$\mathcal{H}_2$ optimal model reduction for general domains

Alessandro Borghi, Tobias Breiten

Technische Universität Berlin

Nonlinear model reduction for control
Conference at Virginia Tech, 2023

DAEDALUS
RTG 2433

Technische Universität Berlin
Freie Universität Berlin

The objective

DAEDALUS
RTG 2433

- Consider a SISO FOM system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) = \mathbf{c}^*\mathbf{x}(t), \end{cases} \qquad H(s) = \mathbf{c}^*(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b},$$

where $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{c}, \mathbf{b} \in \mathbb{C}^n$.

- We want to design a reduced order system

$$\begin{cases} \dot{\hat{\mathbf{x}}}_r(t) = \hat{\mathbf{A}}_r\hat{\mathbf{x}}_r(t) + \hat{\mathbf{b}}_r u(t) \\ \widehat{y}_r(t) = \hat{\mathbf{c}}_r^*\hat{\mathbf{x}}_r(t), \end{cases} \qquad \hat{H}(s) = \hat{\mathbf{c}}_r^*(s\mathbf{I} - \hat{\mathbf{A}}_r)^{-1}\hat{\mathbf{b}}_r,$$

where $\hat{\mathbf{A}}_r \in \mathbb{C}^{r\times r}$ and $\hat{\mathbf{c}}_r, \hat{\mathbf{b}}_r \in \mathbb{C}^r$ for $r \ll n$ such that $\widehat{y}_r(t) \approx y(t)$.

The objective

DAEDALUS RTG 2433

- Consider a SISO FOM system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) = \mathbf{c}^*\mathbf{x}(t), \end{cases} \qquad H(s) = \mathbf{c}^*(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b},$$

  where $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{c}, \mathbf{b} \in \mathbb{C}^n$.

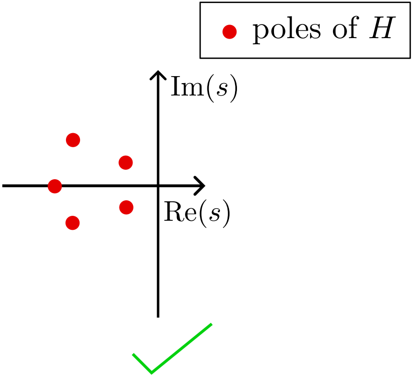
- We want to design a reduced order system

$$\begin{cases} \dot{\hat{\mathbf{x}}}_r(t) = \hat{\mathbf{A}}_r\hat{\mathbf{x}}_r(t) + \hat{\mathbf{b}}_r u(t) \\ \widehat{y}_r(t) = \hat{\mathbf{c}}_r^*\hat{\mathbf{x}}_r(t), \end{cases} \qquad \hat{H}(s) = \hat{\mathbf{c}}_r^*(s\mathbf{I} - \hat{\mathbf{A}}_r)^{-1}\hat{\mathbf{b}}_r,$$

  where $\hat{\mathbf{A}}_r \in \mathbb{C}^{r\times r}$ and $\hat{\mathbf{c}}_r, \hat{\mathbf{b}}_r \in \mathbb{C}^r$ for $r \ll n$ such that $\widehat{y}_r(t) \approx y(t)$.

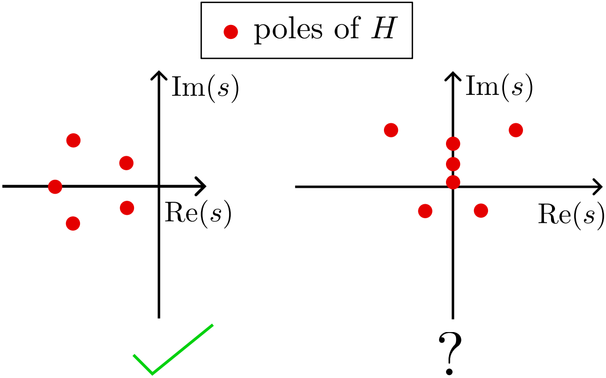
The objective

- Having $H(s) = \mathbf{c}^*(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ find $\hat{H}(s) = \hat{\mathbf{c}}_r^*(s\mathbf{I} - \hat{\mathbf{A}}_r)^{-1}\hat{\mathbf{b}}_r$ with $r \ll n$ such that $\widehat{y}_r(t) \approx y(t)$.



The objective

- Having $H(s) = \mathbf{c}^*(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ find $\hat{H}(s) = \hat{\mathbf{c}}_r^*(s\mathbf{I} - \hat{\mathbf{A}}_r)^{-1}\hat{\mathbf{b}}_r$ with $r \ll n$ such that $\widehat{y}_r(t) \approx y(t)$.



Interpolatory model order reduction

DAEDALUS RTG 2433

Theorem ([Gugercin et al., 2008][Gerstner et al., 2010])

Consider the transfer functions $H(s) = \mathbf{c}^*(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ *of the FOM and the interpolation points* $\{\sigma_j\}_{j=1}^r$ *such that* $(\sigma_j \mathbf{I} - \mathbf{A})$ *and* $(\sigma_j \mathbf{I} - \hat{\mathbf{A}}_r)$ *are both nonsingular. Let the two projection matrices* $\mathbf{V}_r$ *and* $\mathbf{W}_r$ *be chosen such that*

$$\mathrm{Ran}(\mathbf{V}_r) = \mathrm{span}\left\{(\sigma_1 \mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \dots, (\sigma_r \mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\right\},$$
$$\mathrm{Ran}(\mathbf{W}_r) = \mathrm{span}\left\{(\sigma_1^* \mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{c}, \dots, (\sigma_r^* \mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{c}\right\}.$$

Then the reduced order model $\hat{H}$ *will match* H *as follows*

$$\hat{H}(\sigma_j) = H(\sigma_j) \text{ and } \hat{H}'(\sigma_j) = H'(\sigma_j) \text{ for } j = 1, \dots, r.$$

- How do we **optimally** choose σ?

Interpolatory model order reduction

DAEDALUS RTG 2433

Theorem ([Gugercin et al., 2008][Gerstner et al., 2010])

Consider the transfer functions $H(s) = \mathbf{c}^*(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ of the FOM and the interpolation points $\{\sigma_j\}_{j=1}^r$ such that $(\sigma_j\mathbf{I} - \mathbf{A})$ and $(\sigma_j\mathbf{I} - \hat{\mathbf{A}}_r)$ are both nonsingular. Let the two projection matrices $\mathbf{V}_r$ and $\mathbf{W}_r$ be chosen such that

$$\mathrm{Ran}(\mathbf{V}_r) = \mathrm{span}\left\{(\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \ldots, (\sigma_r\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\right\},$$
$$\mathrm{Ran}(\mathbf{W}_r) = \mathrm{span}\left\{(\sigma_1^*\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{c}, \ldots, (\sigma_r^*\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{c}\right\}.$$

Then the reduced order model $\hat{H}$ will match H as follows

$$\hat{H}(\sigma_j) = H(\sigma_j) \text{ and } \hat{H}'(\sigma_j) = H'(\sigma_j) \text{ for } j = 1, \ldots, r.$$

- How do we **optimally** choose σ?

Let F, G be analytic in $\mathbb{C}_+$. Then their $\mathcal{H}_2$ inner product is

$$\langle F, G \rangle_{\mathcal{H}_2(\mathbb{C}_+)} := \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\imath\omega)^* G(\imath\omega) \mathrm{d}\omega.$$

The $\mathcal{H}_2$ space is then defined as

$$\mathcal{H}_2(\mathbb{C}_+) := \left\{ G \colon \mathbb{C}_+ \to \mathbb{C} \text{ analytic } \middle| \|G\|_{\mathcal{H}_2(\mathbb{C}_+)} < \infty \right\}.$$

where $\|G\|^2_{\mathcal{H}_2(\mathbb{C}_+)} = \langle G, G \rangle_{\mathcal{H}_2(\mathbb{C}_+)}$.

The objective is to find a reduced order model $\hat{H}$ that satisfies

$$\hat{H} = \arg \min_{\substack{\tilde{H} \text{ is as. stable} \\ \dim(\tilde{H})=r}} \|H - \tilde{H}\|_{\mathcal{H}_2(\mathbb{C}_+)}.$$

Theorem (Meier-Luenberger conditions [Meier et al., 1967])

Given a stable SISO system H, let $\hat{H}$ be a local minimizer of dimension r for the optimal $\mathcal{H}_2$ model reduction problem with poles $\{\hat{\lambda}_j\}_{j=1}^r$. Then we have that

$$\hat{H}(\underbrace{-\hat{\lambda}_j^*}_{\sigma}) = H(\underbrace{-\hat{\lambda}_j^*}_{\sigma}) \text{ and } \hat{H}'(\underbrace{-\hat{\lambda}_j^*}_{\sigma}) = H'(\underbrace{-\hat{\lambda}_j^*}_{\sigma}) \text{ for } j = 1, \ldots, r.$$

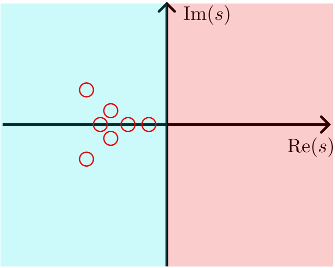
The roots of the idea [Kubalinska, 2008]

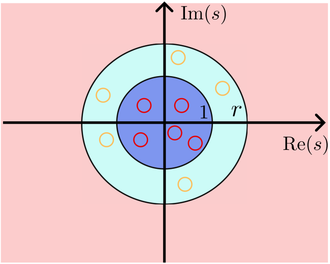
- $\mathcal{H}_2$ optimality conditions

$$\hat{H}(-\hat{\lambda}^*) = H(-\hat{\lambda}^*),$$
$$\hat{H}'(-\hat{\lambda}^*) = H'(-\hat{\lambda}^*).$$

- $h_{2,r}$ optimality conditions

$$\hat{H}\left(r^2/\hat{\lambda}^*\right) = H\left(r^2/\hat{\lambda}^*\right),$$
$$\hat{H}'\left(r^2/\hat{\lambda}^*\right) = H'\left(r^2/\hat{\lambda}^*\right).$$





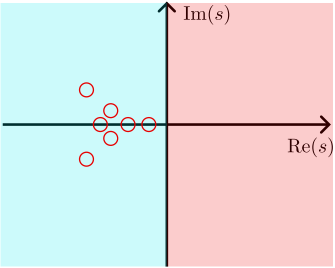
The roots of the idea

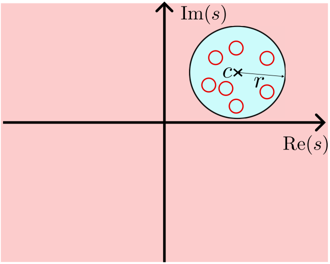
- $\mathcal{H}_2$ optimality conditions

$$\hat{H}(-\hat{\lambda}^*) = H(-\hat{\lambda}^*),$$
$$\hat{H}'(-\hat{\lambda}^*) = H'(-\hat{\lambda}^*).$$

- $h_{2,r,c}$ optimality conditions

$$\hat{H}\left(\frac{r^2}{\hat{\lambda}^* - c^*} + c\right) = H\left(\frac{r^2}{\hat{\lambda}^* - c^*} + c\right),$$
$$\hat{H}'\left(\frac{r^2}{\hat{\lambda}^* - c^*} + c\right) = H'\left(\frac{r^2}{\hat{\lambda}^* - c^*} + c\right).$$





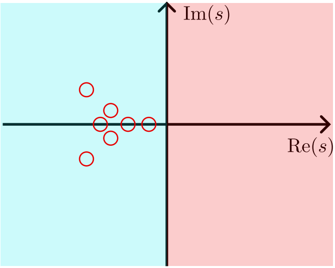
The roots of the idea

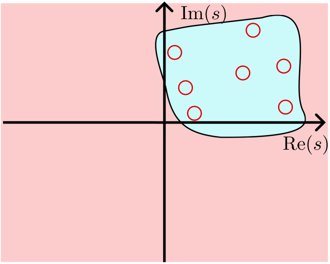
- $\mathcal{H}_2$ optimality conditions

$$\hat{H}(-\hat{\lambda}^*) = H(-\hat{\lambda}^*),$$
$$\hat{H}'(-\hat{\lambda}^*) = H'(-\hat{\lambda}^*).$$

- $\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ optimality conditions

?



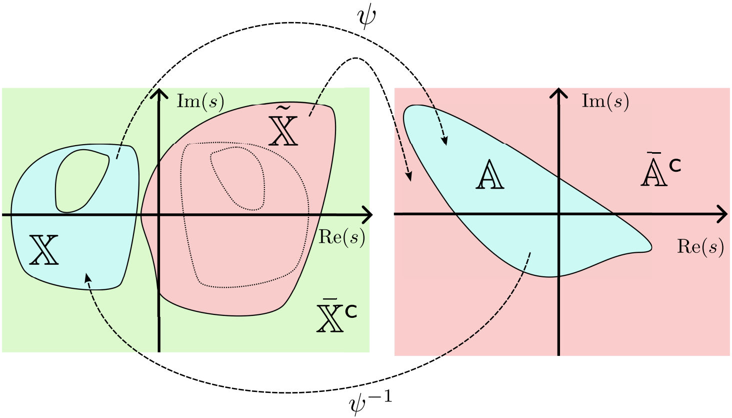


Preliminaries

DAEDALUS
RTG 2433

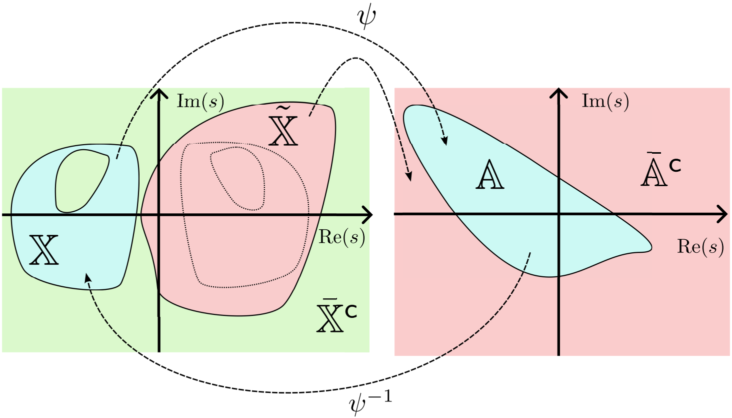
Theorem (conformal map [Wegert, 2012])

Let $\psi\colon \mathbb{X} \to \mathbb{Y}$*, with* $\mathbb{X}, \mathbb{Y} \subset \mathbb{C}$ *open, be Fréchet differentiable as a function of two real variables. The mapping* ψ *is conformal in* $\mathbb{X}$ *if and only if it is analytic in* $\mathbb{X}$ *and* $\psi'(s_0) \neq 0$ *for every* $s_0 \in \mathbb{X}$*.*



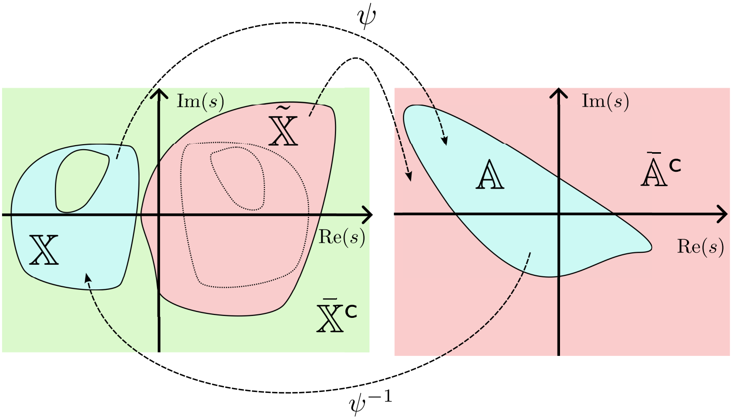
- $\psi\colon \mathbb{C} \to \mathbb{C}$ meromorphic.
- $\psi\colon \mathbb{X} \to \mathbb{A}$, with $\mathbb{X} \subseteq \mathbb{C}_-$ open, is bijective conformal.
- $\tilde{\mathbb{X}} \subseteq \bar{\mathbb{X}}^c$ open such that $\{s \in \mathbb{C} | -s^* \in \mathbb{X}\} \subseteq \tilde{\mathbb{X}}$. Then $\psi\colon \tilde{\mathbb{X}} \to \bar{\mathbb{A}}^c$.
- ψ' zero in a finite amount of points in $\bar{\mathbb{X}}^c$.

DAEDALUS RTG 2433

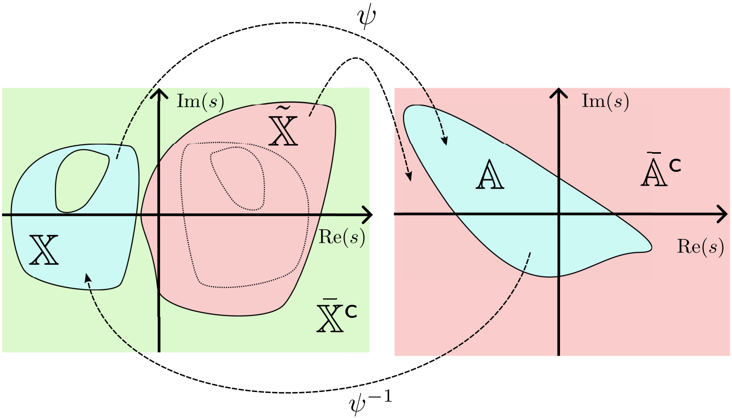


- $\psi\colon \mathbb{C} \to \mathbb{C}$ meromorphic.
- $\psi\colon \mathbb{X} \to \mathbb{A}$, with $\mathbb{X} \subseteq \mathbb{C}_{-}$ open, is bijective conformal.
- $\tilde{\mathbb{X}} \subseteq \bar{\mathbb{X}}^{\mathsf{c}}$ open such that $\{s \in \mathbb{C} \big| -s^* \in \mathbb{X}\} \subseteq \tilde{\mathbb{X}}$. Then $\psi\colon \tilde{\mathbb{X}} \to \bar{\mathbb{A}}^{\mathsf{c}}$.
- ψ' zero in a finite amount of points in $\bar{\mathbb{X}}^{\mathsf{c}}$.

DAEDALUS RTG 2433

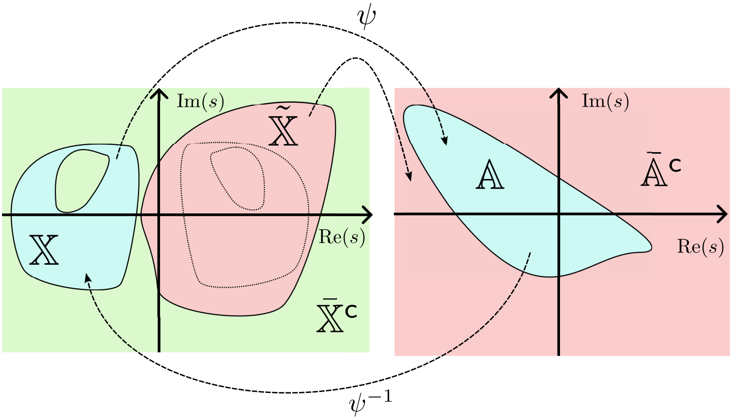


- $\psi\colon \mathbb{C} \to \mathbb{C}$ meromorphic.
- $\psi\colon \mathbb{X} \to \mathbb{A}$, with $\mathbb{X} \subseteq \mathbb{C}_-$ open, is bijective conformal.
- $\tilde{\mathbb{X}} \subseteq \bar{\mathbb{X}}^{\mathsf{c}}$ open such that $\{s \in \mathbb{C} \big| -s^* \in \mathbb{X}\} \subseteq \tilde{\mathbb{X}}$. Then $\psi\colon \tilde{\mathbb{X}} \to \bar{\mathbb{A}}^{\mathsf{c}}$.
- ψ' zero in a finite amount of points in $\bar{\mathbb{X}}^{\mathsf{c}}$.



- $\psi\colon \mathbb{C} \to \mathbb{C}$ meromorphic.
- $\psi\colon \mathbb{X} \to \mathbb{A}$, with $\mathbb{X} \subseteq \mathbb{C}_-$ open, is bijective conformal.
- $\tilde{\mathbb{X}} \subseteq \bar{\mathbb{X}}^{\mathsf{c}}$ open such that $\{s \in \mathbb{C} \mid -s^* \in \mathbb{X}\} \subseteq \tilde{\mathbb{X}}$. Then $\psi\colon \tilde{\mathbb{X}} \to \bar{\mathbb{A}}^{\mathsf{c}}$.
- ψ' zero in a finite amount of points in $\bar{\mathbb{X}}^{\mathsf{c}}$.

Preliminaries



- $\psi\colon \mathbb{C} \to \mathbb{C}$ meromorphic.
- $\psi\colon \mathbb{X} \to \mathbb{A}$, with $\mathbb{X} \subseteq \mathbb{C}_-$ open, is bijective conformal.
- $\tilde{\mathbb{X}} \subseteq \bar{\mathbb{X}}^{\mathsf{c}}$ open such that $\{s \in \mathbb{C} \big| -s^* \in \mathbb{X}\} \subseteq \tilde{\mathbb{X}}$. Then $\psi\colon \tilde{\mathbb{X}} \to \bar{\mathbb{A}}^{\mathsf{c}}$.
- ψ' zero in a finite amount of points in $\bar{\mathbb{X}}^{\mathsf{c}}$.

DAEDALUS RTG 2433

Definition [$\mathcal{H}_2(\bar{\mathbb{A}}^c)$ space]

- Let $\mathfrak{H}_F(s) = F(\psi(s))\psi'(s)^{\frac{1}{2}}$
- Let F, G be analytic in $\bar{\mathbb{A}}^c$. Then the $\mathcal{H}_2(\bar{\mathbb{A}}^c)$ inner product is

$$\begin{aligned}\langle F, G\rangle_{\mathcal{H}_2(\bar{\mathbb{A}}^c)} &:= \langle \mathfrak{H}_F, \mathfrak{H}_G\rangle_{\mathcal{H}_2(\mathbb{C}_+)} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathfrak{H}_F(\imath\omega)^* \mathfrak{H}_G(\imath\omega) \mathrm{d}\omega\end{aligned}$$

- $\mathcal{H}_2(\bar{\mathbb{A}}^c)$ norm is $\|F\|_{\mathcal{H}_2(\bar{\mathbb{A}}^c)} := \|\mathfrak{H}_F\|_{\mathcal{H}_2(\mathbb{C}_+)}$

$$\boxed{\mathcal{H}_2(\bar{\mathbb{A}}^c) := \left\{ F\colon \bar{\mathbb{A}}^c \to \mathbb{C} \text{ analytic} \;\middle|\; \|F\|_{\mathcal{H}_2(\bar{\mathbb{A}}^c)} < \infty \right\}}$$

Definition [$\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ space]

- Let $\mathfrak{H}_F(s) = F(\psi(s))\psi'(s)^{\frac{1}{2}}$
- Let F, G be analytic in $\bar{\mathbb{A}}^{\mathsf{c}}$. Then the $\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ inner product is

$$\begin{aligned}\langle F, G\rangle_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} &:= \langle \mathfrak{H}_F, \mathfrak{H}_G\rangle_{\mathcal{H}_2(\mathbb{C}_+)} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathfrak{H}_F(\imath\omega)^* \mathfrak{H}_G(\imath\omega)\mathrm{d}\omega\end{aligned}$$

- $\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ norm is $\|F\|_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} := \|\mathfrak{H}_F\|_{\mathcal{H}_2(\mathbb{C}_+)}$

$$\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}}) := \left\{ F : \bar{\mathbb{A}}^{\mathsf{c}} \to \mathbb{C} \text{ analytic } \middle| \|F\|_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} < \infty \right\}$$

Definition [$\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ space]

- Let $\mathfrak{H}_F(s) = F(\psi(s))\psi'(s)^{\frac{1}{2}}$
- Let F, G be analytic in $\bar{\mathbb{A}}^{\mathsf{c}}$. Then the $\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ inner product is

$$\begin{aligned}\langle F, G\rangle_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} &:= \langle \mathfrak{H}_F, \mathfrak{H}_G\rangle_{\mathcal{H}_2(\mathbb{C}_+)} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathfrak{H}_F(\imath\omega)^* \mathfrak{H}_G(\imath\omega)\mathrm{d}\omega\end{aligned}$$

- $\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ norm is $\|F\|_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} := \|\mathfrak{H}_F\|_{\mathcal{H}_2(\mathbb{C}_+)}$

$$\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}}) := \left\{ F : \bar{\mathbb{A}}^{\mathsf{c}} \to \mathbb{C} \text{ analytic} \;\middle|\; \|F\|_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} < \infty \right\}$$

DAEDALUS RTG 2433

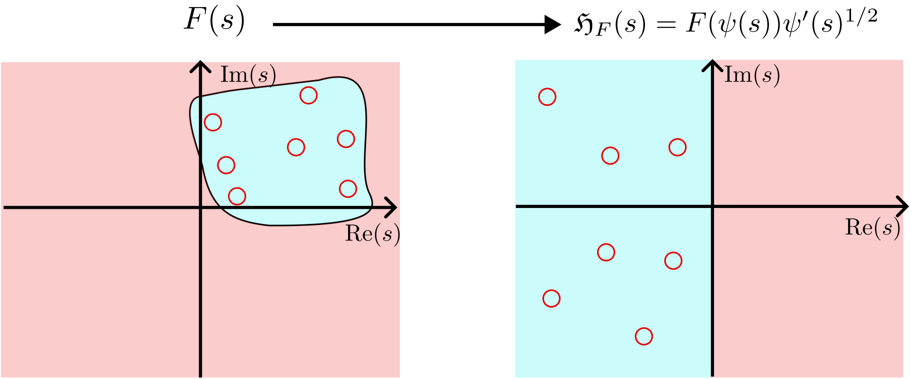
Definition [$\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ space]

- Let $\mathfrak{H}_F(s) = F(\psi(s))\psi'(s)^{\frac{1}{2}}$
- Let F, G be analytic in $\bar{\mathbb{A}}^{\mathsf{c}}$. Then the $\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ inner product is

$$\begin{aligned}\langle F, G\rangle_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} &:= \langle \mathfrak{H}_F, \mathfrak{H}_G\rangle_{\mathcal{H}_2(\mathbb{C}_+)} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathfrak{H}_F(\imath\omega)^*\mathfrak{H}_G(\imath\omega)\mathrm{d}\omega\end{aligned}$$

- $\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ norm is $\|F\|_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} := \|\mathfrak{H}_F\|_{\mathcal{H}_2(\mathbb{C}_+)}$

$$\boxed{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}}) := \left\{F\colon \bar{\mathbb{A}}^{\mathsf{c}} \to \mathbb{C} \text{ analytic } \middle| \|F\|_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} < \infty\right\}}$$



$F(s)$
$\mathfrak{H}_F(s) = F(\psi(s))\psi'(s)^{1/2}$
$\mathrm{Im}(s)$
$\mathrm{Re}(s)$
$\mathrm{Im}(s)$
$\mathrm{Re}(s)$

$\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ optimality conditions

DAEDALUS RTG 2433

- Let $H \in \mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$. We want $\hat{H} \in \mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ such that

$$\hat{H} = \arg \min_{\tilde{H} \in \mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} \|H - \tilde{H}\|_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})}.$$

- Let $\mu \in \mathbb{A}$, then

$$\left\langle H, \frac{1}{\cdot - \mu} \right\rangle_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} = \mathfrak{F}_H(\mu) \quad \text{and} \quad \left\langle H, \frac{1}{(\cdot - \mu)^2} \right\rangle_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} = \mathfrak{F}'_H(\mu).$$

with

$$\mathfrak{F}_H(s) := \overline{\mathfrak{H}}_H(-\psi^{-1}(s))\psi'(\psi^{-1}(s))^{-\frac{1}{2}} + \sum_{\ell=1}^{m} \overline{\mathfrak{H}}_H(-\gamma_\ell)\frac{\text{res}\left[\psi'(s)^{\frac{1}{2}}, \gamma_\ell\right]}{\psi(\gamma_\ell) - s}.$$

Theorem

Let $\mathbb{A}$ be a domain. Let $\hat{H}(s) = \hat{\mathbf{c}}_r^*(s\mathbf{I} - \hat{\mathbf{A}}_r)^{-1}\hat{\mathbf{b}}_r$ be a local minima of the $\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ optimization problem with poles $\{\hat{\lambda}_j\}_{j=1}^r \in \mathbb{A}$. Then the following interpolation conditions hold for $j = 1, \ldots, r$

$$\mathfrak{F}_{\hat{H}}(\hat{\lambda}_j) = \mathfrak{F}_H(\hat{\lambda}_j) \quad \textit{and} \quad \mathfrak{F}'_{\hat{H}}(\hat{\lambda}_j) = \mathfrak{F}'_H(\hat{\lambda}_j).$$

$\mathcal{H}_2(\bar{\mathbb{A}}^c)$ optimality conditions

DAEDALUS RTG 2433

- Let $H \in \mathcal{H}_2(\bar{\mathbb{A}}^c)$. We want $\hat{H} \in \mathcal{H}_2(\bar{\mathbb{A}}^c)$ such that
$$\hat{H} = \arg \min_{\tilde{H} \in \mathcal{H}_2(\bar{\mathbb{A}}^c)} \|H - \tilde{H}\|_{\mathcal{H}_2(\bar{\mathbb{A}}^c)}.$$
- Let $\mu \in \mathbb{A}$, then
$$\left\langle H, \frac{1}{\cdot - \mu} \right\rangle_{\mathcal{H}_2(\bar{\mathbb{A}}^c)} = \mathfrak{F}_H(\mu) \quad \text{and} \quad \left\langle H, \frac{1}{(\cdot - \mu)^2} \right\rangle_{\mathcal{H}_2(\bar{\mathbb{A}}^c)} = \mathfrak{F}'_H(\mu).$$
with
$$\mathfrak{F}_H(s) := \overline{\mathfrak{H}}_H(-\psi^{-1}(s))\psi'(\psi^{-1}(s))^{-\frac{1}{2}} + \sum_{\ell=1}^m \overline{\mathfrak{H}}_H(-\gamma_\ell)\frac{\text{res}\left[\psi'(s)^{\frac{1}{2}}, \gamma_\ell\right]}{\psi(\gamma_\ell) - s}.$$

Theorem

Let $\mathbb{A}$ be a domain. Let $\hat{H}(s) = \hat{\mathbf{c}}_r^*(s\mathbf{I} - \hat{\mathbf{A}}_r)^{-1}\hat{\mathbf{b}}_r$ be a local minima of the $\mathcal{H}_2(\bar{\mathbb{A}}^c)$ optimization problem with poles $\{\hat{\lambda}_j\}_{j=1}^r \in \mathbb{A}$. Then the following interpolation conditions hold for $j = 1, \ldots, r$

$$\mathfrak{F}_{\hat{H}}(\hat{\lambda}_j) = \mathfrak{F}_H(\hat{\lambda}_j) \quad \text{and} \quad \mathfrak{F}'_{\hat{H}}(\hat{\lambda}_j) = \mathfrak{F}'_H(\hat{\lambda}_j).$$

- Let $H \in \mathcal{H}_2(\bar{\mathbb{A}}^\mathsf{c})$. We want $\hat{H} \in \mathcal{H}_2(\bar{\mathbb{A}}^\mathsf{c})$ such that

$$\hat{H} = \arg \min_{\tilde{H} \in \mathcal{H}_2(\bar{\mathbb{A}}^\mathsf{c})} \|H - \tilde{H}\|_{\mathcal{H}_2(\bar{\mathbb{A}}^\mathsf{c})}.$$

- Let $\mu \in \mathbb{A}$, then

$$\left\langle H, \frac{1}{\cdot - \mu} \right\rangle_{\mathcal{H}_2(\bar{\mathbb{A}}^\mathsf{c})} = \mathfrak{F}_H(\mu) \quad \text{and} \quad \left\langle H, \frac{1}{(\cdot - \mu)^2} \right\rangle_{\mathcal{H}_2(\bar{\mathbb{A}}^\mathsf{c})} = \mathfrak{F}'_H(\mu).$$

with

$$\mathfrak{F}_H(s) := \overline{\mathfrak{H}}_H(-\psi^{-1}(s))\psi'(\psi^{-1}(s))^{-\frac{1}{2}} + \textstyle\sum_{\ell=1}^m \overline{\mathfrak{H}}_H(-\gamma_\ell)\frac{\text{res}\left[\psi'(s)^{\frac{1}{2}}, \gamma_\ell\right]}{\psi(\gamma_\ell) - s}.$$

Theorem

Let $\mathbb{A}$ be a domain. Let $\hat{H}(s) = \hat{\mathbf{c}}_r^*(s\mathbf{I} - \hat{\mathbf{A}}_r)^{-1}\hat{\mathbf{b}}_r$ be a local minima of the $\mathcal{H}_2(\bar{\mathbb{A}}^\mathsf{c})$ optimization problem with poles $\{\hat{\lambda}_j\}_{j=1}^r \in \mathbb{A}$. Then the following interpolation conditions hold for $j = 1, \dots, r$

$$\mathfrak{F}_{\hat{H}}(\hat{\lambda}_j) = \mathfrak{F}_H(\hat{\lambda}_j) \quad \textit{and} \quad \mathfrak{F}'_{\hat{H}}(\hat{\lambda}_j) = \mathfrak{F}'_H(\hat{\lambda}_j).$$

Practical consideration

DAEDALUS
RTG 2433

- Let $\varphi(s) = \overline{\psi}(-\psi^{-1}(s))^* = \psi(-\psi^{-1}(s)^*)$
- With some additional assumptions we get

$$\hat{H}\underbrace{\left(\varphi(\hat{\lambda}_j)\right)}_{\sigma} = H\underbrace{\left(\varphi(\hat{\lambda}_j)\right)}_{\sigma} \quad \text{and} \quad \hat{H}'\underbrace{\left(\varphi(\hat{\lambda}_j)\right)}_{\sigma} = H'\underbrace{\left(\varphi(\hat{\lambda}_j)\right)}_{\sigma},$$

  for $j = 1, \ldots, r$
- The assumptions made work for some rational functions

Practical consideration

DAEDALUS
RTG 2433

- Let $\varphi(s) = \overline{\psi}(-\psi^{-1}(s))^* = \psi(-\psi^{-1}(s)^*)$
- With some additional assumptions we get

$$\hat{H}\underbrace{\left(\varphi(\hat{\lambda}_j)\right)}_{\sigma} = H\underbrace{\left(\varphi(\hat{\lambda}_j)\right)}_{\sigma} \quad \text{and} \quad \hat{H}'\underbrace{\left(\varphi(\hat{\lambda}_j)\right)}_{\sigma} = H'\underbrace{\left(\varphi(\hat{\lambda}_j)\right)}_{\sigma},$$

for $j = 1, \ldots, r$

- The assumptions made work for some rational functions

Practical consideration

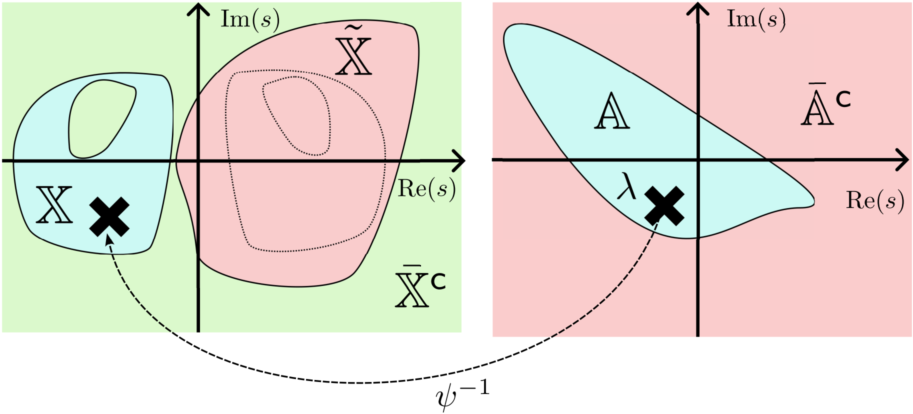
DAEDALUS RTG 2433

- Let $\varphi(s) = \overline{\psi}(-\psi^{-1}(s))^* = \psi(-\psi^{-1}(s)^*)$
- With some additional assumptions we get

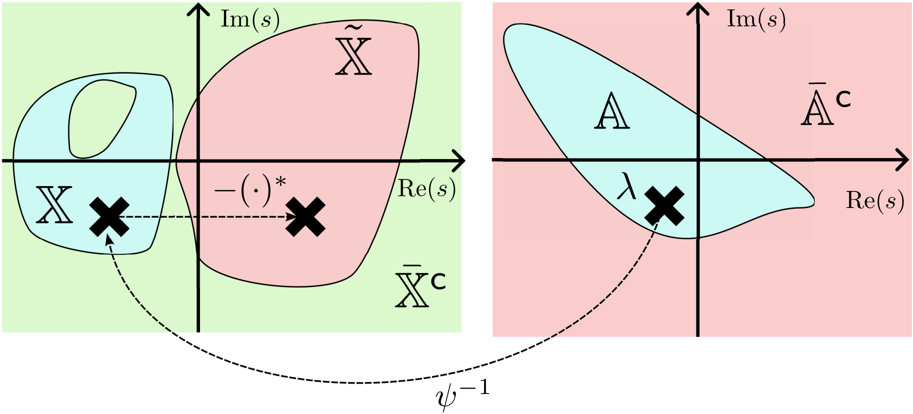
$$\hat{H}\underbrace{\left(\varphi(\hat{\lambda}_j)\right)}_{\sigma} = H\underbrace{\left(\varphi(\hat{\lambda}_j)\right)}_{\sigma} \quad \text{and} \quad \hat{H}'\underbrace{\left(\varphi(\hat{\lambda}_j)\right)}_{\sigma} = H'\underbrace{\left(\varphi(\hat{\lambda}_j)\right)}_{\sigma},$$

  for $j = 1, \ldots, r$
- The assumptions made work for some rational functions

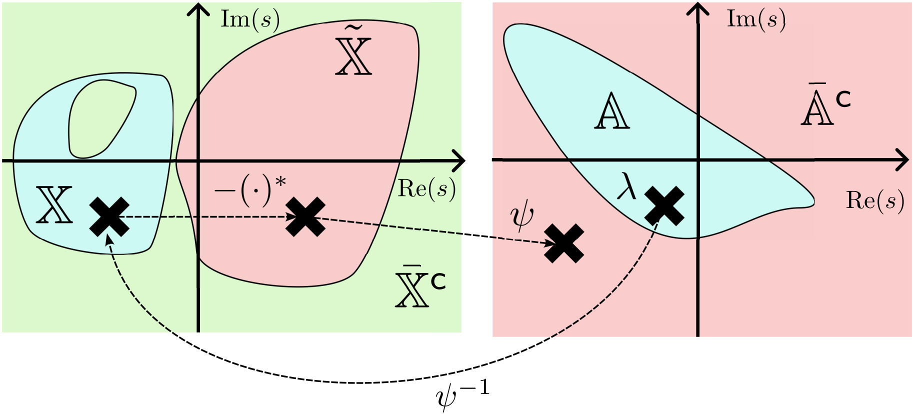
DAEDALUS RTG 2433

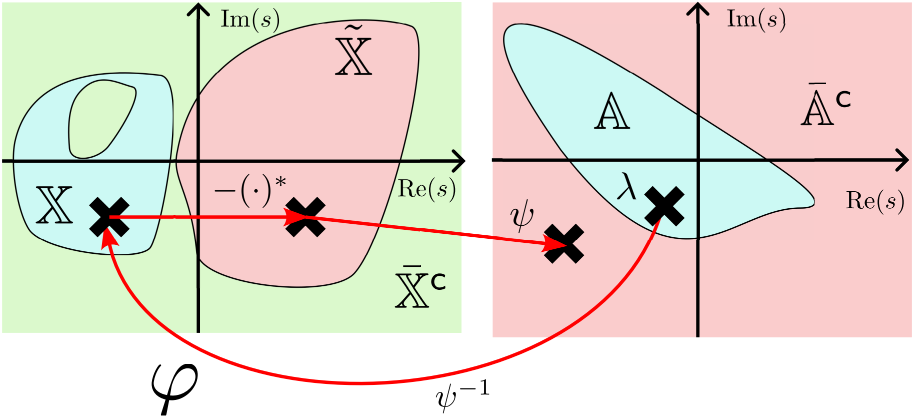


Practical consideration



Practical consideration




Im(s)
X̃
X
−(·)*
Re(s)
X̄ᶜ
φ
ψ⁻¹
Im(s)
A
Āᶜ
λ
ψ
Re(s)

Algorithm

1. initial guess $\boldsymbol{\sigma}_0$
2. construct $\mathbf{V}_r$ and $\mathbf{W}_r$ such that

$$\mathsf{Ran}(\mathbf{V}_r) = \mathsf{span}\left\{(\sigma_1\mathbf{I}-\mathbf{A})^{-1}\mathbf{b},\dots,(\sigma_r\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}\right\},$$
$$\mathsf{Ran}(\mathbf{W}_r) = \mathsf{span}\left\{(\sigma_1^*\mathbf{I}-\mathbf{A}^*)^{-1}\mathbf{c},\dots,(\sigma_r^*\mathbf{I}-\mathbf{A}^*)^{-1}\mathbf{c}\right\}.$$

3. `while` $\|\boldsymbol{\sigma}_{i+1}-\boldsymbol{\sigma}_i\|/\|\boldsymbol{\sigma}_i\| >$ tol
   - $\widehat{\mathbf{A}}_r = (\mathbf{W}_r^*\mathbf{V}_r)^{-1}\mathbf{W}_r^*\mathbf{A}\mathbf{V}_r$
   - $\widehat{\mathbf{A}}_r\mathbf{v} = \widehat{\lambda}\mathbf{v}$
   - $\boldsymbol{\sigma}_{i+1} = -\widehat{\lambda}^*$
   - Update $\mathbf{V}_r$ and $\mathbf{W}_r$
4. $\hat{\mathbf{A}}_r = (\mathbf{W}_r^*\mathbf{V}_r)^{-1}\mathbf{W}_r^*\mathbf{A}\mathbf{V}_r$, $\hat{\mathbf{b}}_r = (\mathbf{W}_r^*\mathbf{V}_r)^{-1}\mathbf{W}_r^*\mathbf{b}$, and $\hat{\mathbf{c}}_r^* = \mathbf{c}^*\mathbf{V}_r$.

DAEDALUS RTG 2433

Algorithm

1. initial guess $\boldsymbol{\sigma}_0$
2. construct $\mathbf{V}_r$ and $\mathbf{W}_r$ such that

$$\mathsf{Ran}(\mathbf{V}_r) = \mathsf{span}\left\{(\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \ldots, (\sigma_r\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\right\},$$
$$\mathsf{Ran}(\mathbf{W}_r) = \mathsf{span}\left\{(\sigma_1^*\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{c}, \ldots, (\sigma_r^*\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{c}\right\}.$$

3. `while` $\|\boldsymbol{\sigma}_{i+1} - \boldsymbol{\sigma}_i\| / \|\boldsymbol{\sigma}_i\| > \mathsf{tol}$
   - $\widehat{\mathbf{A}}_r = (\mathbf{W}_r^*\mathbf{V}_r)^{-1}\mathbf{W}_r^*\mathbf{A}\mathbf{V}_r$
   - $\widehat{\mathbf{A}}_r\mathbf{v} = \widehat{\lambda}\mathbf{v}$
   - $\boldsymbol{\sigma}_{i+1} = \varphi(\widehat{\lambda})$
   - Update $\mathbf{V}_r$ and $\mathbf{W}_r$
4. $\hat{\mathbf{A}}_r = (\mathbf{W}_r^*\mathbf{V}_r)^{-1}\mathbf{W}_r^*\mathbf{A}\mathbf{V}_r$, $\hat{\mathbf{b}}_r = (\mathbf{W}_r^*\mathbf{V}_r)^{-1}\mathbf{W}_r^*\mathbf{b}$, and $\hat{\mathbf{c}}_r^* = \mathbf{c}^*\mathbf{V}_r$.

Schrödinger equation for the free particle

DAEDALUS
RTG 2433

We have the PDE

$$
\begin{aligned}
\frac{\partial w(x,t)}{\partial t} &= -\mathrm{i}\frac{\partial^2 w(x,t)}{\partial x^2}, && \text{on } (0,1)\times(0,T),\\
w(0,t) = 0, \quad w(1,t) &= u(t), && \text{on } (0,T),\\
y(t) &= \int_0^1 w(x,t)\,\mathrm{d}x, && \text{on } (0,T),\\
w(x,0) &= 0, && \text{in } (0,1).
\end{aligned}
$$

- $H(s) = \mathbf{c}^*(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$
- $n = 1000$

Schrödinger equation for the free particle

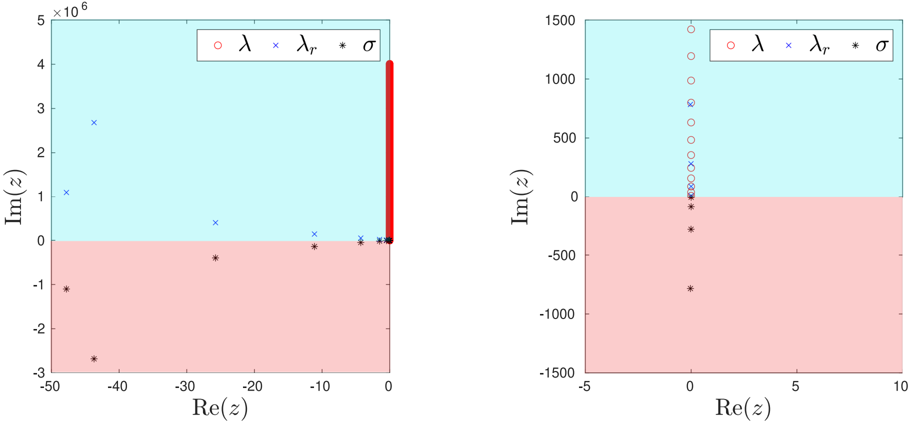
DAEDALUS RTG 2433

We have the PDE

$$\begin{aligned} \frac{\partial w(x,t)}{\partial t} &= -\mathrm{i}\frac{\partial^2 w(x,t)}{\partial x^2}, && \text{on } (0,1)\times(0,T), \\ w(0,t) = 0, \quad w(1,t) &= u(t), && \text{on } (0,T), \\ y(t) &= \int_0^1 w(x,t)\,\mathrm{d}x, && \text{on } (0,T), \\ w(x,0) &= 0, && \text{in } (0,1). \end{aligned}$$

- $H(s) = \mathbf{c}^*(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$
- $n = 1000$

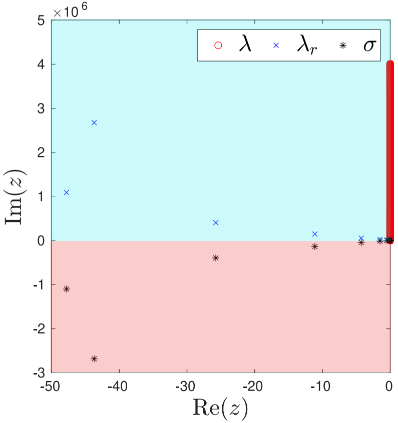
Schrödinger equation for the free particle

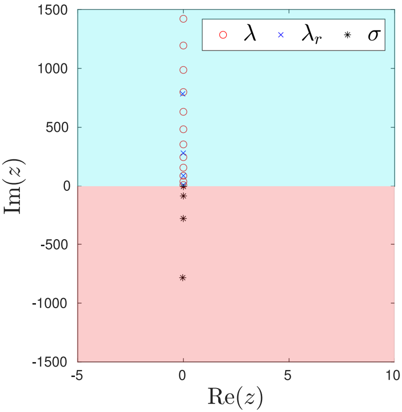


$$\psi(s) = -\mathrm{i}s, \quad r = 15$$

Schrödinger equation for the free particle

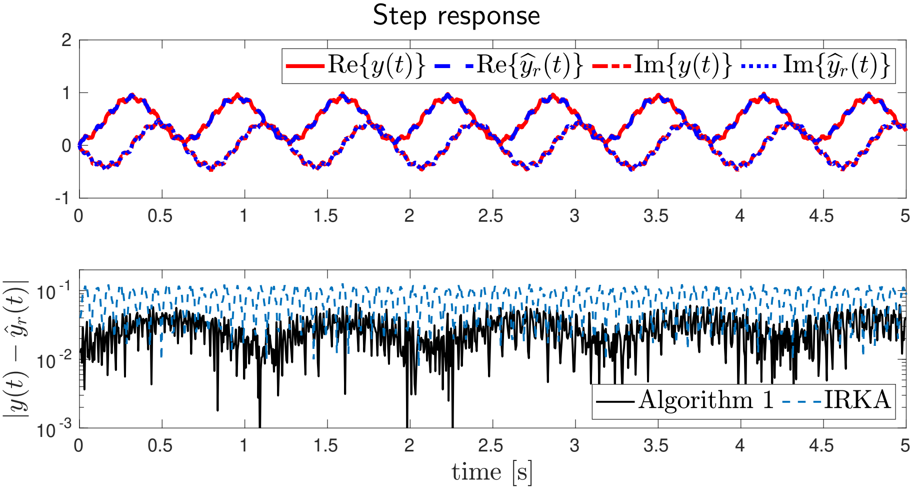
DAEDALUS RTG 2433





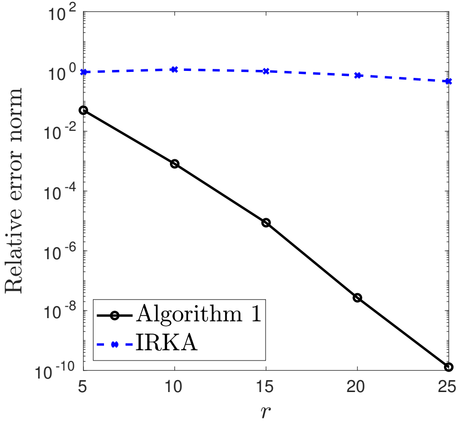
$$\psi(s) = -\mathrm{i}s, \quad r = 15$$

Schrödinger equation for the free particle



Schrödinger equation for the free particle

$$\frac{\|H - \hat{H}\|_{\mathcal{H}_2(\bar{\mathbb{A}}^c)}}{\|H\|_{\mathcal{H}_2(\bar{\mathbb{A}}^c)}}$$



We have the PDE

$$
\begin{aligned}
\frac{\partial^2 w(x,t)}{\partial t} &= \frac{\partial^2 w(x,t)}{\partial x^2} + \chi_{[0.6,0.7]} u(t), && \text{on } (0,1)\times(0,T),\\
w(0,t) &= 0, \quad w(1,t) = 0, && \text{on } (0,T),\\
y(t) &= \int_{0.1}^{0.4} w(x,t)\,\mathrm{d}x, && \text{on } (0,T),\\
w(x,0) &= 0, && \text{in } (0,1),
\end{aligned}
$$

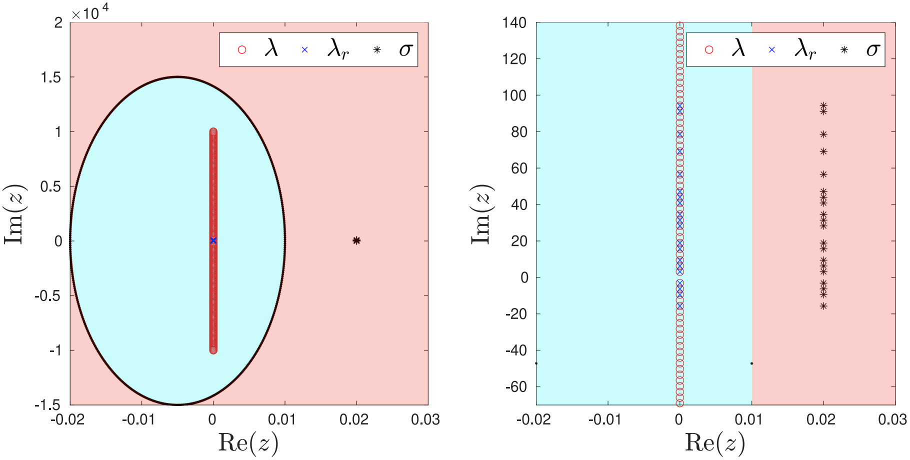
- $H(s) = \mathbf{c}^*(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$
- $n = 10000$

We have the PDE

$$\begin{aligned}
\frac{\partial^2 w(x,t)}{\partial t} &= \frac{\partial^2 w(x,t)}{\partial x^2} + \chi_{[0.6,0.7]} u(t), && \text{on } (0,1) \times (0,T), \\
w(0,t) &= 0, \quad w(1,t) = 0, && \text{on } (0,T), \\
y(t) &= \int_{0.1}^{0.4} w(x,t)\,\mathrm{d}x, && \text{on } (0,T), \\
w(x,0) &= 0, && \text{in } (0,1),
\end{aligned}$$

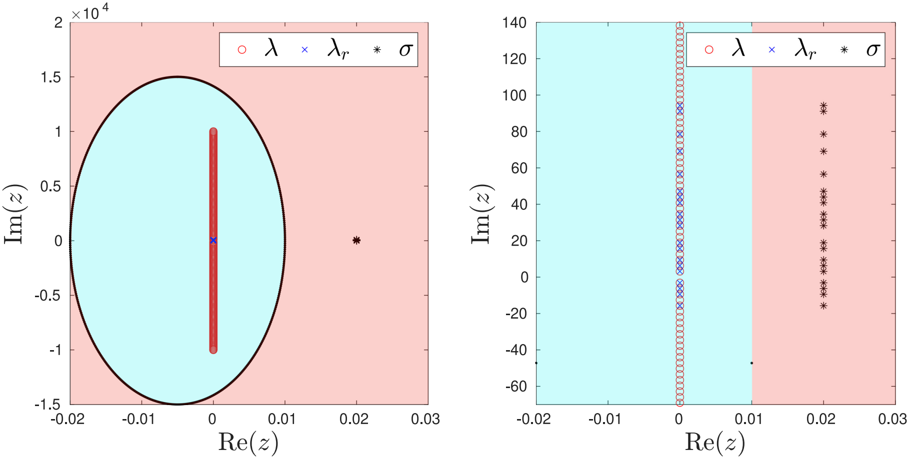
- $H(s) = \mathbf{c}^*(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$
- $n = 10000$

Wave equation

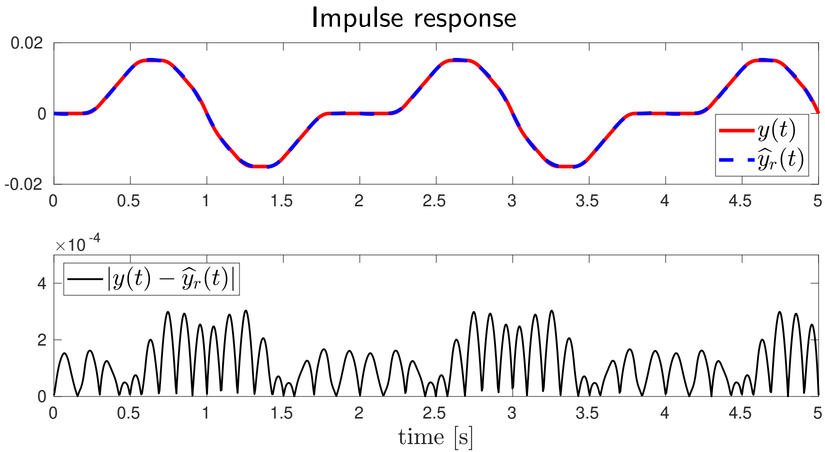


$$\psi(s) = \frac{1}{2}(s + s^{-1}), \qquad r = 20$$

Wave equation



$$\psi(s) = \frac{1}{2}(s + s^{-1}), \quad r = 20$$



Impulse response
0.02
0
-0.02
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
$y(t)$
$\widehat{y}_r(t)$
$\times 10^{-4}$
4
2
0
$|y(t) - \widehat{y}_r(t)|$
time [s]

Conclusions

DAEDALUS
RTG 2433

Summary

- Generalization of the $\mathcal{H}_2$ optimality conditions
- Developed an IRKA-based algorithm

Next Steps

- Develop a TF-IRKA-based algorithm.
- Relate the error $\|H - \hat{H}\|_{\mathcal{H}_2(\bar{\mathbb{A}}^c)}$ with $|y - \widehat{y}_r|$
- Extend the theory to $\mathcal{H}_\infty$ model reduction
- Use concepts from computational conformal mapping

Thank you for your attention!

[1] A. Borghi, T. Breiten, $\mathcal{H}_2$ optimal model reduction on general domains, *arXiv:2305.01511* (2023)

References

Gugercin, S. and Antoulas, A.C. and Beattie, C.
$\mathcal{H}_2$ model reduction for large-scale linear dynamical systems
SIAM J. Matrix Anal. & Appl., 30(2), 609-638, 2008.

Bunse-Gerstner, A. and Kubalinska, D. and Vossen, G. and Wilczek, D.
h_2-norm optimal model reduction for large scale discrete dynamical MIMO systems
Journal of Computational and Applied Mathematics, 233(5), 1202-1216, 2010.

Meier, L. and Luenberger, D.G.
Approximation of linear constant systems
IEEE Trans. Automat. Control, 12, 585-588, 1967.

Kubalinska, D.
Optimal interpolation-based model reduction
PhD thesis, Universität Bremen, Germany, 2008.

Wegert, E.
Visual complex functions
Springer, Basel, Switzerland 2012.

Duren, P.L.
Theory of Hp spaces
Academic Press, New York and London, 1970.

- For $G \in \mathcal{H}_2(\mathbb{C}_+)$ and $\mu \in \mathbb{C}_-$

$$\left\langle G, \frac{1}{\cdot - \mu} \right\rangle_{\mathcal{H}_2(\mathbb{C}_+)} = G(-\mu^*) \quad \text{and} \quad \left\langle G, \frac{1}{(\cdot - \mu)^2} \right\rangle_{\mathcal{H}_2(\mathbb{C}_+)} = G'(-\mu^*).$$

- Let $\hat{H}^{(\varepsilon)}$ be perturbation of $\hat{H}$

$$\|H - \hat{H}\|^2_{\mathcal{H}_2(\mathbb{C}_+)} \leq \|H - \hat{H}^{(\varepsilon)}\|^2_{\mathcal{H}_2(\mathbb{C}_+)}$$

$$0 \leq 2\mathrm{Re}\left\{\left\langle H - \hat{H}, \hat{H} - \hat{H}^{(\varepsilon)} \right\rangle_{\mathcal{H}_2(\mathbb{C}_+)}\right\} + \|\hat{H} - \hat{H}^{(\varepsilon)}\|^2_{\mathcal{H}_2(\mathbb{C}_+)}.$$

- For $\varepsilon \to 0$ and a specific direction of ε we get the interpolation conditions

Proof for Meier-Luenberger conditions [Antoulas et al., 2020]

- For $G \in \mathcal{H}_2(\mathbb{C}_+)$ and $\mu \in \mathbb{C}_-$

$$\left\langle G, \frac{1}{\cdot - \mu} \right\rangle_{\mathcal{H}_2(\mathbb{C}_+)} = G(-\mu^*) \quad \text{and} \quad \left\langle G, \frac{1}{(\cdot - \mu)^2} \right\rangle_{\mathcal{H}_2(\mathbb{C}_+)} = G'(-\mu^*).$$

- Let $\hat{H}^{(\varepsilon)}$ be perturbation of $\hat{H}$

$$\|H - \hat{H}\|^2_{\mathcal{H}_2(\mathbb{C}_+)} \leq \|H - \hat{H}^{(\varepsilon)}\|^2_{\mathcal{H}_2(\mathbb{C}_+)}$$

$$0 \leq 2\mathrm{Re}\left\{ \left\langle H - \hat{H}, \hat{H} - \hat{H}^{(\varepsilon)} \right\rangle_{\mathcal{H}_2(\mathbb{C}_+)} \right\} + \|\hat{H} - \hat{H}^{(\varepsilon)}\|^2_{\mathcal{H}_2(\mathbb{C}_+)}.$$

- For $\varepsilon \to 0$ and a specific direction of ε we get the interpolation conditions

- For $G \in \mathcal{H}_2(\mathbb{C}_+)$ and $\mu \in \mathbb{C}_-$

$$\left\langle G, \frac{1}{\cdot - \mu} \right\rangle_{\mathcal{H}_2(\mathbb{C}_+)} = G(-\mu^*) \quad \text{and} \quad \left\langle G, \frac{1}{(\cdot - \mu)^2} \right\rangle_{\mathcal{H}_2(\mathbb{C}_+)} = G'(-\mu^*).$$

- Let $\hat{H}^{(\varepsilon)}$ be perturbation of $\hat{H}$

$$\|H - \hat{H}\|^2_{\mathcal{H}_2(\mathbb{C}_+)} \leq \|H - \hat{H}^{(\varepsilon)}\|^2_{\mathcal{H}_2(\mathbb{C}_+)}$$

$$0 \leq 2\text{Re}\left\{ \left\langle H - \hat{H}, \hat{H} - \hat{H}^{(\varepsilon)} \right\rangle_{\mathcal{H}_2(\mathbb{C}_+)} \right\} + \|\hat{H} - \hat{H}^{(\varepsilon)}\|^2_{\mathcal{H}_2(\mathbb{C}_+)}.$$

- For $\varepsilon \to 0$ and a specific direction of ε we get the interpolation conditions

Schrödinger equation for the free particle

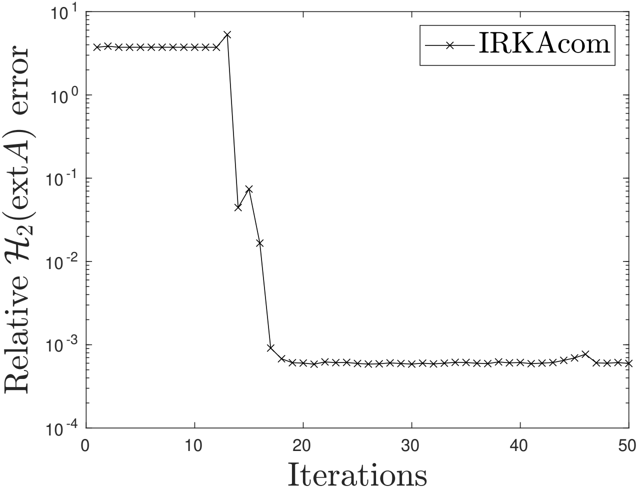
DAEDALUS
RTG 2433

$$
\begin{aligned}
\frac{\partial w(x,t)}{\partial t} &= -\mathrm{i}\frac{\partial^2 w(x,t)}{\partial x^2}, && \text{on } (0,1)\times(0,T),\\
w(0,t) &= 0, \quad w(1,t) = u(t), && \text{on } (0,T),\\
y(t) &= \int_0^1 w(x,t)\,\mathrm{d}x, && \text{on } (0,T),\\
w(x,0) &= 0, && \text{in } (0,1).
\end{aligned}
$$

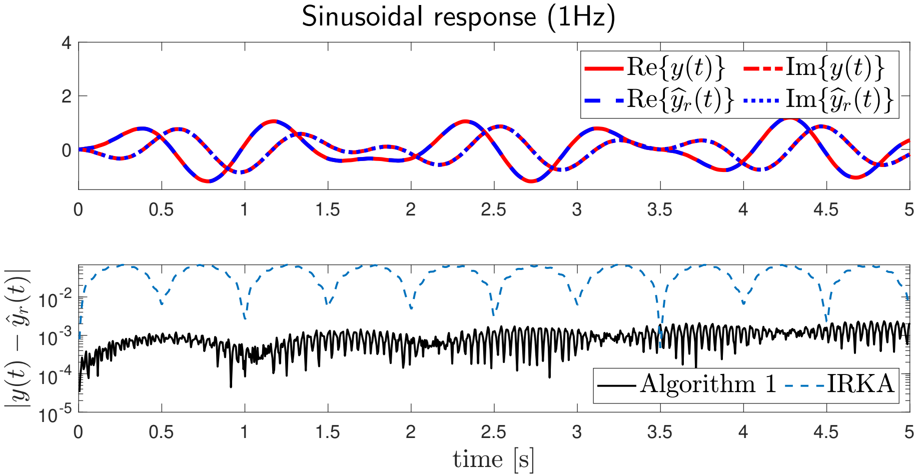
$$\downarrow$$

$$n = 1000, \psi(s) = -\mathrm{i}s$$

Schrödinger equation for the free particle



Schrödinger equation for the free particle



Wave equation

$$
\begin{aligned}
\frac{\partial^2 w(x,t)}{\partial t} &= \frac{\partial^2 w(x,t)}{\partial x^2} + \chi_{[0.6,0.7]} u(t), && \text{on } (0,1)\times(0,T),\\
w(0,t) &= 0, \quad w(1,t) = 0, && \text{on } (0,T),\\
y(t) &= \int_{0.1}^{0.4} w(x,t)\,\mathrm{d}x, && \text{on } (0,T),\\
w(x,0) &= 0, && \text{in } (0,1),
\end{aligned}
$$

$$\downarrow$$

$$n = 10000, \quad \psi(s) = \frac{1}{2}\left(s + \frac{1}{s}\right)$$

- Let $\overline{\psi}(-z)^* = \psi(z)$ for $z = \imath\omega$, $\omega \in \mathbb{R}$ and $\varphi(s) = \overline{\psi}(-\psi^{-1}(s))^*$,
- Let $|\psi'(z)|$ have poles $\{\gamma_\ell\}_{\ell=1}^m \in \mathbb{C}_-$,
- For $H \in \mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$ define

$$\mathfrak{F}_H(s) := \overline{\mathfrak{H}}_H(-\psi^{-1}(s))\psi'(\psi^{-1}(s))^{-\frac{1}{2}} + \sum_{\ell=1}^{m} \overline{\mathfrak{H}}_H(-\gamma_\ell)\frac{\text{res}\left[\psi'(s)^{\frac{1}{2}}, \gamma_\ell\right]}{\psi(\gamma_\ell) - s}.$$

- Let $H \in \mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$, $\frac{1}{\cdot - \mu} \in \mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$, and $\frac{1}{(\cdot - \mu)^2} \in \mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})$. Let $H(\overline{\psi}(-s)^*)^*$ be analytic in a neighborhood of $\psi^{-1}(\mu) \in \mathbb{C}_-$, then

$$\left\langle H, \frac{1}{\cdot - \mu} \right\rangle_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} = \mathfrak{F}_H(\mu) \quad \text{and} \quad \left\langle H, \frac{1}{(\cdot - \mu)^2} \right\rangle_{\mathcal{H}_2(\bar{\mathbb{A}}^{\mathsf{c}})} = \mathfrak{F}'_H(\mu).$$