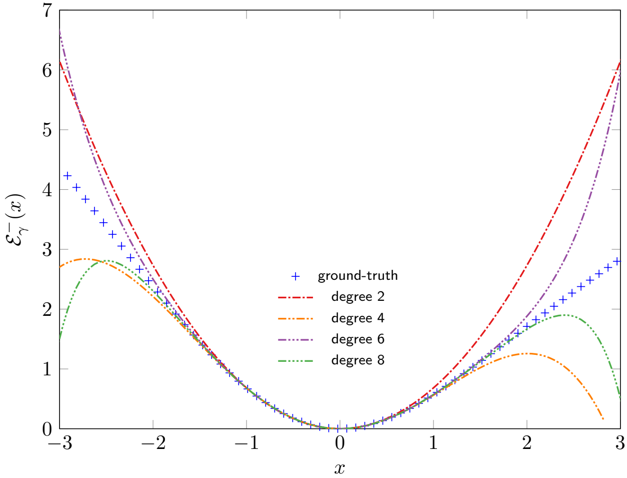
$$0 = \mathrm{a}(x) q(x)^2 + \mathrm{b}(x) q(x) + \mathrm{c}(x)$$

whose roots are given by

$$q(x) = \frac{-\mathrm{b}(x) \pm \sqrt{\mathrm{b}(x)^2 - 4\mathrm{a}(x)\mathrm{c}(x)}}{2\mathrm{a}(x)} \quad \rightarrow \quad \mathcal{E}_\gamma^-(x) = \int \frac{-\mathrm{b}(x) \pm \sqrt{\mathrm{b}(x)^2 - 4\mathrm{a}(x)\mathrm{c}(x)}}{2\mathrm{a}(x)} \mathrm{d}x.$$

1D Example: energy function plots

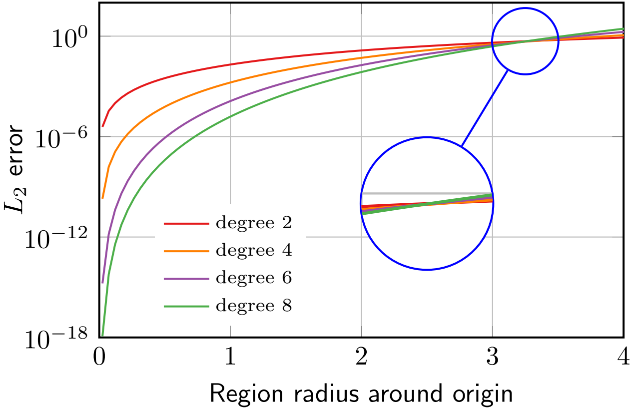
Results for $a = -2$, $n = 1$, $b = 2$, $g_1 = -0.2$, $g_2 = 0.2$, $c = 2$, and $\eta = 0.5$.



$$\begin{aligned}\mathcal{E}_\gamma^-(x) \approx\ & 1.3660x^2 - 1.1925 \times 10^{-1}x^3 \\ & - 1.2475 \times 10^{-1}x^4 + 1.0509 \times 10^{-2}x^5 \\ & + 1.4282 \times 10^{-2}x^6 - 1.5279 \times 10^{-4}x^7 \\ & - 1.6133 \times 10^{-3}x^8\end{aligned}$$

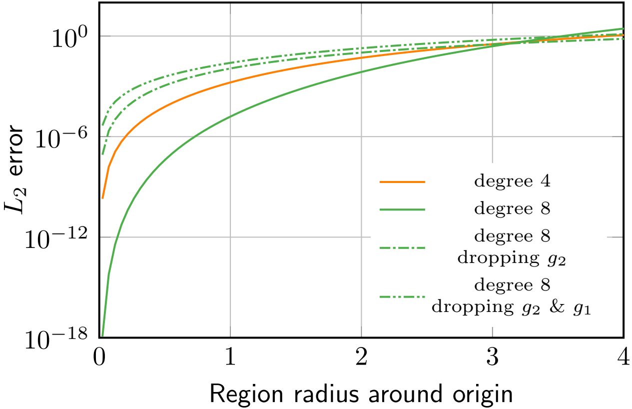
- The energy functions are accurate locally, and adding degrees aids accuracy locally
- As is often the case with polynomial approximation, higher-order approximations also diverge more severely beyond the region of convergence

1D Example: energy function L_2 error



- Adding higher-order terms increases accuracy locally.
- It does not necessarily increase the radius of convergence!

1D Example: is it better to discard G_ξ terms?



$$\dot{x} = ax + nx^2 + bu + \cancel{g_1 xu} + \cancel{g_2 x^2 u},$$
$$y = cx.$$

- Probably not.
- Better to approximate the *right* energy function to lower order than the *wrong* energy function to higher order.

2D Example: energy functions and residuals

$$RES = \left| \frac{\partial \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + \frac{1}{2} \frac{\partial \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} \mathbf{g}(\mathbf{x}) \mathbf{g}(\mathbf{x})^\top \frac{\partial^\top \mathcal{E}_\gamma^-(\mathbf{x})}{\partial \mathbf{x}} - \frac{\eta}{2} \mathbf{h}(\mathbf{x})^\top \mathbf{h}(\mathbf{x}) \right|$$

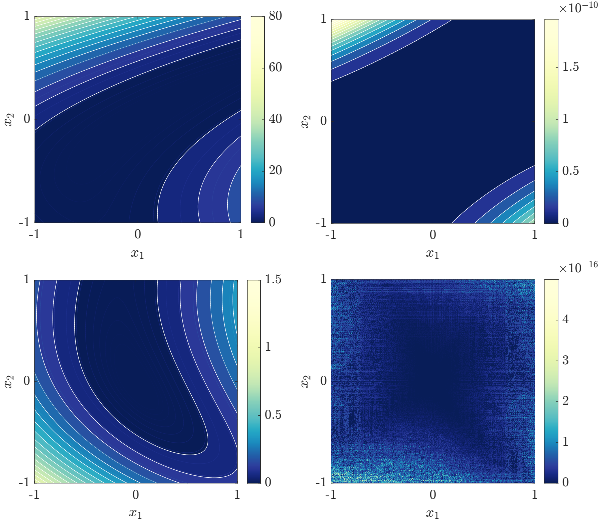
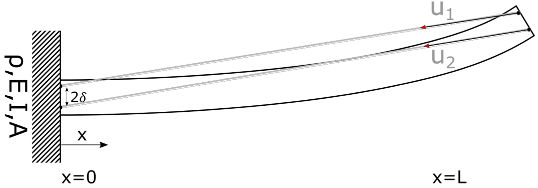


Figure: Left: nonlinear balancing energy functions. Right: HJB residuals.

Nonlinear beam

Consider a cable-actuated cantilever beam. We model this as a nonlinear Euler-Bernoulli beam subject to Von Kármán strains but linear elastic material response [Reddy, 2004].

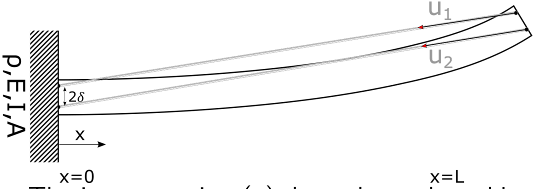


$$
\begin{aligned}
0 &= \rho A \frac{\partial^2 w}{\partial t^2} - \frac{\partial N_{xx}}{\partial x}, \\
0 &= \rho A \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial v}{\partial x}\right) - \rho I \frac{\partial^4 v}{\partial t^2 \partial x^2} + \frac{\partial^2 M_{xx}}{\partial x^2},
\end{aligned}
$$

where

$$
N_{xx} = EA\left[\frac{\partial w}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right] \qquad \text{and} \qquad M_{xx} = EI\frac{\partial^2 v}{\partial x^2}.
$$

Nonlinear beam: finite element discretization



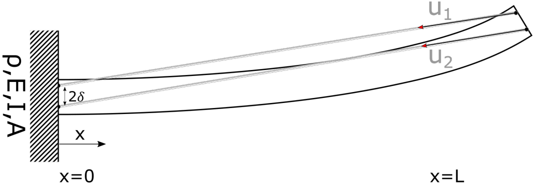
The semi-discretized truncated system can be written in state-space form as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F}_2\mathbf{x}^{②} + \mathbf{F}_3\mathbf{x}^{③} + \mathbf{g}(\mathbf{x})\mathbf{u},$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

The input matrix $\mathbf{g}(\mathbf{x})$ depends on the cable angle $\theta(\mathbf{x})$, which changes with the state. We can model the angular dependence using simple geometry and Taylor series expansion as

$$\cos(\theta(v,w)) = \frac{L + w(L,t)}{\sqrt{(L+w(L,t))^2 + v(L,t)^2}}, \quad \rightarrow \quad \approx \left(1 - \frac{v(L,t)^2}{2L^2} + \frac{w(L,t)v(L,t)^2}{L^3}\right),$$

$$\sin(\theta(v,w)) = \frac{v(L,t)}{\sqrt{(L+w(L,t))^2 + v(L,t)^2}}, \quad \rightarrow \quad \approx \left(\frac{v(L,t)}{L} - \frac{w(L,t)v(L,t)}{L^2} + \frac{(2w(L,t)^2 v(L,t) - v(L,t)^3)}{2L^3}\right).$$



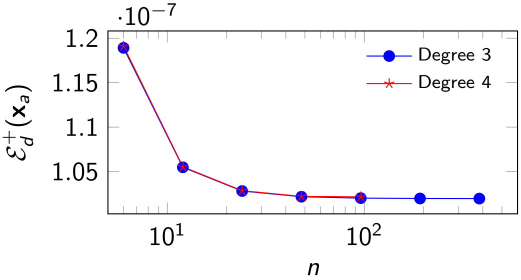


Figure: Convergence with respect to mesh size

With the degree 3 Taylor approximation to $\mathbf{g}(\mathbf{x})$ and discarding the cubic drift term, the model is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F}\mathbf{x}^{②} + \sum_{\xi=1}^{3} \mathbf{G}_\xi \left(\mathbf{x}^{ⓧ} \otimes \mathbf{u}\right) + \mathbf{B}\mathbf{u},$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where $\mathbf{x} \in \mathbb{R}^{6N}$ and N is the number of elements.

By varying the number of elements, we can investigate convergence and scaling performance.

Energy function convergence for the beam

Table: $n = 18$, scaling and convergence w.r.t. d

d	CPU sec	$\mathcal{E}_d^+(\mathbf{x}_a)$	$\mathcal{E}_d^+(\mathbf{x}_b)$
2	$2.18 \cdot 10^{-2}$	$1.012561 \cdot 10^{-7}$	$1.012561 \cdot 10^{-5}$
3	$9.04 \cdot 10^{-3}$	$1.034814 \cdot 10^{-7}$	$1.235090 \cdot 10^{-5}$
4	$1.89 \cdot 10^{-1}$	$1.035239 \cdot 10^{-7}$	$1.277524 \cdot 10^{-5}$
5	$4.19 \cdot 10^{0}$	$1.035240 \cdot 10^{-7}$	$1.278660 \cdot 10^{-5}$
6	$9.82 \cdot 10^{1}$	$1.035240 \cdot 10^{-7}$	$1.276495 \cdot 10^{-5}$

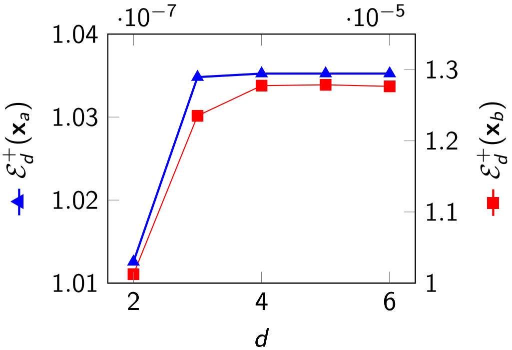


Figure: Energy function convergence as d increases for initial conditions $\mathbf{x}_a, \mathbf{x}_b$

Algorithm scaling for beam example

The solution implementation in [Kramer et al., 2022] for these types of linear systems scales as roughly $O(n^{d+1})$, though their results perform more like $O(n^d)$.

Table: $d = 3$, scaling and convergence w.r.t. n

No. of Elements	n	n^3	CPU sec	$\mathcal{E}_3^+(\mathbf{x}_a)$
1	6	$2.1600 \cdot 10^2$	$1.37 \cdot 10^{-3}$	$1.189137 \cdot 10^{-7}$
2	12	$1.7280 \cdot 10^3$	$2.80 \cdot 10^{-3}$	$1.054830 \cdot 10^{-7}$
4	24	$1.3824 \cdot 10^4$	$1.30 \cdot 10^{-2}$	$1.028107 \cdot 10^{-7}$
8	48	$1.1059 \cdot 10^5$	$9.57 \cdot 10^{-2}$	$1.021709 \cdot 10^{-7}$
16	96	$8.8474 \cdot 10^5$	$9.34 \cdot 10^{-1}$	$1.020124 \cdot 10^{-7}$
32	192	$7.0779 \cdot 10^6$	$1.05 \cdot 10^1$	$1.019728 \cdot 10^{-7}$
64	384	$5.6623 \cdot 10^7$	$3.89 \cdot 10^2$	$1.019625 \cdot 10^{-7}$

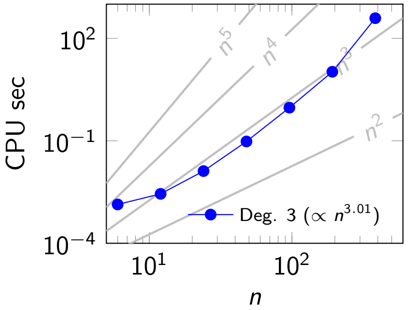


Figure: Scaling of CPU time as n increases

Algorithm scaling for beam example

The solution implementation in [Kramer et al., 2022] for these types of linear systems scales as roughly $O(n^{d+1})$, though their results perform more like $O(n^d)$.

Table: $d = 4$, scaling and convergence w.r.t. n

No. of Elements	n	n^4	CPU sec	$\mathcal{E}_4^+(\mathbf{x}_a)$
1	6	$1.2960{\cdot}10^3$	$2.00{\cdot}10^{-2}$	$1.191000{\cdot}10^{-7}$
2	12	$2.0736{\cdot}10^4$	$3.94{\cdot}10^{-2}$	$1.055358{\cdot}10^{-7}$
4	24	$3.3178{\cdot}10^5$	$5.08{\cdot}10^{-1}$	$1.028498{\cdot}10^{-7}$
8	48	$5.3084{\cdot}10^6$	$8.32{\cdot}10^0$	$1.022069{\cdot}10^{-7}$
16	96	$8.4935{\cdot}10^7$	$1.83{\cdot}10^2$	$1.021582{\cdot}10^{-7}$

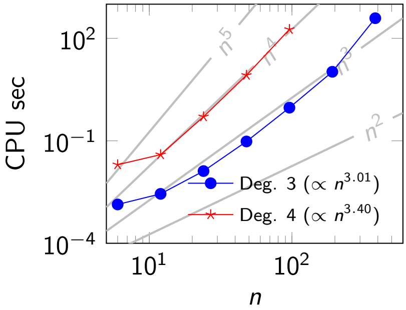


Figure: Scaling of CPU time as n increases

Conclusions

- Presented generalization of [Kramer et al., 2022] to polynomial inputs $\mathbf{g}(\mathbf{x})$
- Linear systems remain similar, only different right-hand sides (allows to inherit solvability, solution algorithms, scaling, etc.)

Future work:

- Extend to polynomial $\mathbf{f}(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ for full polynomial systems
- State feedback control (using these energy functions and others)
- Soft robot examples, model reduction, and output feedback control
- Optimize implementation (current speed OK but also RAM usage), maybe tensor_toolbox [2]

[2] [Bader et al., 2023]

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Thieffry, M., Kruszewski, A., Duriez, C., and Guerra, T.-m. (2018).
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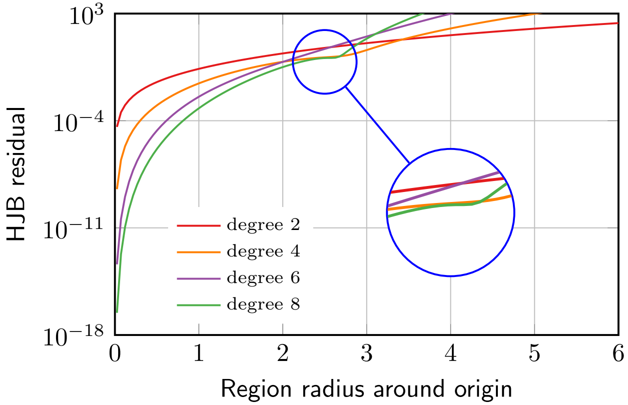
Kinematic modeling and observer based control of soft robot using real-time finite element method.
In *2016 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE.

1D Example: HJB residual as a metric



- Maybe a better measure of a 'good' energy function, since it involves a) the HJB equation and b) gradients of the energy

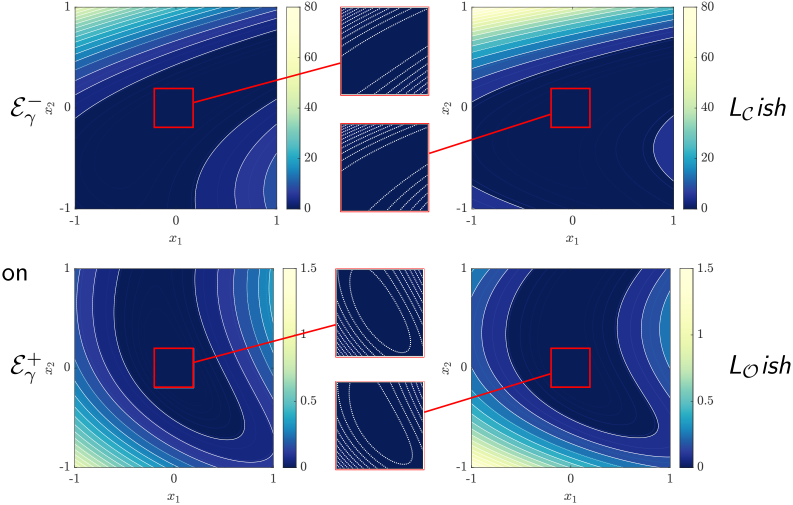
2D Example: nonlinear balancing vs. differential balancing

Next we take the 2D quadratic-bilinear system from [Kawano and Scherpen, 2017]:

$$\dot{x}_1 = -x_1 + x_2 - x_2^2 + u + 2x_2 u$$
$$\dot{x}_2 = -x_2 + u$$
$$y = x_1$$

We use a degree 4 approximation with $\eta = 0$ for open-loop balancing.

$$L_{\mathcal{C}} = \frac{1}{2}\delta \mathbf{x}_0^\top \mathbf{Q}(\mathbf{x}_0)\delta \mathbf{x}_0$$
$$L_{\mathcal{C}} ish = \frac{1}{2}\mathbf{x}_0^\top \mathbf{Q}(\mathbf{x}_0)\mathbf{x}_0$$



Nonlinear balancing for polynomial systems

Theorem (Past energy polynomial coefficients)

For $3 \leq k \leq d$, let $\tilde{\mathbf{v}}_k \in \mathbb{R}^{n^k}$ solve the linear system

$$\mathcal{L}_k\left(\mathbf{A}+\mathbf{B}\mathbf{B}^\top\mathbf{V}_2\right)^\top \tilde{\mathbf{v}}_k = -\sum_{\substack{i,\xi\geq 2\\ \xi+i=k+1}} \mathcal{L}_i(\mathbf{F}_\xi)^\top \mathbf{v}_i - \frac{1}{4}\sum_{\substack{i,j>2\\ i+j=k+2}} ij \operatorname{vec}\left[\mathbf{V}_i^\top \mathbf{B}\mathbf{B}^\top \mathbf{V}_j\right] + \eta \sum_{\substack{p,q\geq 1\\ p+q=k}} \operatorname{vec}\left[\mathbf{H}_p^\top \mathbf{H}_q\right]$$
$$-\frac{1}{4}\sum_{o=1}^{2\ell}\left(\sum_{\substack{p,q\geq 0\\ p+q=o}}\left(\sum_{\substack{i,j\geq 2\\ i+j=k-o+2}} ij \operatorname{vec}\left[\left(\mathbf{I}_{n^p}\otimes \operatorname{vec}\left[\mathbf{I}_m\right]^\top\right)\left(\operatorname{vec}\left[\mathbf{G}_q^\top \mathbf{V}_j\right]^\top \otimes \left(\mathbf{G}_p^\top \mathbf{V}_i \otimes \mathbf{I}_m\right)\right)\times\right.\right.\right.$$
$$\left.\left.\left.\left(\mathbf{I}_{n^{j-1}}\otimes \mathbf{S}_{n^{i-1}\times n^q m}\otimes \mathbf{I}_m\right)\left(\mathbf{I}_{n^{k-p}}\otimes \operatorname{vec}\left[\mathbf{I}_m\right]\right)\right]\right)\right) \tag{16}$$

Then the $\mathbf{v}_k = \operatorname{vec}(\mathbf{V}_k) \in \mathbb{R}^{n^k}$ for $3 \leq k \leq d$ is obtained by symmetrization of $\tilde{\mathbf{v}}_k$.

2D Example: approximate input normal transformations

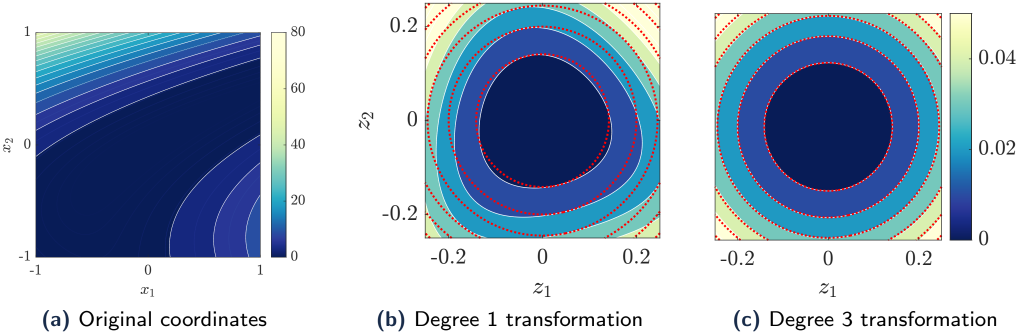


Figure: Past energy function under approximate input normal transformation