RG Modelling, Simulation and
Optimization of Real Processes

~~Nonlinear~~ Bilinear control and model reduction

Simple enough, yet too complicated...

Tobias Breiten

Tutorial Sessions
Nonlinear Model Reduction for Control
Workshop and Conference @ Virginia Tech

May 22-26, 2023

Overview

1. Bilinear control systems
   - Finite and infinite approximations
   - Basics from bilinear system theory
2. Model reduction of bilinear systems
   - Interpolatory model reduction
   - Balancing-based model reduction
3. Optimal control of bilinear systems
   - Open loop control
   - Closed loop control

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Bilinear control systems

We consider

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} N_k x(t) u_k(t) + Bu(t), \quad x(0) = x_0$$
$$y(t) = Cx(t) + Du(t),$$

where for fixed t, we call

- $x(t) \in \mathbb{R}^n$ the state,
- $u(t) \in \mathbb{R}^m$ the input/control,
- $y(t) \in \mathbb{R}^p$ the output/observation.

Throughout this talk, we assume

- **always** $D = 0$,
- **often** $x_0 = 0$,
- **often** $N_1 = N, B = b \in \mathbb{R}^n, C = c^\top \in \mathbb{R}^{1\times n}$.

A bilinearly controlled heat equation

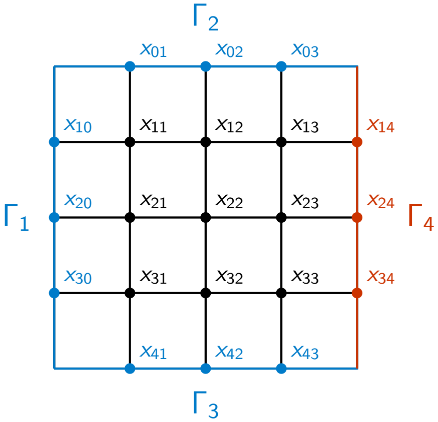
- 2-dimensional heat distribution
- boundary control by spraying intensities of a cooling fluid

$$
\begin{aligned}
\Omega &= (0,1) \times (0,1) & & \\
x_t &= \Delta x & & \text{in } \Omega \\
\nu \cdot \nabla x &= u_{1,2,3}(x-1) & & \text{on } \Gamma_1, \Gamma_2, \Gamma_3 \\
x &= u_4 & & \text{on } \Gamma_4
\end{aligned}
$$

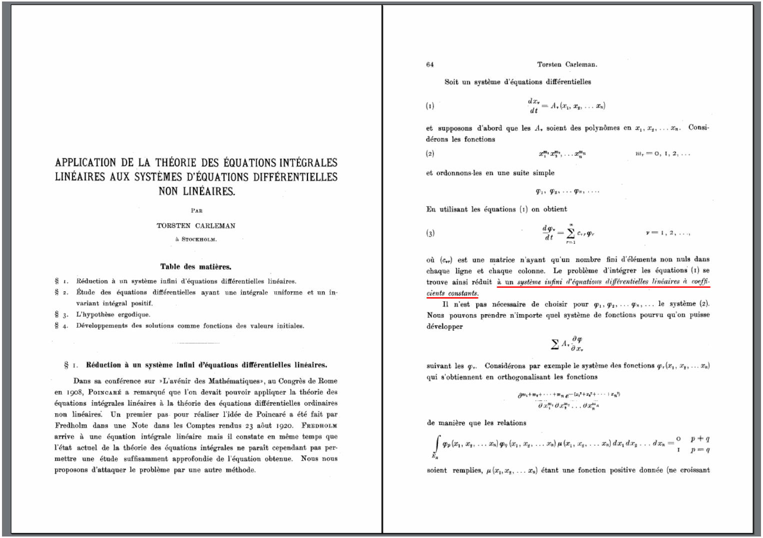
- spatial discretization $k \times k$-grid

$$\Rightarrow \dot{x} \approx A_1 x + \sum_{i=1}^{3} N_i x u_i + Bu$$

- output: $y = \frac{1}{k^2} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$



Carleman linearization



APPLICATION DE LA THÉORIE DES ÉQUATIONS INTÉGRALES LINÉAIRES AUX SYSTÈMES D'ÉQUATIONS DIFFÉRENTIELLES NON LINÉAIRES.

PAR

TORSTEN CARLEMAN

à STOCKHOLM.

Table des matières.

§ 1. Réduction à un système infini d'équations différentielles linéaires.

Dans sa conférence sur »L'avénir des Mathématiques», au Congrès de Rome en 1908, POINCARÉ a remarqué que l'on devait pouvoir appliquer la théorie des équations intégrales linéaires à la théorie des équations différentielles ordinaires non linéaires. Un premier pas pour réaliser l'idée de Poincaré a été fait par Fredholm dans une Note dans les Comptes rendus 23 août 1920. FREDHOLM arrive à une équation intégrale linéaire mais il constate en même temps que l'état actuel de la théorie des équations intégrales ne paraît cependant pas permettre une étude suffisamment approfondie de l'équation obtenue. Nous nous proposons d'attaquer le problème par une autre méthode.

64 Torsten Carleman.

Soit un système d'équations différentielles

$$\frac{dx_\nu}{dt} = A_\nu(x_1, x_2, \dots x_n) \qquad (1)$$

et supposons d'abord que les A_ν soient des polynômes en $x_1, x_2, \dots x_n$. Considérons les fonctions

$$x_1^{m_1} x_2^{m_2}, \dots x_n^{m_n} \qquad m_\nu = 0, 1, 2, \dots \qquad (2)$$

et ordonnons-les en une suite simple

$$\varphi_1, \varphi_2, \dots \varphi_n, \dots .$$

En utilisant les équations (1) on obtient

$$\frac{d\varphi_\nu}{dt} = \sum_{r=1}^{\infty} c_{\nu r} \varphi_r \qquad \nu = 1, 2, \dots, \qquad (3)$$

où $(c_{\nu r})$ est une matrice n'ayant qu'un nombre fini d'éléments non nuls dans chaque ligne et chaque colonne. Le problème d'intégrer les équations (1) se trouve ainsi réduit *à un système infini d'équations différentielles linéaires à coefficients constants.*

Il n'est pas nécessaire de choisir pour $\varphi_1, \varphi_2, \dots \varphi_n, \dots$ le système (2). Nous pouvons prendre n'importe quel système de fonctions pourvu qu'on puisse développer

$$\sum A_\nu \frac{\partial \varphi}{\partial x_\nu}$$

suivant les φ_ν. Considérons par exemple le système des fonctions $\varphi_\nu(x_1, x_2, \dots x_n)$ qui s'obtiennent en orthogonalisant les fonctions

$$\frac{\partial^{m_1+m_2+\dots+m_n} e^{-(x_1^2+x_2^2+\dots+x_n^2)}}{\partial x_1^{m_1} \partial x_2^{m_2} \dots \partial x_n^{m_n}}$$

de manière que les relations

$$\int_{E_n} \varphi_p(x_1, x_2, \dots x_n)\, \varphi_q(x_1, x_2, \dots x_n)\, \mu(x_1, x_2, \dots x_n)\, dx_1 dx_2 \dots dx_n = \begin{matrix} 0 & p \neq q \\ 1 & p = q \end{matrix}$$

soient remplies, $\mu(x_1, x_2, \dots x_n)$ étant une fonction positive donnée (ne croissant

Carleman linearization

Question: why should we care about bilinear control systems?

Consider a linear-analytic control affine system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = 0$$

with (convergent) Taylor series around 0

$$\begin{aligned} f(x) &= A_1 x + A_2 x \otimes x + \cdots + A_k x \otimes \cdots \otimes x + \cdots \\ g(x) &= B_0 + B_1 x + B_2 x \otimes x + \cdots + B_{k-1} x \otimes \cdots \otimes x + \cdots \end{aligned}$$

where $A_i, B_i \in \mathbb{R}^{n \times n^i}$.

Carleman linearization cont'd

Let us introduce

$$x^{\otimes} := \begin{bmatrix} x^\top & x^\top \otimes x^\top & \cdots & \underbrace{x^\top \otimes \cdots \otimes x^\top}_{k} \end{bmatrix}^\top$$

and consider the bilinear approximation

$$\frac{\mathrm{d}}{\mathrm{d}t} x^{\otimes} \approx \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ 0 & A_{2,1} & \cdots & A_{2,k-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{k,1} \end{bmatrix} x^{\otimes} + \begin{bmatrix} B_1 & \cdots & B_{k-1} & 0 \\ B_{2,0} & \ddots & \vdots & 0 \\ 0 & \vdots & B_{k-1,1} & 0 \\ 0 & 0 & B_{k,0} & 0 \end{bmatrix} x^{\otimes} u + \begin{bmatrix} B_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

where

$$A_{i,j} = A_j \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes A_j$$
$$B_{i,j} = B_j \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes B_j$$

Pros: better approximations than linearization
Cons: exponential increase of unknowns $\rightsquigarrow$ curse of dimensionality

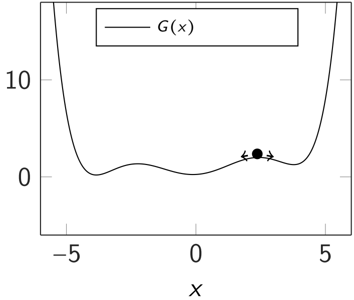
The Fokker-Planck equation

A dragged Brownian particle

Consider stochastic particle $X_t \in \Omega \subset \mathbb{R}^n$ and its motion given by

$$\mathrm{d}X_t = -\nabla V(X_t, t)\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}W_t, \quad X_{t=0} = X_0,$$

- $W_t \in \mathbb{R}^n$ a Wiener process, ν (dimensionless) temperature,
- particle confined by potential $V(X_t, t) = G(X_t)$,



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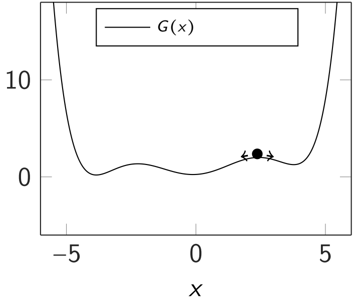

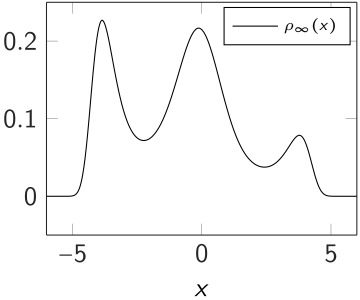

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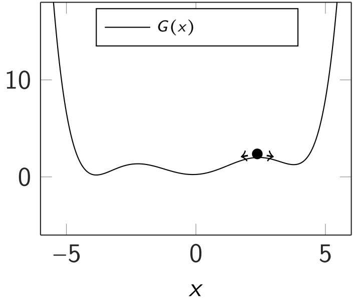

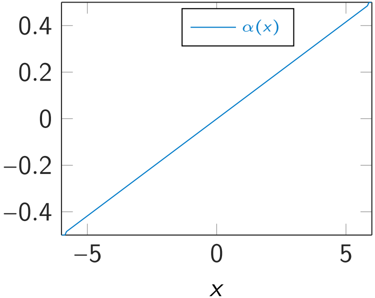

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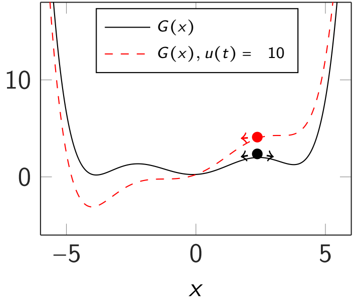

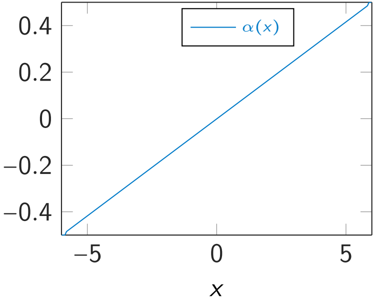

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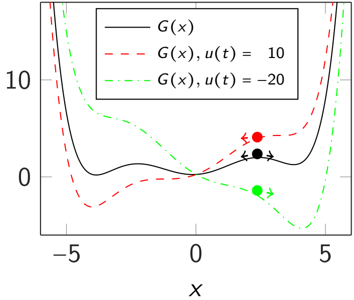

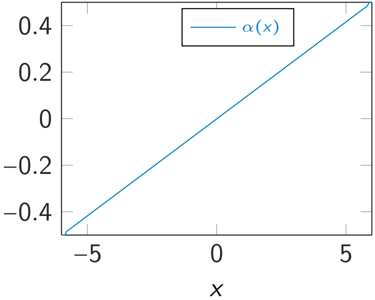

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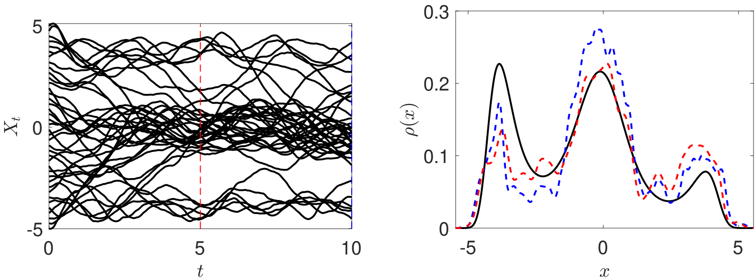

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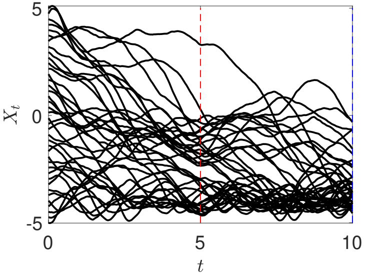

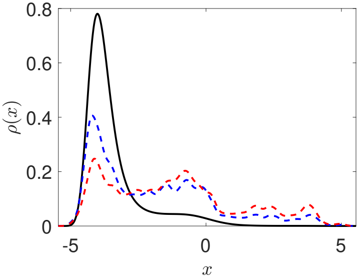

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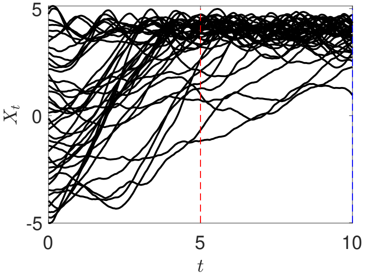

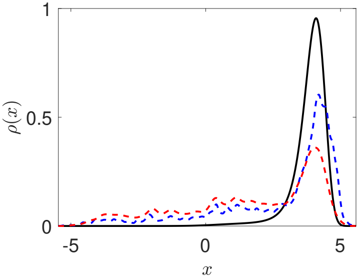

The Fokker-Planck equation cont'd

Consider probability distribution function

$$\rho(x,t)\mathrm{d}x = \mathbb{P}[X_t \in [x, x+\mathrm{d}x)]$$

Fokker-Planck equation

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nu \Delta \rho + \nabla \cdot (\rho \nabla V) && \text{in } \Omega \times (0,\infty), \\ 0 &= (\nu \nabla \rho + \rho \nabla V) \cdot \vec{n} && \text{on } \Gamma \times (0,\infty), \\ \rho(x,0) &= \rho_0(x) && \text{in } \Omega, \end{aligned}$$

- $\Omega \subset \mathbb{R}^n$ bounded open set with boundary $\Gamma = \partial\Omega$,
- ρ_0 initial probability distribution with $\int_\Omega \rho_0(x)\mathrm{d}x = 1$,
- $V(x,t) = G(x) + \alpha(x)u(t)$.

The Fokker-Planck equation cont'd

An infinite-dimensional bilinear control system

Consider the bilinear control system

$$\dot{\rho}(t) = A\rho(t) + N\rho(t)u(t),$$
$$\rho(0) = \rho_0,$$

where the operators A and N are defined as follows

$$A\colon \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega),$$
$$\mathcal{D}(A) = \left\{\rho \in H^2(\Omega) \,\middle|\, (\nu\nabla\rho + \rho\nabla G) \cdot \vec{n} = 0 \text{ on } \Gamma\right\},$$
$$A\rho = \nu\Delta\rho + \nabla \cdot (\rho\nabla G),$$
$$N\colon H^1(\Omega) \to L^2(\Omega), \quad N\rho = \nabla \cdot (\rho\nabla\alpha).$$

The Fokker-Planck equation cont'd

An infinite-dimensional bilinear control system

Consider the bilinear control system

$$\dot{\rho}(t) = A\rho(t) + N\rho(t)u(t),$$
$$\rho(0) = \rho_0,$$

its $L^2(\Omega)$-adjoints are given by

$$A^*: \mathcal{D}(A^*) \subset L^2(\Omega) \to L^2(\Omega),$$
$$\mathcal{D}(A^*) = \left\{ \varphi \in H^2(\Omega) \,\middle|\, (\nu \nabla \varphi) \cdot \vec{n} = 0 \text{ on } \Gamma \right\},$$
$$A^* \phi = \nu \Delta \varphi - \nabla G \cdot \nabla \varphi,$$

$$N^*: H^1(\Omega) \to L^2(\Omega), \quad N^* \varphi = -\nabla \varphi \cdot \nabla \alpha.$$

The Fokker-Planck equation cont'd

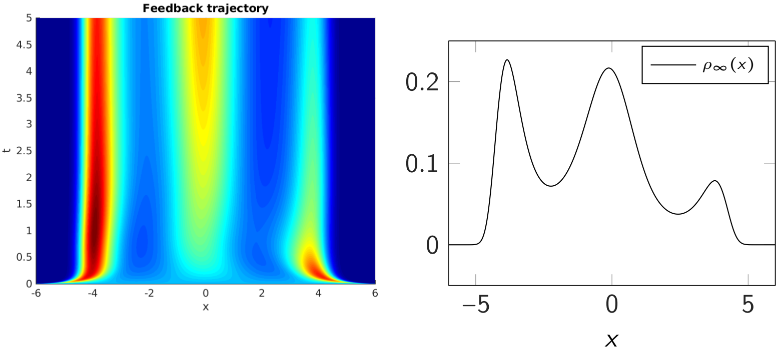

Figure: 1D Fokker-Planck equation, $n = 1024$.

The Fokker-Planck equation ...

... and its deterministic counterpart

Consider motion of ~~stochastic~~ deterministic particle $X_t \in \Omega \subset \mathbb{R}^n$

$$\mathrm{d}X_t = -\nabla V(X_t,t)\mathrm{d}t + \cancel{\sqrt{2\nu}\,\mathrm{d}W_t}, \quad X_{t=0} = X_0,$$

where $V(X_t,t) = G(X_t) + \alpha(X_t)u(t)$.

We then obtain

$$\dot{\rho}(t) = A\rho(t) + N\rho(t)u(t), \quad \rho(0) = \rho_0,$$

where

$$A\rho = \cancel{\nu\Delta\rho} + \nabla\cdot(\rho\nabla G), \quad A^*\phi = \cancel{\nu\Delta\varphi} - \nabla G\cdot\nabla\varphi$$

$$N\rho = \nabla\cdot(\rho\nabla\alpha), \quad N^*\varphi = -\nabla\varphi\cdot\nabla\alpha$$

Note: A and A^* generate Perron-Frobenius and Koopman operator!

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Finite and infinite dimensional approximations

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Overview

The Volterra series

The (nonlinear) time domain mapping $u \mapsto x$

Back to a (simple) bilinear control system

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + bu(t), \quad x(0) = 0$$

whose solution is given by a Volterra series of the form

$$x(t) = \sum_{i=1}^{\infty} \int_0^t \int_0^{\sigma_1} \cdots \int_0^{\sigma_{i-1}} g_i(t, \sigma_1, \ldots, \sigma_{i-1}) u(\sigma_i) \cdots u(\sigma_1) \, \mathrm{d}\sigma_i \cdots \mathrm{d}\sigma_1$$

where $g_i(t, \sigma_1, \ldots, \sigma_{i-1}) = \underbrace{e^{A(t-\sigma_1)} N \cdots e^{A(\sigma_{i-2}-\sigma_{i-1})} N}_{i-1 \text{ times}} e^{A(\sigma_{i-1}-\sigma_i)} b$

Regular kernels: change of variables lead to $e^{At_i} N \cdots e^{At_2} N e^{At_1} b$

Proof idea: successive approximations (via Picard-Lindelöf) related to

$$\begin{aligned} \dot{x}_1(t) &= Ax_1(t) + bu(t), & x_1(0) &= 0, \\ \dot{x}_i(t) &= Ax_i(t) + Nx_{i-1}(t)u(t) + bu(t), & x_i(0) &= 0 \end{aligned}$$

Generalized transfer functions

The (nonlinear) frequency domain mapping $u \mapsto y$

$\Rightarrow$ input-output map depends on $h(t_1,\ldots,t_n) = c^\top e^{At_i} N \cdots e^{At_2} N e^{At_1} b$

For f dep. on $(t_1,\ldots,t_n)$ consider multivariate Laplace transformation

$$\widetilde{f}(s_1,\ldots,s_n) = \mathcal{L}[f](s_1,\ldots,s_n) = \int_0^\infty \cdots \int_0^\infty e^{-s_1 t_1} \cdots e^{-s_n t_n} f(t_1,\ldots,t_n)\,\mathrm{d}t_1 \cdots \mathrm{d}t_n$$

We obtain generalized transfer functions of the form

$$G_1(s_1) = c^\top (s_1 I - A)^{-1} b$$
$$G_k(s_1,\ldots,s_k) = c^\top (s_k I - A)^{-1} N \cdots (s_2 I - A)^{-1} N (s_1 I - A)^{-1} b$$

Pros: may be used as abstract input-output mappings (for MOR)

Cons: lacks physical meaning/interpretation/measurement

Stability notions

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + bu(t)$$

If

- A is asymptotically stable, i.e., $\sigma(A) \subset \mathbb{C}_-$
- u is uniformly bounded on $[0, \infty)$, i.e., $|u(t)| \leq M$ for all $t > 0$
- $\|N\|$ is sufficiently small

then

- the Volterra series converges on $[0, \infty)$

Reachability, observability and algebraic Gramians

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + bu(t), \quad y(t) = c^\top x(t)$$

Consider

$$P_1(t_1) = \int_0^\infty e^{At_1} b, \quad P_i(t_1, \ldots, t_i) = e^{At_i} N P_{i-1}, \quad i = 2, 3, \ldots$$

$$Q_1(t_1) = \int_0^\infty e^{A^\top t_1} c, \quad Q_i(t_1, \ldots, t_i) = e^{A^\top t_i} N^\top Q_{i-1}, \quad i = 2, 3, \ldots$$

If

$$P = \sum_{i=1}^{\infty} \int_0^\infty \cdots \int_0^\infty P_i P_i^\top \mathrm{d}t_1 \cdots \mathrm{d}t_i, \quad Q = \sum_{i=1}^{\infty} \int_0^\infty \cdots \int_0^\infty Q_i Q_i^\top \mathrm{d}t_1 \cdots \mathrm{d}t_i$$

exist, then

- $\sigma(I \otimes A + A \otimes I + N \otimes N) \subset \mathbb{C}_-$
- $AP + PA^\top + NPN^\top + bb^\top = 0, \quad P > 0 \Leftrightarrow$ system reachable from 0
- $A^\top Q + QA + N^\top QN + cc^\top = 0, \quad Q > 0 \Leftrightarrow$ system is observable

A generalized $\mathcal{H}_2$-norm

Recall: for linear systems, the $\mathcal{H}_2$-norm is defined as

$$\|(A,b,c)\|^2_{\mathcal{H}_2(\mathbb{C}_+)} := \sup_{\sigma>0} \int_{-\infty}^{\infty} \|G_1(\sigma+\imath\omega)\|_{\mathrm{F}}^2 \,\mathrm{d}\omega = c^\top Pc$$

with $AP + PA^\top + bb^\top = 0 \Rightarrow \;\; \|G\|^2_{\mathcal{H}_2(\mathbb{C}_+)} = c^\top \left(\int_0^\infty (e^{At}b)(e^{At}b)^\top \,\mathrm{d}t\right) c$

Natural idea: use regular Volterra kernels and define

$$\|(A,N,b,c)\|_{\mathcal{H}_2} := \sum_{k=1}^{\infty} \int_0^\infty \cdots \int_0^\infty g_k^{(\ell_1,\ldots,\ell_k)} (g_k^{(\ell_1,\ldots,\ell_k)})^\top \mathrm{d}t_1 \cdots \mathrm{d}t_k$$

where $g_k^{(\ell_1,\ldots,\ell_k)} = c^\top e^{At_k} N \cdots e^{At_2} N e^{At_1} b$.

Note: $\|(A,N,b,c)\|^2_{\mathcal{H}_2} = c^\top Pc$ with $AP + PA^\top + NPN^\top + bb^\top = 0$

A generalized $\mathcal{H}_2$-norm cont'd

If

$$\sigma(I \otimes A + A \otimes I + N \otimes N) \subset \mathbb{C}_-$$

then for

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + bu(t), \quad x(0) = 0, \; y(t) \quad = c^\top x(t)$$

we have

$$\sup_{t \geq 0} |y(t)| \leq \|(A, N, b, c)\|_{\mathcal{H}_2} \exp\left(0.5 \|u^0\|_{L^2}^2\right) \|u\|_{L^2}$$

where $u^0 \equiv 0$ if $N \equiv 0$.

Note: $\mathcal{H}_2$-norm relates L^2 and L^∞ (as in the linear case)

A link to linear stochastic control systems

Consider the linear stochastic systems

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + NX_t\,\mathrm{d}W_t, \quad X_{t=0} = X_0.$$

Then the following are equivalent:

- $\sigma(I \otimes A + A \otimes I + N \otimes N) \subset \mathbb{C}_-$
- The system is exponentially mean square stable, i.e.,

$$\mathbb{E}\|X_t(x_0)\|_2^2 \leq M\|x_0\|_2^2 e^{-ct},$$

for some $M \geq 1$ and $c > 0$.

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Bilinear control theory

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Overview

Model reduction by projection

Given

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + bu(t), \quad x(0) = x_0$$

we seek an approximation $\widetilde{x}(t) \in \mathcal{V} \subseteq \mathbb{R}^n$ with $\dim(\mathcal{V}) = r$.

Consequence

$$\dot{\widetilde{x}}(t) \approx A\widetilde{x}(t) + N\widetilde{x}(t)u(t) + bu(t)$$

Petrov-Galerkin condition

$$\dot{\widetilde{x}}(t) - A\widetilde{x}(t) - N\widetilde{x}(t)u(t) - bu(t) = \mathrm{res}(t) \perp \mathcal{W}$$

where $\mathcal{W} \subseteq \mathbb{R}^n, \dim(\mathcal{W}) = r$ is another (test) subspace.

Model reduction by projection cont'd

Given

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + bu(t), \quad x(0) = x_0$$

consider bases $\{v_1, \ldots, v_r\}$ and $\{w_1, \ldots, w_r\}$ of $\mathcal{V}, \mathcal{W}$.

Approximation $\widetilde{x}(t)$ characterized by coordinate vector $x_r(t)$

$$x(t) \approx \widetilde{x}(t) = Vx_r(t), \quad V = [v_1, \ldots, v_r] \in \mathbb{R}^{n \times r}, \quad x_r(t) \in \mathbb{R}^r$$

Petrov-Galerkin condition in vector form reads

$$\langle \dot{\widetilde{x}}(t) - A\widetilde{x}(t) - N\widetilde{x}(t)u(t) - bu(t), w_i \rangle = 0, \quad i = 1, \ldots, r$$

and in matrix form

$$W^\top (\dot{\widetilde{x}}(t) - A\widetilde{x}(t) - Nx(t)u(t) - bu(t)) = 0.$$

Model reduction by projection cont'd

Given biorthogonal $V, W \in \mathbb{R}^{n\times r}$, i.e., $W^\top V = I$, replace

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Nx(t)u(t) + bu(t),\\ y(t) &= c^\top x(t)\end{aligned}$$

by a reduced-order model

$$\begin{aligned}\dot{x}_r(t) &= \underbrace{(W^\top AV)}_{=:A_r} x_r(t) + \underbrace{(W^\top NV)}_{=:N_r} x_r(t)u(t) + \underbrace{(W^\top b)}_{=:b_r} u(t),\\ y_r(t) &= \underbrace{(c^\top V)}_{=:c_r^\top} x_r(t)\end{aligned}$$

Goals: $r \ll n$ and $y_r \approx y$, but how?

Overview

Interpolatory model reduction in a nutshell

Krylov spaces and moment matching

Consider w.l.o.g. the SISO case

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = 0$$
$$y(t) = c^\top x(t)$$

and observe that for s such that $\|\frac{1}{s}A\| < 1$

$$\begin{aligned} G(s) &= c^\top (sI - A)^{-1} b = c^\top (s(I - \tfrac{1}{s}A))^{-1} b \\ &= \tfrac{1}{s} c^\top (I - \tfrac{1}{s}A)^{-1} b = \tfrac{1}{s} c^\top \sum_{i=0}^{\infty} (s^{-1}A)^i b \\ &= s^{-1} c^\top b + s^{-2} c^\top A b + s^{-3} c^\top A^2 b + \cdots \end{aligned}$$

where we used the Neumann series.

The terms $c^\top A^k b$ are called Markov parameters.

Interpolatory model reduction in a nutshell

Krylov spaces and moment matching cont'd

(How) can we construct $G_r(s) = c_r^\top (sI - A_r)^{-1} b_r$ such that

$$c^\top A^k b = c_r^\top A_r^k b_r, \quad k = 0, \ldots, q-1. \tag{1}$$

This process is called moment matching.

Consider Krylov subspace $\mathcal{V} = \mathcal{K}_q(A, b) = \mathrm{span}\{b, Ab, \ldots, A^{q-1}b\}$.

If $V = [v_1, \ldots, v_q]$ basis of $\mathcal{V}$ and $W \in \mathbb{R}^{n \times q}$ s.t. $W^\top V = I_q$ then

$$A_r = W^\top AV, \quad b_r = W^\top b, \quad c_r = V^\top c$$

defines G_r satisfying (1).

Proof uses projection $P = VW^\top$ onto $\mathcal{V}$.

If additionally $\mathcal{W} = \mathcal{K}_q(A^\top, c)$, then (1) holds up to $k = 2q - 1$.

Interpolatory model reduction in a nutshell

Rational interpolation by projection

Note that

$$G(s) = c^\top (sI - A)^{-1} b, \quad G'(s) = -c^\top (sI - A)^{-2} b, \ldots$$

New goal: for $s = \sigma$ not an eigenvalue of A

$$G^{(k)}(\sigma) = G_r^{(k)}(\sigma), \quad k = 0, 1, \ldots, q-1.$$

This constitutes a rational interpolation problem.

Can be achieved by rational Krylov subspaces of the form

$$\mathcal{V} = \mathcal{K}_q(A, b; \sigma) = \mathrm{span}\{(\sigma I - A)^{-1} b, \ldots, (\sigma I - A)^{-q} b\},$$
$$\mathcal{W} = \mathcal{K}_q(A^\top, c; \sigma) = \mathrm{span}\{(\sigma I - A^\top)^{-1} c, \ldots, (\sigma I - A^\top)^{-q} c\}.$$

Note: $x = (\sigma I - A)^{-1} b \Leftrightarrow Ax - x\sigma + b \cdot 1 = 0 \Rightarrow \boxed{AX - X\Lambda + b\mathbb{1}^\top = 0}$

Back to the bilinear case

Multimoments

As in the linear case, we may expand G_k based on

$$\begin{aligned}
G_k(s_1,\dots,s_k) &= c^\top \left(\prod_{j=2}^{k} (s_j I - A)^{-1} N \right) (s_1 I - A)^{-1} b \\
&= (-1)^k c^\top \left(\prod_{j=2}^{k} (A - \sigma_j I - (s_j - \sigma_j) I)^{-1} N \right) \\
&\quad \cdot (A - \sigma_1 I - (s_1 - \sigma_1) I)^{-1} b \\
&= (-1)^k c^\top \left(\prod_{j=2}^{k} (I - (s_j - \sigma_j)(A - \sigma_j I)^{-1})^{-1} (A - \sigma_j I)^{-1} N \right) \\
&\quad \cdot (I - (s_1 - \sigma_1)(A - \sigma_1 I)^{-1})^{-1} (A - \sigma_1 I)^{-1} b
\end{aligned}$$

Multimoments cont'd

$$G_k(s_1,\ldots,s_k) = (-1)^k c^\top \left(\prod_{j=2}^{k} (I-(s_j-\sigma_j)(A-\sigma_j I)^{-1})^{-1}(A-\sigma_j I)^{-1} N \right) \cdot (I-(s_1-\sigma_1)(A-\sigma_1 I)^{-1})^{-1}(A-\sigma_1 I)^{-1} b$$

Using Neumann series for s_j around σ_j we can substitute

$$\left(I-(s_j-\sigma_j)\,(A-\sigma_j I)^{-1}\right)^{-1} = \sum_{i=0}^{\infty}(s_j-\sigma_j)^i(A-\sigma_j I)^{-i}$$

and obtain

$$G_k(s_1,\ldots,s_k) = (-1)^k c^\top \left(\prod_{j=2}^{k} \left(\sum_{i=0}^{\infty}(s_j-\sigma_j)^i(A-\sigma_j I)^{-(i+1)} \right) N \right) \cdot \left(\sum_{i=0}^{\infty}(s_j-\sigma_j)^i(A-\sigma_j I)^{-(i+1)} \right) b\,.$$

Multimoments cont'd

A multivariable power series notation leads to

$$G_k(s_1, \ldots, s_k) = \sum_{l_k=1}^{\infty} \ldots \sum_{l_1=1}^{\infty} m(l_1, \ldots, l_k)(s_1 - \sigma_1)^{l_1 - 1} \ldots (s_k - \sigma_k)^{l_k - 1},$$

where

$$m(l_1, \ldots, l_k) = (-1)^k c^T (A - \sigma_k I)^{-l_k} N \ldots (A - \sigma_2 I)^{-l_2} N (A - \sigma_1 I)^{-l_1} b$$

are multimoments associated with the k-th transfer function.

Analogously, expansions around $s_i = \infty$ lead to

$$m(l_1, \ldots, l_k) = c^\top A^{l_k - 1} N \ldots A^{l_2 - 1} N A^{l_1 - 1} b$$

Question: how to construct a ROM with $m(l_1, \ldots, l_k) = \widehat{m}(l_1, \ldots, l_k)$?

Multimoment matching

Construct a ROM by Petrov-Galerkin projection $P = VW^\top$

$$\widehat{A} = W^\top AV, \quad \widehat{N} = W^\top NV, \quad \widehat{b} = W^\top b, \quad \widehat{c} = W^\top c$$

such that

$$\begin{aligned} \operatorname{span}\{V^{(1)}\} &= \mathcal{K}_q((A-\sigma_1 I)^{-1}, (A-\sigma_1 I)^{-1} b), \\ \operatorname{span}\{V^{(k)}\} &= \mathcal{K}_q((A-\sigma_k I)^{-1}, (A-\sigma_k I)^{-1} NV^{(k-1)}), \quad k = 2,\ldots,r \\ \operatorname{span}\{V\} &= \operatorname{span}\left\{\bigcup_{k=1}^{r} \operatorname{span}\{V^{(k)}\}\right\}. \end{aligned}$$

Then $m(l_1,\ldots,l_k) = \widehat{m}(l_1,\ldots,l_k)$, for $k = 1,\ldots,r, l_1,\ldots,l_k = 1,\ldots,q$.

This process is called multimoment matching.

Pros: easy to implement

Cons: local approach, "good" choice of σ_i nontrivial

$\mathcal{H}_2$-optimal model reduction

Bilinear $\mathcal{H}_2$-optimal MOR: $G_r = \underset{\substack{\widetilde{G}\in\mathcal{H}_2\\ \dim(\widetilde{G})=r}}{\arg\min} \|G - \widetilde{G}\|_{\mathcal{H}_2}$

Define

$$A_{\mathrm{err}} = \begin{pmatrix} A & 0 \\ 0 & \widetilde{A} \end{pmatrix}, \quad N_{\mathrm{err}} = \begin{pmatrix} N & 0 \\ 0 & \widetilde{N} \end{pmatrix}, \quad b_{\mathrm{err}} = \begin{pmatrix} b \\ \widetilde{b} \end{pmatrix}, \quad c_{\mathrm{err}} = \begin{pmatrix} c \\ -\widetilde{c} \end{pmatrix}$$

$$P_{\mathrm{err}} = \begin{pmatrix} P & X \\ X^\top & \widetilde{P} \end{pmatrix}, \quad Q_{\mathrm{err}} = \begin{pmatrix} Q & Y \\ Y & \widetilde{Q} \end{pmatrix}$$

$$0 = A_{\mathrm{err}}P_{\mathrm{err}} + P_{\mathrm{err}}A_{\mathrm{err}}^\top + N_{\mathrm{err}}P_{\mathrm{err}}N_{\mathrm{err}}^\top + b_{\mathrm{err}}b_{\mathrm{err}}^\top,$$

$$0 = A_{\mathrm{err}}^\top Q_{\mathrm{err}} + Q_{\mathrm{err}}A_{\mathrm{err}} + N_{\mathrm{err}}^\top Q_{\mathrm{err}}N_{\mathrm{err}} + c_{\mathrm{err}}c_{\mathrm{err}}^\top$$

Optimality conditions:

$$0 = Y^\top b + \widetilde{Q}\widetilde{b}, \quad 0 = \widetilde{c}^\top \widetilde{P} - c^\top X,$$

$$0 = X^\top Y + \widetilde{P}\widetilde{Q}, \quad 0 = X^\top N Y + \widetilde{Q}\widetilde{N}\widetilde{P}.$$

Volterra series interpolation

(How) does this relate to multimoments/Volterra series?

If $(\widehat{A}, \widehat{N}, \widehat{b}, \widehat{c})$ is a locally $\mathcal{H}_2$-optimal ROM, then

$$\sum_{k=1}^{\infty} \sum_{\ell_1=1}^{r} \cdots \sum_{\ell_k=1}^{r} \widehat{\Phi}_{\ell_1,\ldots,\ell_k} G_k(-\widehat{\lambda}_1, \ldots, -\widehat{\lambda}_k)$$
$$= \sum_{k=1}^{\infty} \sum_{\ell_1=1}^{r} \cdots \sum_{\ell_k=1}^{r} \widehat{\Phi}_{\ell_1,\ldots,\ell_k} \widehat{G}_k(-\widehat{\lambda}_1, \ldots, -\widehat{\lambda}_k)$$

where

- $\widehat{\lambda}_i$ are the eigenvalues of $\widehat{A}$
- $\widehat{\Phi}_{\ell_1,\ldots,\ell_k} := \lim\limits_{s_k \to \widehat{\lambda}_{\ell_k}} (s_k - \widehat{\lambda}_k) \cdots \lim\limits_{s_1 \to \widehat{\lambda}_{\ell_1}} (s_1 - \widehat{\lambda}_1) \; \widehat{G}_k(s_1, \ldots, s_k)$

Note 1: optimality char. by multipoint Volterra series interpolation

Note 2: if $N = 0$, we have $G_1(-\widehat{\lambda}_i) = \widehat{G}_1(-\widehat{\lambda}_i) \Rightarrow$ IRKA

An iterative algorithm

Algorithm Generalized Sylvester iteration (B-IRKA)

Input: (A, N_k, B, C), $(\widehat{A}, \widehat{N}_k, \widehat{B}, \widehat{C})$
Output: $(\widehat{A}, \widehat{N}_k, \widehat{B}, \widehat{C})$ satisyfing 1st order $\mathcal{H}_2$ opt. cond.

repeat

Solve $AX + X\widehat{A}^\top + \sum_{k=1}^{m} N_k X \widehat{N}_k^\top + B\widehat{B}^\top = 0.$

Solve $A^\top Y + Y\widehat{A} + \sum_{k=1}^{m} N_k^\top Y \widehat{N}_k - C^\top \widehat{C} = 0.$

$V = \mathrm{orth}(X),\ W = \mathrm{orth}(Y),\ Z = W(V^\top W)^{-1}$

$\widehat{A} = Z^\top AV,\ \ \widehat{N}_k = Z^\top N_k V,\ \ \widehat{B} = Z^\top B,\ \ \widehat{C} = CV$

until convergence

Overview

1. Bilinear control systems
   - Finite and infinite approximations
   - Basics from bilinear system theory
2. Model reduction of bilinear systems
   - Interpolatory model reduction
   - Balancing-based model reduction
3. Optimal control of bilinear systems
   - Open loop control
   - Closed loop control

Balancing algebraic Gramians

Basic idea

- (A, N, b, c), is called balanced, if solutions P, Q of

$$AP + PA^\top + NPN^\top + bb^\top = 0, \quad A^\top Q + QA + N^\top QN + cc^\top = 0$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of G.

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- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of G.
- Compute balanced realization via state-space transformation

$$\begin{aligned} \mathcal{T}: (A, N, b, c) &\mapsto (TAT^{-1}, TNT^{-1}, Tb, T^{-\top}c) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right). \end{aligned}$$

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- Truncation $\rightsquigarrow (\widehat{A}, \widehat{N}, \widehat{b}, \widehat{c}) = (A_{11}, N_{11}, b_1, c_1)$.

Balanced truncation for bilinear systems cont'd

Do we have an energy interpretation similar to the linear case?

We need the dual, antistable bilinear system

$$\dot{\xi} = -A^\top \xi - N^\top \xi u + cu.$$

For $x_0 \in \mathbb{R}^n$ let $u = u_{x_0}$ be L^2 minimal, s.t. $\lim_{t\to\infty} \xi(t, x_0, u) = 0$.

Define the energy functionals

$$E_\mathrm{c}(x_0) = \min_{\substack{u \in L^2((-\infty,0]) \\ x(-\infty,x_0,u)=0}} \|u\|^2_{L^2((-\infty,0])}, \quad E_\mathrm{o}(x_0) = \|y(\cdot, x_0, u_{x_0})\|^2_{L^2([0,\infty))}.$$

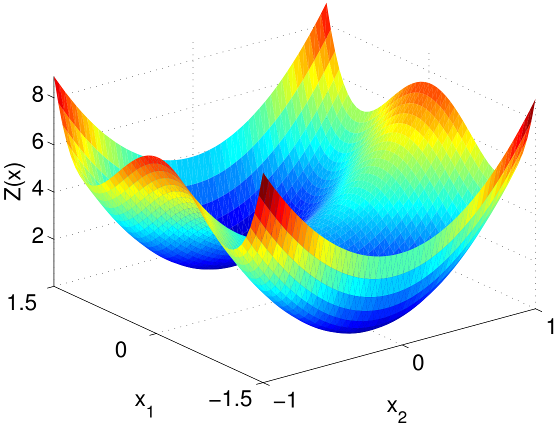
Energy bounds

If G is a balanced bilinear system with $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, then there exists $\varepsilon > 0$ s.t. for all canonical unti vectors e_j it holds

$$E_\mathrm{c}(\varepsilon e_j) > \varepsilon^2 \sigma_j^{-1}, \quad E_\mathrm{o}(\varepsilon e_j) < \varepsilon^2 \sigma_j.$$

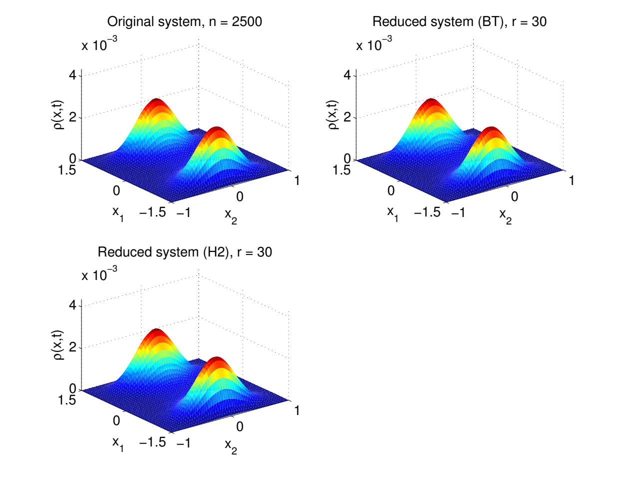
Fokker-Planck equation - Numerical results

Potential $G(x) = \frac{5}{2}(x_1^2 - 1)^2 + 5x_2^2$



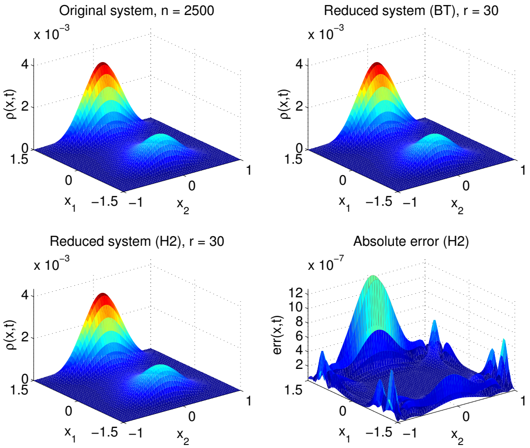
Fokker-Planck equation - Numerical results

Evolution of $\rho(x,t)$ on a 50×50-grid, $u(t) = 5\sin(2\pi t)$, $t = 0s$



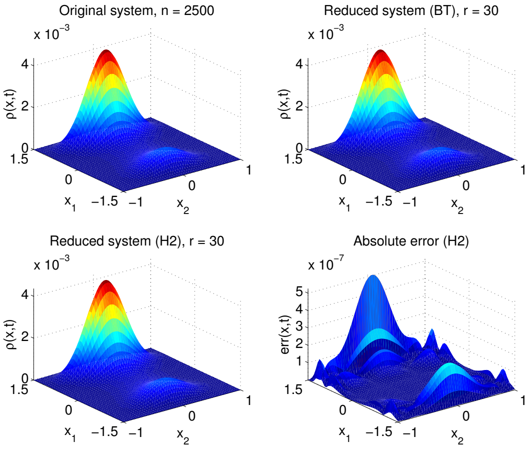
Fokker-Planck equation - Numerical results

Evolution of $\rho(x,t)$ on a 50×50-grid, $u(t) = 5\sin(2\pi t)$, $t = 0.25s$



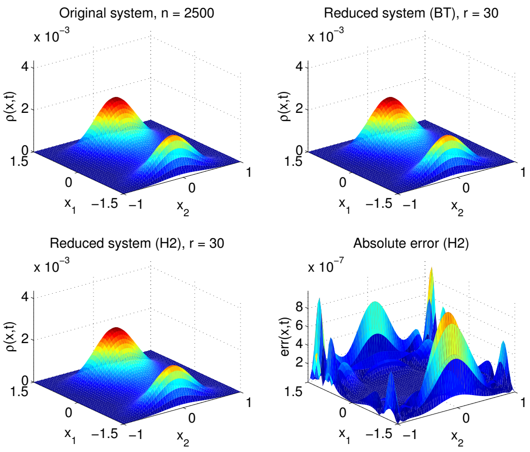
Fokker-Planck equation - Numerical results

Evolution of $\rho(x,t)$ on a 50×50-grid, $u(t) = 5\sin(2\pi t)$, $t = 0.5s$



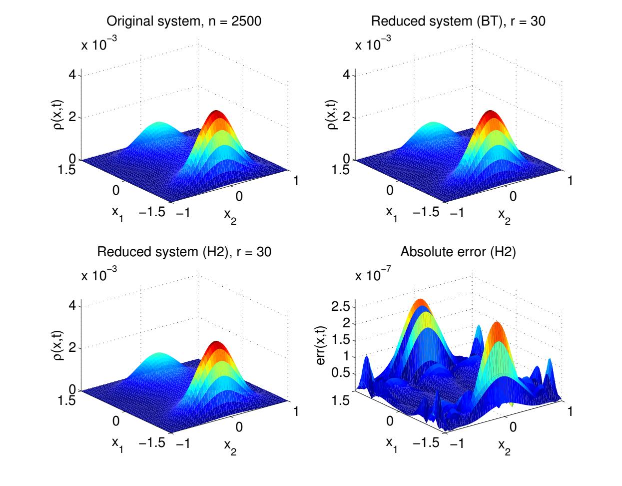
Fokker-Planck equation - Numerical results

Evolution of $\rho(x, t)$ on a 50×50-grid, $u(t) = 5 \sin(2\pi t)$, $t = 0.75s$



Fokker-Planck equation - Numerical results

Evolution of $\rho(x,t)$ on a 50×50-grid, $u(t) = 5\sin(2\pi t)$, $t = 1s$



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Overview

A general optimal control problem

Given a general nonlinear control system

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) &= g(t, x(t), u(t)) \end{aligned} \tag{2}$$

and a cost functional $\mathcal{J}\colon \mathcal{U}_{\mathrm{ad}} \to \mathbb{R}$

$$\mathcal{J}(u) := h_f(x(t_f, u)) + \int_{t_0}^{t_f} h(t, x(t,u), y(t,u), u(t))\,\mathrm{d}t$$

consider the optimal control problem

$$\inf_{u \in \mathcal{U}_{\mathrm{ad}}} \mathcal{J}(u) \quad \text{s.t. } (2)$$

Here: $\mathcal{U}_{\mathrm{ad}}$ set of admissible controls, e.g., $\mathcal{U}_{\mathrm{ad}} = L^2((t_0, t_f); \mathbb{R}^m)$

Lagrange function and Hamiltonian

Recall from constrained optimization

$$\min_{x\in\mathbb{R}^n} j(x) \quad \text{s.t.} \quad f(x) = 0$$

the Lagrange function

$$\mathcal{L}(x,\lambda) = j(x) + \lambda^\top f(x)$$

with Lagrange multiplier $\lambda \in \mathbb{R}^m$. Optimality via $\mathcal{L}_\lambda = 0, \mathcal{L}_x = 0$.

Dynamical systems: introduce Hamiltonian $\mathcal{H}$

$$\mathcal{H}(x(t),u(t),p(t)) = h(x(t),u(t)) + p(t)^\top f(x(t),u(t))$$

with co-state $p: [t_0, t_f] \mapsto \mathbb{R}^n$

Note: co-state/adjoint takes role of Lagrange multiplier

Pontryagin's maximum principle

Assume $(\tilde{u}, \tilde{x})$ is an optimal pair, then

$$\dot{\tilde{x}}(t) = \mathcal{H}_p(\tilde{x}(t), \tilde{u}(t), p(t))$$

$$\mathcal{H}(\tilde{x}(t), \tilde{u}(t), p(t)) = \inf_u \mathcal{H}(\tilde{x}(t), u(t), p(t)) \quad \forall t \in [t_0, t_f]$$

$$\dot{p}(t) = -\mathcal{H}_x(\tilde{x}(t), \tilde{u}(t), p(t))$$

$$p(t_f) = \nabla h_f(x(t_f))$$

First order opt. conditions called Pontryagin's maximum principle.

Linear-quadratic optimal control

For the special linear-quadratic case

$$\min_{u \in \mathcal{U}_{\mathrm{ad}}} \mathcal{J}(u) := \frac{1}{2}\left(x(t_f)^\top M x(t_f) + \int_{t_0}^{t_f} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^\top \begin{pmatrix} Q(t) & S(t) \\ S(t)^\top & R(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t \right)$$

s.t. $\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0$

Pontryagin's maximum principle yields

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}(t) \\ \dot{p}(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} A & 0 & B \\ Q & A^\top & S \\ S^\top & B^\top & R \end{pmatrix} \begin{pmatrix} x(t) \\ p(t) \\ u(t) \end{pmatrix}$$

with boundary conditions

$$x(t_0) = x_0, \quad p(t_f) = Mx(t_f).$$

Optimal feedback control

Assumption: for simplicity $Q, S = 0$

Ansatz: $p(t) = P(t)x(t)$ with $P(t) \in \mathbb{R}^{n\times n}$ and $P(t_f) = M$

$$\dot{p}(t) = \dot{P}(t)x(t) + P(t)\dot{x}(t)$$
$$p(t_f) = P(t_f)x(t_f)$$

After some algebraic manipulations

$$\dot{x} = (A - BR^{-1}B^\top P)x$$
$$\dot{P}x = -(A^\top P + PA - PBR^{-1}B^\top P + Q)x$$

We obtain the differential Riccati equation

$$\dot{P} = -(A^\top P + PA - PBR^{-1}B^\top P + Q), \quad P(t_f) = M$$

and the optimal (linear) feedback law

$$u(t) = -R(t)^{-1}B(t)^\top P(t)x(t)$$

Optimal feedback control cont'd

If we consider the time-invariant infinite-horizon problem

$$\min_{u\in\mathcal{U}_{\mathrm{ad}}} \mathcal{J}(u) := \frac{1}{2}\left(\int_0^\infty x(t)^\top Qx(t) + u(t)^\top Ru(t)\,\mathrm{d}t\right)$$

$$\text{s.t. } \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

we obtain the algebraic Riccati equation

$$0 = A^\top P + PA - PBR^{-1}B^\top P + Q$$

and the static optimal (linear) feedback law

$$u(t) = -R^{-1}B^\top Px(t)$$

Bilinear infinite-horizon optimal control

Let us go back to a bilinear control system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Nx(t)u(t) + bu(t), \quad x(0) = x_0, \\ y(t) &= c^\top x(t), \end{aligned}$$

- $A, N \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$,
- control $u\colon [0, \infty) \to \mathbb{R}$ and
- output $y\colon [0, \infty) \to \mathbb{R}$ of the system,
- (A, b) stabilizable.

For this system, we introduce the minimal value function

$$\mathcal{V}(x_0) = \inf_{u \in L^2(0,\infty)} \frac{1}{2} \int_0^\infty \|y(t)\|^2 \mathrm{d}t + \frac{\beta}{2} \int_0^\infty u(t)^2 \mathrm{d}t.$$

The dynamic programming principle

By the dynamic programming principle, for any x_0 and $\tau > 0$:

$$\mathcal{V}(x_0) = \inf_{u \in L^2(0,\tau)} \int_0^\tau \ell(y(u, x_0; t), u(t))\,\mathrm{d}t + \mathcal{V}(x(u, x_0; \tau)),$$

where $\ell(y, u) = \frac{1}{2}\|y\|^2 + \frac{\beta}{2}u^2$.

Under smoothness assumptions on $\mathcal{V}$, we obtain

$$\min_{u \in \mathbb{R}} \left[(Ax + (Nx + b)u)^\top \nabla\mathcal{V}(x) + \frac{1}{2}\|c^\top x\|^2 + \frac{\beta}{2}u^2 \right] = 0, \quad \mathcal{V}(0) = 0.$$

The Hamilton-Jacobi-Bellman equation

Consider again

$$\min_{u\in\mathbb{R}}\left[(Ax+(Nx+b)u)^\top\nabla\mathcal{V}(x)+\frac{1}{2}\|y\|^2+\frac{\beta}{2}u^2\right]=0,\quad \mathcal{V}(0)=0.$$

Minimization yields Hamilton-Jacobi-Bellman (HJB) equation

$$x^\top A^\top\nabla\mathcal{V}(x)+\frac{1}{2}\|c^\top x\|^2-\frac{1}{2\beta}((Nx+b)^\top\nabla\mathcal{V}(x))^2=0,\quad \mathcal{V}(0)=0.$$

Optimal feedback law via solving HJB equation

$$u_{\mathrm{opt}}(x)=-\frac{1}{\beta}(Nx+b)^\top\nabla\mathcal{V}(x).$$

Problem: The HJB equation is a nonlinear *n*-dimensional PDE...

Taylor expansions – basic idea

Assume that $\mathcal{V}$ can be expanded around 0 as follows

$$\mathcal{V}(x) = \underbrace{\mathcal{V}(0)}_{\in\mathbb{R}} + \underbrace{D\mathcal{V}(0)}_{\in\mathbb{R}^n}(x) + \tfrac{1}{2!}\underbrace{D^2\mathcal{V}(0)}_{\in\mathbb{R}^{n\times n}}(x,x) + \tfrac{1}{3!}\underbrace{D^3\mathcal{V}(0)}_{\in\mathbb{R}^{n\times n\times n}}(x,x,x) + \ldots$$

Approximate feedback law can be determined via

$$u_d = -\frac{1}{\beta}\sum_{k=2}^{d}\frac{1}{(k-1)!}D^k\mathcal{V}(0)(Nx+b,x,\ldots,x)$$

Question: what can be said about the quality of such u_d?

Smoothness and error estimates

Smoothness of the value function

There ex. $\varepsilon > 0$ s.t. $\mathcal{V}$ is infinitely differentiable on $\mathcal{B}_\varepsilon := \{x \in \mathbb{R}^n \mid \|x\| < \varepsilon\}$.

Smoothness and error estimates

Smoothness of the value function

There ex. $\varepsilon > 0$ s.t. $\mathcal{V}$ is infinitely differentiable on $\mathcal{B}_\varepsilon := \{x \in \mathbb{R}^n \mid \|x\| < \varepsilon\}$.

Estimates for polynomial feedback laws

There exists $\widehat{\varepsilon}$ s.t. $\forall x_0 \in \mathcal{B}_{\widehat{\varepsilon}}$ it holds that:

$$\max\left(\|u_{\text{opt}} - u_d\|_{L^2(0,\infty)}, \|x_{\text{opt}} - x_d\|_{H^1(0,\infty;\mathbb{R}^n)}\right) \le M\|x_0\|^d,$$

where

$$\begin{aligned} \dot{x}_d &= Ax_d + (Nx_d + b)u_d, \quad x_d(0) = x_0, \\ u_d &= -\frac{1}{\beta}\sum_{j=2}^{d} \frac{1}{(j-1)!} D^j\mathcal{V}(0)(Nx_d + b, x_d, \ldots, x_d). \end{aligned}$$

Smoothness of $\mathcal{V}$: proof idea

Sensitivity analysis, inverse function theorem

Define the space $H := \mathbb{R}^n \times L^2(0,\infty;\mathbb{R}^n) \times L^2(0,\infty;\mathbb{R}^n) \times L^2(0,\infty)$.

Consider $\Phi\colon H^1(0,\infty;\mathbb{R}^n) \times L^2(0,\infty) \times H^1(0,\infty;\mathbb{R}^n) \to X$ defined by

$$\Phi(x,u,p) = \begin{pmatrix} x(0) \\ \dot{x} - Ax - Nxu - bu \\ -\dot{p} - A^\top p - uN^\top p - cc^\top x \\ \beta u + p^\top(Nx+b) \end{pmatrix}.$$

Key ingredient: $\Phi(x_{\mathrm{opt}}, u_{\mathrm{opt}}, p) = (x_0, 0, 0, 0)$.

Proposition

There exist $\delta' > 0$ and three C^∞-mappings

$$x_0 \in B_{\delta'} \mapsto \big(\mathcal{X}(x_0), \mathcal{U}(x_0), \mathcal{P}(x_0)\big) \in H^1(0,\infty;\mathbb{R}^n) \times L^2(0,\infty) \times H^1(0,\infty;\mathbb{R}^n)$$

s.t. $(\mathcal{X}(x_0), \mathcal{U}(x_0))$ is the unique optimal state and $\mathcal{P}(x_0)$ is the unique associated costate.

Estimates for polynomial feedback laws: proof idea

Consider the nonlinear closed-loop system (CL)

$$\dot{x}_d = Ax_d + (Nx_d + b)\left(-\frac{1}{\beta}\sum_{j=2}^{d}\frac{1}{(j-1)!}D^j\mathcal{V}(0)(Nx_d + b, x_d, \dots, x_d)\right)$$

$$= \left(Ax_d - \frac{1}{\beta}bD^2\mathcal{V}(0)(b, x_d)\right) - \frac{1}{\beta}Nx_d D^2\mathcal{V}(0)(Nx_d + b, x_d)$$

$$+ (Nx_d + b)\left(-\frac{1}{\beta}\sum_{j=3}^{d}\frac{1}{(j-1)!}D^j\mathcal{V}(0)(Nx_d + b, x_d, \dots, x_d)\right)$$

The proof is based on the following results

- $D^2\mathcal{V}(0) \widehat{=} \Pi$ where Π solves algebraic Riccati equation
- local well-posed of (CL) by fixed point argument
- feedback formulation $u_{\mathrm{opt}}(x_{\mathrm{opt}}) = -\frac{1}{\beta}D\mathcal{V}(x_{\mathrm{opt}})(Nx_{\mathrm{opt}} + b)$
- Taylor remainder term for error system $\dot{e} = \dot{x}_{\mathrm{opt}} - \dot{x}_d = \dots$

A 1D Fokker-Planck equation

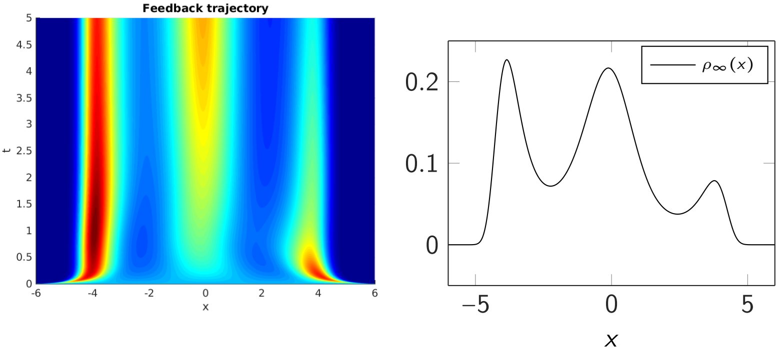

Figure: 1D Fokker-Planck equation

A 1D Fokker-Planck equation

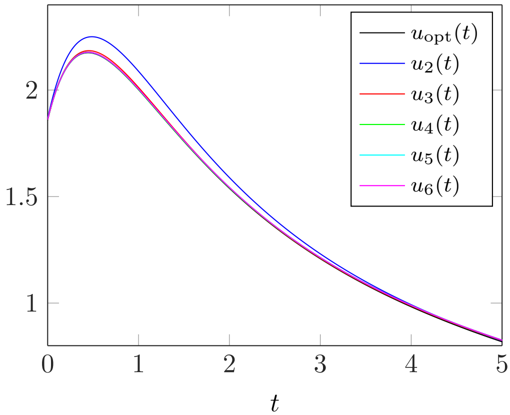

Figure: 1D Fokker-Planck equation, $\beta = 10^{-3}$.

A 1D Fokker-Planck equation

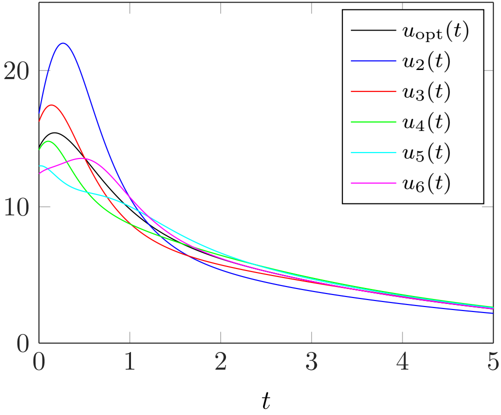

Figure: 1D Fokker-Planck equation, $\beta = 10^{-4}$.

A 1D Fokker-Planck equation

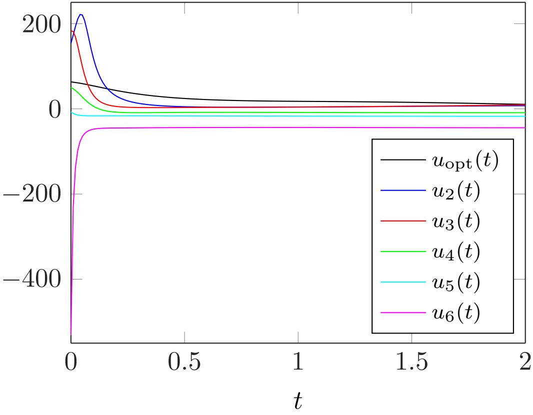

Figure: 1D Fokker-Planck equation, $\beta = 10^{-5}$.

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