Large Eddy Simulation for Turbulent Flows

by

Traian Iliescu

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FACULTY OF ARTS AND SCIENCES

This dissertation was presented

by

Traian Iliescu

It was defended on

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and approved by

Prof. William J. Layton (Chairman)

Prof. Thomas Porsching

Prof. Patrick Rabier

Prof. Anne Robertson

Committee Chairperson

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Traian Iliescu, Ph.D.

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This thesis is concerned with one of the most promising approaches to the numerical simulation of turbulent flows, the Large Eddy Simulation (LES) in which the large scales of the flow are calculated directly, while the interactions with the small scales are modeled. Specifically, we analyze two new LES models, introduced in [30] and [51].

First, we sketch the derivation of the LES model introduced by Galdi and Layton in [30], pointing out its main improvement over traditional LES models: the modeling of the interactions between small and large scales uses a closure approximation in the Fourier space which attenuates the high wave number components (or equivalently the small scales) of the flow , instead of increasing them. We then present the mathematical analysis of this new model, proving existence, uniqueness and stability of weak solutions.

Second, we present three new models for the turbulent fluctuations modeling the interactions between the small scales in the flow. These models, introduced in [51], are based on approximations for the distribution of kinetic energy in the small scales in terms of the mean flow. We also prove existence of weak solutions for one of these models. We then show how this model can be implemented in finite element procedures and prove that its action is no larger than that of the popular Smagorinsky subgrid-scale model.

Third, we consider "numerical-errors" in LES. Specifically, for one filtered flow model, we show convergence of the semidiscrete finite element approximation of the model and give an estimate of the error.

Finally, we provide a careful numerical assessment and comparison of a classical LES model and the Galdi-Layton model. We are focusing herein on *global*, *quantitative* properties of the above models vis à vis those of the mean flow variables. Direct numerical simulation (DNS), the classical LES model, two variants of the

Galdi-Layton LES model and the Smagorinsky model are compared using the two–dimensional driven cavity problem for Reynolds numbers 400 and 10000.

To my family.

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$\mathbf{u}$	fluid velocity
$\overline{\mathbf{u}}$	mean component of the fluid velocity
$\mathbf{u}'$	turbulent component of the fluid velocity
p	fluid pressure
$\overline{p}$	mean component of the fluid pressure
p'	turbulent component of the fluid pressure
$\mathbf{x}$	space coordinates
t	time coordinate
Re	Reynolds number
$\mathbf{f}$	external force
$\overline{\mathbf{f}}$	mean component of the external force
Ω	domain; usually open, bounded, simply connected subset of $\mathbb{R}^2$ or $\mathbb{R}^3$
$\partial\Omega$	boundary of Ω
δ	radius of the spatial filter
γ	constant in the Gaussian filter (usually $\gamma = 6$)
μ	exponent in the Smagorinsky term
$L^p(\Omega)$	space of functions whose p-th power is integrable
L^∞	space of functions bounded almost everywhere
$W^{m,p}(\Omega)$	space of functions in L^p whose distributional derivatives up to order m are in L^p
$W^{m,\infty}(\Omega)$	space of functions in L^∞ whose distributional derivatives up to order m are in L^∞
$\hat{\tau}_j$	orthonormal set of tangent vectors
$\hat{n}$	exterior unit normal
$\beta(\delta, Re)$	slip coefficient
τ	Cauchy stress tensor
$\mathbb{D}(\cdot)$	deformation tensor
ν_T	turbulent viscosity

X	continuous velocity space
Y	continuous pressure space
X^h	discrete velocity spaces
Y^h	discrete pressure spaces
$(\mathbf{u}^h, p^h)$	solution to the discrete LES models
$b(.;.,.)$	trilinear form associated with the non-linear term of the Navier-Stokes equations
h	spatial mesh size
Δt	time step
LES	Large Eddy Simulation
DNS	Direct Numerical Simulation

Chapter 1
Introduction

1.1 Turbulence

Turbulence has been a long standing challenge for human mind. Five centuries after the first studies of Leonardo da Vinci, understanding turbulence continues to be many scientists' dream.

Figure 1.1: Leonardo da Vinci's manuscript

Turbulence is part of everyday life. Atmospheric flows, water currents below the ocean's surface and rivers are turbulent. Fluid flows around cars, ships and airplanes are turbulent. Many other examples of turbulent flows arise in aeronautics, hydraulics, nuclear and chemical engineering, environmental sciences, oceanography, meteorology, astrophysics and internal geophysics (see [71]). These are not only scientific challenges: predicting hurricanes, global climate change calculations, pollution dispersal estimation, and energy consumption optimization are some of the most important practical challenges at present (see [32]).

The Navier-Stokes equations probably contain all of turbulence. They describe the motion of every incompressible newtonian (i.e. with a linear stress-strain relation) fluid, since they are derived directly from conservation laws without further assumptions. However, except for very simple flows, there is no analytical solution for these equations. Moreover, the mathematical theory the Navier-Stokes equations is not complete. Thus, it is the author's belief that, even though the Navier-Stokes equations represent the cornerstone to the understanding of turbulence, we need more insight coming from physics and numerical simulations. In 1949, John von Neumann wrote in one of his reports, privately circulated for many years (see [26]):

> These considerations justify the view that a considerable mathematical effort towards a detailed understanding of the mechanism of turbulence is called for. The entire experience with the subject indicates that the purely analytical approach is beset with difficulties, which at this moment are still prohibitive. The reason for this is probably as was indicated above: That our intuitive relationship to the subject is still too loose - not having succeeded at anything like deep mathematical penetration in any part of the subject, we are still quite disoriented as to the relevant factors, and as to the proper analytical machinery to be used.
>
> Under these conditions there might be some hope to 'break the deadlock' by extensive, well-planned, computational efforts. It must be admitted that the problems in question are too vast to be solved by a direct computational attack, that is, by an outright calculation of a representative family of special cases. There are, however, strong indications that one could name certain strategic points in this complex, where relevant information must be obtained by direct calculations. If this is properly done, and then the operation is repeated on the basis of broader information then becoming available, etc., there is a reasonable chance of effecting real penetrations in this complex of problems and gradually developing a useful, intuitive relationship to it. This should, in the end, make an attack with analytical methods, that is truly more mathematical, possible.

This is so very true at present, too!

1.1.1 Mathematical Description of Turbulence - the Navier-Stokes Equations

The equations describing the motion of any incompressible newtonian fluid in a bounded domain are the well-known Navier-Stokes equations:

$$\begin{cases} \partial_t \mathbf{u} - Re^{-1}\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \end{cases} \tag{1.1.1}$$

where $\mathbf{u}$ is the velocity of the fluid flow, p is the pressure, $\mathbf{f}$ is the external force, and Ω is a bounded simply connected domain with smooth boundary Γ. The equations (1.1.1) are the Navier-Stokes equations in the non-dimensional form; the control parameter of the flow is the **Reynolds number**, Re, which is defined as:

$$Re = \frac{LV}{\nu},$$

L and V being respectively a characteristic scale and velocity of the flow, and ν its (kinematic) viscosity. From the physical point of view, Re represents the ratio between the inertial forces and the viscous forces. In (1.1.1), $-Re^{-1}\Delta\mathbf{u}$ is usually called the "viscous term", and $\mathbf{u}\cdot\nabla\mathbf{u} = \left(\sum_{j=1}^{d}\mathbf{u}_j\dfrac{\partial\mathbf{u}_i}{\partial\mathbf{x}_j}\right)_{i=1,d}$ the "convective term".

The Navier-Stokes equations (1.1.1) must be supplemented by boundary and initial conditions. Also, to ensure the uniqueness of the pressure, we impose $\int_\Omega p(\mathbf{x},t)d\mathbf{x} = 0$.

As mentioned before, the Navier-Stokes equations are derived directly from conservation laws: the first equation in (1.1.1) represents the conservation of momentum, and the second equation in (1.1.1) the conservation of mass. Thus, these equations do not represent a model - every incompressible newtonian fluid flow (laminar or turbulent) must satisfy the Navier-Stokes equations.

In the mathematical setting of the Navier-Stokes equations, the only control parameter, Re, makes the difference between laminar and turbulent flows: laminar flows occur at low Reynolds numbers, whereas turbulent flows occur at high Reynolds numbers. Thus, since all turbulent flows satisfy the Navier-Stokes equations, it seems natural to use a mathematical approach in trying to understand turbulence.

However, the present state of the mathematical theory of the Navier-Stokes equations is not encouraging. First of all, except for very simple settings like Couette flow or Poiseuille flow, we do not have an analytical solution. Moreover, existence and uniqueness for weak solutions (introduced by Jean Leray in the celebrated paper [70] in 1934) have been proven only in two dimensions. In the three dimensional case, existence and uniqueness for the weak solutions have been proven ([59]) only for small Reynolds numbers (laminar flows) or, equivalently, for a small time interval ("small" considered with respect to the initial data). The coupling between the nonlinearity of the convective term ($\mathbf{u}\cdot\nabla\mathbf{u}$) and the absence of symmetry in ($\mathbf{u}, p$) are considered (see G.P. Galdi [27]) to be the core difficulty in the mathematical theory of the Navier-Stokes equations. Despite the efforts of many famous scientists, much remains to be done since Jean Leray's first attempt in 1934. It is the author's belief that, even with a complete mathematical theory for the Navier-Stokes equations, we will still need extra insight from the physical (and numerical) world in order to understand turbulence!

In this respect, the quotes from von Neumann's paper given in the beginning of the chapter are as true as ever!

1.1.2 Physical Description of Turbulence

Since we are still far from a mathematically rigorous understanding of turbulence, of great interest is the physical (experimentalist) approach.

However, this path is by no means easier. This is apparent when we try to define turbulence. To the author's knowledge, there is no widely accepted definition of turbulence. As Frisch noted in [26], a good way to enter the rich world of turbulence phenomena is through the book of Van Dyke (1982) "An Album of Fluid Motion", presenting pictures of varied turbulent flows. Another way to describe turbulence is by listing its characteristic features (for a detailed presentation, the reader is referred to Lesieur [71], Frisch [26], and Hinze [44]). We are presenting now some of these characteristic features:

- Turbulent flows are **irregular**. This is a very important feature, appearing in almost any definition of turbulence. Because of irregularity, the deterministic approach to turbulence becomes impractical, in that it appears impossible to describe the turbulent motion in all details as a function of time and space coordinates. However, it is believed possible to indicate average (with respect to space and time) values of velocity and pressure.

- Turbulent flows are **diffusive**. This causes rapid mixing and increased rates of momentum, heat and mass transfer. Turbulent flows should be able to mix transported quantities much more rapidly than if only molecular diffusion processes were involved. For example, if a passive scalar is being transported by the flow, a certain amount of mixing will occur due molecular diffusion. In a turbulent flow, the same sort of mixing is observed, but in a much greater amount than predicted by molecular diffusion. From the practical viewpoint, diffusivity is very important: the engineer, for instance, is mainly concerned with the knowledge of turbulent heat diffusion coefficients, or the turbulent drag (depending on turbulent momentum diffusion in the flow).

- Turbulent flows are **rotational**. For a large class of flows, turbulence arises due to the presence of boundaries or obstacles, which create vorticity inside a flow which was initially irrotational. Turbulence is thus associated with vorticity, and it is impossible to imagine a turbulent irrotational flow.

- Turbulent flows occur at **high Reynolds numbers**. Turbulence often arises as an instability of laminar flows when the Reynolds number becomes too high. This instability is related to the complex interaction of viscous and convective (inertial) terms.

- Turbulent flows are **dissipative**. Viscosity effects will result in the conversion of kinetic energy of the flow into heat. If there is no external source of energy to make up for this kinetic energy loss, the turbulent motion will decay (see [26]).

- Turbulent flows are **continuum** phenomena. As noticed in [44] , even the smallest scales occuring in a turbulent flow are ordinarily far larger than any molecular length scale.

- Turbulence is a feature of **fluid flows**, and not of fluids. If the Reynolds number is high enough, most of the dynamics of turbulence is the same in all fluids (liquids or gases). The main characteristics of turbulent flows are not controlled by the molecular properties of the particular fluid.

1.1.3 Numerical Approach to Turbulence

As we have seen from the previous two sections, both approaches (mathematical and physical) are pretty far from giving a complete answer to the understanding of turbulence. However, mainly due to the efforts in the engineering and geophysics communities, the numerical simulation of turbulent flows emerged as an essential approach in tackling turbulence. Even though the numerical approach has undeniable accomplishments ("We Flew to the Moon!"), it is by no means an easy and straightforward one.

The most natural approach to the numerical simulation of turbulent flows is the Direct Numerical Simulation (DNS), in which all the scales of the motion are simulated using solely the Navier-Stokes equations. Motivated by Kolmogorov's theory (see [3] p. 8, [77] p. 25), small scales exist down to $O(Re^{-3/4})$. Thus, in order to capture them on our mesh, we need a meshsize $h \approx Re^{3/4}$, and consequently (in 3D) $N = Re^{9/4}$ mesh points. To give the flavor of the Reynolds number, here are some examples (see [77] p. 7)

- model airplane (characteristic length 1 m, characteristic velocity 1 m/s)

  $Re \approx 7 \cdot 10^4$

- cars (characteristic speed 3 m/s)

  $Re \approx 6 \cdot 10^5$

- airplanes (characteristic speed 30 m/s)

  $Re \approx 2 \cdot 10^7$

- atmospheric flows

  $Re \approx 10^{20}$

So for $Re = 10^6$ (a reasonable number for many industrial applications), the number of meshpoints would be $N = 10^{13.5}$. The present computational resources make such a calculation impossible!

Even though the DNS is obviously unsuited for the numerical simulation of turbulent flows, it can be useful to validate turbulence models. Moreover, even if DNS were feasible for turbulent flows, a major hurdle would be defining precise initial and boundary conditions. At high Reynolds numbers the flow is unstable. Thus, even small boundary perturbations may excite the already existing small scales. This results in unphysical noise being introduced in the system, and in the random character of the flow. Indeed, as observed in [3], the uncontrollable nature of the boundary conditions (in terms of wall roughness, wall vibration, differential heating or cooling, etc.) forces the analyst to characterize them as "random forcings" which, consequently, produce random responses.

Since turbulent flows must be simulated numerically with or without mathematical support ("Airplanes Must Fly!!!"), and since DNS approach, based solely on the Navier-Stokes equations, is not suitable (at least at present) for turbulent flows, scientists had to find different approaches . Most of these approaches to the numerical simulation of high Reynolds numbers flow problems are based on the insight gained from the phenomenological (physical) description of turbulence.

As we pointed out in subsection 1.1.2, irregularity is one of the most important features of turbulent flows. Even though it seems impossible to describe the turbulent motion in all details as a function of time and space coordinates, it appears possible to indicate **average** values of flow variables (velocity and pressure). As pointed out in [44], mere observation of turbulent flows and oscillograms of quantities varying turbulently shows that these averages exist, because:

1. At a given point in the turbulent domain a distinct pattern is repeated more or less regularly in time.

2. At a given instant a distinct pattern is repeated more or less regularly in space; so turbulence, broadly speaking has the same over-all structure throughout the domain considered.

Moreover, the details of the motion at the level of small scales are not of interest for most applications in engineering and geophysics. Also, the very data used in practice is an average, too: for example the weather forecasting centers are usually hundreds of kilometers apart.

Motivated by this, Osborne Reynolds developed a statistical approach in 1895 and derived the famous equations that bear his name to describe the dynamics of the "mean" (average) flow.

Formally, the Reynolds equations are obtained from the Navier-Stokes equations by decomposing the velocity $\mathbf{u}$ and the pressure p into a mean (average) component, $\overline{\mathbf{u}}$ and $\overline{p}$ respectively, and a turbulent component (fluctuation), $\mathbf{u}'$ and p' respectively:

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}' \qquad\qquad p = \overline{p} + p' \tag{1.1.2}$$

It is interesting to note that, while this decomposition into means and fluctuations was developed by Reynolds, it was advanced much earlier by, for example, da Vinci in 1510 in his description of vortices trailing a blunt body (as translated by Piomelli): *"Observe the motion of the water surface, which resembles that of hair, that has two motions: One due to the weight of the shaft, the other due to the shape* *of the curls; thus water has eddying motions, one part of which is due to the principal current, the other to the random and reverse motion."*

We have to define the "mean component". There are esentially three ways of defining the mean component, corresponding to the way we average the Navier-Stokes equations.

1. Ensemble Averaging. This is done by performing many physical experiments on the same problem, measuring the velocity and pressure at every time and at every point in the domain, and then averaging over this set of experimental data. This can be done via realization of a physical experiment or a computational simulation with white noise introduced in the problem data. The mean $\overline{\mathbf{u}}$ is well defined as there are the fluctuations $\mathbf{u}' = \mathbf{u} - \overline{\mathbf{u}}$. Ensemble averaging satisfies: $\overline{\overline{\mathbf{u}}} = \overline{\mathbf{u}}$ and $\overline{\mathbf{u}'} = 0$.

2. Time Averaging. This was Reynolds' original idea. Choosing a time scale T, we define the mean flow variables by:

$$\overline{\mathbf{u}}(\mathbf{x}, t) := \frac{1}{T}\int_t^{t+T} \mathbf{u}(\mathbf{x}, \tau) d\tau \qquad\qquad \overline{p}(\mathbf{x}, t) := \frac{1}{T}\int_t^{t+T} p(\mathbf{x}, \tau) d\tau$$

This time scale has to be (see [44] p. 6) sufficiently large compared with the time scale of turbulence, and small compared with the time scale of any slow variations in the flow that we do not want to consider as part of turbulence. For "stationary turbulence", the means are time independent and defined by:

$$\overline{\mathbf{u}} = \lim_{T\to\infty} \frac{1}{T}\int_0^{T} \mathbf{u}(\mathbf{x}, \tau) d\tau.$$

In the limit as $T \to \infty$ the following hold:

$$\overline{\overline{\mathbf{u}}} = \overline{\mathbf{u}}, \qquad \overline{\overline{\mathbf{f}}} = \overline{\mathbf{f}} \qquad \overline{\overline{p}} = \overline{p}, \qquad \overline{\mathbf{u}'} = 0, \qquad \overline{p'} = 0, \qquad \overline{\mathbf{f}'} = 0. \tag{1.1.3}$$

Thus, they are often imposed as an approximation for T large.

Substituting decomposition (1.1.2) into the Navier-Stokes equations, averaging the resulting equations and using (1.1.3) we get the famous **Reynolds equations**:

$$\begin{cases} \partial_t \overline{\mathbf{u}} - Re^{-1}\Delta\overline{\mathbf{u}} + \overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{u}} - \nabla\cdot\tau + \nabla p = \overline{\mathbf{f}} & \text{in } \Omega, \\ \nabla\cdot\overline{\mathbf{u}} = 0 & \text{in } \Omega, \end{cases} \tag{1.1.4}$$

where $\tau = (\tau_{ij})_{ij},\ \tau_{ij} = \overline{\mathbf{u}\mathbf{u}}_{ij} - \overline{\mathbf{u}}_i\, \overline{\mathbf{u}}_j$ is the **Reynolds stress tensor** representing the influence of the energy contained in the small scales upon the mean flow variables. Since τ is symmetric, it contains six unknown variables. Thus, to get a closed system in (1.1.4), we have to model the Reynolds stress tensor τ in terms of $\overline{\mathbf{u}}$ and $\overline{p}$. A presentation of the corresponding models is delayed until Chapter 3.

3. Spatial Averaging This approach uses a space filtering operation applied on the Navier-Stokes equations. As in the time averaging approach, we obtain a system of equations for the mean flow variables. This system is not closed. Thus, as before, we have to use some modeling (approximation) techniques (usually in the Fourier space) in order to get a closed system. The resulting models are called **Large Eddy Simulation (LES)** models. We will present the evolution of these models, as well as an anlysis of two new models in the next chapter.

A good survey of the spatial filters commonly used in LES is given in [3], [16], [83].

Let $\mathbf{f}(\mathbf{x}, t)$ be an instantaneous flow variable (velocity or pressure) in the Navier-Stokes equations, and h denote an averaging kernel. Specifically, $h(0) > 0$, $\int_{\mathbb{R}^d} h(\mathbf{x}) d\mathbf{x} = 1$ with $h(\mathbf{x}) \to 0$ rapidly as $|\mathbf{x}| \to \infty$. The corresponding filtered flow variable is defined by convolution:

$$\overline{\mathbf{f}}(\mathbf{x}, t) := \int_{\mathbb{R}^d} \mathbf{h}(\mathbf{x} - \mathbf{x}')\ \mathbf{f}(\mathbf{x}', t)\ d\mathbf{x}'. \tag{1.1.5}$$

The effect of the filtering operation becomes clear by taking the Fourier transform of expression (1.1.5). By definition, the Fourier transform of $\mathbf{f}$ is:

$$\hat{\mathbf{f}}(\mathbf{k}, t) := \int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x}, t)\ e^{-i\mathbf{k}\mathbf{x}}\ d\mathbf{x}, \tag{1.1.6}$$

where $\mathbf{k}$ represents the wave number vector. As a notation convention, from now on we will denote the Fourier transform of $\mathbf{f}$ by either $\hat{\mathbf{f}}$, or $\mathcal{F}(\mathbf{f})$. By the convolution theorem, we get:

$$\hat{\overline{\mathbf{f}}}(\mathbf{k}, t) = \hat{\mathbf{h}}(\mathbf{k})\ \hat{\mathbf{f}}(\mathbf{k}, t). \tag{1.1.7}$$

Thus, if $\hat{\mathbf{h}} = 0$, for $|\ \mathbf{k}_i\ | > \mathbf{k}_c,\ 1 \le i \le d$, where $\mathbf{k}_c$ is a "cut-off" wave number, all the high wave number components of $\mathbf{f}$ are filtered out by convoluting $\mathbf{f}$ with $\mathbf{h}$. In 1958 Holloway [45] denoted a filter with these characteristics an "Ideal Low Pass Filter". However, if $\hat{\mathbf{h}}$ falls off rapidly (exponentially, say), a cut-off wave number can also be defined for all practical purposes. Several researchers have investigated the properties of different filters in connection with their applicability to the numerical simulation of turbulent flows ([69], [58], [14]).

In addition to the ideal low pass filter, most commonly box filters and Gaussian filters have been used ([24]). The box filter (also known as "moving average" or "top hat filter") is commonly used in practice for experimental or field data.

- **Ideal Low Pass Filter**

$$\mathbf{h}(\mathbf{x}) := \prod_{j=1}^{d} \frac{\sin \dfrac{2\pi \mathbf{x}_j}{\delta}}{\pi \mathbf{x}_j} \tag{1.1.8}$$

$$\hat{\mathbf{h}}(\mathbf{k}) = \begin{cases} 1 & \text{if } \mid \mathbf{k}_j \mid \leq \dfrac{2\pi}{\delta}, \quad \forall\, 1 \leq j \leq d, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.9}$$

- **Box Filter**

$$\mathbf{h}(\mathbf{x}) := \begin{cases} \dfrac{1}{\delta^3} & \text{if } \mid \mathbf{x}_j \mid \leq \dfrac{\delta}{2}, \quad \forall\, 1 \leq j \leq d, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.10}$$

$$\hat{\mathbf{h}}(\mathbf{k}) = \prod_{j=1}^{d} \frac{\sin \dfrac{\delta \mathbf{k}_j}{2}}{\dfrac{\delta \mathbf{k}_j}{2}} \tag{1.1.11}$$

- **Gaussian Filter**

$$\mathbf{h}(\mathbf{x}) := \left(\frac{\gamma}{\pi}\right)^{3/2} \frac{1}{\delta^3} e^{-\dfrac{\gamma \mid \mathbf{x} \mid^2}{\delta^2}} \tag{1.1.12}$$

$$\hat{\mathbf{h}}(\mathbf{k}) = e^{-\dfrac{\delta^2 \mid \mathbf{k} \mid^2}{4\gamma}} \tag{1.1.13}$$

In formulas (1.1.8)–(1.1.13), δ represents the radius of the spatial filter $\mathbf{h}$, and γ is a parameter. For the ideal low pass filter, it can be defined a clear cut-off wave number, equal to $\dfrac{2\pi}{\delta}$. In contrast, the Fourier transform the box filter is a damped sinusoid and thus, spurious "amplitude reversals" are produced by its use in the Fourier space. Finally, the Fourier transform the Gaussian filter is also a Gaussian and decays very rapidly. In fact, for all practical purposes, it is essentially contained in the range $\left[-\dfrac{2\pi}{\delta}, \dfrac{2\pi}{\delta}\right]$.

In the light of the above discussion, we conclude that spatial filtering tends to eliminate from the filtered varaibles the rapidly fluctuating (in space) components, usually characterized as "turbulence". The spatial filter $\mathbf{h}$ and the the spatial as well as temporal derivatives commute (in the absence of boundaries and for constant filter radius). Thus, the space filtered Navier-Stokes equations are:

$$\begin{cases} \partial_t \overline{\mathbf{u}} - Re^{-1}\Delta\overline{\mathbf{u}} + \nabla\cdot(\overline{\mathbf{u}\mathbf{u}}) + \nabla p = \overline{\mathbf{f}} & \text{in } \Omega, \\ \nabla\cdot\overline{\mathbf{u}} = 0 & \text{in } \Omega, \end{cases} \tag{1.1.14}$$

The above system is not closed. Thus, we have to use different techniques, such as approximation and modeling in Fourier space, in order to obtain a closed system for the mean flow variables. The resulting models are called **Large Eddy Simulation (LES) models**. In these models, the motions and interactions of large eddies are computed directly, while the effects of the small eddies on those large eddies are modeled. This approach is motivated by one of the most important features of turbulent flows, **irregularity**. Indeed, homogeneous, isotropic turbulence is (at least when fully developed and sufficiently far away from walls) widely believed to have a random structure. This is consistent with experiments and observations of the mixing property. The fact that it is random suggests that it has an universal character and the effects of the small scales on the larger ones should be modelable and (in the mean at least) predictable. On the other hand, the large eddies in a turbulent flow are widely believed to be <u>deterministic</u>, hence predictable once the effects of the smaller eddies on them is known. It is also widely believed (and evidence to date is in accord) that these large eddies do NOT exhibit exponential sensitivity to perturbations. Further, these large eddies are often the most important flow structures and carry the most energy. LES is based on this idea: model the mean effects of the small scales on the larger ones using their universal features, and then simulate via DNS the motion of the larger ones.

The fundamental questions in LES are:

1. The famous **closure problem**. Writing the velocity as $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$, we get:

$$\overline{\mathbf{u}\mathbf{u}} = \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} + \overline{\overline{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\overline{\mathbf{u}}} + \overline{\mathbf{u}'\mathbf{u}'},$$

representing the decomposition of the averaged nonlinear interactions. In order to obtain a closed system in (**??**) – (**??**), the LES model has to model $\overline{\mathbf{u}\mathbf{u}}$ in terms of $\overline{\mathbf{u}}$ only.

2. Modeling the turbulent fluctuations, $\overline{\mathbf{u}'\mathbf{u}'}$. Although $\overline{\mathbf{u}'\mathbf{u}'}$ represents the interaction of small scales, numerous experiments, both physical and experimental, have shown that it plays an important role in modeling $\overline{\mathbf{u}\mathbf{u}}$. The model used for $\overline{\mathbf{u}'\mathbf{u}'}$ should be faithful to the physics of the turbulent flow.

3. Boundary conditions. Since spatial filtering is NOT a pointwise operation (it involves integration), special care has to be taken in imposing the boundary conditions on $\overline{\mathbf{u}}$. Actually, many studies have reported a serious loss of accuracy in LES near walls. Since the behavior of turbulent motion is crucial for important applications (such as aerospace industry), obtaining realistic boundary conditions is one of the main challenges in LES. A new approach, more consistent with the physics of the turbulent flow, is presented in [30] and [83].

4. Mathematical Foundations. Even though LES is a highly developed field in the engineering and geophysics communities, its mathematical foundations are yet to be set. This is a stringent challenge, since many LES models, based solely on physical intuition (and data fitting!), are pretty far from giving satisfactory answers to the understanding and prediction of turbulence.

5. Numerical Algorithms. By their very nature, turbulent flows are strongly unstable in physical as well as numerical experiments. Thus, advancing the the understanding and prediction of turbulence requires specialized numerical algorithms.

5. Numerical Validation and Testing. This is a very important and subtle issue – it is the ultimate test to assess the quality of our numerical solution.

This thesis is concerned with some of these challenges.

1.2 Chapter Description

Chapter 2 of this thesis analyzes the Galdi-Layton LES model. Introduced in [30], this model is an improvement over the LES model introduced by Clark, Ferziger and Reynolds in [14] in that it uses a closure approximation which better attenuates the small scales in the flow. First we present the evolution of the LES models. Then, we briefly introduce the Galdi-Layton model, pointing out the improvement over [14] in the closure approximation: instead of using a Taylor series expansion in the Fourier space (which actually <u>increases</u> the high wave number components), in [30] a rational (Padé) approximation (<u>attenuating</u> the high wave number components) was used. This different approach results in a different model for the cross terms $\overline{\overline{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\overline{\mathbf{u}}}$. We also present the mathematical analysis (existence, uniqueness and stability of the weak solutions) of the corresponding continuum model, where the turbulent fluctuations $\overline{\mathbf{u}'\mathbf{u}'}$ are modeled by the commonly used Smagorinsky term.

Chapter 3 presents three new models for the turbulent fluctuations $\overline{\mathbf{u}'\mathbf{u}'}$. Introduced in [51] and motivated by Boussinesq assumption, these new models are more faithful to the physics of the turbulent flows. Specifically, the turbulent diffusion vanishes for linear mean velocities and the magnitudes of the turbulent diffusion is proportional to a consistent approximation of the turbulent kinetic energy. For one of this new models we also prove existence of weak solutions for the resulting system (NSE plus the proposed subgrid-scale term). Finally, we show how it can be implemented using finite element methods and prove that its action is no larger than that of the popular Smagorinsky subgrid-scale model.

In chapter 4 we consider the "numerical errors" in LES. Specifically, for one space filtered flow model, we show convergence of the semidiscrete finite element approximation of the model and give an estimate of the error.

Chapter 5 provides a numerical assesment of the Galdi-Layton LES model. Specifically, for the 2D Driven Cavity test problem, for Reynolds numbers ranging between 400 and 10000, we present the numerical results (including graphs of the kinetic energy as well as plots of the streamlines) corresponding to the Galdi-Layton model, Direct Numerical Simulation, the model in [14], and the benchmark results in [34].

Finally, Chapter 6 consists of conclusions and future research.

Chapter 2
The Galdi-Layton LES Model

2.1 Evolution and Present State of LES Models

Developed by the engineering and geophysics communities, LES has emerged as one of the most promising approaches in the numerical simulation of turbulent flows. A detailed presentation of the evolution of the LES models is given in [3], [32] and [71]. We will just mention now the main developements related to the LES models considered in this thesis.

LES was introduced in 1970 by Deardorff [20], who carried out a numerical simulation of the turbulent flow in a channel at infinite Reynolds number. Leonard in 1974 [69], Kwak, Reynolds and Ferziger in 1975 [58], and Clark, Ferziger and Reynolds in 1979 [14] have applied different spatial filters to turbulent flow simulations. Moin, Reynolds and Ferziger in 1978 [78], and Moin and Kim in 1982 [79], studied the near wall region in numerical simulations of turbulent channel flows.

One important class of LES models are the "Scale similarity models", where the subgrid-sclae (SGS) velocity is approximated by the difference between the filtered and twice filtered velocities $\overline{\mathbf{u}} - \overline{\overline{\mathbf{u}}}$. These models were introduced in 1980 by Bardina, Ferziger and Reynolds [6].

But probably the most commonly used LES model is the "dynamic eddy viscosity model" introduced in 1991 by Germano, Piomelli, Moin and Cabot [33]. In this model, the eddy viscosity coefficient is computed dynamically as the numerical computations evolve rather than imposed *a priori*, and depends on the energy contained in the smallest resolved scales. The approximation to the SGS stresses, so computed, are often observed to nearly vanish in laminar flows and at solid boundaries. It has been used in LES of transitional and fully-developed turbulent channel flows and extended to compressible flows, too.

From the above presentation, it is clear that LES for turbulent flows has a very rich history, being used in a wide range of applications and having associated with, as a result, a well documented data base. However, it is the author's belief that despite the undeniable achievements of the engineering and geophysics communities in developing and using LES, there is an urgent need for a rigorous, more mathematical approach to LES. Specifically, although physical insight has to be used in developing LES models, it is desirable to devise models with greater universality, meaning, which work more generally and contain fewer problem dependent parameters.

This chapter is concerned with the analysis of such a model.

2.2 The Galdi-Layton LES Model

This section briefly presents a traditional LES model (introduced by Clark, Ferziger and Reynolds in [14], analyzed mathematically by Coletti in [15], [16], and numerically Cantekin, Westerink and Luetich in [10]) and the new LES model introduced by Galdi and Layton in [30], pointing out the essential difference in their derivation.

Consider an incompressible viscous fluid flowing in a bounded domain Ω in $\mathbb{R}^3$ and driven by body forces and/or boundary velocities. In nondimensionalized terms, its velocity $\mathbf{u}$ and pressure p are solutions of the Navier-Stokes equations, given by:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \mathrm{Re}^{-1}\Delta \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega \times [0,T], \\ \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \Omega \times [0,T], \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{g} & \text{on} \quad \partial\Omega, \end{cases}$$

where $\int_\Omega p(\mathbf{x})d\mathbf{x} = 0$. The spatial averages of the flow variables are obtained through convolution with a spatial filter; one common filter, which we select herein, is the Gaussian filter:

$$g_\delta(\mathbf{x}) := \left(\frac{\gamma}{\pi}\right)^{3/2} \frac{1}{\delta^3}\, e^{-\gamma \frac{|\mathbf{x}|^2}{\delta^2}}, \tag{2.2.1}$$

where γ is a constant (often $\gamma = 6$) and δ is the averaging radius. Extending all variables by zero outside Ω, the convolution $\overline{\mathbf{u}} = g_\delta * \mathbf{u}$ represents the eddies of size $O(\delta)$ or larger. For constant filter width δ, differentiation and convolution commute. Thus, applying this averaging operator to the Navier-Stokes equations gives the set of space-filtered Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \overline{\mathbf{u}}}{\partial t} - \mathrm{Re}^{-1}\Delta\overline{\mathbf{u}} + \nabla\cdot(\overline{\mathbf{u}\mathbf{u}}) + \nabla\overline{p} = \overline{\mathbf{f}} & \text{in} \quad \Omega\times[0,T], \\ \nabla\cdot\overline{\mathbf{u}} = 0 & \text{in} \quad \Omega\times[0,T], \\ \overline{\mathbf{u}}(\mathbf{x},0) = \overline{\mathbf{u}}_0(\mathbf{x}) & \text{in} \quad \Omega, \\ \overline{\mathbf{u}}(\mathbf{x},t) = (g_\delta * \mathbf{u})(\mathbf{x},t) & \text{on} \quad \partial\Omega. \end{cases} \tag{2.2.2}$$

Letting $\overline{\mathbf{u}} = g_\delta * \mathbf{u}$, $\mathbf{u}' = \mathbf{u} - \overline{\mathbf{u}}$, the nonlinear interaction term $\overline{\mathbf{u}\mathbf{u}} = \overline{(\overline{\mathbf{u}}+\mathbf{u}')(\overline{\mathbf{u}}+\mathbf{u}')}$ can be decomposed into three parts:

$$\overline{\mathbf{u}\mathbf{u}} = \overline{(\overline{\mathbf{u}}+\mathbf{u}')(\overline{\mathbf{u}}+\mathbf{u}')} = \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} + \overline{\overline{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\overline{\mathbf{u}}} + \overline{\mathbf{u}'\mathbf{u}'} \tag{2.2.3}$$

Thus, developing a continuum model for the motion of large eddies has minimally two essential ingredients: an approximation for the outer convolution and an approximation for $\mathbf{u}'$ in terms of $\overline{\mathbf{u}}$. The system (2.2.2) cannot be directly solved due to the well known closure problem. Continuum models used for LES are an aproximation to $\overline{\mathbf{u}}$ in (2.2.2). Thus, it is worthwile to consider the essential properties of solutions of (2.2.2) which we seek (in so far as possible) to be retained in solutions of LES models.

Proposition 2.2.1 *Let $(\mathbf{u}(\mathbf{x},t), p(\mathbf{x},t))$ be weak solutions of the Navier-Stokes equations. Then,*

(i) $\overline{\mathbf{u}}(\mathbf{x},t)$ is infinitely differentiable in space, $\overline{\mathbf{u}}$ in $C^\infty(\Omega)$,
(ii) $\overline{\mathbf{u}} \to \mathbf{u}$ in $L^2(\Omega)$ as $\delta \to 0$, and
(iii) the kinetic energy in $\overline{\mathbf{u}}$ is bounded by that of $\mathbf{u}$:

$$\frac{1}{2}\int_\Omega |\,\overline{\mathbf{u}}\,|^2\, d\mathbf{x} \leq C\int_\Omega |\,\mathbf{u}\,|^2\, d\mathbf{x} \leq C(Re,\mathbf{f},\mathbf{u}_0) < \infty \tag{2.2.4}$$

Proof: (i) and (ii) follow by standard properties of convolution operators (see, e.g., [46]), while (iii) follows from Young's inequality for convolutions:

$$\begin{aligned} \frac{1}{2}\int_\Omega |\,\overline{\mathbf{u}}\,|^2\, d\mathbf{x} \;&\leq\; \frac{1}{2}\int_{\mathbb{R}^3} |\,\overline{\mathbf{u}}\,|^2\, d\mathbf{x} \leq C\int_{\mathbb{R}^3} |\,\mathbf{u}\,|^2\, d\mathbf{x} = C\int_\Omega |\,\mathbf{u}\,|^2\, d\mathbf{x} \\ &\leq\; C(Re,\mathbf{f},\mathbf{u}_0) < \infty \end{aligned}$$

■

To perform a large eddy simulation, closure in (2.2.2) must be addressed to obtain a system whose solution approximates $(\overline{\mathbf{u}}, \overline{p})$. Many approximations are possible see, e.g., Aldama [3], Sagaut [82]. Proposition 2.2.1 gives a quantitative criterion for comparing different models: if the global kinetic energy in the model approximating $\overline{\mathbf{u}}$ is significantly larger than that of a DNS approximation to $\mathbf{u}$, the model needs to be reconsidered. Further the distribution of energy is also important. By Proposition 2.2.1 (i) it should be concentrated in the low frequencies with corresponding attenuation of high frequencies.

With this in mind, consider the closure problem in (2.2.2). Equation (2.2.3) is a decomposition of the averaged nonlinear interactions into "resolved sclaes", $\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}}$, "cross terms", $\overline{\overline{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\overline{\mathbf{u}}}$ describing the interaction of large and small eddies, and "subgrid scale" term $\overline{\mathbf{u}'\mathbf{u}'}$, describing the effects of the interaction of small eddies on the mean flow.

Proposition 2.2.2 *For smooth functions* $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\begin{aligned} \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} &= O(1), \\ \overline{\overline{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\overline{\mathbf{u}}} &= O(\delta^2), \quad \textit{and} \\ \overline{\mathbf{u}'\mathbf{u}'} &= O(\delta^4), \quad \textit{specifically} \quad \|\overline{\mathbf{u}'\mathbf{u}'}\| \le C\delta^4 \|\nabla\nabla\mathbf{u}\|^2 \end{aligned}$$

Proof: The proofs of all three are by similar Fourier methods. Thus, for compactness, we will give only the last one. Since $\mathbf{u}' = \mathbf{u} - \overline{\mathbf{u}}$,

$$\begin{aligned} \|\overline{\mathbf{u}'\mathbf{u}'}\| &= \|\mathcal{F}(\overline{\mathbf{u}'\mathbf{u}'})\| = \|\hat{g}_\delta \hat{\mathbf{u}}' * \hat{\mathbf{u}}'\| \\ &= \|\hat{g}_\delta (\hat{\mathbf{u}} - \hat{\overline{\mathbf{u}}}) * (\hat{\mathbf{u}} - \hat{\overline{\mathbf{u}}})\| \end{aligned}$$

Since $\max \mid \hat{g}_\delta \mid \le 1$, by standard properties of convolution operators (see Corollary 4.5.2 in [46]), we have $\|\overline{\mathbf{u}'\mathbf{u}'}\| \le \|\hat{\mathbf{u}} - \hat{\overline{\mathbf{u}}}\|^2$.

Now consider $\|\hat{\mathbf{u}} - \hat{\overline{\mathbf{u}}}\| = \|(1 - \hat{g}_\delta)\hat{\mathbf{u}}\|$. Expanding $1 - \hat{g}_\delta$ in a Taylor series in $\mathbf{k}$, note that $\hat{g}_\delta(\mathbf{k}) = \mathbf{1} - \frac{\delta^{\mathbf{2}}}{\mathbf{4}\gamma} \mid \mathbf{k} \mid^{\mathbf{2}} (+\mathbf{O}(\delta^{\mathbf{4}} \mid \mathbf{k} \mid^{\mathbf{4}}))$.

Thus, by a standard approximation theoretic argument $\|\hat{\mathbf{u}} - \hat{\overline{\mathbf{u}}}\| \le C\delta^2 \|\nabla\nabla\mathbf{u}\|$. We finally obtain

$$\|\overline{\mathbf{u}'\mathbf{u}'}\| \le C\delta^4 \|\nabla\nabla\mathbf{u}\|^2,$$

completing the proof.

■

This proposition gives insight into the selection of subgrid scale models. If the subgrid scale model for $\nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'})$ is $O(\delta^2)$ on the smooth components of the flow, then it is not a faithful model of turbulent fluctuations. An $O(\delta^2)$ subgrid scale model will dominate the model of the larger cross terms in calculations. Accordingly, it is important to perform experiments in which the subgrid scale model is either absent or exceedingly small to test the model of the cross terms.

We start now presenting the derivation of the classical LES model. It is worthwile to note that this model evolved in several steps. First, in 1974 Leonard [69] developed a continuum model of $\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}}$:

$$\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} = \overline{\mathbf{u}}\,\overline{\mathbf{u}} + \frac{\delta^2}{4\gamma}\Delta(\overline{\mathbf{u}}\,\overline{\mathbf{u}}) + O(\delta^4)$$

Using this in (2.2.3) and dropping the second, third and fourth terms on the RHS of (2.2.3) gives a first LES model. Next, in 1979 Clark, Ferziger and Reynolds [14] developed an anlogous model for the cross terms $\overline{\overline{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\overline{\mathbf{u}}}$. These two models were combined and rederived in a unified manner in Aldama [3]. We will refer to this combination as the "clasical" model. Typically, the last term $\overline{\mathbf{u}'\mathbf{u}'}$ is modeled by a nonlinear diffusion mechanism.

The modeling technique used in the derivation of the classical LES model is employing closure approximation in Fourier space in (2.2.3) (see , e.g., [69], [14], [3]). For example, for the first term on the RHS of (2.2.3), we have:

$$\begin{aligned}
\mathcal{F}(\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}}) &= \hat{g_\delta} * \mathcal{F}(\overline{\mathbf{u}}\overline{\mathbf{u}}) = \hat{g_\delta}(\hat{\overline{\mathbf{u}}} * \hat{\overline{\mathbf{u}}}) \\
&= e^{-\frac{\delta^2}{4\gamma}|\mathbf{k}|^2}(\hat{\overline{\mathbf{u}}} * \hat{\overline{\mathbf{u}}}).
\end{aligned}$$

Using a Taylor series approximation to the exponential, gives:

$$\mathcal{F}(\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}}) = \left(1 - \frac{\delta^2}{4\gamma} \mid \mathbf{k} \mid^2\right)(\hat{\overline{\mathbf{u}}} * \hat{\overline{\mathbf{u}}}) + O(\delta^4)$$

Proceeding in a similar manner for the other three terms, dropping all terms (formally) of order $O(\delta^4)$ or higher, and then taking the inverse Fourier transform (see [3] for details), we get the classical space-filtered LES model used in many studies, e.g., [3], [16], [10], etc.:

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} - \mathrm{Re}^{-1}\Delta \mathbf{w} + \nabla\cdot(\mathbf{w}\mathbf{w}) + \nabla q + \nabla\cdot\left(\dfrac{\delta^2}{2\gamma}\nabla\mathbf{w}\nabla\mathbf{w}\right) = \mathbf{f} & \text{in} \quad \Omega\times[0,T], \\ \nabla\cdot\mathbf{w} = 0 & \text{in} \quad \Omega\times[0,T], \\ \mathbf{w}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) & \text{in} \quad \Omega, \\ \mathbf{w}(\mathbf{x},t) = (g_\delta * \mathbf{u})(\mathbf{x},t) & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2.5}$$

where $(\mathbf{w}, q)$ are an approximation to $(\overline{\mathbf{u}}, \overline{p})$ and $\nabla\mathbf{w}\nabla\mathbf{w}$ is shorthand for $\sum_l \frac{\partial \mathbf{w}_i}{\partial \mathbf{x}_l}\frac{\partial \mathbf{w}_j}{\partial \mathbf{x}_l}$ and model the cross terms in (2.2.3). The turbulent fluctuations $\overline{\mathbf{u}'\mathbf{u}'}$ in (2.2.3) are usually modeled by a Smagorinsky term of the form $C_s\delta^2|\mathbb{D}(\overline{\mathbf{u}})|\mathbb{D}(\overline{\mathbf{u}})$, which is added to the LHS of the first equation in (2.2.5) (here $\mathbb{D}(\overline{\mathbf{u}}) := \frac{1}{2}(\nabla\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^t)$ is the deformation tensor associated with $\overline{\mathbf{u}}$). A detailed presentation of the Smagorinsky term is given in Section 3.1. The Smagorinsky [84] model is also common and popular due to its simplicity and good stability properties. Thus, the classical LES model becomes:

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} - \mathrm{Re}^{-1}\Delta \mathbf{w} + \nabla\cdot(\mathbf{w}\mathbf{w}) + \nabla q + \nabla\cdot\left(\dfrac{\delta^2}{2\gamma}\nabla\mathbf{w}\nabla\mathbf{w}\right) & \\ \qquad\qquad - \nabla\cdot\left(C_s\delta^2|\nabla\mathbf{w}|\nabla\mathbf{w}\right) = \bar{\mathbf{f}} & \text{in} \quad \Omega\times[0,T], \\ \nabla\cdot\mathbf{w} = 0 & \text{in} \quad \Omega\times[0,T], \\ \mathbf{w}(\mathbf{x},0) = \overline{\mathbf{u}}_0(\mathbf{x}), & \text{in} \quad \Omega, \\ \mathbf{w}(\mathbf{x},t) = (g_\delta * \mathbf{u})(\mathbf{x},t), & \text{on} \quad \partial\Omega. \end{cases}$$

The Taylor series approximation used in the derivation of the above model is, however, *inconsistent* with Proposition 2.2.1 (i)'s required attenuation of high frequencies in $\overline{\mathbf{u}}$. It actually *increases* the high wave number components (large $|\ \mathbf{k}\ |$), whereas the original function $\hat{g_\delta} = e^{-\frac{\delta^2}{4\gamma}|\mathbf{k}|^2}$ *decreases* the high wave number components, see figure below.

This incorrect stimulation of high frequencies plays an important role in numerical calculations at high Reynolds numbers, since increasing the high wave number components (large $|\ \mathbf{k}\ |$) in the Fourier space is equivalent to increasing the small scales (small $|\ \mathbf{x}\ |$) in the physical space ($\mathbb{R}^3$). We believe that this is the true

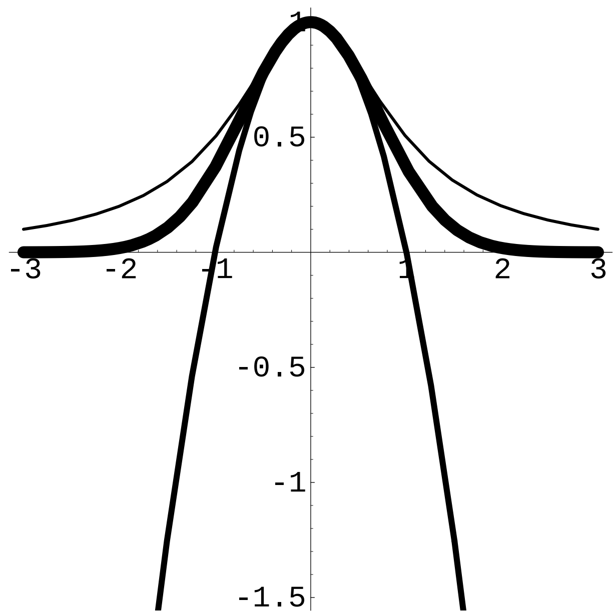

Figure 2.1: (a) Fourier transform of the Gaussian filter - thickest line; (b) Traditional approximation by Taylor series - second to thickest line; (c) New approximation by rational function - thinest line.

cause of the need for powerful dissipative mechanisms, such as the $O(\delta^2)$ Smagorinsky model for $\overline{\mathbf{u}'\mathbf{u}'}$.

Motivated by the above observation, Galdi and Layton proposed in [30] a modified model which is consistent with the required attenuation of high frequencies in $\overline{\mathbf{u}}$. This model is based upon a rational approximation to $\hat{g_\delta}$, such as the (0,1) Padé:

$$\hat{g_\delta} = e^{-\frac{\delta^2}{4\gamma}|\mathbf{k}|^2} = \frac{1}{1 + \frac{\delta^2}{4\gamma} \mid \mathbf{k} \mid^2} + O(\delta^4). \tag{2.2.6}$$

The resulting model is given by:

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} - \mathrm{Re}^{-1}\Delta \mathbf{w} + \nabla \cdot (\mathbf{w}\mathbf{w}) + \nabla q + \nabla \cdot \left[\left(-\dfrac{\delta^2}{4\gamma}\Delta + I \right)^{-1} \left(\dfrac{\delta^2}{2\gamma} \nabla \mathbf{w} \nabla \mathbf{w} \right) \right] = \bar{\mathbf{f}}, & \text{in} \quad \Omega \times [0,T], \\ \nabla \cdot \mathbf{w} = 0, & \text{in} \quad \Omega \times [0,T], \\ \mathbf{w}(\mathbf{x}, 0) = \mathbf{w}_0(\mathbf{x}), & \text{in} \quad \Omega, \\ \mathbf{w}(\mathbf{x}, t) = (g_\delta * \mathbf{w})(\mathbf{x}, t), & \text{on} \quad \partial\Omega. \end{cases}$$

Actually, there are two natural variants of this model:
1) Galdi-Layton with convolution, where the smoothing operator $\left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1}$ is replaced by smoothing by direct convolution with the Gaussian filter;
2) Galdi-Layton with auxiliary problem, where the inverse operator is calculated directly, solving a discrete Poisson problem.

As for the classical LES model, the turbulent fluctuations are modeled by a Smagorinsky term. Thus, the Galdi-Layton LES model we consider is:

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} - \mathrm{Re}^{-1}\Delta\mathbf{w} + \nabla\cdot(\mathbf{w}\mathbf{w}) + \nabla q + \nabla\cdot\left[\left(-\dfrac{\delta^2}{4\gamma}\Delta + I\right)^{-1}\left(\dfrac{\delta^2}{2\gamma}\nabla\mathbf{w}\nabla\mathbf{w}\right)\right] & \\ \qquad\qquad - \nabla\cdot\left(C_s\delta^2|\nabla\mathbf{w}|\nabla\mathbf{w}\right) = \bar{\mathbf{f}}, \qquad \text{in} \quad \Omega, & \\ \nabla\cdot\mathbf{w} = 0, & \text{in} \quad \Omega, \\ \mathbf{w}(\mathbf{x},0) = \overline{\mathbf{u}}_0(\mathbf{x}), & \text{in} \quad \Omega, \\ \mathbf{w}(\mathbf{x},t) = (g_\delta * \mathbf{u})(\mathbf{x},t), & \text{on} \quad \partial\Omega. \end{cases}$$

Before moving to the mathematical analysis of the Galdi-Layton model, we wish to note one peculiar feature of the smooth solutions of the classical model (2.2.5) for 2D flows with periodic in space boundary conditions.

Lemma 2.2.1 *Let* $\mathbf{w}$ *be a* C^1 *L-periodic in space function in 2 dimensions (i.e.* $\mathbf{w}(\mathbf{x}+L\mathbf{e}_i,t) = \mathbf{w}(\mathbf{x},t), \quad i=1,2, \quad \forall\mathbf{x}\in\mathbb{R}^2, \quad \forall t>0,$ *where* $\mathbf{e}_1, \mathbf{e}_2$ *is the canonical basis of* $\mathbb{R}^2$*, and* L *is the period in the i-th direction), with* $\nabla\cdot\mathbf{w} = 0$*. Then,*

$$\sum_{i,j,l=1}^{2}\int_0^L\int_0^L \frac{\partial\mathbf{w}_i}{\partial\mathbf{x}_l}\frac{\partial\mathbf{w}_j}{\partial\mathbf{x}_l}\frac{\partial\mathbf{w}_i}{\partial\mathbf{x}_j}d\mathbf{x}d\mathbf{y} = 0.$$

The same property is <u>*not*</u> *true in 3 dimensions:* $(\nabla\mathbf{w}\nabla\mathbf{w}, \nabla\mathbf{w})$ *does not vanish identically in 3D.*

Proof: The first claim is a simple index calculation. Indeed:

$$\begin{aligned} \sum_{i,j,l=1}^{2}\frac{\partial\mathbf{w}_i}{\partial\mathbf{x}_l}\frac{\partial\mathbf{w}_j}{\partial\mathbf{x}_l}\frac{\partial\mathbf{w}_i}{\partial\mathbf{x}_j} &= \left(\frac{\partial\mathbf{w}_1}{\partial\mathbf{x}_1}\frac{\partial\mathbf{w}_1}{\partial\mathbf{x}_1} + \frac{\partial\mathbf{w}_1}{\partial\mathbf{x}_2}\frac{\partial\mathbf{w}_1}{\partial\mathbf{x}_2}\right)\frac{\partial\mathbf{w}_1}{\partial\mathbf{x}_1} + \left(\frac{\partial\mathbf{w}_2}{\partial\mathbf{x}_1}\frac{\partial\mathbf{w}_1}{\partial\mathbf{x}_1} + \frac{\partial\mathbf{w}_2}{\partial\mathbf{x}_2}\frac{\partial\mathbf{w}_1}{\partial\mathbf{x}_2}\right)\frac{\partial\mathbf{w}_2}{\partial\mathbf{x}_1} \\ &+ \left(\frac{\partial\mathbf{w}_1}{\partial\mathbf{x}_1}\frac{\partial\mathbf{w}_2}{\partial\mathbf{x}_1} + \frac{\partial\mathbf{w}_1}{\partial\mathbf{x}_2}\frac{\partial\mathbf{w}_2}{\partial\mathbf{x}_2}\right)\frac{\partial\mathbf{w}_1}{\partial\mathbf{x}_2} + \left(\frac{\partial\mathbf{w}_2}{\partial\mathbf{x}_1}\frac{\partial\mathbf{w}_2}{\partial\mathbf{x}_1} + \frac{\partial\mathbf{w}_2}{\partial\mathbf{x}_2}\frac{\partial\mathbf{w}_2}{\partial\mathbf{x}_2}\right)\frac{\partial\mathbf{w}_2}{\partial\mathbf{x}_2} \end{aligned}$$

Multiplying out, the R.H.S. of the above expression becomes a sum of 8 terms. Using

$$\nabla \cdot \mathbf{w} = \frac{\partial \mathbf{w}_1}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{w}_2}{\partial \mathbf{x}_2} = 0,$$

the first and the eighth term on the R.H.S. cancel out; the second and the sixth term on the R.H.S. cancel out; the third and the seventh term on the R.H.S. cancel out; the fourth and the fifth term on the R.H.S. cancel out. Thus, the whole R.H.S. cancels out.

For the second claim, it is straightforward to simply choose smooth, periodic vector functions $\mathbf{w}$, calculate $(\nabla\mathbf{w}\nabla\mathbf{w}, \nabla\mathbf{w})$, and verify that it does not identically vanish. Indeed, choosing:

$$\begin{aligned}
\mathbf{w}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \\
&= \{Sin(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3),\ Cos(\mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_3), \\
&\quad -2Cos(\mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_3) - Sin(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)\},
\end{aligned}$$

it is obvious that $\mathbf{w}$ is a periodic function of period 2π in 3D. Moreover, $\nabla \cdot \mathbf{w} = 0$. However, $(\nabla\mathbf{w}\nabla\mathbf{w}, \nabla\mathbf{w})$ does not vanish identically (for example, $(\nabla\mathbf{w}\nabla\mathbf{w}, \nabla\mathbf{w})(0, 1, 0)$ $= -3.23722$, roughly).

■

Using the above lemma, we can prove an interesting bound on the kinetic energy of smooth 2D solutions of the classical model under periodic in space boundary conditions.

Proposition 2.2.3 *Let* $(\mathbf{w}, q)$ *be a smooth, classical solution of the model (2.2.5) under* periodic in space boundary conditions *in* two dimensions*. Then, the kinetic energy in* $\mathbf{w}$ *is bounded by problem data:*

$$\frac{1}{2}\int_\Omega |\ \mathbf{w}(\mathbf{x}, t)\ |^2\, d\mathbf{x} \leq \frac{1}{2}\int_\Omega |\ \mathbf{w}(\mathbf{x}, 0)\ |^2\, d\mathbf{x} + C(Re, \mathbf{f}).$$

Proof: Multiply (2.2.5) by $\mathbf{w}$, integrate over Ω, integrate by parts as necessary, and use Lemma 2.2.1. This gives:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\ \mathbf{w}(\mathbf{x}, t)\ |^2\, d\mathbf{x} + Re^{-1}\int_\Omega |\ \nabla\mathbf{w}(\mathbf{x}, t)\ |^2\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{w} d\mathbf{x},$$

from which the result follows.

■

Lemma 2.2.1 describes an exact cancellation property of the kinetic energy contribution to the large eddies by their interaction with small eddies in the classical model for 2D smooth periodic in space solutions of the classical model. How is this to be reconciled with the clear picture developed of the model as (incorrectly) stimulating the kinetic energy in high frequencies of $\mathbf{w}$? Our hypothesis (tested numerically in Chapter 5) is clear. The classical model redistributes (incorrectly) the kinetic energy in 2D and 3D, but perhaps also augments it in 3D.

The simplest test problem which fits Proposition 2.2.3's assumption in every respect except the boundary conditions is the 2D driven cavity. We thus choose this as being most favorable to the classic model and would anticipate the failure of the classical model to be more severe in 3D.

2.3 Mathematical Analysis of the New Model

First, let us give a short survey of the corresponding mathematical analysis for the classical LES model (for a detailed analysis, see Coletti [15], [16]). For the classical model with the turbulent fluctuations modeled by a Smagorinsky term, Coletti has proved (Theorem 16 in [16]) that there exists a weak solution, provided that the power μ of the norm of the deformation tensor in the Smagorinsky term satisfies $\mu > 0.5$, and that the coeficient C_s is "large" compared with δ. If these conditions are satisfied, Coletti also proved the uniqueness (Theorem 17 in [16]) and stability (Theorem 19 in [16]) of the weak solution.

2.3.1 Existence of Weak Solutions

In this section we will prove the existence of a weak solution of the Galdi-Layton LES model with the turbulent fluctuations modeled by a Smagorinsky term.

$$
\begin{cases}
\dfrac{\partial \mathbf{w}}{\partial t} - \mathrm{Re}^{-1}\Delta \mathbf{w} + \nabla\cdot(\mathbf{w}\mathbf{w}) + \nabla q + \nabla\cdot\left[\left(-\dfrac{\delta^2}{4\gamma}\Delta + I\right)^{-1}\left(\dfrac{\delta^2}{2\gamma}\nabla\mathbf{w}\nabla\mathbf{w}\right)\right] \\
\qquad\qquad\qquad - \nabla\cdot\left(C_s\delta^2|\mathbb{D}(\mathbf{w})|\mathbb{D}(\mathbf{w})\right) = \bar{\mathbf{f}}, \qquad \text{in} \quad \Omega, \\
\nabla\cdot\mathbf{w} = 0, \qquad \text{in} \quad \Omega, \\
\mathbf{w}(\mathbf{x},0) = \bar{\mathbf{u}}_0(\mathbf{x}), \qquad \text{in} \quad \Omega, \\
\mathbf{w}(\mathbf{x},t) = (g_\delta * \mathbf{u})(\mathbf{x},t), \quad \text{on} \quad \partial\Omega.
\end{cases} \tag{2.3.7}
$$

For clarity of presentation, in the sequel we shall replace $\mathbb{D}(\mathbf{w})$ by $\nabla\mathbf{w}$ and consider periodic in space boundary conditions, as defined in Lemma 2.2.1. According to a note in [86], p.4, periodic in space boundary conditions lead to "a simpler functional setting, while many of the mathematical difficulties remain unchanged (except, of course those related to the boundary layer difficulty, which vanish)".

We prove existence of weak solutions for a small μ, specifically for $\mu \geq 0.1$. This reduction of μ is an improvement over the restriction for the classical model ($\mu \geq 0.5$).

We start by proving two a priori estimates. Due to the highly nonlinear term occuring in LES models, more a priori bounds are needed than in the case of the Navier-Stokes equations.

Lemma 2.3.1 *(First a priori estimate). Assume* $||\bar{\mathbf{u}}_0||^2 \leq C_s/3c$, *and* $\frac{Re}{2}\int_0^T ||\bar{\mathbf{f}}||^2 ds \leq C_s/3c$. *Any weak solution of (2.3.7) satisfies*

$$
||\mathbf{w}(t)||^2 + \frac{Re^{-1}}{2}\int_0^t ||\nabla\mathbf{w}||^2 ds \leq \frac{2C_s}{c}, \ \forall t \in [0,T], \tag{2.3.8}
$$

where c *is a positive constant depending only on the size of the domain* Ω *and on the constants of our problem* (C_s, Re, μ, δ *and* γ).

Proof: Let $s = 1+\mu$, and $\mathbf{v} := \left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1}\left(\frac{\delta^2}{2\gamma}\nabla\mathbf{w}\nabla\mathbf{w}\right)$. Multiplying (2.3.7) by $\mathbf{w}$, and integrating over Ω, we get:

$$
\frac{d}{dt}||\mathbf{w}||^2 = -Re^{-1}||\nabla\mathbf{w}||^2 - (\mathbf{v},\nabla\mathbf{w}) + (\bar{\mathbf{f}},\mathbf{w}) - C_s||\nabla\mathbf{w}||_{2s}^{2s} \tag{2.3.9}
$$

Case 1: $1 < s < \dfrac{3}{2}$. By the definition of $||\cdot||_{-1,\left(\dfrac{3s}{3-s}\right)'}$, we have:

$$|(\mathbf{v}, \nabla \mathbf{w})| \leq ||\mathbf{v}||_{1,\dfrac{3s}{3-s}} \; ||\nabla \mathbf{w}||_{-1,\left(\dfrac{3s}{3-s}\right)'} \tag{2.3.10}$$

By the Sobolev Embedding theorem and Elliptic Regularity, we get:

$$||\mathbf{v}||_{1,\dfrac{3s}{3-s}} \leq c||\mathbf{v}||_{2,s} \leq c||\nabla \mathbf{w} \nabla \mathbf{w}||_s = c||\nabla \mathbf{w}||_{2s}^2 \tag{2.3.11}$$

By the definition of the spaces involved, we also have:

$$||\nabla \mathbf{w}||_{-1,(\frac{3s}{3-s})'} \leq ||\mathbf{w}||_{(\frac{3s}{3-s})'} \tag{2.3.12}$$

Using (2.3.10), (2.3.11), (2.3.12), and the Cauchy-Schwarz inequality, we get:

$$\frac{d}{dt}||\mathbf{w}||^2 \leq \frac{-Re^{-1}}{2}||\nabla \mathbf{w}||^2 - C_s||\nabla \mathbf{w}||_{2s}^{2s} + c||\nabla \mathbf{w}||_{2s}^2 ||\mathbf{w}||_{\frac{3s}{4s-3}} + \frac{Re}{2}||\bar{\mathbf{f}}||^2 \tag{2.3.13}$$

Since $1 < s < \dfrac{3}{2}$, we have:

$$||\mathbf{w}||_{\frac{3s}{4s-3}} = ||\mathbf{w}||_{\frac{3s}{4s-3}}^{2(s-1)} \; ||\mathbf{w}||_{\frac{3s}{4s-3}}^{3-2s} \tag{2.3.14}$$

We now distinguish the following two subcases:
Subcase (i) $\dfrac{3s}{4s-3} \leq 2$ or, equivalently, $s \geq \dfrac{6}{5}$.
Thus,

$$||\mathbf{w}||_{\frac{3s}{4s-3}}^{3-2s} \leq c \; ||\mathbf{w}||_2^{3-2s} \tag{2.3.15}$$

Using the Sobolev Embedding theorem and Poincaré's inequality for periodic functions with zero mean gives:

$$||\mathbf{w}||_{\frac{3s}{4s-3}}^{2(s-1)} \leq c \; ||\mathbf{w}||_{1,\frac{3s}{5s-3}}^{2(s-1)} \leq c \; ||\mathbf{w}||_{1,2s}^{2(s-1)} \leq c \; ||\nabla \mathbf{w}||_{2s}^{2(s-1)} \tag{2.3.16}$$

By (2.3.14), (2.3.15) and (2.3.16), we have

$$||\mathbf{w}||_{\frac{3s}{4s-3}} \leq c \; ||\nabla \mathbf{w}||_{2s}^{2(s-1)} \; ||\mathbf{w}||_2^{3-2s} \tag{2.3.17}$$

Putting together (2.3.13) and (2.3.17), we get:

$$\frac{d}{dt} \; ||\mathbf{w}||^2 \leq -\frac{Re^{-1}}{2} \; ||\nabla \mathbf{w}||^2 - C_s||\nabla \mathbf{w}||_{2s}^{2s}(1 - c||\mathbf{w}||^{3-2s}) + \frac{Re}{2}||\bar{\mathbf{f}}||^2, \tag{2.3.18}$$

for all $s \in \left(\frac{6}{5}, \frac{3}{2}\right)$.

Subcase (ii) $\dfrac{3s}{4s-3} > 2$ or, equivalently, $s < \dfrac{6}{5}$.
Applying the interpolation inequality given by Lemma 2.2' [G94], we get:

$$|\mathbf{w}|_{0,\frac{3s}{4s-3}} \leq c\ |\mathbf{w}|_{1,2s}^{\frac{5s-6}{3-5s}}\ ||\mathbf{w}||_2^{1-\frac{5s-6}{3-5s}}. \tag{2.3.19}$$

Now, for $s \in \left[\frac{11}{10}, \frac{6}{5}\right)$, we also have:

$$\frac{5s-6}{3-5s} \leq 2(s-1). \tag{2.3.20}$$

Thus, from (4.3.11) and (4.3.12), we get:

$$||\mathbf{w}||_{\frac{3s}{4s-3}} \leq c\ ||\nabla\mathbf{w}||_{2s}^{2(s-1)}\ ||\mathbf{w}||_2^{1-\frac{5s-6}{3-5s}}. \tag{2.3.21}$$

By (2.3.13), (4.3.12), and using $s = 1+\mu$, we get:

$$\frac{d}{dt}||\mathbf{w}||^2 \leq -\frac{Re^{-1}}{2}\ ||\nabla\mathbf{w}||^2 - C_s||\nabla\mathbf{w}||_{2s}^{2s}\left(1 - c||\mathbf{w}||^{1-\frac{5s-6}{3-5s}}\right) + \frac{Re}{2}||\bar{\mathbf{f}}||^2, \tag{2.3.22}$$

for all $s \in \left(\frac{11}{10}, \frac{6}{5}\right)$.

Case 2: $s \geq \dfrac{3}{2}$ Using Hölder's inequality, gives:

$$|(\mathbf{v}, \nabla\mathbf{w})| \leq\ ||\mathbf{v}||_{\frac{2s}{2s-1}}\ ||\nabla\mathbf{w}||_{2s}. \tag{2.3.23}$$

Elliptic Regularity thus implies:

$$||\mathbf{v}||_{\frac{2s}{2s-1}} \leq ||\mathbf{v}||_{2,\frac{2s}{2s-1}} \leq c||\nabla\mathbf{w}\nabla\mathbf{w}||_{\frac{2s}{2s-1}} = c||\nabla\mathbf{w}||^2_{\frac{4s}{2s-1}}. \tag{2.3.24}$$

Thus, by (4.3.15) and (4.3.16), there follows:

$$|(\mathbf{v}, \nabla\mathbf{w})| \leq ||\nabla\mathbf{w}||^2_{\frac{4s}{2s-1}}\ ||\nabla\mathbf{w}||_{2s}. \tag{2.3.25}$$

But, since $\dfrac{4s}{2s-1} \leq 2s$ for $s \geq \dfrac{3}{2}$, we have:

$$||\nabla\mathbf{w}||^2_{\frac{4s}{2s-1}} \leq ||\nabla\mathbf{w}||^2_{2s}. \tag{2.3.26}$$

Inequalities (4.3.17) and (4.3.18) imply:

$$|(\mathbf{v}, \nabla\mathbf{w})| \leq c||\nabla\mathbf{w}||^2_{2s} \leq c||\nabla\mathbf{w}||^{2s}_{2s} \leq c||\nabla\mathbf{w}||^{2s}_{2s}\ ||\mathbf{w}||_2 \tag{2.3.27}$$

Therefore, using (2.3.13) and (2.3.27), we get:

$$\frac{d}{dt}\ ||\mathbf{w}||^2 \leq -\frac{Re^{-1}}{2}\ ||\nabla\mathbf{w}||^2 - C_s||\nabla\mathbf{w}||^{2+2\mu}_{2+2\mu}(1-c||\mathbf{w}||_2) + \frac{Re}{2}\ ||\bar{\mathbf{f}}||^2 \qquad (2.3.28)$$

Now, putting together (4.3.10), (4.3.14) and (4.3.19), gives:

$$\frac{d}{dt}\ ||\mathbf{w}||^2 \leq -\frac{Re^{-1}}{2}||\nabla\mathbf{w}||^2 - C_s||\nabla\mathbf{w}||^{2+2\mu}_{2+2\mu}(1-c||\mathbf{w}||^{1-\beta}) + \frac{Re}{2}||\bar{\mathbf{f}}||^2, \qquad (2.3.29)$$

where β is a nonnegative number. Using the hypothesis on the smallness of the data, and (2.3.29), we can now easily prove (by contradiction) that:

$$||\mathbf{w}(t)||^2 + \frac{Re^{-1}}{2}\ \int_0^t ||\nabla\mathbf{w}(s)||^2\ ds \leq \frac{2C_s}{c}\ \forall\ t \in [0,T].$$

■

Remark: Another way of phrasing Lemma 2.3.1 is that, for small data, $\mathbf{w} \in L^\infty(0,T;L^2(\Omega))$, and $\nabla\mathbf{w} \in L^2(0,T;L^2(\Omega))$.

Lemma 2.3.2 *(Second a priori estimate). Assume*

$$\bar{\mathbf{u}}_0 \in L^2(\Omega),\ \partial_t\bar{\mathbf{u}}_0 \in L^2(\Omega),\ \nabla\bar{\mathbf{u}}_0 \in L^{2+2\mu}(\Omega),$$
$$\bar{\mathbf{f}} \in L^2(0,T;L^2(\Omega)),\ \textit{and}\ \partial_t\bar{\mathbf{f}} \in L^2(0,T;L^2(\Omega)).$$

Any weak solution of (2.3.7)satisfies:

$$\begin{aligned}
&||\mathbf{w}_t||^2_{L^\infty(L^2)} + ||\nabla\mathbf{w}||^2_{L^\infty(L^2)} + ||\nabla\mathbf{w}||^{2+2\mu}_{L^\infty(L^2+2\mu)} + ||\nabla\mathbf{w}_t||^2_{L^2(L^2)}\\
&\int_0^T\int_\Omega |\nabla\mathbf{w}_t|^2\ |\nabla\mathbf{w}|^{2\mu}d\mathbf{x}dt + \int_0^T\int_\Omega (\nabla\mathbf{w}\cdot\nabla\mathbf{w}_t)^2\ |\nabla\mathbf{w}|^{2\mu-2}d\mathbf{x}dt\\
&\leq c[||\partial_t\bar{\mathbf{u}}_0||^2_{L^2} + ||\nabla\bar{\mathbf{u}}_0||^2_{L^2} + ||\nabla\bar{\mathbf{u}}_0||^{2+2\mu}_{2+2\mu} + ||\bar{\mathbf{f}}||^2_{L^2(L^2)} + ||\partial_t\bar{\mathbf{f}}||^2_{L^2(L^2)}]e^{cT}
\end{aligned}$$

Notation. $\partial_t\bar{\mathbf{u}}_0$ means:

$$\begin{aligned}
\partial_t\bar{\mathbf{u}}_0 :\ &=\ -\bar{\mathbf{u}}_{oj}\partial_j\bar{\mathbf{u}}_0 + \partial_j[(Re^{-1} + C_s|\nabla\bar{\mathbf{u}}_0|^{2\mu})\partial_j\bar{\mathbf{u}}_0] -\\
&\quad \partial_j\left[\left(-\frac{\delta^2}{4\gamma}\Delta + \dot{I}\right)^{-1}\left(\frac{\delta^2}{2\gamma}\partial_\ell\bar{\mathbf{u}}_0\partial_\ell\bar{\mathbf{u}}_{0j}\right)\right] + \bar{\mathbf{f}}|_{t=0} - \nabla q_0
\end{aligned}$$

To get the initial value of pressure q_0, we apply the divergence operator to the above equation:

$$\begin{aligned}
\Delta q_0\ &=\ -\partial_i\bar{\mathbf{u}}_{0j}\partial_j\bar{\mathbf{u}}_{0i} + C_s\partial_j\left[\partial_i|\nabla\bar{\mathbf{u}}_0|^{2\mu}\partial_j\bar{\mathbf{u}}_{0i}\right] -\\
&\quad \partial_i\partial_j\left[\left(-\frac{\delta^2}{4\gamma}\Delta + \dot{I}\right)^{-1}\left(\frac{\delta^2}{2\gamma}\partial_\ell\bar{\mathbf{u}}_{0i}\ \partial_\ell\bar{\mathbf{u}}_{0j}\right)\right]\ \text{in}\ \Omega.
\end{aligned}$$

The natural boundary conditions are obtained integrating by parts:

$$\int_\Omega -\partial_i \bar{\mathbf{u}}_{0j}\partial_j \bar{\mathbf{u}}_{0i} + C_s \partial_j[\partial_i |\nabla \bar{\mathbf{u}}_0|^{2\mu} \partial_j \bar{\mathbf{u}}_{0i}] - \partial_i \partial_j \left[\left(-\frac{\delta^2}{4\gamma}\Delta + \dot{I} \right)^{-1} \left(\frac{\delta^2}{2\gamma} \partial_\ell \bar{\mathbf{u}}_{0i} \partial_\ell \bar{\mathbf{u}}_{0j} \right) \right] d\mathbf{x}$$

$$= \int_{\partial\Omega} \partial_j [C_s \mid \nabla \bar{\mathbf{u}}_0 \mid^{2\mu} \partial_j \bar{\mathbf{u}}_0] \cdot \mathbf{n} - \partial_j \left[\left(-\frac{\delta^2}{4\gamma}\Delta + \dot{I} \right)^{-1} \left(\frac{\delta^2}{2\gamma} \partial_\ell \bar{\mathbf{u}}_{0i} \partial_\ell \bar{\mathbf{u}}_{0j} \right) \right] \cdot \mathbf{n} \; d\sigma,$$

and thus we take:

$$\partial_{\mathbf{n}} q_0 := \partial_j [C_s |\nabla \bar{\mathbf{u}}_0|^{2\mu} \partial_j \bar{\mathbf{u}}_0] - \partial_j \left[\left(-\frac{\delta^2}{4\gamma}\Delta + \dot{I} \right)^{-1} \left(\frac{\delta^2}{2\gamma} \partial_\ell \bar{\mathbf{u}}_{0i} \partial_\ell \bar{\mathbf{u}}_{0j} \right) \right]$$

In the above derivations we have used the consistency conditions on the initial data: $\nabla \cdot \bar{\mathbf{u}}_0 = 0$ in Ω and $\bar{\mathbf{u}}_0 = 0$ on $\partial\Omega$, as well as the Helmholtz-Weyl decomposition for $\bar{\mathbf{f}}$.

Proof: We differentiate in time the first equation of (2.3.7), multiply by $\mathbf{w}_t$, and integrate over Ω:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}||\mathbf{w}_t||^2_{L^2} + \int_\Omega \partial_t \mathbf{w}_j \partial_j \mathbf{w}_i \partial_t \mathbf{w}_i d\mathbf{x} + C_s \int_\Omega |\nabla \mathbf{w}_t|^2 |\nabla \mathbf{w}|^{2\mu} d\mathbf{x} + \\ &Re^{-1} ||\nabla \mathbf{w}_t||^2_{L^2} + 2\mu C_s \int_\Omega (\nabla \mathbf{w} \cdot \nabla \mathbf{w}_t)^2 |\nabla \mathbf{w}|^{2\mu-2} d\mathbf{x} = \\ &\frac{\delta^2}{2\gamma} \int_\Omega (\partial_\ell \partial_t \mathbf{w}_i \partial_\ell \mathbf{w}_j + \partial_\ell \mathbf{w}_i \partial_\ell \partial_t \mathbf{w}_j) \left[\left(-\frac{\delta^2}{4\gamma}\Delta + I \right)^{-1} (\partial_j \partial_t \mathbf{w}_i) \right] d\mathbf{x} + \\ &\int_\Omega \bar{\mathbf{f}}_t \cdot \mathbf{w}_t d\mathbf{x} \leq \frac{\delta^2}{\gamma} \int_\Omega |\nabla \mathbf{w}_t| \; |\nabla \mathbf{w}| \; | \left(-\frac{\delta^2}{4\gamma}\Delta + I \right)^{-1} \nabla \mathbf{w}_t | d\mathbf{x} + \int_\Omega \bar{\mathbf{f}}_t \cdot \mathbf{w}_t d\mathbf{x} \end{aligned} \tag{2.3.30}$$

Then, we multiply the first equation of (2.3.7) by $\mathbf{w}_t$, and integrate over Ω:

$$\begin{aligned} &||\mathbf{w}_t||^2_{L^2} + \int_\Omega \mathbf{w}_j \partial_j \mathbf{w}_i \partial_t \mathbf{w}_i d\mathbf{x} + \frac{Re^{-1}}{2} \frac{d}{dt} ||\nabla \mathbf{w}||^2_{L^2} + \\ &C_s \int_\Omega (\nabla \mathbf{w} \cdot \nabla \mathbf{w}_t) |\nabla \mathbf{w}|^{2\mu} d\mathbf{x} = \\ &\frac{\delta^2}{2\gamma} \int_\Omega \partial_\ell \mathbf{w}_i \partial_\ell \mathbf{w}_j \left[\left(-\frac{\delta^2}{4\gamma}\Delta + I \right)^{-1} (\partial_j \partial_t \mathbf{w}_i) \right] d\mathbf{x} + \int_\Omega \bar{\mathbf{f}} \cdot \mathbf{w}_t d\mathbf{x} \end{aligned} \tag{2.3.31}$$

Summing up (2.3.30) and (2.3.31), and using

$$C_s \int_\Omega (\nabla \mathbf{w} \cdot \nabla \mathbf{w}_t) |\nabla \mathbf{w}|^{2\mu} d\mathbf{x} = \frac{C_s}{2\mu + 2} \frac{d}{dt} \int_\Omega |\nabla \mathbf{w}|^{2\mu+2} d\mathbf{x},$$

we get:

$$\frac{d}{dt}\left(\frac{1}{2}||\mathbf{w}_t||_{L^2}^2+\frac{Re^{-1}}{2}||\nabla\mathbf{w}||_{L^2}^2+\frac{C_s}{2+2\mu}\;||\nabla\mathbf{w}||_{2+2\mu}^{2+2\mu}\right)+C_s\int_\Omega\;|\nabla\mathbf{w}_t|^2|\nabla\mathbf{w}|^{2\mu}d\mathbf{x}+$$
$$2\mu C_s\int_\Omega(\nabla\mathbf{w}\cdot\nabla\mathbf{w}_t)^2|\nabla\mathbf{w}|^{2\mu-2}d\mathbf{x}+Re^{-1}||\nabla\mathbf{w}_t||_{L^2}^2+||\mathbf{w}_t||_{L^2}^2\le$$
$$-\int_\Omega\partial_t\mathbf{w}_j\partial_j\mathbf{w}_i\partial_t\mathbf{w}_id\mathbf{x}-\int_\Omega\mathbf{w}_j\partial_j\mathbf{w}_i\partial_t\mathbf{w}_id\mathbf{x}+$$
$$-\frac{\delta^2}{\gamma}\int_\Omega|\nabla\mathbf{w}_t||\nabla\mathbf{w}|\;|\left(-\frac{\delta^2}{4\gamma}\Delta+I\right)^{-1}\nabla\mathbf{w}_t|d\mathbf{x}+$$
$$\frac{\delta^2}{2\gamma}\int_\Omega|\nabla\mathbf{w}|^2|\left(-\frac{\delta^2}{4\gamma}\Delta+I\right)^{-1}\nabla\mathbf{w}_t|d\mathbf{x}+\int_\Omega\bar{\mathbf{f}}_t\cdot\mathbf{w}_td\mathbf{x}+\int_\Omega\bar{\mathbf{f}}\cdot\mathbf{w}_td\mathbf{x}\qquad(2.3.32)$$

We now try to estimate the “bad” terms on the RHS of the above relation so that we can apply Gronwall’s lemma.

By the Sobolev Embedding theorem, Elliptic Regularity, and Lemma 2.2’ in [G94], we have:

$$||\left(-\frac{\delta^2}{4\gamma}\Delta+I\right)^{-1}\nabla\mathbf{w}_t||_{L^\infty}\le c||\left(-\frac{\delta^2}{4\gamma}\Delta+I\right)^{-1}\nabla\mathbf{w}_t||_{\frac{3}{2}+s,2}\le$$
$$c||\nabla\mathbf{w}_t||_{-\frac{1}{2}+s,2}\le c||\mathbf{w}_t||_{\frac{1}{2}+s,2}\le c||\nabla\mathbf{w}_t||^{\frac{1}{2}+s}||\mathbf{w}_t||^{\frac{1}{2}-s}\qquad(2.3.33)$$

for any $s\in(0,1/2)$. Using Young’s inequality, we evaluate the third term on the RHS of (2.3.32) as follows:

$$\frac{\delta^2}{\gamma}\int_\Omega|\nabla\mathbf{w}_t||\nabla\mathbf{w}||\left(-\frac{\delta^2}{4\gamma}\Delta+I\right)^{-1}\nabla\mathbf{w}_t|d\mathbf{x}=$$
$$\frac{\delta^2}{\gamma}\int_\Omega|\nabla\mathbf{w}_t||\nabla\mathbf{w}|^{\mu}|\nabla\mathbf{w}|^{1-\mu}|\left(-\frac{\delta^2}{4\gamma}\Delta+I\right)^{-1}\nabla\mathbf{w}_t|d\mathbf{x}\le$$
$$\varepsilon\int_\Omega|\nabla\mathbf{w}_t|^2|\mathbf{w}|^{2\mu}d\mathbf{x}+c\int_\Omega|\nabla\mathbf{w}|^{2-2\mu}|\left(-\frac{\delta^2}{4\gamma}\Delta+I\right)^{-1}\nabla\mathbf{w}_t|^2d\mathbf{x}\quad(2.3.34)$$

Using (2.3.33) and Young’s inequality the last term on the RHS of the above inequality can be further evaluated as:

$$\int_\Omega|\nabla\mathbf{w}|^{2-2\mu}|\left(-\frac{\delta^2}{4\gamma}\Delta+I\right)^{-1}\nabla\mathbf{w}_t|^2d\mathbf{x}\le||\left(-\frac{\delta^2}{4\gamma}\Delta+I\right)^{-1}\nabla\mathbf{w}_t||_{L^\infty}^2\int_\Omega|\nabla\mathbf{w}|^{2-2\mu}d\mathbf{x}$$
$$\le c||\nabla\mathbf{w}_t||^{1+2s}||\mathbf{w}_t||^{1-2s}\int_\Omega|\nabla\mathbf{w}|^{2-2\mu}d\mathbf{x}\le$$
$$\varepsilon||\nabla\mathbf{w}_t||^2\int_\Omega|\nabla\mathbf{w}|^{2-2\mu}d\mathbf{x}+c||\mathbf{w}_t||^2\int_\Omega|\nabla\mathbf{w}|^{2-2\mu}d\mathbf{x}\qquad(2.3.35)$$

Now, since the above inequality is true for any positive ϵ, and since $\int_\Omega|\nabla\mathbf{w}|^{2-2\mu}d\mathbf{x}$ is bounded in $L^1(0,T)$, (by Lemma 2.3.1), (2.3.34) and (2.3.35) imply:

$$\frac{\delta^2}{4\gamma}\int_\Omega|\nabla\mathbf{w}_t|\;|\nabla\mathbf{w}|\;|\left(-\frac{\delta^2}{4\gamma}\Delta+I\right)^{-1}\nabla\mathbf{w}_t|d\mathbf{x}$$

is a "good" term (we can apply Gronwall's inequality). (Note that we used $\mu \leq 1$; the case $\mu > 1$ is trivial).

We treat similarly (using Young's inequality and (2.3.33)) the fourth term on the RHS of (2.3.32):

$$\frac{\delta^2}{2\gamma}\int_\Omega |\nabla \mathbf{w}|^2| \left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1} \nabla \mathbf{w}_t|d\mathbf{x} \leq \frac{\delta^2}{2\gamma}|| \left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1} \nabla \mathbf{w}_t||_{L^\infty} \int_\Omega |\nabla \mathbf{w}|^2 d\mathbf{x} \leq$$
$$c||\nabla \mathbf{w}_t||^{\frac{1}{2}+s}||\mathbf{w}_t||^{\frac{1}{2}-s}\int_\Omega |\nabla \mathbf{w}|^2 d\mathbf{x} \leq \varepsilon||\nabla \mathbf{w}_t||^2 \int_\Omega |\nabla \mathbf{w}|^2 d\mathbf{x} + c||\mathbf{w}_t||^{\frac{2-4s}{3-2s}}\int_\Omega |\nabla \mathbf{w}|^2 d\mathbf{x}.$$

Using now the same argument as above ($\nabla \mathbf{w} \in L^2(0,T;L^2(\Omega))$ by Lemma 2.3.1), we get:

$$\frac{\delta^2}{2\gamma}\int_\Omega |\nabla \mathbf{w}|^2| \left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1} \nabla w_t|d\mathbf{x}$$

is a "good" term, too.

We now evaluate the first term on the RHS of (2.3.32). Using Hölder's inequality, Lemma 2.2' in [G94] and Young's inequality, we get:

$$-\int_\Omega \partial_t \mathbf{w}_j \partial_j \mathbf{w}_i \partial_t \mathbf{w}_i d\mathbf{x} \leq \int_\Omega |\mathbf{w}_t|^2 |\nabla \mathbf{w}| d\mathbf{x} \leq ||\mathbf{w}_t||^2_{L^4}||\nabla \mathbf{w}||$$
$$\leq c||\mathbf{w}_t||^{1/2}||\nabla \mathbf{w}_t||^{3/2}||\nabla \mathbf{w}|| \leq \epsilon||\nabla \mathbf{w}||\ ||\nabla \mathbf{w}_t||^2 + c||\nabla \mathbf{w}||\ ||\mathbf{w}_t||^2.$$

Since $||\nabla \mathbf{w}|| \in L^2(0,T)$ (Lemma 2.3.1), we get that $-\int_\Omega \partial_t \mathbf{w}_j \partial_j \mathbf{w}_i \partial_t \mathbf{w}_i$ is a "good" term, too. The second term on the RHS of (2.3.32) can be estimated exactly in the same way (it can also be estimated in a better way, but it is worthless here):

$$-\int_\Omega \mathbf{w}_j \partial_j \mathbf{w}_i \partial_t \mathbf{w}_i d\mathbf{x} \leq \int_\Omega |\mathbf{w}|\ |\nabla \mathbf{w}|\ |\mathbf{w}_t| d\mathbf{x} \leq ||\nabla \mathbf{w}|| \left(\int_\Omega |\mathbf{w}|^2 |\mathbf{w}_t|^2\right)^{1/2}$$
$$\leq ||\nabla \mathbf{w}||\ ||\mathbf{w}||_{L^4}||\mathbf{w}_t||_{L^4} \leq c||\nabla \mathbf{w}|| \left(||\mathbf{w}||^{1/4}||\nabla \mathbf{w}||^{3/4}\right)\left(||\mathbf{w}_t||^{1/4}||\nabla \mathbf{w}_t||^{3/4}\right)$$
$$\leq c||\nabla \mathbf{w}||^{7/4}||\mathbf{w}_t||^{1/4}||\nabla \mathbf{w}_t||^{3/4} \leq \varepsilon||\nabla \mathbf{w}||^{7/4}||\nabla \mathbf{w}_t|| + c||\nabla \mathbf{w}||^{7/4}||\mathbf{w}_t||$$

Using the Cauchy-Schwarz inequality, we can trivially estimate $\int_\Omega \overline{\mathbf{f}} \cdot \mathbf{w}_t\, d\mathbf{x}$ and $\int_\Omega \overline{\mathbf{f}}_t \cdot \mathbf{w}_t d\mathbf{x}$.

Now, applying Gronwall's Lemma for (2.3.32), the lemma is proven.

■

Theorem 2.3.1 *(Existence of Weak Solutions) If the conditions in Lemma 2.3.1 and Lemma 2.3.2 are satisfied, then there exists a weak solution to (2.3.7) in* $L^\infty(0,T;L^2(\Omega)) \cap L^{2+2\mu}(0,T;W^{1,2+2\mu}_{0,div}(\Omega))$.

Proof: We shall use a so called "Faedo-Galerkin" method. Let $\{a_\ell\} \subseteq W^{1,2+2\mu}_{0,\mathrm{div}}(\Omega)$ be an orthonormal basis. We can assume, without loss of generality, that $a_1 = \bar{\mathbf{u}}_0$. Consider now the sequence of functions

$$V^n = \sum_{\ell=1}^{n} c_{\ell n}(t) a^\ell(\mathbf{x}),$$

where the coefficients $c_{\ell n}$ are chosen to satisfy the following system of differential equations:

$$\begin{aligned}&\int_\Omega \left(\partial_t V^n a^\ell + (Re^{-1} + C_s|\nabla V^n|^{2\mu})\nabla V^n \cdot \nabla a^\ell + V^n_j \partial_j V^n a^\ell\right) d\mathbf{x} = \\ &+\frac{\delta^2}{2\gamma}\int_\Omega \left[\left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1} (\partial_\ell V^n_i \partial_\ell V^n)\right] \partial_j a^\ell d\mathbf{x} + \int_\Omega \bar{\mathbf{f}} a^\ell d\mathbf{x},\end{aligned} \tag{2.3.36}$$

with the initial condition

$$c_{\ell n}(0) = \int_\Omega \bar{\mathbf{u}}_0 a^\ell d\mathbf{x}. \tag{2.3.37}$$

Note that the a priori estimate of Lemmas 2.3.1 and 2.3.2 hold for (2.3.36), too. Also note that (2.3.36) and (2.3.37) is an autonomous system of differential equations with $c_{\ell n}(t)$ as unknowns, and from the first a priori estimate (Lemma 2.3.1) we have that

$$\max_{t\in[0,T]} \sum_{\ell=1}^{n} c^2_{\ell n}(t) = \|V^n\|^2_{L^\infty(L^2)} \tag{2.3.38}$$

is bounded uniformly in n. Thus, from the elementary theory of differential equations, it follows the existence and uniqueness of $c_{\ell n}$.

From the sequence $\{V^n\}$ we shall choose subsequences which converge in some sense. For simplicity, these subsequences will be still denoted by $\{V^n\}$. Thus, using the first a priori estimate given by Lemma 2.3.1, by the usual technique (see [38]) we get a subsequence (still denoted by $\{V^n\}$) converging strongly in $L^2(0,T;L^2(\Omega))$, and weakly in $L^\infty(0,T;L^2(\Omega)) \cap L^{2+2\mu}(0,T;W^{1,2+2\mu}_{0,\mathrm{div}}(\Omega))$ to a function V. Using Lemma 2.2' in [27], the Cauchy-Schwarz inequality, and the Sobolev Embedding theorem, we get:

$$\begin{aligned}\int_0^T ||V^n - V||^4_{L^4} dt &\le c\int_0^T ||V^n - V||_{L^2} ||\nabla(V^n - V)||^3_{L^2} dt \le \\ &\le c\left(\int_0^T ||V^n - V||^2_{L^2}\right)^{1/2} \left[\left(\int_0^T ||\nabla(V^n - V)||^6_{L^2}\right)^{1/6}\right]^3 \\ &\le c\left(\int_0^T ||V^n - V||^2_{L^2}\right)^{1/2} \left[\left(\int_0^T ||\nabla(V^n - V)_t||^2_{L^2}\right)^{1/2}\right]^3,\end{aligned}$$

which can be written as:

$$||V^n - V||_{L^4(L^4)} \leq c||\nabla(V^n - V)_t||^{3/4}_{L^2(L^2)}||V^n - V||^{1/4}_{L^2(L^2)}$$

Using the a priori estimates given by Lemma 2.3.1 and Lemma 2.3.2, and the above inequality, we get the strong convergence of V^n to V in $L^q(0,T;L^q(\Omega))$ for any $1 \leq q \leq 4$.

Multiplying (2.3.36) by $d_{\ell n}$, summing over ℓ, and integrating from 0 to T, we get:

$$\int_0^T \int_\Omega (\partial_t V^n + V_j^n \partial_j V^n)\Phi + (Re^{-1} + C_s|\nabla V^n|^{2\mu})\nabla V^n \cdot \nabla\Phi \; d\mathbf{x}\; dt =$$
$$= \frac{\delta^2}{2\gamma}\int_0^T \int_\Omega \left[\left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1}(\partial_\ell V_i^n \partial_\ell V^n)\right]\partial_j \Phi d\mathbf{x}\; dt + \int_0^T \int_\Omega \bar{\mathbf{f}}\Phi d\mathbf{x}\; dt, \quad (2.3.39)$$

where Φ is an arbitrary function obtained as a linear combination of $a^\ell(x)$ with coefficients $d_\ell(t)$, which are absolutely continuous functions on time with square summable first derivatives. Now it is easy to verify that (2.3.39) is valid for any $\Phi \in L^\infty(0,T;L^2(\Omega)) \cap L^{2+2\mu}(0,T;W^{1,2+2\mu}_{0,\text{div}}(\Omega))$. For fixed Φ, we pass to the limit in (2.3.39) as $n \to \infty$. Using the a priori estimates in Lemmas 1 and 2, we can pass to the limit in the first and last term by the usual technique (see [38] Section 3). For the strongly nonlinear second and third terms, we use an idea of Minty and Browder. We introduce the functions:

$$A_i^k(\nabla V^n) = (Re^{-1} + C_s|\nabla V^n|^{2\mu})\partial_k V_i^n - \frac{\delta^2}{2\gamma}\sum_\ell \left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1}\partial_\ell V_i^n \partial_\ell V_k^n \quad (2.3.40)$$

which are uniformly bounded (by Lemmas 2.3.1 and 2.3.2), and therefore converge weakly to functions $B_i^k(\mathbf{x},t)$. Thus, the limiting equation of (2.3.39) is:

$$\int_0^T \int_\Omega [(\partial_t V_i + V_j \partial_j V_i)\Phi_i + B_i^k \partial_k \Phi_i] d\mathbf{x}\; dt = \int_0^T \int_\Omega \bar{\mathbf{f}}_i \Phi_i d\mathbf{x}\; dt \quad (2.3.41)$$

We now need the following lemma:

Lemma 2.3.3 *For any two functions* $\mathbf{v}', \mathbf{v}'' \in L^\infty(0,T;L^2(\Omega)) \cap L^{2+2\mu}(0,T;W^{1,2+2\mu}_{0,div}(\Omega))$ *we have:*

$$\sum_{i,k}\int_0^T \int_\Omega [A_i^k(\nabla \mathbf{v}') - A_i^k(\nabla \mathbf{v}'')](\partial_k \mathbf{v}'_i - \partial_k \mathbf{v}''_i) d\mathbf{x}\; dt \geq 0 \quad (2.3.42)$$

Proof: Letting $\mathbf{w} := \mathbf{v}' - \mathbf{v}''$, and using the strong monotonicity of the μ-Laplacian, the Sobolev Embedding Theorem, elliptic regularity, Lemma 2.2' in [27], Poincaré's inequality, and Lemmas 2.3.1 and 2.3.2, we get:

$$\sum_{i,k} \int_0^T \int_\Omega \left[A_i^k(\nabla \mathbf{v}') - A_i^k(\nabla \mathbf{v}'')\right] (\partial_k \mathbf{v}_i' - \partial_k \mathbf{v}_i'') d\mathbf{x}\ dt \geq$$

$$Re^{-1}||\nabla \mathbf{w}||^2 + c\int_0^T ||\nabla \mathbf{w}||_{2+2\mu}^{2+2\mu} dt - \frac{\delta^2}{2\gamma}\int_0^T \int_\Omega \left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1} [\nabla \mathbf{v}'\nabla \mathbf{v}' - \nabla \mathbf{v}''\nabla \mathbf{v}''] \nabla \mathbf{w}\ d\mathbf{x}\ dt$$

$$Re^{-1}||\nabla \mathbf{w}||^2 + c\int_0^T ||\nabla \mathbf{w}||_{2+2\mu}^{2+2\mu} dt - \frac{\delta^2}{2\gamma}\int_0^T \int_\Omega [\nabla \mathbf{v}'\nabla \mathbf{v}' - \nabla \mathbf{v}''\nabla \mathbf{v}''] \left[\left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1} \nabla \mathbf{w}\right] d\mathbf{x}$$

$$Re^{-1}||\nabla \mathbf{w}||^2 + c\int_0^T ||\nabla \mathbf{w}||_{2+2\mu}^{2+2\mu} dt - c\frac{\delta^2}{2\gamma}\int_0^T || \left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1} \nabla \mathbf{w}||_{L^\infty} \int_\Omega |\nabla \mathbf{v}'\nabla \mathbf{w} + \nabla \mathbf{w}\nabla \mathbf{v}''|$$

$$Re^{-1}||\nabla \mathbf{w}||^2 + c\int_0^T ||\nabla \mathbf{w}||_{2+2\mu}^{2+2\mu} dt - c\frac{\delta^2}{2\gamma}\int_0^T || \left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1} \nabla \mathbf{w}||_{\frac{3}{2}+\epsilon,2} \int_\Omega |\nabla \mathbf{v}'\nabla \mathbf{w} + \nabla \mathbf{w}\nabla \mathbf{v}'$$

$$Re^{-1}||\nabla \mathbf{w}||^2 + c\int_0^T ||\nabla \mathbf{w}||_{2+2\mu}^{2+2\mu} dt - \frac{c}{2\gamma}\int_0^T ||\mathbf{w}||_{\frac{1}{2}+\epsilon,2} \int_\Omega |\nabla \mathbf{v}'\nabla \mathbf{w} + \nabla \mathbf{w}\nabla \mathbf{v}''| d\mathbf{x}\ dt \geq$$

$$Re^{-1}||\nabla \mathbf{w}||^2 + c\int_0^T ||\nabla \mathbf{w}||_{2+2\mu}^{2+2\mu} dt - \frac{c}{2\gamma}\int_0^T ||\nabla \mathbf{w}||^{\frac{1}{2}+\epsilon} ||\mathbf{w}||^{\frac{1}{2}-\epsilon} \int_\Omega |\nabla \mathbf{v}'\nabla \mathbf{w} + \nabla \mathbf{w}\nabla \mathbf{v}''| d\mathbf{x}\ dt \geq$$

$$Re^{-1}||\nabla \mathbf{w}||^2 + c\int_0^T ||\nabla \mathbf{w}||_{2+2\mu}^{2+2\mu} dt - \frac{c}{2\gamma}\int_0^T ||\nabla \mathbf{w}||^{\frac{3}{2}+\epsilon} ||\mathbf{w}||^{\frac{1}{2}-\epsilon} \{\nabla \mathbf{v}'\nabla \mathbf{w} + \nabla \mathbf{w}\nabla \mathbf{v}''\} dt \geq$$

$$Re^{-1}||\nabla \mathbf{w}||^2 + c\int_0^T ||\nabla \mathbf{w}||_{2+2\mu}^{2+2\mu} dt - \frac{c}{2\gamma}\int_0^T ||\nabla \mathbf{w}||^2 \{||\nabla \mathbf{v}'\nabla \mathbf{w} + \nabla \mathbf{w}\nabla \mathbf{v}''\} dt \geq$$

$$Re^{-1}||\nabla \mathbf{w}||^2 + c\int_0^T ||\nabla \mathbf{w}||_{2+2\mu}^{2+2\mu} dt - \frac{c}{2\gamma}\int_0^T ||\nabla \mathbf{w}||^2 (||\nabla \mathbf{v}'|| + ||\nabla \mathbf{v}''||) dt \geq 0,$$

by Lemma 2.3.2 and the smallness of data with respect to Re.

■

Subtracting (2.3.39) from (2.3.42), we get:

$$-\int_0^T \int_\Omega (\partial_t V_i^n + V_k^n \partial_k V_i^n - \bar{\mathbf{f}}_i)(V_i^n - \eta_i) + A_i^k(\nabla \eta)(\partial_k V_i^n - \partial_k \eta_i) d\mathbf{x} dt \geq 0 \quad (2.3.43)$$

for any function $\eta \in L^\infty(0,T;L^2(\Omega)) \cap L^{2+2\mu}(0,T;W_{0,\mathrm{div}}^{1,2+2\mu}(\Omega))$. Passing to the limit as $n \to \infty$ in the above relation does not present any problem except in the second term. Thus, since $\nabla V^n \in L^{2+2\mu}(0,T;L^{2+2\mu}(\Omega))$ (by Lemma 2.3.2), we have to verify that $V_k^n V^n$ converges strongly in the $L^{\frac{2+2\mu}{1+2\mu}}$ norm. Letting $p = (2+2\mu)/(1+2\mu)$, we get:

$$||V_k^n V_\ell^m - V_k V_\ell||_{L^p(L^p)} \leq ||(V_k^n - V_k)V_\ell^n||_{L^p(L^p)} + ||V_k(V_\ell^n - V_\ell)||_{L^p(L^p)} \leq$$
$$||V_k^n - V_k||_{L^{2p}(L^{2p})}||V_\ell^n||_{L^{2p}(L^{2p})} + ||V_k||_{L^{2p}(L^{2p})}||V_\ell^n - V_\ell||_{L^{2p}(L^{2p})}$$

Since $2p < 4$, the RHS of the above inequality tends to zero as $n \to \infty$. Therefore, we can pass to the limit in (2.3.43), and we get:

$$-\int_0^T \int_\Omega (\partial_t V_i + V_k \partial_k V_i - \bar{\mathbf{f}}_i)(V_i - \eta_i) + A_i^k(\nabla \eta)(\partial_k V_i - \partial_k \eta_i) d\mathbf{x}\, dt \geq 0 \quad (2.3.44)$$

Letting $\eta = V - \varepsilon\xi$ in the above relation, where $\varepsilon > 0$ and $\xi \in L^\infty(0,T;L^2(\Omega)) \cap L^{2+2\mu}(0,T;W^{1,2+2\mu}_{0,\mathrm{div}}(\Omega))$, and $\Phi = \varepsilon\xi$ in (2.3.41), and then summing up these two relations, we get:

$$\int_0^T \int_\Omega [B_i^k - A_i^k(\nabla V - \varepsilon \nabla \xi)] \partial_k \xi_i d\mathbf{x}\, dt \geq 0 \quad (2.3.45)$$

Doing the same as above, but now for $\eta = V + \varepsilon\xi$, we get:

$$\int_0^T \int_\Omega [B_i^k - A_i^k(\nabla V + \varepsilon \nabla \xi)] \partial_k \xi_i d\mathbf{x}\, dt \geq 0 \quad (2.3.46)$$

Since $\varepsilon > 0$ is arbitrary, relations (2.3.45) and (2.3.46) imply that:

$$\int_0^T \int_\Omega [B_i^k - A_i^k(\nabla V)] \partial_k \xi_i d\mathbf{x}\, dt = 0 \quad (2.3.47)$$

for any $\xi \in L^\infty(0,T;L^2(\Omega) \cap L^{2+2\mu}(0,T;W^{1,2+2\mu}_{0,\mathrm{div}}(\Omega))$. Relations (2.3.41) and (2.3.47) imply the existence of a weak solution to (2.3.7):

$$\int_0^T \int_\Omega (\partial_t V + V_j \partial_j V)\Phi + (Re^{-1} + C_s|\nabla V|^{2\mu})\nabla V \cdot \nabla \Phi d\mathbf{x}\, dt =$$
$$\frac{\delta^2}{2\gamma}\int_0^T \int_\Omega \left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1} (\partial_\ell V_j \partial_\ell V)\partial_j \Phi \; d\mathbf{x}\, dt + \int_0^T \int_\Omega \bar{\mathbf{f}}\Phi d\mathbf{x}\, dt.$$

■

2.3.2 Uniqueness and Stability of Weak Solutions

Using the same assumptions as in the previous section, we shall prove uniqueness for the weak solution of (2.3.7)in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2+2\mu}_{0,\mathrm{div}}(\Omega))$, and stability.

Theorem 2.3.2 *(Uniqueness) With the same assumptions as in the existence theorem, problem (2.3.7)has a unique weak solution in* $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2+2\mu}_{0,div}(\Omega))$.

Proof: Let V', V'' be two weak solutions of (2.3.7)in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,2+2\mu}_{0,\mathrm{div}}(\Omega))$, and $U = V' - V''$. Then U satisfies, in a distributional sense:

$$\frac{d}{dt}U - \partial_k[A^k(\nabla V') - A^k(\nabla V'')] + V'\nabla V' - V''\nabla V'' = 0$$

with $U(0) = 0$. Multiplying by U and integrating over Ω, we get:

$$\frac{1}{2}\frac{d}{dt}||U||^2_{L^2} + \int_\Omega [A^k(\nabla V') - A^k(\nabla V'')]\partial_k U d\mathbf{x} + \int_\Omega (V'\nabla V'U - V''\nabla V''U)d\mathbf{x} = 0$$

Adding and subtracting $V''\nabla V'U$ to the above relation, and using the fact that $\int_\Omega V''\nabla UU = 0$, we get:

$$\frac{1}{2}\frac{d}{dt}||U||^2_{L^2} + \int_\Omega [A^k(\nabla V') - A^k(\nabla V'')]\partial_k U d\mathbf{x} + \int_\Omega U\nabla V'U \; d\mathbf{x} = 0 \qquad (2.3.48)$$

Using Hölder's inequality, and the Sobolev Embedding theorem, we get:

$$\begin{aligned}\int_\Omega U\nabla V'U d\mathbf{x} &\leq ||U||_{L^6}||U||_{L^3}||\nabla V'||_{L^2} \leq c||U||^2_{L^6}||\nabla V'||_{L^2} \\ &\leq c||\nabla U||^2_{L^2}||\nabla V'||_{L^2} \qquad (2.3.49)\end{aligned}$$

Noting that in Lemma 2.3.3 we actually proved that

$$\int_\Omega [A^k(\nabla V') - A^k(\nabla V'')]\nabla U d\mathbf{x} \geq \alpha||\nabla U||^2_{L^2},$$

with α positive, and using Lemma 2.3.2 and relations (2.3.48) and (2.3.49), we get:

$$\frac{1}{2}\frac{d}{dt}||U||^2_{L^2} + \alpha||\nabla U||^2_{L^2} \leq \frac{\alpha}{2}||\nabla U||^2_{L^2}$$

Since $U(0) = 0$, the above relation implies that $U \equiv 0$.

■

Theorem 2.3.3 *(Stability) With the same assumptions as in the existence theorem, two solutions V' and V'' in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,2+2\mu}_{0,div}(\Omega))$ with different initial data V'_0, V''_0, and different external forces $\mathbf{f}'$ and $\mathbf{f}''$, satisfy:*

$$||V' - V''||_{L^\infty(L^2)} \leq \left(||V'_0 - V''_0|| + c||\mathbf{f}' - \mathbf{f}''||_{L^1(L^2)}\right) e^c, \qquad (2.3.50)$$

where c is a positive constant depending on all the parameters of the fluid, including T.

Proof: Let $\mathbf{f} = \mathbf{f}' - \mathbf{f}''$, and $U = V' - V''$. Using the same approach as in the previous theorem's proof, we get:

$$\frac{1}{2}\frac{d}{dt}||U||_{L^2}^2 + \alpha||\nabla U||_{L^{2+2\mu}}^{2+2\mu} \leq c||U||_{L^2}^2||\nabla V'||_{L^{2+2\mu}}^2 + c||f||_{L^2}^2 + \frac{\alpha}{2}||\nabla U||_{L^{2+2\mu}}^{2+2\mu}$$

Applying Gronwall's inequality and the a priori estimates in Lemma 2.3.2, we get (2.3.50).

■

Remark. In Theorem 2.3.1's proof, we have used the smallness of data with respect to Re. For μ small enough, i.e. $32\mu^3(4 + 2\mu) < 1/4$ or, equivalently $\mu < 0.122547$ roughly, we can avoid this by multiplying (2.3.7) by $A\mathbf{u} := \nabla \cdot [(1 + |\nabla \mathbf{u}|^{2\mu})\nabla \mathbf{u}]$, and then using the inequality $\int_\Omega |A\mathbf{u}|^2 d\mathbf{x} \geq (1 - 2\sqrt{32\mu^3(4 + 2\mu)}) \int_\Omega (1 + |\nabla \mathbf{u}|^{2\mu})^2 |\Delta \mathbf{u}|^2 d\mathbf{x}$. Of course, we cannot use the Faedo-Galerkin method anymore, and thus we follow the approach in [38].

Chapter 3

New LES Models for the Turbulent Fluctuations

3.1 Introduction

In 1877 Boussinesq (and others) put forward the basic analogy between the mixing effects of turbulent fluctuations and molecular diffusion: $-\nabla \cdot \overline{(\mathbf{u}'\mathbf{u}')} \sim -\nabla \cdot (\nu_T(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^t))$. This assumption lies at the heart of essentially all turbulence models and subgridscale models. By revisiting the original arguments of Boussinesq, Saint-Venant, Kelvin, Reynolds and others, we give three new approximations for the turbulent viscosity coefficient ν_T in terms of the mean flow based on approximation for the distribution of kinetic energy in $\mathbf{u}'$ in terms of the mean flow $\overline{\mathbf{u}}$. We prove existence of weak solutions for one of the corresponding models. Finite difference implementations of the new eddy viscosity/subgrid-scale model are transparent. We show how it can be implemented in finite element procedures and prove its action is no larger than that of the popular Smagorinsky-subgrid-scale model.

We start with the space-filtered Navier Stokes equations, derived in subsection 1.1.3:

$$\begin{cases} \partial_t \overline{\mathbf{u}} - Re^{-1}\Delta\overline{\mathbf{u}} + \nabla\cdot(\overline{\mathbf{u}\mathbf{u}}) + \nabla p = \overline{\mathbf{f}} & \text{in } \Omega\times[0,T], \\ \nabla\cdot\overline{\mathbf{u}} = 0 & \text{in } \Omega\times[0,T], \end{cases}$$

where $\overline{\mathbf{u}\mathbf{u}} = \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} + \overline{\overline{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\overline{\mathbf{u}}} + \overline{\mathbf{u}'\mathbf{u}'}$. We consider herein the modeling of the turbulent stresses.

The fundamental relation still used for modeling turbulent stresses was put forward by Boussinesq in 1877, [9], stating, in effect, that the interaction of turbulent fluctuations are dissipative in the mean. This "Boussinesq model" is based upon an analogy between the mixing effects of turbulent fluctuations and molecular diffusion and is written as:

$$\overline{\mathbf{u}'\mathbf{u}'} \sim -\nu_T(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^t) + \frac{2}{3}k\delta_{ij}, \quad \nu_T := \text{ turbulent diffusion parameter,} \quad (3.1.1)$$

where k is the mean turbulent kinetic energy, which can be included in the pressure term. Relations like (3.1.1) have been verified for some simplified settings such as convection of a passive scalar under various assumptions. This work is presented and surveyed well in Part III of Mohammadi and Pironneau [77]. However, we emphasize that the Boussinesq "model/approximation" (3.1.1) is not strictly speaking an approximation but rather a physical analogy.

Since (3.1.1) is the first step and very heart of turbulence models, subgrid-scale models and even the shallow water equations, it is useful first to review the reasoning of Boussinesq behind (3.1.1) and a few representative attempts to determine $\nu_T = \nu_T(\mathbf{x}, t, \overline{\mathbf{u}}, \overline{p}, Re, \cdots)$.

The assumption behind (3.1.1) is that in the mean the small eddies or turbulent fluctuations are isotropic and collide elastically and exchange momentum like molecules. Eddies do not, of course, interact perfectly elastically, see Frisch [26] or Corrsin [18]. Furthermore, there are also turbulent flows for which the turbulent fluctuations do not even seem to be isotropic. See Section 9.6 of Frisch [26] for more on the physical reasoning underlying (3.1.1).

There are numerous approaches to calculating the coefficient ν_T. Turbulence models (algebraic, mixing length, one equation, two equation...) typically give formulas of increasing complexity which are solved approximately to determine ν_T – see, for example, [16], [39], [77], [72] for good surveys of these approaches. Most recently, the trend has been away from these approaches to much simpler subgrid-scale models combined with resolution and adaptivity. The most common such subgrid-scale model is due to Smagorinsky [84] in which:

$$\nu_T \sim C_s\delta^2|\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^t|, \ \delta := \text{ length scale of resolvable eddies,}$$

so that

$$\nabla\cdot\tau \sim \nabla\cdot(C_s\delta^2|\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^t|(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^t)) - \frac{2}{3}\nabla k; \quad (3.1.2)$$

see also [84], [33], [59], [62], [63], [22], [39] for more on the mathematical foundation of this model. The parameter C_s in (3.1.2) was originally taken a constant; in the dynamic eddy viscosity model of [33], $C_s = C_s(\mathbf{x}, t)$ is determined by a clever extrapolation procedure. The form of (3.1.2) is consistent with Kraichnan's [57] extension of of Kolmogorov "$K-42$" theory to 2D turbulence and, accordingly, (3.1.2) performs well in 2D simulations (as reported, e.g., by [77]). The most straightforward ([62], [63]) extension to 3D is

$$\nu_T \sim C_s \delta^2 |\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^t|^{2/3}, \tag{3.1.3}$$

so that

$$\nabla \cdot \overline{\mathbf{u}'\mathbf{u}'} \sim -\nabla \cdot (C_s \delta^2 |\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^t|^{2/3} (\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^t)) + \frac{2}{3} \nabla k.$$

To explain relations like (3.1.3), recall that the "$K-42$" theory of isotropic turbulence developed by Kolmogorov (see Frisch [26]) predicts that the smallest length scale of persistent eddies is $O(Re^{-3/4})$. When a formula like (3.1.3) is used on a computational mesh, the smallest resolvable eddy occurs when $|\nabla u + \nabla u^t| \sim O(\delta^{-1})$. Thus, the effective local turbulent viscosity in these eddies is $\nu_T \sim C_s \delta^2 \delta^{-2/3} = C_s \delta^{4/3}$. By the "$K-42$" theory, eddies below $O((C_s \delta^{4/3})^{3/4}) = O(\delta)$ thus decay exponentially due to the turbulent dissipative term (3.1.3). Thus, the only persistent eddies are precisely those resolvable: $O(\delta)$ or larger.

Again, C_s in (3.1.3) can be regarded as a constant or a distribution to be determined by dynamic extrapolation methods, following [33].

3.2 A new Boussinesq-type subgridscale model

To motivate the need for another subgrid-scale model, note that the models (3.1.2) and (3.1.3) have at least two intuitive shortcomings. First, for flows with linear velocity profiles, the formulas (3.1.2) and (3.1.3) would still introduce significant amounts of turbulent diffusion even though the flow field is laminar. Second, accepting the reasoning of Boussinesq, the amount of turbulent diffusion by small eddies should depend on the kinetic energy in those small eddies, so that $\nu_T = \nu_T \left(\frac{1}{2} \rho_0 |\mathbf{u}'|^2 \right)$ or $\nu_T = \nu_T \left(\overline{\frac{1}{2} \rho_0 |\mathbf{u}'|^2} \right)$.

This section presents a subgridscale model similar to (3.1.2), (3.1.3) which meets these two conditions based on a more complete elaboration of the analogy of Boussinesq. The space filter used in the sequel is a Gaussian:

$$g_\delta(\mathbf{x}) := \left(\frac{\gamma}{\pi} \right)^{3/2} \frac{1}{\delta^3} \, e^{-\gamma \frac{|\mathbf{x}|^2}{\delta^2}},$$

where γ is a constant (often $\gamma = 6$) and δ is the averaging radius.

Herein, we consider the term describing turbulent fluctuations:

$$-\nabla \cdot (g_\delta * (\mathbf{u}'\mathbf{u}')). \tag{3.2.4}$$

We use a closure approximation introduced in [30]. Specifically, since

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}' \Rightarrow \overline{\mathbf{u}} = g_\delta * \overline{\mathbf{u}} + g_\delta * \mathbf{u}'.$$

Extending all functions by zero outside Ω, the Fourier transform of this last equation gives:

$$\hat{\overline{\mathbf{u}}} = \hat{g}_\delta \hat{\overline{\mathbf{u}}} + \hat{g}_\delta \hat{\mathbf{u}}',$$

from which

$$\hat{\mathbf{u}}'(\mathbf{k}) = ((1/\hat{g}_\delta(\mathbf{k})) - 1)\ \hat{\overline{\mathbf{u}}}(\mathbf{k}), \tag{3.2.5}$$

where $\hat{g}_\delta(\mathbf{k}) = \exp\left(-\dfrac{\delta^2}{4\gamma}(\mathbf{k}_1^2 + \mathbf{k}_2^2 + \mathbf{k}_3^2)\right)$.
Using the approximation

$$\hat{g}_\delta(\mathbf{k}) \doteq \frac{1}{1 + \frac{\delta^2}{4\gamma}|\mathbf{k}|^2} + O\left(\frac{\delta^4}{16\gamma^2}|\mathbf{k}|^4\right).$$

in (3.2.5) gives:

$$\hat{\mathbf{u}}'(\mathbf{k}) \doteq \frac{\delta^2}{4\gamma}|\mathbf{k}|^2 \hat{\overline{\mathbf{u}}}(\mathbf{k}).$$

After taking the inverse Fourier transform, we arrive at the approximation

$$\mathbf{u}' \doteq -\frac{\delta^2}{4\gamma}\Delta\overline{\mathbf{u}} \quad (+O(\delta^4) \text{ terms}). \tag{3.2.6}$$

The term involving the turbulent fluctuations

$$\nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'})$$

is formally $O(\delta^4)$. However, turbulent fluctuations play an important role in dissipation of energy from large eddies to smaller eddies. Thus, in turbulent flow simulations, models of the turbulent fluctuations are normally included in the simulation via some variant of Smagorinsky's model as discussed in the introduction.

To model these turbulent fluctuations, reconsider Boussinesq's idea (3.1.1). Following the analogy between turbulent mixing and molecular diffusion to its logical conclusion, ν_T must be considered as a function of either the local kinetic energy in $\mathbf{u}'$ or its local average

$$\nu_T = \nu_T\left(\frac{1}{2}\rho_0|\mathbf{u}'|^2\right), \text{ or } \nu_T = \nu_T\left(\overline{\frac{1}{2}\rho_0|\mathbf{u}'|^2}\right).$$

A classical dimensional argument suggests the correct form of ν_T to be given by the, so-called, Kolmogorov-Prandtl expression:

$$\nu_T \doteq c_\mu \ell_m \sqrt{k'}, \quad k' := \frac{1}{2}\rho_0 |\mathbf{u}'|^2, \quad \ell_m := \text{"mixing length"}. \tag{3.2.7}$$

In spatial filtering, the critical length scale is the filter width δ. The kinetic energy in the small eddies can be related back to using the approximation (3.2.6). Indeed, the approximation (3.2.6) for $\mathbf{u}'$ in (3.2.7) gives the relation:

$$\nu_T \doteq c_\mu \delta \sqrt{\frac{1}{2}\rho_0 \left(\frac{\delta^2}{4\gamma}\right)^2 |\Delta \overline{\mathbf{u}}|^2} = c'_\mu \, \frac{\delta^3}{\gamma} \, |\Delta \overline{\mathbf{u}}|. \tag{3.2.8}$$

This yields the subgridscale model for turbulent fluctuations given by

$$\nabla \cdot \overline{\mathbf{u}'\mathbf{u}'} \sim -\nabla \cdot \left[c'_\mu \, \frac{\delta^3}{\gamma} \, |\Delta \overline{\mathbf{u}}| (\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^t) \right] + \frac{2}{3} \nabla k. \tag{3.2.9}$$

An important issue arises in how to use (3.2.9). The model (3.2.9) can be added to the models used for large eddy motion, such as in [10], [3], [30]. However, there has recently been a trend in the direction of simple fluids models composed of the Navier-Stokes equations augmented by a subgridscale model such as (3.2.9). These simpler models are paired with highly resolution in simulations by adaptive algorithms, or with dynamical modeling of the parameters, following [33].

Accordingly, as a first step in the analytical understanding of (3.2.9) we consider the Navier-Stokes equations supplemented by (3.2.9):

$$\begin{cases} \mathbf{w}_t + \nabla \cdot (\mathbf{w}\mathbf{w}) + \nabla q - Re^{-1} \Delta \mathbf{w} - \nabla \cdot \left[c'_\mu \dfrac{\delta^3}{\gamma} |\Delta \mathbf{w}| (\nabla \mathbf{w} + \nabla \mathbf{w}^t) \right] = \overline{\mathbf{f}}, & \text{in } \Omega, \\ \nabla \cdot \mathbf{w} = 0, & \text{in } \Omega. \end{cases} \tag{3.2.10}$$

The subgridscale model (3.2.9) has the clear features that the turbulent diffusion vanishes for linear mean velocities and the magnitude of the turbulent diffusion is proportional to a consistent approximation of the turbulent kinetic energy.

If ν_T is taken to be a function of the local average of the kinetic energy in the small eddies, the previous derivation is changed only slightly, replacing $|\Delta \overline{\mathbf{u}}|$ in (3.2.9) by $g_\delta * |\Delta \overline{\mathbf{u}}|$. Another important approximation step is $g_\delta * |\Delta \overline{\mathbf{u}}| \sim |g_\delta * \Delta \overline{\mathbf{u}}|$, which is the magnitude of the "Gaussian Laplacian". The resulting subgridscale model (which replaces (3.2.9) in (3.2.10)) is given by:

$$\nabla \cdot \overline{\mathbf{u}'\mathbf{u}'} \sim -\nabla \cdot \left[c'_\mu \frac{\delta^3}{\gamma} |g_\delta * \Delta \overline{\mathbf{u}}| (\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^t) \right] + \frac{2}{3} \nabla k. \tag{3.2.11}$$

Interestingly, if we begin with (3.2.11) and *reverse* the steps in the derivation of (3.2.9), (3.2.11) a *third* subgridscale model results. Indeed, by (3.2.6), $-\mathbf{u}' \doteq \Delta\mathbf{u}$. Thus, $|g_\delta * \Delta\bar{\mathbf{u}}| \doteq |g_\delta * (-\mathbf{u}')| = |g_\delta * (\bar{\mathbf{u}} - \mathbf{u})| = |\bar{\mathbf{u}} - g_\delta * \bar{\mathbf{u}}|$. The third model reads:

$$\nabla \cdot \overline{\mathbf{u}'\mathbf{u}'} \sim -\nabla \cdot \left[c''_\mu \frac{\delta^3}{\gamma} |\bar{\mathbf{u}} - g_\delta * \bar{\mathbf{u}}|(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^t)\right] + \frac{2}{3}\nabla k. \tag{3.2.12}$$

This third model can arise in a particularly simple way. Indeed, as $\mathbf{u}' = \mathbf{u} - \bar{\mathbf{u}}$, $|\mathbf{u}'|^2 = |\mathbf{u} - \bar{\mathbf{u}}|^2$. A scale similarity approximation can be used, specifically, $|\mathbf{u} - \bar{\mathbf{u}}|^2 \sim |\bar{\mathbf{u}} - \bar{\bar{\mathbf{u}}}|^2$. This scale similarity approximation to the turbulent kinetic energy gives (3.2.12).

Nonlinear diffusion equations whose coefficients depend upon local averages of $\bar{\mathbf{u}}$ (as in (3.2.12)) have recently also used for image smoothing, e.g., [75], [11].

3.3 Implementation Issues

The subgrid-scale term (3.2.9) can, of course, be added to any discretization of any convection dominated problem to enhance numerical stability. (This is the viewpoint of subgridscale modeling adopted in, for example, [62], [63].) For this use, (3.2.9) is computationally attractive with $\delta = h$ (the local meshwidth) since the extra term vanishes when $\Delta\bar{\mathbf{u}} = 0$ and is $O(h^3)$ when $|\Delta\bar{\mathbf{u}}| = O(1)$. When $\bar{\mathbf{u}}$ fluctuates across a few mesh points, $|\Delta\bar{\mathbf{u}}| = O(h^{-2})$ so that in this case (3.2.9) introduces only $O(h)$ artificial viscosity.

If the convection-dominated problem is discretized by a finite difference method, the implementation of the model (3.2.8) is clear: $|\Delta\bar{\mathbf{u}}|$ is calculated by the usual five point discrete Laplacian. If a finite element method *using C^1-elements* is used, then the implementation of (3.2.8) is equally clear. If the most usual case of C^0 elements, e.g., conforming linears on triangles or tetrahedrons, is used, then (3.2.9) must be interpreted correctly as $\Delta\phi^h$ does not exist for such functions ϕ^h.

Specifically, let Ω be a polyhedral domain with a finite element mesh $\Pi^h(\Omega)$ constructed which divides Ω into conforming d-simplices. Thus, in 2D the triangles are edge to edge and in 3D the tetrahedra are face to face. Let X^h denote the usual space of conforming, C^0, piecewise linears

$$X^h := \{\mathbf{v}^h(\mathbf{x}) : \mathbf{v}^h \in C^0(\Omega) \cap H^1_\circ(\Omega) \text{ and } \mathbf{v}^h \text{ is affine on each simplex}\}.$$

For such functions $\mathbf{v}^h \in X^h, \Delta_h$ can be correctly interpreted in terms of jumps in $\nabla\mathbf{v}^h \cdot \hat{n}$ across edges (2D) or faces (3D) of $\Pi^h(\Omega)$. This interpretation was (to our knowledge) first pointed out and exploited in the work of Eriksson and Johnson [23] (see also [54] for another use of edge jump in a discretization). For K a triangle (2D) or tetrahedron (3D), following [23], and $\mathbf{v}^h \in X^h$, define the piecewise constant, discrete Laplacian as

$$|\Delta_h \mathbf{v}^h||_K := \max_{f \subset \partial K \cap \Omega} |[\nabla \mathbf{v}^h \cdot \mathbf{n}_f]_f| / \text{ diameter } (K),$$

where $\mathbf{n}_f$ is the exterior unit normal to K on the edge or face f of K, and $[\cdot]_f$ denotes the jump of the indicated (discontinuous) quantity across f. The function $|\Delta_h \mathbf{v}^h|$ is piecewise constant for a linear finite element space. For higher order spaces, $\Delta \mathbf{v}^h$ inside K is added to the above edge jump definition of Δ_h.

For locally quasi-uniform meshes, $|\Delta_h \mathbf{v}^h|$ satisfies the following inverse estimates, [23] for all $\mathbf{v}^h \in X^h$,

$$||h \Delta_h \mathbf{v}^h||_{L^2(\Omega)} \leq C ||\nabla \mathbf{v}^h||_{L^2(\Omega)},$$

and

$$||h_K \Delta_h \mathbf{v}^h||_{L^2(K)} \leq C ||\nabla \mathbf{v}^h||_{L^2(\omega(K))}, \tag{3.3.13}$$

where $\omega(K)$ is the union of all the elements sharing a common edge/face with K in $2D/3D$.

The next proposition shows that the effects of the SGS model (3.2.8) are, *at their largest*, comparable to those of the Smagorinsky model (3.1.2), provided $\delta \sim h_K$.

Proposition 3.3.1 *Suppose the inverse estimates (3.3.13) hold. Then, for all* $\mathbf{u}^h \in X^h$

$$\int_\Omega \delta^3 |\Delta^h \mathbf{u}^h| \nabla \mathbf{u}^h \cdot \nabla \mathbf{u}^h d\mathbf{x} \leq C \int_\Omega (\delta^3 / h_K) |\nabla \mathbf{u}^h| \nabla \mathbf{u}^h \cdot \nabla \mathbf{u}^h d\mathbf{x},$$

where C depends on the smallest angle in the triangulation.

Proof: Consider one triangle K. Since both $\nabla \mathbf{u}^h$ and $\Delta^h \mathbf{u}^h$ are constant on K,

$$\int_K \delta^3 |\Delta^h \mathbf{u}^h| \nabla \mathbf{u}^h \cdot \nabla \mathbf{u}^h d\mathbf{x} = \delta^3 |\Delta^h \mathbf{u}^h| \; ||\nabla \mathbf{u}^h||^2_{L^2(K)} \leq$$

$$\text{(using (3.3.13))} \;\; \leq C \; \delta^3 h_K^{-1} \Big(\sum_{K' \subset \omega(K)} ||\nabla \mathbf{u}^h||^2_{L^2(K')} \Big)^{1/2} ||\nabla \mathbf{u}^h||^2_{L^2(K)}$$

$$\leq C \delta^3 h_K^{-1} \Big(\sum_{K' \subset \omega(K)} ||\nabla \mathbf{u}^h||^2_{L^2(K)} \Big)^{1/2} \Big(\sum_{K' \subset \omega(K)} ||\nabla \mathbf{u}^h||^2_{L^2(K)} \Big) \tag{3.3.14}$$

Now, $\sum_{K'\subset\omega(K)}$ is a sum over a fixed number of terms bounded by $C(\theta_{\min})$, a constant only depending on the minimum angle $\theta_{\min}$ in the mesh. Thus, using Hölder's inequality, we get

$$\left[\sum_{K'\subset\omega(K)} ||\nabla \mathbf{u}^h||^2_{L^2(K')}\right]^{1/2} \leq C(\theta_{\min}) \left[\sum_{K'\subset\omega(K)} ||\nabla \mathbf{u}^h||^3_{L^2(K')}\right]^{1/3}.$$

Therefore,

$$(3.3.14) \leq C(\theta_{\min})\delta^3 h_K^{-1}\left[\sum_{K'\subset\omega(K)} ||\nabla \mathbf{u}^h||_{L^2(K')}\right]^{1/3}\left[\sum_{K'\subset\omega(K)} ||\nabla \mathbf{u}^h||^3_{L^2(K')}\right]^{2/3}.$$

Since $\nabla \mathbf{u}^h_{|_{K'}}$ is a constant vector,

$$||\nabla \mathbf{u}^h||_{L^2(K')} \leq meas(K')^{-1/6}||\nabla \mathbf{u}^h||_{L^3(K')} \leq ||\nabla \mathbf{u}^h||_{L^3(K')},$$

provided that $meas(K') = \int_{K'} 1 d\mathbf{x} \leq 1$ (this condition holds if, for example, $0 < h << 1$). Thus,

$$\int_K \delta^3|\Delta^h \mathbf{u}^h| \cdot |\nabla \mathbf{u}^h|^2 \leq C(\theta_{\min})\delta^3 h_K^{-1}\left[\sum_{K'\subset\omega(K)} ||\nabla \mathbf{u}^h||^3_{L^3(K')}\right].$$

Summing this inequality over $K \in \Pi^h(\Omega)$ and again using that $\Pi^h(\Omega)$ satisfies a minimum angle condition gives the result.

■

This proposition thus establishes that the SGS model (3.2.8) is more selective than the good and accepted Smagorinsky SGS model in that the former vanishes in cases where the latter is significant while (in the sense of the quadratic form) the largest action of the former is bounded by the latter.

3.4 Existence of Solutions

This section considers the question of existence of weak solutions to the system (3.2.10) with the modification (3.2.11). Thus, we seek $(\mathbf{w}, q)$ satisfying:

$$\begin{cases} \mathbf{w}_t + \nabla\cdot(\mathbf{w}\mathbf{w}) + \nabla q - Re^{-1}\Delta\mathbf{w} - \nabla\cdot(\delta^3|g_\delta * \Delta\mathbf{w}|(\nabla\mathbf{w} + \nabla\mathbf{w}^t)) = \overline{\mathbf{f}}, & \text{in } \Omega, t>0 \\ \nabla\cdot\mathbf{w} = 0, & \text{in } \Omega, t>0, \\ \mathbf{w}(\mathbf{x},0) = g_\delta * \mathbf{u}_0(\mathbf{x}), & \text{in } \Omega, \\ \mathbf{w}(\mathbf{x},t) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4.15}$$

The Dirichlet boundary condition we take in (3.4.15) provides a convenient condition on Γ. It is known, however, that for modeling accuracy near Γ it should be replaced by a slip with friction type condition which recovers no-slip as $\delta \to 0$, [30], [83].

Theorem 3.4.1 *Let $T > 0$, and Ω be a bounded domain in $\mathbb{R}^n$. Then, for any given*

$$\mathbf{u}_0 \in L^2(\Omega),\ \mathbf{f} \in L^2(\Omega \times (0,T)),$$

there exists at least one weak solution to (3.4.15) in $\Omega \times (0,T)$.

Remark: The model (3.2.9) without regularization is more difficult due to the unbounded coefficient $|\Delta \mathbf{u}|$ in (3.2.9). Appropriate mathematical tools for such problems are in their early stages of development, see, e.g., [31].

Proof: We shall use the Faedo–Galerkin method. Let $\mathcal{D}(\Omega) := \{\psi \in C_0^\infty(\Omega) : \nabla \cdot \psi = 0 \text{ in } \Omega\}$, $H(\Omega)$ the completion of $\mathcal{D}(\Omega)$ in $L^2(\Omega), H^1(\Omega)$ the completion of $\mathcal{D}(\Omega)$ in $W^{1,2}(\Omega)$ and $\{\psi_r\} \subset \mathcal{D}(\Omega)$ be the orthonormal basis of $H(\Omega)$ given in Lemma 2.3 in [38]. We shall look for approximating solutions v_k of the form:

$$v_k(\mathbf{x},t) = \sum_{r=1}^{k} c_{kr}(t)\psi_r(\mathbf{x}) \quad ,\ k \in \mathbb{N}, \tag{3.4.16}$$

where the coefficients c_{kr} are required to satisfy the following system of ordinary differential equations

$$\begin{aligned}&\frac{dc_{kr}}{dt} + \sum_{i=1}^{k} Re^{-1}(\nabla\psi_i, \nabla\psi_r)c_{ki} + \sum_{i,s=1}^{k}(\psi_i\nabla\psi_s, \psi_r)c_{ki}c_{ks} + \\ &\gamma^3 \sum_{i=1}^{k} c_{ki}\left(\sum_{j=1}^{k} |c_{kj}(g_\delta * \Delta\psi_r)|(\nabla\phi_i + \nabla\phi_i^t), \nabla\psi_r + \nabla\psi_r^t)\right) = (\overline{\mathbf{f}}, \psi_r), \\ &\qquad\qquad r = 1, \cdots, k\end{aligned} \tag{3.4.17}$$

with the initial condition

$$c_{kr}(0) = (g_\delta * v_0, \psi_r) \tag{3.4.18}$$

Multiplying (3.4.17) by c_{kr}, and summing over r, we get:

$$\begin{aligned}||v_k(t)||_2^2 \quad &+2Re^{-1}\int_0^t ||\nabla v_k(\zeta)||_2^2 d\zeta + 2\delta^3\int_0^t |g_\delta * \Delta v_k(\zeta)|\ ||\nabla v_k(\zeta) + \nabla v_k^t(\zeta)||_2^2 d\zeta = \\ &2\int_0^t (v_k(\zeta), \overline{f}(\zeta))d\zeta + ||v_k(0)||_2^2 \quad \forall\ t \in [0,T)\end{aligned} \tag{3.4.19}$$

Using the Cauchy-Schwarz inequality, Korn's inequality, and Gronwall's lemma, we get:

$$||v_k(t)||_2^2 + \int_0^t ||\nabla v_k(\zeta)||_2^2 d\zeta \leq M, \quad \forall\, t \in [0,T), \tag{3.4.20}$$

with M independent of t and k. Thus,

$$|c_{kr}(t)| \leq M^{1/2}, \quad \forall\, r = 1, \cdots, k. \tag{3.4.21}$$

From the elementary theory of partial differential equations, (3.4.21) implies that (3.4.17) admits a unique solution $c_{kr} \in W^{1,2}(0,T)$ for all $k \in N$.
Using the same approach as the one in [38], from these a priori bounds we get the existence of $v \in L^2(0,T,H^1(\Omega))$ such that

$$\lim_{k\to\infty} (v_k(t) - v(t), \mathbf{w}) = 0 \text{ uniformly in } t \in [0,T],\ \forall\, \mathbf{w} \in L^2(\Omega) \tag{3.4.22}$$

$$\lim_{k\to\infty} \int_0^T (\partial_i(v_k - v), \mathbf{w}) d\zeta = 0 \ \ \forall\, \mathbf{w} \in L^2(\Omega \times [0,T]),\ i = 1, \cdots, k. \tag{3.4.23}$$

Now we shall prove the strong convergence of $\{g_\delta * \Delta v_k\}$ to $g_\delta * \Delta v$ in $L^2(\omega \times [0,T])$ for all $\omega \subset\subset \Omega$. To show this, we need the following Friederichs' inequality (see, e.g, [27] Lemma II. 4.2): Let C be a cube in $\mathbb{R}^n$, then for any $\eta > 0$, there exists $K(\eta, C)$ functions $\varphi_i \in L^\infty(C), i = 1, \cdots, K$ such that:

$$\int_0^T ||\mathbf{w}(t)||_{2,C}^2 dt \leq \sum_{i=1}^k \int_0^T (\mathbf{w}(t), \varphi_i)_C^2 dt + \eta \int_0^t ||\nabla \mathbf{w}(t)||_{2,C}^2 dt \tag{3.4.24}$$

Applying the above inequality with $w := g_\delta * \Delta v_k - g_\delta * \Delta \mathbf{w}_k$, we get

$$\begin{aligned}
\int_0^t ||g_\delta * \Delta v_k - g_\delta * \Delta v||_{2,C}^2 dt \ &\leq\ \sum_{i=1}^k \int_0^T (g_\delta * \Delta v_k - g_\delta * \Delta v, \varphi_i)_C^2 dt + \\
&+\ \eta \int_0^T ||\nabla (g_\delta * \Delta v_k - g_\delta * \Delta v)||_{2,C}^2 dt \\
&=\ -\sum_{i=1}^k \int_0^T (\nabla v_k - \nabla v, g_\delta * \nabla \varphi_i)_C^2 dt + \\
&\quad\ \eta \int_0^T ||\nabla (g_\delta * \Delta v_k - g_\delta * \Delta v)||_{2,C}^2 dt
\end{aligned}$$

Using (3.4.23) and the fact that $||\nabla(g_\delta * \Delta v_k - g_\delta * \Delta v)||_{2,C}^2 \leq C(g,\delta)||v_k - v||_{2,C}^2$, we get

$$\lim_{k\to\infty} \int_0^T ||g_\delta * \Delta v_k - g_\delta * \Delta v||_{2,C}^2 = 0 \tag{3.4.25}$$

Applying (3.4.24) with $w := v_k - v$, and using (3.4.23), we get

$$\int_0^T ||v_k(t) - v(t)||^2_{2,C} dt = 0 \tag{3.4.26}$$

Now we shall prove that v is a weak solution of (3.4.15). Integrating (3.4.17) from 0 to $t \le T$, we get:

$$\int_0^t -Re^{-1}(\nabla v_k, \nabla\psi_r) - (v_k \cdot \nabla v_k, \psi_r) d\zeta = -\int_0^t (\overline{\mathbf{f}}, \psi_r) d\zeta + \tag{3.4.27}$$
$$\delta^3 \int_0^t (|g_\delta * \Delta v_k|(\nabla v_k + \nabla v_k^t), \nabla\psi_r + \nabla\psi_r^t) + (v_k(t), \psi_r) - (v_k(0), \psi_r)$$

From (3.4.22) and (3.4.23) we get

$$\lim_{k\to\infty} (v_k(t) - v(t), \psi_r) = 0, \quad \lim_{k\to\infty} \int_0^t (\nabla v_k(\zeta) - \nabla v(\zeta), \nabla\psi_r) d\zeta = 0 \tag{3.4.28}$$

Let C be a cube containing the support of ψ_r. Then:

$$\begin{aligned} |\int_0^t (v_k \cdot \nabla v_k, \psi_r) - (v \cdot \nabla v, \psi_r) d\zeta| &\le |\int_0^t ((v_k - v) \cdot \nabla v_k, \psi_r)_C d\zeta)| \\ &+ |\int_0^t (v \cdot \nabla(v_k - v), \psi_r)_C d\zeta| \end{aligned} \tag{3.4.29}$$

Setting $S := \max_{x\in C} |\psi_r(x)|$, and using (3.4.20), we also have:

$$\begin{aligned} |\int_0^t ((v_k - v) \cdot \nabla v_k, \psi_r)_C d\zeta| &\le S \left(\int_0^t ||v_k - v||^2_{2,C} d\zeta\right)^{1/2} \left(\int_0^t ||\nabla v_k||^2_{2,C} d\zeta\right)^{1/2} \\ &\le SM^{1/2} \left(\int_0^t ||v_k - v||^2_{2,C} d\zeta\right)^{1/2} \end{aligned}$$

Thus, using (3.4.26), we get:

$$\lim_{k\to\infty} |\int_0^t ((v_k - v) \cdot \nabla v_k, \psi_r)_C d\zeta| = 0 \tag{3.4.30}$$

We also have:

$$|\int_0^t (v \cdot \nabla(v_k - v), \psi_r)_C d\zeta| \le \sum_{i=1}^n |\int_0^t (\partial_i(v_k - v), v_i\psi_r)_C d\zeta|$$

and since $v_i\psi_r \in L^2(\Omega \times [0,T])$, (3.4.23) implies:

$$\lim_{k\to\infty} |\int_0^t (v \cdot \nabla(v_k - v), \psi_r)_C d\zeta| = 0 \tag{3.4.31}$$

Relations (3.4.29) – (3.4.31) yield:

$$\lim_{k\to\infty} |\int_0^t (v_k \cdot \nabla v_k - v \cdot \nabla v, \psi_r) d\zeta| = 0. \tag{3.4.32}$$

Also,

$$\begin{aligned}
&\int_0^t (|g_\delta * \Delta v_k|(\nabla v_k + \nabla v_k^t) - |g_\delta * \Delta v|(\nabla v_\Delta + \nabla v^t), \nabla\psi_r + \nabla\psi_r^t) d\zeta \leq \\
&|\int_0^t (|g_\delta * \Delta v|(\nabla(v_k - v) + \nabla(v_k - v)^t), \nabla\psi_r + \nabla\psi_r^t) d\zeta| + \\
&|\int_0^t (|g_\delta * \Delta v_k - g_\delta * \Delta v|)(\nabla v_k, + \nabla v_k^t), \nabla\psi_r + \nabla v_k^t)\nabla\psi_r^t) d\zeta|
\end{aligned} \tag{3.4.33}$$

We have:

$$|\int_0^t (|g_\delta + \Delta v|(\nabla(v_k - v) + \nabla(v_k - v)^t), \nabla\psi_r + \nabla\psi_r^t) d\zeta| \leq \sum_{i=1}^n |\int_0^t (\partial_i(v_k - v), |g_\delta * \Delta v|\nabla\psi_r) d\zeta|$$

and since $|g_\delta * \Delta v|\nabla\psi_r \in L^2(\Omega \times [0,T])$, (3.4.23) implies:

$$\lim_{k\to\infty} |\int_0^t (|g_\delta * \Delta v|(\nabla(v_k - v) + \nabla(v_k - v)^t), \nabla\psi_r + \nabla\psi_r^t) d\zeta| = 0 \tag{3.4.34}$$

On the other hand, setting $\tilde{S} := \max_{x\in C} |\nabla\psi_r(x)|$, and using (3.4.20), we get:

$$\begin{aligned}
&|\int_0^t (|g_\delta * \Delta v_k - g_\delta * \Delta v|)(\nabla v_k, + \nabla v_k^t), \nabla\psi_r + \nabla\psi_r^t) d\zeta| \leq \\
&S\left(\int_0^t ||g_\delta * \Delta v_k - g_\delta * \Delta v||_{2,C}^2 d\zeta\right)^{1/2} \left(\int_0^t ||\nabla v_k||_{2,C}^2 d\zeta\right)^{1/2} \leq \\
&\tilde{S} M^{1/2} \left(\int_0^t ||g_\delta * \Delta v_k - g_\delta * \Delta v||_{2,C}^2 d\zeta\right)^{1/2}
\end{aligned}$$

Thus, using (3.4.26), we get:

$$\lim_{k\to\infty} |\int_0^t (|g_\delta * \Delta v_k - g_\delta * \Delta v|)(\nabla v_k + \nabla v_k^t), \nabla\psi_r + \nabla\psi_r^t) d\zeta| = 0 \tag{3.4.35}$$

Relations (3.4.33) - (3.4.35) yield:

$$\lim_{k\to\infty} |\int_0^t (|g_\delta * \Delta v_k)(\nabla v_k + \nabla v_k^t - |g_\delta * \Delta v|(\nabla v + \nabla v^t), \nabla\psi_r + \nabla\psi_r^t) d\zeta| = 0 \tag{3.4.36}$$

Therefore, taking the limit over $k \to \infty$ in (3.4.28), and using (3.4.28), (3.4.32) and (3.4.36), we get:

$$\begin{aligned}
\int_0^t \{-Re^{-1}(\nabla v, \nabla v_r) - (v \cdot \nabla v, \psi_r\} d\zeta = &- \int_0^t (\overline{\mathbf{f}}, \psi_r) d\zeta + (v(t), \psi_r) - (v(0), \psi_r) \\
&+ \delta^3 \int_0^t (|g_\delta * \Delta v|(\nabla v + \nabla v^t), \nabla\psi_r + \nabla\psi_r^t)
\end{aligned}$$

However, from Lemma 2.3 in [38] we know that every function $\psi \in \mathcal{D}(\Omega)$ can be uniformly approximated in $C^2(\overline{\Omega})$ by functions of the form

$$\psi_N(x) = \sum_{r=1}^{N} \gamma_r \psi_r(x) \quad , N \in \mathbb{N}, \gamma_r \in \mathbb{R}$$

So, writing (3.4.35) with ψ_N instead of ψ_r, and passing to the limit as $N \to \infty$, we get the validity of (3.4.35) for all $\psi \in \mathcal{D}(\Omega)$. Thus, v is a weak solution of (3.4.15).

$\blacksquare$

Chapter 4

Numerical Analysis of LES Models

4.1 Convergence of Finite Element Approximations of Large Eddy Motion

This chapter considers "numerical-errors" in LES. Specifically, for one filtered flow model, we show convergence of the semidiscrete finite element approximation of the model and give an estimate of the error.

Motivated by the presentations in Chapter 2 and Chapter 3, this chapter considers a class of LES models, including the classical LES model (developed in [14]) and the Galdi-Layton LES model (developed in [30]). This class of LES models can be written generically as: find $(\mathbf{w}, q)$, where $\mathbf{w} : \Omega(\subset \mathbb{R}^d)\times[0,T] \to \mathbb{R}^d, p : \Omega\times(0,T] \to \mathbb{R}$ satisfying:

$$\begin{cases} \mathbf{w}_t + \nabla\cdot(\mathbf{w}\mathbf{w}) - Re^{-1}\Delta\mathbf{w} + \nabla q + \delta^2\nabla\cdot(A^{-1}(\nabla\mathbf{w}\nabla\mathbf{w})) & \\ \qquad\qquad - \nabla\cdot(\nu_T(\mathbf{w})\ \nabla\mathbf{w}) = \bar{\mathbf{f}}, & \text{in } \Omega\times(0,T], \\ \nabla\cdot\mathbf{w} = 0, & \text{in } \Omega\times(0,T], \\ \mathbf{w}(\mathbf{x},0) = \bar{\mathbf{u}}_0(\mathbf{x}) & \text{in } \Omega, \\ +\ \text{Boundary conditions on } \Gamma = \partial\Omega. & \end{cases} \tag{4.1.1}$$

The notation and terms in (4.1.1) require some explanation. The operator A^{-1} denotes a regularization operator, described below. The term $\nabla\mathbf{w}\nabla\mathbf{w}$ is shorthand for the 2-tensor:

$$(\nabla\mathbf{w}\nabla\mathbf{w})_{ij} := \sum_{\ell=1}^{d} \frac{\partial \mathbf{w}_i}{\partial \mathbf{x}_\ell}\, \frac{\partial \mathbf{w}_j}{\partial \mathbf{x}_\ell}$$

The function $\nu_T(\mathbf{w})$ is the "turbulent viscosity" coefficient arising from the subgridscale model employed for turbulent fluctuations. A detailed presentation of the choices for ν_T is given in Chapter 3. In this chapter we will use the most commonly used Smagorinsky [84] model (described in Section 3.1), in which, for clarity, we replace $\nabla\mathbf{w} + \nabla\mathbf{w}^t$ by $\nabla\mathbf{w}$:

$$\nu_T(\mathbf{w}) = C_s\delta^2|\nabla\mathbf{w}|.$$

The domain Ω is assumed to be bounded, simply connected and have C^1 boundary Γ. The question of boundary conditions for (4.1.1) is a fundamental question in LES. There are various proposals; we impose a boundary condition suggested in [30] and developed in [83]. If the fluid particles adhere to the walls, it does not follow that the large eddies also "stick". (In fact, it is clear that large eddies do move slip along walls and lose energy as they slip.) The conditions we impose are no-penetration and slip with resistance. Specifically,

$$\begin{cases} \mathbf{w}\cdot\hat{n} = 0, & \text{on } \partial\Omega, \\ \mathbf{w}\cdot\hat{\tau}_j + \beta^{-1}(\delta, Re)t\cdot\hat{\tau}_j = 0, & \text{on } \partial\Omega,\ j = 1, d-1. \end{cases} \tag{4.1.2}$$

Here $\beta(\delta, Re)$ is the friction coefficient, and the vectors $\hat{n}$ and $\hat{\tau}_j$ (where $j = 1$ if $d = 2$ and if $d = 3,\ j = 1, 2$) denote the unit normal and tangent vectors to Γ where, if $d = 3,\ \hat{\tau}_1 \perp \hat{\tau}_2$. If $d = 3$, all terms in which $\hat{\tau}_j$ occurs should (by understanding) be summed from $j = 1, 2$; for example, $||\mathbf{w}\cdot\hat{\tau}_j||_\Gamma^2$ means $\sum_{j=1}^{2} ||\mathbf{w}\cdot\hat{\tau}_j||_\Gamma^2$. Also t represents the Cauchy stress vector associated with $\mathbf{w}$. Specifically,

$$t = \hat{n}\cdot[-qI - \delta^2 A^{-1}(\nabla\mathbf{w}\nabla\mathbf{w}) + Re^{-1}\nabla\mathbf{w} + \nu_t(\mathbf{w})\nabla\mathbf{w}].$$

There are several natural choices for the regularization A^{-1} in (4.1.1). The most commonly used model was with no regularization, i.e. $A^{-1} \equiv I$ (see [69], [14], [3], [16]). A more careful derivation of the LES model in [30] suggests the inclusion of the regularization operator A^{-1} in the system (4.1.1). One choice of A^{-1} is simply to reapply the spatial filter underlying (4.1.1): $A^{-1}\mathbf{v} = g_\delta * \mathbf{v}$; another possibility is $A^{-1}\mathbf{v} = (-\delta^2\Delta + I)^{-1}\mathbf{v}$. The convergence analysis in this chapter is for the classical model of [69], [14] with no regularization, $A^{-1} \equiv I$.

We now introduce the notation for the functional setting. The $L^2(\Omega)$ norm and inner product are denoted $||\cdot||$ and $(\cdot,\cdot)$. The $L^2(\Gamma)$ norm and inner product are denoted $||\cdot||_\Gamma$ and $(\cdot,\cdot)_\Gamma$. The $L^3(\Omega)$ norm is $||\cdot||_{L^3}$ and the Sobolev $W^{k,p}(\Omega)$ norm is denoted $||\cdot||_{k,p}$, with p omitted if $p=2$. See, e.g., [27] for a clear development of Sobolev spaces focusing on those important for the Navier-Stokes equations.

The velocity space is $X := \{\mathbf{v} \in W^{1,2}(\Omega)^d : \mathbf{v}\cdot\hat{n} = 0 \text{ on } \Gamma\}$. The pressure space is $Q := L^2_0(\Omega)$ where $L^2_0(\Omega) := \{\lambda(\mathbf{x}) \in L^2(\Omega) : (\lambda, 1) = 0\}$. Mathematical properties of such velocity-pressure spaces are developed in, e.g., [35], [38], [39], [65], [73].

There are two fundamental issues in large eddy simulation: assessment of "modeling errors" and "numerical errors". The modeling error refers to the question of how close $\mathbf{w}(\mathbf{x},t)$ is to the true flow averages: $|||\mathbf{w}-\bar{\mathbf{u}}|||$ for some norm $|||\cdot|||$. To our knowledge, there are no analytical results to date on this question, but there are experimental results comparing various <u>averages</u> of w to those same <u>averages</u> of $\bar{\mathbf{u}}$ (i.e. averages of averages of $\mathbf{u}$). Accepting $\mathbf{w}(\mathbf{x},t)$ as an interesting model for $\bar{\mathbf{u}}$, "numerical errors" describe how close an approximation $\mathbf{w}^h$ is to $\mathbf{w}$. This leads to classical questions of stability, consistency and convergence for approximations of (4.1.1).

This chapter considers precisely this question for finite element approximations of (4.1.1). In Theorem 4.3.1 we show that the usual, continuous in time, finite element approximation to (4.1.1), $\mathbf{w}^h$, converges to $\mathbf{w}$ as the meshwidth $h \to 0$ for the Reynolds number Re and averaging radius δ fixed.

This analysis leads to interesting questions beyond the case of the usual Navier-Stokes equations (pioneered by Heywood and Rannacher in a series of papers [40], [41], [42], [43]), including: the case of slip with friction boundary conditions (4.1.2) (see e.g., [65], [73] for some work related to this case), the degeneracy of the μ-Laplacian based subgridscale model in (4.1.1) (see, e.g., [21], [62] for numerical analysis of the equilibrium model composed of NSE + μ-Laplacian), the "cross-term" $\delta^2\nabla\cdot(\nabla\mathbf{w}\nabla\mathbf{w})$ in (4.1.1) which is non-monotone, nonlinear and higher order, and the dependence of the error on the Reynolds number Re, and the averaging radius δ.

Our convergence analysis comes to grips with some of these questions but not all. In particular, we prove convergence as $h \to 0$ for fixed Re. In some sense, Theorem 4.3.1 shows that the parameter δ does not degrade convergence. Naturally, it is hoped and expected that a sharper analysis would show that its presence in the model results in improved estimates. The degeneracy in the Smagorinsky [84] subgridscale model is not an essential difficulty but (surprisingly) its polynomial growth, which must match that of the cross term to ensure boundedness of the kinetic energy in $\mathbf{w}$, seems to cause suboptimality in the resulting error estimates. This issue has recently been studied in a simplified setting in [50].

Nevertheless, convergence $\mathbf{w}^h \to \mathbf{w}$ as $h \to 0$ is proven. The long term analytical goals in the numerical analysis of large eddy simulation are then to sharpen this result especially with respect to error dependence on δ and Re, where possible, and complement it with analysis of the modeling error. Preliminary steps in this last direction have recently been made in [52] for a different class of LES model.

4.2 Variational Formulation of the Model

A detailed presentation of existence results for weak solutions of (4.1.1) has been given in Chapter 2 and Chapter 3. Although existence of strong solutions is still an open problem, we shall nevertheless assume that (4.1.1), (4.1.2) has a unique strong solution. Any additional required smoothness on $(\mathbf{w}, q)$ will be explicitly stated as it is used. Since the boundary conditions on $\mathbf{w}$ are not simple Dirichlet conditions, extra care must be taken in developing a variational formulation of (4.1.1) in (X, Q).

Consider the following term, for $\mathbf{v} \in X$:

$$
\begin{aligned}
&\int_\Omega \nabla \cdot [qI + \delta^2(\nabla\mathbf{w}\nabla\mathbf{w}) - (Re^{-1} + \nu_T(\mathbf{w}))\nabla\mathbf{w}] \cdot \mathbf{v}\ d\mathbf{x} = \\
&\int_\Gamma \hat{n} \cdot [qI + \delta^2(\nabla\mathbf{w}\nabla\mathbf{w}) - (Re^{-1} + \nu_T(\mathbf{w}))\nabla\mathbf{w}] \cdot \mathbf{v}\ ds - \\
&\int_\Omega q\nabla \cdot \mathbf{v} + [\delta^2(\nabla\mathbf{w}\nabla\mathbf{w}) - (Re^{-1}\nabla\mathbf{w} + \nu_T(\mathbf{w}))\nabla\mathbf{w}] : \nabla\mathbf{v}\ d\mathbf{x}.
\end{aligned}
$$

Decomposing $\mathbf{v} = (\mathbf{v}\cdot\hat{\tau}_j)\hat{\tau}_j + (\mathbf{v}\cdot\hat{n})\hat{n} = (\mathbf{v}\cdot\hat{\tau}_j)\hat{\tau}_j$ in the first integral, cancelling the obvious terms and using (4.1.2), gives:

$$
\begin{aligned}
&\int_\Omega \nabla \cdot [qI + \delta^2(\nabla\mathbf{w}\nabla\mathbf{w}) - (Re^{-1} + \nu_T(\mathbf{w}))\nabla\mathbf{w}] \cdot \mathbf{v}\ d\mathbf{x} = \\
&\beta(\delta, Re) \int_\Gamma (\mathbf{w} \cdot \hat{\tau}_j)(\mathbf{v} \cdot \hat{\tau}_j)\ ds - \\
&\int_\Omega q\nabla \cdot \mathbf{v} + [\delta^2(\nabla\mathbf{w}\nabla\mathbf{w}) - (Re^{-1} + \nu_T(\mathbf{w}))\nabla\mathbf{w}] : \nabla\mathbf{v}\ d\mathbf{x}
\end{aligned}
\tag{4.2.3}
$$

With the integral identity (4.2.3) in mind, the strong solution of (4.1.1), (4.1.2) is a differentiable map $\mathbf{w} : [0,T] \to X, q : (0,T] \to Q$ satisfying:

$$\begin{cases} (\mathbf{w}_t, \mathbf{v}) - (q, \nabla \cdot \mathbf{v}) + \beta(\delta, Re)(\mathbf{w} \cdot \hat{\tau}_j, \mathbf{v} \cdot \hat{\tau}_j)_\Gamma + ((Re^{-1} + \nu_T(\mathbf{w}))\nabla \mathbf{w}, \nabla \mathbf{v}) \\ \qquad\qquad - (\mathbf{w} \cdot \nabla \mathbf{v}, \mathbf{w}) - \delta^2((\nabla \mathbf{w} \nabla \mathbf{w}), \nabla \mathbf{v}) = (\bar{\mathbf{f}}, \mathbf{v}), \\ (\lambda, \nabla \cdot \mathbf{w}) = 0, \end{cases} \tag{4.2.4}$$

for all $(\mathbf{v}, \lambda) \in (X, Q)$. The next lemma is fundamental to energy estimation. Its proof is the same index calculation as in the case of the no-slip boundary condition.

Lemma 4.2.1 *For any* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X$ *satisfying* $\nabla \cdot \mathbf{w} = 0$

$$(\mathbf{w} \cdot \nabla \mathbf{v}, \mathbf{u}) = -(\mathbf{w} \cdot \nabla \mathbf{u}, \mathbf{v}).$$

Thus, $(\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{v}) = 0$ *for any such* $\mathbf{v}$ *and*

$$(\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{w}) = \frac{1}{2}(\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{w}) - \frac{1}{2}(\mathbf{v} \cdot \nabla \mathbf{w}, \mathbf{v}).$$

Proof: Using the divergence theorem, $\mathbf{w} \cdot \hat{n} = 0$ on Γ for all $\mathbf{w} \in X$, and $\nabla \cdot \mathbf{w} = 0$, we get:

$$\begin{aligned} (\mathbf{w} \cdot \nabla \mathbf{v}, \mathbf{u}) + (\mathbf{w} \cdot \nabla \mathbf{u}, \mathbf{v}) &= \int_\Omega \sum_{i=1}^d \left(\mathbf{w}_i \frac{\partial \mathbf{v}_j}{\partial \mathbf{x}_i} \mathbf{u}_j + \mathbf{w}_i \frac{\partial \mathbf{u}_j}{\partial \mathbf{x}_i} \mathbf{v}_j \right) \, d\mathbf{x} \\ &= \int_\Omega \sum_{i=1}^d \mathbf{w}_i \frac{\partial}{\partial \mathbf{x}_i} (\mathbf{v}_j \mathbf{u}_j) \, d\mathbf{x} \\ &= \int_\Gamma \sum_{i=1}^d \frac{\partial}{\partial \mathbf{x}_i} (\mathbf{v}_j \mathbf{u}_j) \mathbf{w}_i \hat{n}_i \, ds - \int_\Omega \sum_{i=1}^d \frac{\partial \mathbf{w}_i}{\partial \mathbf{x}_i} \mathbf{v}_j \mathbf{u}_j \, d\mathbf{x} \\ &= 0, \end{aligned}$$

which completes the proof.

$\blacksquare$

The next two technical lemmas quantify the control the model of turbulent diffusion exerts over the interaction of large and small eddies. They are also the key for proving existence of weak solutions, as we have seen in Chapter 2 and Chapter 3. Define, for compactness,

$$F(\mathbf{w}) := (Re^{-1} + \nu_T(\mathbf{w}))\nabla \mathbf{w} - \delta^2(\nabla \mathbf{w} \nabla \mathbf{w}). \tag{4.2.5}$$

Lemma 4.2.2 *Let $\nu_T(\mathbf{w}) := C_s\delta^2|\nabla \mathbf{w}|$, where $C_s = C_s(\Omega)$ is large enough. Then, there is a constant $\underline{C}$ such that for any $\mathbf{v}_1, \mathbf{v}_2 \in X$:*

$$(F(\mathbf{v}_1) - F(\mathbf{v}_2), \nabla(\mathbf{v}_1 - \mathbf{v}_2)) + \beta(\delta, Re)((\mathbf{v}_1 - \mathbf{v}_2)\cdot\hat{\tau}_j, (\mathbf{v}_1 - \mathbf{v}_2)\cdot\hat{\tau}_j)_\Gamma \geq$$
$$Re^{-1}||\nabla(\mathbf{v}_1 - \mathbf{v}_2)||^2 + \underline{C}C_s\delta^2||\nabla(\mathbf{v}_1 - \mathbf{v}_2)||^3_{L^3} + \beta(\delta, Re)||(\mathbf{v}_1 - \mathbf{v}_2)\cdot\hat{\tau}_j||^2_\Gamma.$$

Remark: This lemma does *not* include the $\nabla\cdot(\mathbf{ww})$ nonlinearity describing how the large eddies convect themselves. Due to this $\nabla\cdot(\mathbf{ww})$ term the nonlinearity in (4.1.1) is *not* monotonic.

Proof: Let $\tilde{F}(\mathbf{w}) := (Re^{-1} + C_s\delta^2|\nabla\mathbf{w}|)\,\nabla\mathbf{w} - \delta^2(\nabla\mathbf{w}\nabla\mathbf{w})$.
Note that $F(\mathbf{w}) = \tilde{F}(\mathbf{w}) + C_s\delta^2|\nabla\mathbf{w}|\nabla\mathbf{w}$.
Letting $\mathbf{v}^\gamma := \gamma\mathbf{v}_1 + (1-\gamma)\mathbf{v}_2,\ \gamma\in[0,1]$, and using the μ-Laplacian's strong monotonicity (see, e.g., [62]) and the approach in [16], we get for $C_s \geq 4$:

$$(F(\mathbf{v}_1) - F(\mathbf{v}_2), \nabla(\mathbf{v}_1 - \mathbf{v}_2)) + \beta(\delta, Re)((\mathbf{v}_1 - \mathbf{v}_2)\cdot\hat{\tau}_j, (\mathbf{v}_1 - \mathbf{v}_2)\cdot\hat{\tau}_j)_\Gamma =$$
$$\beta(\delta, Re)\|(\mathbf{v}_1 - \mathbf{v}_2)\cdot\hat{\tau}_j\|^2_\Gamma + (F(\mathbf{v}_1) - F(\mathbf{v}_2), \nabla(\mathbf{v}_1 - \mathbf{v}_2)) \geq$$
$$\beta(\delta, Re)\|(\mathbf{v}_1 - \mathbf{v}_2)\cdot\hat{\tau}_j\|^2_\Gamma + \int_\Omega\left(\int_0^1 \frac{d}{d\gamma}\tilde{F}(\mathbf{v}^\gamma)d\gamma\right)\nabla(\mathbf{v}_1 - \mathbf{v}_2))d\mathbf{x} + \underline{C}C_s\delta^2||\nabla(\mathbf{v}_1 - \mathbf{v}_2)||^3_{L^3} \geq$$
$$\beta(\delta, Re)\|(\mathbf{v}_1 - \mathbf{v}_2)\cdot\hat{\tau}_j\|^2_\Gamma + Re^{-1}||\nabla(\mathbf{v}_1 - \mathbf{v}_2)||^2 + \underline{C}C_s\delta^2||\nabla(\mathbf{v}_1 - \mathbf{v}_2)||^3_{L^3} +$$
$$\int_\Omega\int_0^1\left(\frac{1}{2}C_s\delta^2|\nabla\mathbf{v}^\gamma|\ |\nabla(\mathbf{v}_1 - \mathbf{v}_2)|^2 - \delta^2|\nabla\mathbf{v}^\gamma||\nabla(\mathbf{v}_1 - \mathbf{v}_2)|^2 - \delta^2|\nabla\mathbf{v}^\gamma||\nabla(\mathbf{v}_1 - \mathbf{v}_2)|^2\right)d\gamma d\mathbf{x} \geq$$
$$\beta(\delta, Re)\|(\mathbf{v}_1 - \mathbf{v}_2)\cdot\hat{\tau}_j\|^2_\Gamma + Re^{-1}||\nabla(\mathbf{v}_1 - \mathbf{v}_2)||^2 + \underline{C}C_s\delta^2||\nabla(\mathbf{v}_1 - \mathbf{v}_2)||^3_{L^3}$$

■

The next technical lemma concerns the continuity properties of $F(\cdot)$.

Lemma 4.2.3 *Assume $\nu_T(\mathbf{w}) := C_s\delta^2|\nabla\mathbf{w}|$. Then, there is a constant $\overline{C}$ such that for any $\mathbf{v}_1, \mathbf{v}_2, \phi \in X$ with $||\nabla\mathbf{v}_1||_{L^3} \leq r$ and $||\nabla\mathbf{v}_2||_{L^3} \leq r$,*

$$(F(\mathbf{v}_1) - F(\mathbf{v}_2), \nabla\phi) \leq \bar{C}3r\delta^2||\nabla(\mathbf{v}_1 - \mathbf{v}_2)||_{L^3}||\nabla\phi||_{L^3} + Re^{-1}||\nabla(\mathbf{v}_1 - \mathbf{v}_2)||\ ||\nabla\phi||.$$

Proof: Using the Cauchy-Schwarz inequality and adding and subtracting terms as appropriate, gives:

$$
\begin{aligned}
&(F(\mathbf{v}_1) - F(\mathbf{v}_2), \nabla\phi) = Re^{-1}\|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|\|\nabla\phi\| + \\
&C_s\delta^2(|\nabla\mathbf{v}_1|\nabla\mathbf{v}_1 - |\nabla\mathbf{v}_2|\nabla\mathbf{v}_2, \nabla\phi) + \delta^2(\nabla\mathbf{v}_1\nabla\mathbf{v}_1 - \nabla\mathbf{v}_2\nabla\mathbf{v}_2, \nabla\phi) \leq \\
&Re^{-1}\|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|\|\nabla\phi\| + C_s\delta^2(|\nabla\mathbf{v}_1|\nabla\mathbf{v}_1 - |\nabla\mathbf{v}_1|\nabla\mathbf{v}_2 + |\nabla\mathbf{v}_1|\nabla\mathbf{v}_2 - |\nabla\mathbf{v}_2|\nabla\mathbf{v}_2) + \\
&\delta^2(\nabla\mathbf{v}_1\nabla\mathbf{v}_1 - \nabla\mathbf{v}_1\nabla\mathbf{v}_2 + \nabla\mathbf{v}_1\nabla\mathbf{v}_2 - \nabla\mathbf{v}_2\nabla\mathbf{v}_2, \nabla\phi) \leq \\
&Re^{-1}\|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|\|\nabla\phi\| + C_s\delta^2\|\nabla\mathbf{v}_1\|_{L^3}\|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|_{L^3}\|\nabla\phi\|_{L^3} + \\
&C_s\delta^2\|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|_{L^3}\|\nabla\mathbf{v}_2\|_{L^3}\|\nabla\phi\|_{L^3} + \delta^2\|\nabla\mathbf{v}_1\|_{L^3}\|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|_{L^3}\|\nabla\phi\|_{L^3} + \\
&\delta^2\|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|_{L^3}\|\nabla\mathbf{v}_2\|_{L^3}\|\nabla\phi\|_{L^3},
\end{aligned}
$$

which proves the lemma.

■

Using these lemmas, an energy bound for the solution of the continuous problem (4.1.1), (4.1.2) can be proven. (This first bound is the foundation upon which an existence theory for (4.1.1), (4.1.2) is built).

Proposition 4.2.1 *[Leray's inequality for the Large Eddy Model].*
Let $\mathbf{w}(\mathbf{x}, t)$ *satisfy (4.2.4). Then,* $\mathbf{w}$ *satisfies the energy inequality:*

$$
\begin{aligned}
\frac{1}{2}||\mathbf{w}(t)||^2 + \int_0^t [\beta(\delta, Re)||\mathbf{w}\cdot\hat{\tau}_j||_\Gamma^2 + Re^{-1}||\nabla\mathbf{w}||^2 + \underline{C}C_s\delta^2||\nabla\mathbf{w}||_{L^3}^3]\ dt' \\
\leq \frac{1}{2}||\mathbf{w}(0)||^2 + \int_0^t (\bar{\mathbf{f}}, \mathbf{w})\ dt',
\end{aligned}
$$

for any $t > 0$.

Proof: Set $\mathbf{v} = \mathbf{w}$ and $\lambda = q$ in (4.2.4). Using Lemma 4.2.2 then gives:

$$
\frac{1}{2}\frac{d}{dt}||\mathbf{w}||^2 + Re^{-1}||\nabla\mathbf{w}||^2 + \underline{C}C_s\delta^2||\nabla\mathbf{w}||_{L^3}^3 + \beta(\delta, Re)||\mathbf{w}\cdot\hat{\tau}_j||_\Gamma^2 \leq (\bar{\mathbf{f}}, \mathbf{w}),
$$

from which the energy inequality follows.

■

The Nonlinear Galerkin Projection under Slip with Friction Boundary Conditions

Before proceeding with the error analysis of the nonlinear, time dependent problem, we give estimates of two equilibrium projections. The first (Proposition 4.2.2) gives an estimate of the error in the nonlinear Galerkin projection obtained by dropping time dependence and convection (hence retaining only those terms associated with the turbulence modeling.) This estimate is not optimal - reflecting the quadratic growth in the model's nonlinearity. (Suboptimal estimates similar to this also occur in error analysis of problems, such as the μ-Laplacian [5], which are locally Lipschitz and strongly monotone in the sense of Vainberg [89].) Proposition 4.2.2 thus gives an idea of rates of convergence attainable in more complex settings as well. After that, in Proposition 4.2.3, we give an analysis of the error in the Galerkin approximation to the Stokes problem with slip with friction boundary conditions. This projection is used essentially in the analysis of the time dependent problem in Section 4.3.

We assume that the velocity-pressure space (X^h, Q^h) satisfies the natural ([65], [73]) inf-sup condition associated with slip with friction conditions on Γ:

$$\inf_{\lambda^h \in Q^h} \sup_{\mathbf{v}^h \in X^h} \frac{(\lambda^h, \nabla \cdot \mathbf{v}^h)}{||\lambda^h|| \left[||\nabla \mathbf{v}^h||^2 + ||\mathbf{v}^h \cdot \hat{\tau}_j||^2_{\frac{1}{2},\Gamma} \right]^{1/2}} \geq \alpha > 0. \tag{4.2.6}$$

Under this condition, the space of discretely divergence-free functions V^h

$$V^h := \{\mathbf{v}^h \in X^h : (\lambda^h, \nabla \cdot \mathbf{v}^h) = 0, \ \ \forall \ \lambda^h \in Q^h\}$$

is well defined ([35], [39]).

Proposition 4.2.2 *Let χ^h denote the best approximation of $\mathbf{w}$ in V^h and assume $||\nabla \chi^h||_{L^3} \leq \bar{C} ||\nabla \mathbf{w}||_{L^3}$. Assume also the conditions of Lemma 4.2.2 hold. Let $\tilde{\mathbf{w}} \in V^h$ be defined by*

$$(F(\mathbf{w}) - F(\tilde{\mathbf{w}}), \nabla \mathbf{v}^h) + \beta(\delta, Re)((\mathbf{w} - \tilde{\mathbf{w}}) \cdot \hat{\tau}_j, \mathbf{v}^h \cdot \hat{\tau}_j)_\Gamma = 0, \ \textit{for all } \mathbf{v}^h \in V^h.$$

Then, $\tilde{\mathbf{w}} \in V^h$ exists uniquely and the error $\mathbf{w} - \tilde{\mathbf{w}}$ satisfies:

$$\begin{aligned}
&\beta(\delta, Re)||(\mathbf{w} - \tilde{\mathbf{w}}) \cdot \hat{\tau}_j||^2_\Gamma + Re^{-1}||\nabla(\mathbf{w} - \tilde{\mathbf{w}})||^2 + \underline{C} C_s \delta^2 ||\nabla(\mathbf{w} - \tilde{\mathbf{w}})||^3_{L^3} \leq \\
&\leq C\{(\underline{C} C_s)^{-1/2} (\overline{C} ||\nabla \mathbf{w}||_{L^3})^{3/2} \delta^2 ||\nabla(\mathbf{w} - \chi^h)||^{3/2}_{L^3} + \\
&Re^{-1}||\nabla(\mathbf{w} - \chi^h)||^2 + \beta(\delta, Re)||(\mathbf{w} - \chi^h) \cdot \hat{\tau}_j||^2_\Gamma\},
\end{aligned}$$

where $C, \underline{C}$ and $\bar{C}$ are constants.

Proof: That $\tilde{\mathbf{w}}$ exists uniquely follows from standard arguments using monotonicity following Minty's Lemma ([76], [74]).

Adding and subtracting terms gives:

$$\beta(\delta, Re)((\chi^h - \tilde{\mathbf{w}}) \cdot \hat{\tau}_j, \mathbf{v}^h \cdot \hat{\tau}_j)_\Gamma + (F(\chi^h) - F(\tilde{\mathbf{w}}), \nabla \mathbf{v}^h) =$$
$$\beta(\delta, Re)((\chi^h - \mathbf{w}) \cdot \hat{\tau}_j, \mathbf{v}^h \cdot \tau_j)_\Gamma + (F(\chi^h) - F(\mathbf{w}), \nabla \mathbf{v}^h),\ \forall\ \mathbf{v}^h \in V^h.$$

Setting $\mathbf{v}^h = \chi^h - \tilde{\mathbf{w}}$ and using Lemma 4.2.2, gives:

$$\beta(\delta, Re)||(\chi^h - \tilde{\mathbf{w}}) \cdot \hat{\tau}_j||^2_\Gamma + Re^{-1}||\nabla(\chi^h - \tilde{\mathbf{w}})||^2 + \underline{C} C_s \delta^2 ||\nabla(\chi^h - \tilde{\mathbf{w}})||^3_{L^3} \leq$$
$$\leq (F(\chi^h) - F(\mathbf{w}), \nabla(\chi^h - \tilde{\mathbf{w}})) + \beta(\delta, Re)((\chi^h - \mathbf{w}) \cdot \hat{\tau}_j, (\chi^h - \tilde{\mathbf{w}}) \cdot \hat{\tau}_j)_\Gamma$$

Thus, using the Cauchy-Schwarz inequality, Young's inequality and Lemma 4.2.3, gives:

$$\beta(\delta, Re)||(\chi^h - \tilde{\mathbf{w}}) \cdot \hat{\tau}_j||^2_\Gamma + Re^{-1}||\nabla(\chi^h - \tilde{\mathbf{w}})||^2 + \underline{C} C_s \delta^2 ||\nabla(\chi^h - \tilde{\mathbf{w}})||^3_{L^3} \leq$$
$$\leq \frac{1}{2} Re^{-1}||\nabla(\chi^h - \mathbf{w})||^2 + \frac{1}{2} Re^{-1}||\nabla(\chi^h - \tilde{\mathbf{w}})||^2 +$$
$$\frac{\beta(\delta, Re)}{2}||(\chi^h - \tilde{\mathbf{w}}) \cdot \hat{\tau}_j||^2_\Gamma + \frac{\beta(\delta, Re)}{2}||(\chi^h - \mathbf{w}) \cdot \hat{\tau}_j||^2_\Gamma +$$
$$\bar{C} 3 r \delta^2 ||\nabla(\chi^h - \tilde{\mathbf{w}})||_{L^3} ||\nabla(\mathbf{w} - \chi^h)||_{L^3},$$

where $r = \max\{||\nabla \chi^h||_{L^3}, ||\nabla \mathbf{w}||_{L^3}\}$, which is bounded by $C||\nabla \mathbf{w}||_{L^3}$. Collecting terms, gives:

$$\beta(\delta, Re)||(\chi^h - \tilde{\mathbf{w}}) \cdot \hat{\tau}_j||^2_\Gamma + Re^{-1}||\nabla(\chi^h - \tilde{\mathbf{w}})||^2 + 2\underline{C} C_s \delta^2 ||\nabla(\chi^h - \tilde{\mathbf{w}})||^3_{L^3} \leq$$
$$\delta^2 6 \bar{C} r ||\nabla(\chi^h - \tilde{\mathbf{w}})||_{L^3} ||\nabla(\mathbf{w} - \chi^h)||_{L^3} + Re^{-1}||\nabla(\mathbf{w} - \chi^h)||^2 +$$
$$\beta(\delta, Re)||(\mathbf{w} - \chi^h) \cdot \hat{\tau}_j||^2_\Gamma.$$

Using Young's inequality and the triangle inequality, completes the proof.

■

Remark 4.2.1 *For stability estimates of the L_2 projection, the reader is referred to [19] and [90].*

The Stokes Projection under Slip with Friction Boundary Conditions

We consider the linear projection operator $\Pi(\mathbf{w}, q) = (\tilde{\mathbf{w}}, \tilde{q}) \in (X^h, Q^h)$ defined by solving the following discrete Stokes problem. $(\tilde{\mathbf{w}}, \tilde{q})$ satisfies:

$$Re^{-1}(\nabla(\mathbf{w} - \tilde{\mathbf{w}}), \nabla \mathbf{v}^h) + \beta(\delta, Re)((\mathbf{w} - \tilde{\mathbf{w}}) \cdot \hat{\tau}_j, \mathbf{v}^h \cdot \hat{\tau}_j)_\Gamma - (q - \tilde{q}, \nabla \cdot \mathbf{v}^h) = 0,$$
$$(\nabla \cdot (\mathbf{w} - \tilde{\mathbf{w}}), \lambda^h) = 0, \quad \text{for all } (\mathbf{v}^h, \lambda^h) \in (V^h, Q^h).$$

This is equivalent to the following. Find $\tilde{\mathbf{w}} \in V^h$ satisfying

$$Re^{-1}(\nabla(\mathbf{w}-\tilde{\mathbf{w}}), \nabla \mathbf{v}^h) + \beta(\delta, Re)((\mathbf{w}-\tilde{\mathbf{w}}) \cdot \hat{\tau}_j, \mathbf{v}^h \cdot \hat{\tau}_j)_\Gamma - (q - \lambda^h, \nabla \cdot \mathbf{v}^h) = 0,$$

for all $\mathbf{v}^h \in V^h$ and for any $\lambda^h \in Q^h$.

Proposition 4.2.3 *Suppose the discrete inf-sup condition (4.2.6) holds. Then,* $(\tilde{\mathbf{w}}, \tilde{q})$ *exists uniquely in* (X^h, Q^h)*. The error satisfies:*

$$\begin{aligned}
&Re^{-1}||\nabla(\mathbf{w}-\tilde{\mathbf{w}})||^2 + \beta(\delta, Re)||(\mathbf{w}-\tilde{\mathbf{w}}) \cdot \hat{\tau}_j||^2_\Gamma \leq \\
&\leq C \inf_{\substack{\lambda^h \in Q^h \\ \mathbf{v}^h \in V^h}} \{Re^{-1}||\nabla(\mathbf{w}-\mathbf{v}^h)||^2 + \beta(\delta, Re)||(\mathbf{w}-\mathbf{v}^h) \cdot \hat{\tau}_j||^2_\Gamma + Re||q-\lambda^h||^2\} \leq \\
&\leq C \inf_{\substack{\lambda^h \in Q^h \\ \mathbf{v}^h \in X^h}} \{\max\{Re^{-1}, \beta(\delta, Re)\}(||\nabla(\mathbf{w}-\mathbf{v}^h)||^2 + ||(\mathbf{w}-\mathbf{v}^h) \cdot \hat{\tau}_j||^2_\Gamma) + \\
&\qquad\qquad Re||q-\lambda^h||^2\}.
\end{aligned}$$

Proof: Let $I^h(\mathbf{w})$ denote some approximation of $\mathbf{w}$ in V^h. Decompose the error as $\mathbf{w}-\tilde{\mathbf{w}} = \eta - \phi^h$ where $\eta = \mathbf{w} - I^h(\mathbf{w})$ and $\phi^h = \mathbf{w}^h - I^h(\mathbf{w}) \in V^h$. The error equation can then be rewritten, picking $v^h = \phi^h$, as:

$$\begin{aligned}
&Re^{-1}(\nabla\phi^h, \nabla\phi^h) + \beta(\delta, Re)(\phi^h \cdot \hat{\tau}_j, \phi^h \cdot \hat{\tau}_j)_\Gamma = \\
&Re^{-1}(\nabla\eta, \nabla\phi^h) + \beta(\delta, Re)(\eta \cdot \hat{\tau}_j, \phi^h \cdot \hat{\tau}_j)_\Gamma - (q - \lambda^h, \nabla \cdot \phi^h)
\end{aligned}$$

Using the Cauchy-Schwarz inequality and $||\nabla \cdot \phi^h|| \leq ||\nabla\phi^h||$, we get:

$$Re^{-1}||\nabla\phi^h||^2 + \beta(\delta, Re)||\phi^h \cdot \hat{\tau}_j||^2_\Gamma \leq 2\left(Re^{-1}||\nabla\eta||^2 + \beta(\delta, Re)||\eta \cdot \hat{\tau}_j||^2_\Gamma + Re||q-\lambda^h||^2\right).$$

By the triangle inequality, we get:

$$\begin{aligned}
&Re^{-1}||\nabla(\mathbf{w}-\tilde{\mathbf{w}})||^2 + \beta(\delta, Re)||(\mathbf{w}-\tilde{\mathbf{w}}) \cdot \hat{\tau}_j||^2_\Gamma \leq \\
&\leq C \inf_{\substack{\lambda^h \in Q^h \\ \mathbf{v}^h \in V^h}} \{Re^{-1}||\nabla(\mathbf{w}-\mathbf{v}^h)||^2 + \beta(\delta, Re)||(\mathbf{w}-\mathbf{v}^h) \cdot \hat{\tau}_j||^2_\Gamma + Re||q-\lambda^h||^2\}
\end{aligned}$$

The stated result with infimum taken over X^h follows since, under the discrete inf-sup condition (4.2.6), it is known that if $\nabla \cdot \mathbf{w} = 0$ the infimum over V^h can be replaced by an infimum over X^h (relation (1.12) on p.60 in [35]).

$\blacksquare$

4.3 Finite Element Approximation of Large Eddy Motion

The usual, continuous-in-time, Galerkin finite element approximation of the solution of $(\mathbf{w}, q)$ of the large eddy model (4.1.1), (4.1.2), is defined as follows. First, finite dimensional, finite element subspaces

$$X^h \subset X, \quad Q^h \subset Q$$

are selected satisfying the discrete inf-sup condition:

$$\inf_{\lambda^h \in Q^h} \sup_{\mathbf{v}^h \in X^h} \frac{(\lambda^h, \nabla \cdot \mathbf{v}^h)}{||\lambda^h|| \left[||\nabla \mathbf{v}^h||^2 + ||\mathbf{v}^h \cdot \hat{\tau}_j||^2_{\frac{1}{2},\Gamma} \right]^{1/2}} \geq C > 0. \tag{4.3.7}$$

The continuous-in-time approximations $(\mathbf{w}^h, q^h)$ are maps $\mathbf{w}^h : [0,T] \to X^h, q^h : (0,T] \to Q^h$, satisfying $\mathbf{w}^h(0)$ approximates $\bar{\mathbf{u}}_0$ in X^h, and

$$\begin{cases} (\mathbf{w}^h_t, \mathbf{v}^h) - (q^h, \nabla \cdot \mathbf{v}^h) + \beta(\delta, Re)(\mathbf{w}^h \cdot \hat{\tau}_j, \mathbf{v}^h \cdot \hat{\tau}_j)_\Gamma + (Re^{-1} + \nu_T(\mathbf{w}^h)\nabla \mathbf{w}^h, \nabla \mathbf{w}^h) \\ - \delta^2(\nabla \mathbf{w}^h \nabla \mathbf{w}^h), \nabla \mathbf{w}^h) + \dfrac{1}{2}(\mathbf{w}^h \cdot \nabla \mathbf{w}^h, \mathbf{v}^h) - \dfrac{1}{2}(\mathbf{w}^h \cdot \nabla \mathbf{v}^h, \mathbf{w}^h) = (\bar{\mathbf{f}}, \mathbf{v}^h) \\ (\lambda^h, \nabla \cdot \mathbf{w}^h) = 0, \end{cases} \tag{4.3.8}$$

for all $(\mathbf{v}^h, \lambda^h) \in (X^h, Q^h)$. Using V^h and the nonlinear operator $F(\cdot)$ defined by (4.2.5), the approximation (4.3.8) can be written more compactly. $\mathbf{w}^h : [0,T] \to V^h$ satisfes:

$$\begin{aligned} (\mathbf{w}^h_t, \mathbf{v}^h) + \beta(\delta, Re)(\mathbf{w}^h \cdot \hat{\tau}_j, \mathbf{v}^h \cdot \hat{\tau}_j)_\Gamma \quad &+ \quad (F(\mathbf{w}^h), \nabla \mathbf{v}^h) + \\ &+ \quad b(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h) = (\bar{\mathbf{f}}, \mathbf{v}^h), \text{ for all } \mathbf{v}^h \in V^h, \end{aligned} \tag{4.3.9}$$

where, as before, $b(\cdot,\cdot,\cdot)$ is explicitly skew-symmetrized $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v})$. The method (4.3.9) is stable. It satisfies the same energy inequality as the continuous problem.

Proposition 4.3.1 *[Leray's inequality for $\mathbf{w}^h$]. Let $\mathbf{w}^h$ satisfy (4.3.8). Then, $\mathbf{w}^h$ satisfies:*

$$\begin{aligned} &\frac{1}{2}||\mathbf{w}^h(t)||^2 + \int_0^t \left[\beta(\delta, Re)||\mathbf{w}^h \cdot \hat{\tau}_j||^2_\Gamma + Re^{-1}||\nabla \mathbf{w}^h||^2 + \underline{C} C_s \delta^2 ||\nabla \mathbf{w}^h||^3_{L^3} \right] \, dt' \\ &\leq \frac{1}{2}||\mathbf{w}^h(0)||^2 + \int_0^t (\bar{\mathbf{f}}(t'), \mathbf{w}^h(t')) \, dt'. \end{aligned}$$

In particular,

$$||\mathbf{w}^h(t)||^2 + \int_0^T \left[\beta(\delta, Re)||\mathbf{w}^h \cdot \hat{\tau}_j||_\Gamma^2 + Re^{-1}||\nabla \mathbf{w}^h||^2\right] \ dt \leq$$
$$||\mathbf{w}^h(0)||^2 + C\max\{Re, \beta(\delta, Re)^{-1}\} \int_0^T ||\bar{\mathbf{f}}(t)||^2 \ dt.$$

Proof: The proof is the same as that of Proposition 4.2.1.

■

By a similar argument, we obtain a particularly simple bound on $||\mathbf{w}^h(t)||$, uniform in both Re and δ.

Lemma 4.3.1 *Let $\mathbf{w}^h$ satisfy (4.3.8). Then,*

$$\max_{0\leq t\leq T} ||\mathbf{w}^h(t)|| \leq ||\mathbf{w}^h(0)|| + \int_0^T ||\bar{\mathbf{f}}(t)|| \ dt.$$

Proof: Set $\mathbf{v}^h = \mathbf{w}^h$ and $\lambda^h = q^h$ in (4.3.8). Dropping the non-negative terms resulting on the L.H.S., gives:

$$\frac{1}{2}\frac{d}{dt}||\mathbf{w}^h(t)||^2 \leq (\bar{\mathbf{f}}, \mathbf{w}^h) \leq ||\bar{\mathbf{f}}|| \ ||\mathbf{w}^h||.$$

Thus, $\frac{d}{dt}||\mathbf{w}^h(t)|| \leq ||\bar{\mathbf{f}}(t)||$, and the result follows.

■

Combining this lemma and Proposition 4.2.1, gives an a priori bound on the quantity

$$a^h(t) := ||\mathbf{w}^h(t)||^{1/2}\left(||\nabla \mathbf{w}^h||^2 + ||\mathbf{w}^h \cdot \hat{\tau}_j||_\Gamma^2\right)^{1/4}.$$

Lemma 4.3.2 *With $a^h(t)$ as above, $a^h(t) \in L^4(0,T)$ uniformly in h. Indeed,*

$$||a^h(t)||^4_{L^4(0,T)} \leq \left(||\mathbf{w}^h(0)|| + \int_0^T ||\bar{\mathbf{f}}(t)|| \ dt\right)^2$$
$$\left(\max\{Re, \beta(\delta, Re)^{-1}\}||\mathbf{w}^h(0)||^2 + C\max\{Re, \beta(\delta, Re)^{-1}\}^2 \int_0^T ||\bar{\mathbf{f}}(t)||^2 \ dt\right).$$

Proof:

$$||a^h(t)||^4_{L^4(0,T)} \leq ||\mathbf{w}^h||^2_{L^\infty(0,T)} \int_0^T ||\nabla \mathbf{w}^h||^2 + ||\mathbf{w}^h \cdot \hat{\tau}_j||_\Gamma^2 \ dt.$$

The result now follows from Proposition 4.3.1 and Lemma 4.3.1.

■

The method (4.3.8) reduces existence of $\mathbf{w}^h$ to existence for a system of ordinary differential equations in V^h. The Cauchy-Schwarz inequality and Proposition

4.3.1 give immediately an a priori bound on $\mathbf{w}^h(t)$. Thus, $\mathbf{w}^h(\mathbf{x},t)$ exists uniquely. Using (4.3.7) and standard arguments, (see [35]), q^h does as well.

We now turn to the error in the approximation $\mathbf{w}^h$ of $\mathbf{w}$. There are many interesting and important questions in the error analysis of large eddy simulation. These include dependence of the errors upon Re and δ, including cases in which δ and h are related. In this report we consider only the first without which later steps are not imaginable: we consider convergence of $\mathbf{w}^h$ to $\mathbf{w}$ as $h \to 0$ for Re and δ fixed. For the above reasons, this is already nontrivial. Further, if there were a convergence result for $\mathbf{w}^h$ to $\mathbf{w}$ which was uniformly in δ, this would immediately imply a convergence result $\mathbf{w} \to \mathbf{u}$ (the solution of the underlying Navier-Stokes equations) as $\delta \to 0$. Such a result has never been proven (to the author's knowledge) for any conventional turbulence model and only for the Camassa-Holm's model and one large eddy model (see [66]).

Theorem 4.3.1 *Suppose* $\nabla \mathbf{w} \in L^4(0,T;L^2(\Omega))$, $\mathbf{w}\cdot\hat{\tau}_j \in L^4(0,T;L^2(\Gamma))$, *and the discrete inf-sup condition (4.3.7) holds. Let* $\tilde{\mathbf{w}}$ *denote the Stokes projection of* $\mathbf{w}$ *into* V^h, *and suppose* $||\nabla\tilde{\mathbf{w}}||_{L^3} \leq C||\nabla \mathbf{w}||_{L^3}$. *Then, the error* $e = \mathbf{w} - \mathbf{w}^h$ *satisfies*

$$\begin{aligned}
&||e||^2_{L^\infty(0,T;L^2)} + \beta(\delta, Re)||e\cdot\hat{\tau}_j||^2_{L^2(0,T;L^2(\Gamma))} + Re^{-1}||\nabla e||^2_{L^2(0,T;L^2)} + \underline{C}C_s\delta^2||\nabla e||^3_{L^3(0,T;L^3)}\\
&\leq C^*(T)||\mathbf{w}(0)-\mathbf{w}^h(0)||^2 + \beta(\delta,Re)||(\mathbf{w}-\tilde{\mathbf{w}})\cdot\hat{\tau}_j||^2_{L^2(0,T;L^2(\Gamma))}\\
&+Re^{-1}||\nabla(\mathbf{w}-\tilde{\mathbf{w}})||^2_{L^2(0,T;L^2)} + C^*(T)||\mathbf{w}_t-\tilde{\mathbf{w}}_t||^2_{L^2(0,T;L^2)} +\\
&+C^*(T)(\bar{C}r)^{3/2}(\underline{C}C_s)^{-1/2}\delta^2||\nabla(\mathbf{w}-\tilde{\mathbf{w}})||^{3/2}_{L^{3/2}(0,T;L^3(\Omega))} +\\
&C^*(T)\max\{Re,\beta(\delta,Re)^{-1}\}[||\nabla\mathbf{w}||^2_{L^4(0,T;L^2)} + ||\mathbf{w}\cdot\hat{\tau}_j||^2_{L^4(0,T,L^2(\Gamma))} + ||a^h(t)||^2_{L^4(0,T)}]\cdot\\
&\qquad\cdot[||\nabla(\mathbf{w}-\tilde{\mathbf{w}})||^2_{L^4(0,T;L^2)} + ||(\mathbf{w}-\tilde{\mathbf{w}})\cdot\hat{\tau}_j||^2_{L^4(0,T;L^2(\Gamma))}].
\end{aligned}$$

Remark 4.3.1 *The norm* $||a^h(t)||_{L^4(0,T)}$ *is bounded by problem data in Lemma 4.3.2, and* $C^*(T) = C\ exp\left[C\max\{Re,\beta(\delta,Re)^{-1}\}||\mathbf{w}||^2_{L^4(0,T;H^1)}\right]$. *The error in the Stokes projection* $\tilde{\mathbf{w}}$ *is bounded by approximation theoretic terms in Proposition 4.2.3.*

Remark 4.3.2 *The dependence of constants upon Re (which is* $>> O(1)$, *typically) and* $\beta(\delta, Re)$ *(which* $\to 0$ *as* $Re \to \infty$, *typically, [83]) is as expected. It would be very interesting (and certainly challenging) to sharpen the dependence of these constants on Re. On the other hand, some of the suboptimality (in terms of rates of convergence as* $h \to 0$*) parallels that of the simple nonlinear projection studied in Proposition 4.2.2.*

Proof: Letting $e = \mathbf{w} - \mathbf{w}^h$ and $v^h \in V^h$, an error equation is obtained by subtracting (4.3.8) from (4.2.4). This yields

$$\begin{aligned}(e_t, \mathbf{v}^h) &+ \beta(\delta, Re)(e^h \cdot \hat{\tau}_j, \mathbf{v}^h \cdot \hat{\tau}_j)_\Gamma \\ &+ (F(\mathbf{w}) - F(\mathbf{w}^h), \mathbf{v}^h) + b(\mathbf{w}, \mathbf{w}, \mathbf{v}^h) - b(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h) = (q - \lambda^h, \nabla \cdot \mathbf{v}^h),\end{aligned} \tag{4.3.10}$$

where $\lambda^h \in Q^h$ is arbitrary. Let $\tilde{\mathbf{w}} \in V^h$ denote an approximation to $\mathbf{w}$. Then, with $\eta = \mathbf{w} - \tilde{\mathbf{w}}$ and $\phi^h = (\mathbf{w}^h - \tilde{\mathbf{w}}) \in V^h$, the error equation (4.3.10) can be rewritten as:

$$\begin{aligned}&(\phi^h_t, \mathbf{v}^h) + \beta(\delta, Re)(\phi^h \cdot \hat{\tau}_j, \mathbf{v}^h \cdot \hat{\tau}_j)_\Gamma + (F(\mathbf{w}^h) - F(\tilde{\mathbf{w}}), \nabla \mathbf{v}^h) = \\ &b(\mathbf{w}, \mathbf{w}, \mathbf{v}^h) - b(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h) + (F(\mathbf{w}) - F(\tilde{\mathbf{w}}), \nabla \mathbf{v}^h) - \\ &(q - \lambda^h, \nabla \cdot \mathbf{v}^h) + (\eta_t, \mathbf{v}^h) + \beta(\delta, Re)(\eta \cdot \hat{\tau}_j, \mathbf{v}^h \cdot \hat{\tau}_j)_\Gamma.\end{aligned} \tag{4.3.11}$$

Motivated by (4.3.11), we shall take $\tilde{\mathbf{w}}$ to be the Stokes projection under slip with friction boundary conditions, whose error is estimated in Proposition 4.2.3. Specifically, $\tilde{\mathbf{w}} \in V^h$ satisfies:

$$\begin{aligned}\beta(\delta, Re)((\mathbf{w} - \tilde{\mathbf{w}}) \cdot \hat{\tau}_j, \mathbf{v}^h \cdot \hat{\tau}_j)_\Gamma + Re^{-1}(\nabla(\mathbf{w} - \tilde{\mathbf{w}}) \quad , \quad \nabla \mathbf{v}^h) - (q - \lambda^h, \nabla \cdot \mathbf{v}^h) = 0, \\ \text{for all } \mathbf{v}^h \in V^h.\end{aligned} \tag{4.3.12}$$

With this definiton of $\tilde{\mathbf{w}}$ some terms on the R.H.S. of (4.3.11) vanish, and the R.H.S. of (4.3.11) becomes:

$$\begin{aligned}&\text{R.H.S. of (4.3.11) with this } \tilde{\mathbf{w}} = b(\mathbf{w}, \mathbf{w}, \mathbf{v}^h) - b(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h) \\ &+([\nu_T(\mathbf{w})\nabla \mathbf{w} - \delta^2(\nabla \mathbf{w} \nabla \mathbf{w})] - [\nu_T(\tilde{\mathbf{w}})\nabla(\tilde{\mathbf{w}}) - \delta^2(\nabla \tilde{\mathbf{w}} \nabla \tilde{\mathbf{w}})],\ \nabla \mathbf{v}^h) \\ &+(\eta_t, \mathbf{v}^h).\end{aligned} \tag{4.3.13}$$

In (4.3.11), set $\mathbf{v}^h = \phi^h = \mathbf{w}^h - \tilde{\mathbf{w}} \in V^h$. Consider the resulting (nonlinear) eddy viscosity terms with $\mathbf{v}^h = \phi^h$.

$$\begin{aligned}&([\nu_T(\mathbf{w})\nabla \mathbf{w} - \delta^2(\nabla \mathbf{w} \nabla \mathbf{w})] - [\nu_T(\tilde{\mathbf{w}})\nabla(\tilde{\mathbf{w}}) - \delta^2(\nabla \tilde{\mathbf{w}} \nabla \tilde{\mathbf{w}})], \nabla \phi^h) \\ &\leq \ \text{(using Lemma 4.2.3)} \ \leq \bar{C} 3 r \delta^2 ||\nabla(\mathbf{w} - \tilde{\mathbf{w}})||_{L^3} ||\nabla \phi^h||_{L^3}\end{aligned} \tag{4.3.14}$$

Using (4.3.14) in (4.3.11) with $\mathbf{v}^h = \phi^h$, gives, using Lemma 4.2.2,

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}||\phi^h||^2 + \beta(\delta, Re)||\phi^h \cdot \hat{\tau}_j||^2_\Gamma + Re^{-1}||\nabla \phi^h||^2 + \\ &+\underline{C} C_s \delta^2 ||\nabla \phi^h||^3_{L^3} \leq b(\mathbf{w}, \mathbf{w}, \phi^h) - b(\mathbf{w}^h, \mathbf{w}^h, \phi^h) + \\ &+||\eta_t||\ ||\phi^h|| + \bar{C} 3 r \delta^2 ||\nabla \eta||_{L^3}\ ||\nabla \phi^h||_{L^3}.\end{aligned} \tag{4.3.15}$$

Consider now the convection terms on the R.H.S. of this last inequality. Adding and subtracting terms, gives:

$$b(\mathbf{w},\mathbf{w},\phi^h) - b(\mathbf{w}^h,\mathbf{w}^h,\phi^h) = b(\mathbf{w},e,\phi^h) + b(e,\mathbf{w},\phi^h) - b(e,e,\phi^h).$$

By skew symmetry and $e = \eta - \phi^h$, this reduces to:

$$b(\mathbf{w},\mathbf{w},\phi^h) - b(\mathbf{w}^h,\mathbf{w}^h,\phi^h) = b(\mathbf{w},\eta,\phi^h) + b(e,\mathbf{w},\phi^h) - b(e,\eta,\phi^h). \quad (4.3.16)$$

In the analysis of the trilinear form we will use the following lemma:

Lemma 4.3.3 *(Lemma 2.2 (f) in [68]) For all* $\mathbf{u},\mathbf{v},\mathbf{w} \in X$

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) \leq C\left(||\nabla\mathbf{u}||^2 + ||\mathbf{u}\cdot\hat{\tau}_j||_\Gamma^2\right)^{1/4}||\mathbf{u}||^{1/2}||\nabla\mathbf{v}||\left(||\nabla\mathbf{w}||^2 + ||\mathbf{w}\cdot\hat{\tau}_j||_\Gamma^2\right)^{1/2}$$

We shall use this lemma in (4.3.16). To this end, note that by the energy inequalities in Proposition 4.2.1 and Proposition 4.3.1, we have:

$$||\mathbf{w}^h|| \in L^\infty(0,T),\ ||\nabla\mathbf{w}||,\ ||\nabla\mathbf{w}^h|| \in L^2(0,T), \quad (4.3.17)$$

and their indicated norms are bounded by problem data of (4.1.1) uniformly in h. Thus,

$$|b(\mathbf{w},\eta,\phi^h) + b(\eta,\mathbf{w},\phi^h) - b(e,\eta,\phi^h)| \leq C\left(a^h(t) + a(t)\right)||\nabla\eta||\ ||\nabla\phi^h||,$$

where $a^h(t) = ||\mathbf{w}^h||^{1/2}(||\nabla\mathbf{w}^h||^2 + ||\mathbf{w}^h\cdot\hat{\tau}_j||_\Gamma^2)^{1/4}$ and $a(t) = (||\nabla\mathbf{w}||^2 + ||\mathbf{w}\cdot\hat{\tau}_j||_\Gamma^2)^{1/2}$. The remaining term on the R.H.S. of (4.3.16) is $b(\phi^h,\mathbf{w},\phi^h)$. Using Lemma 4.3.3, gives:

$$\begin{aligned} b(\phi^h,\mathbf{w},\phi^h) &\leq C(||\nabla\phi^h||^2 + ||\phi^h\cdot\hat{\tau}_j||_\Gamma^2)^{3/4}||\phi^h||^{1/2}||\nabla\mathbf{w}|| \\ &\leq \frac{\epsilon}{2}(||\nabla\phi^h||^2 + ||\phi^h\cdot\hat{\tau}_j||_\Gamma^2) + \frac{C}{\epsilon}||\nabla\mathbf{w}||^4||\phi^h||^2. \end{aligned}$$

Using these bounds in (4.3.16), gives for any $\epsilon > 0$:

$$\begin{aligned} &|b(\mathbf{w},\mathbf{w},\phi^h) - b(\mathbf{w}^h,\mathbf{w}^h,\phi^h)| \leq \epsilon(||\nabla\phi^h||^2 + ||\phi^h\cdot\hat{\tau}_j||_\Gamma^2) + \\ &\frac{C}{\epsilon}(a^h(t)^2 + a(t)^2)(||\nabla\eta||^2 + ||\eta\cdot\hat{\tau}_j||_\Gamma^2) + \frac{C}{\epsilon}||\nabla\mathbf{w}||^4\ ||\phi^h||^2. \end{aligned}$$

Note that $a(t) \in L^2(0,T)$ by the energy estimate, and we have assumed the additional regularity in time that $a(t) \in L^4(0,T)$. The form of $a^h(t)$ differs from

$a(t)$. Note that by Lemma 4.3.2, $a^h(t) \in L^4(0,T)$ uniformly in h. Picking $\epsilon = 1/2\ \min\{Re^{-1}, \beta(\delta, Re)\}$ and inserting this last estimate in (4.3.15), gives:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\phi^h||^2 + \frac{\beta(\delta, Re)}{2}||\phi^h \cdot \hat{\tau}_j||_\Gamma^2 + \frac{1}{2}Re^{-1}||\nabla\phi^h||^2 + \underline{C}C_s\delta^2||\nabla\phi^h||_{L^3}^3 \le \\
&C\max\{Re, \beta(\delta, Re)^{-1}\}(a^h(t)^2 + a(t)^2)(||\nabla\eta||^2 + ||\eta\cdot\hat{\tau}_j||_\Gamma^2) + \\
&C\max\{Re, \beta(\delta, Re)^{-1}\}||\nabla\mathbf{w}||^4||\phi^h||^2 + ||\eta_t||\ ||\phi^h|| + \\
&\bar{C}3r\delta^2||\nabla\eta||_{L^3}||\nabla\phi^h||_{L^3}.
\end{aligned} \tag{4.3.18}$$

Applying again Young's inequality, for any $\epsilon > 0$, gives:

$$\begin{aligned}
&\bar{C}3r\delta^2||\nabla\phi^h||_{L^3}||\nabla\eta||_{L^3} \le \\
&\le \frac{\epsilon}{3}\delta^2||\nabla\phi^h||_{L^3}^3 + \frac{2}{3}\epsilon^{-1/2}\delta^2(\bar{C}3r)^{3/2}||\nabla\eta||_{L^3}^{3/2} \le (\text{picking } \epsilon = \underline{C}C_s) \\
&\le \frac{1}{3}\underline{C}C_s\delta^2||\nabla\phi^h||_{L^3}^3 + 2\sqrt{3}(\bar{C}r)^{3/2}(\underline{C}C_s)^{-1/2}\delta^2||\nabla\eta||_{L^3}^{3/2}.
\end{aligned} \tag{4.3.19}$$

Inserting (4.3.19) into (4.3.18) and collecting terms, gives:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\phi^h||^2 + \frac{\beta(\delta, Re)}{2}||\phi^h \cdot \hat{\tau}_j||_\Gamma^2 + \frac{1}{2}Re^{-1}||\nabla\phi^h||^2 + \frac{2}{3}\underline{C}C_s\delta^2||\nabla\phi^h||_{L^3}^3 \le \\
&C\max\{Re, \beta(\delta, Re)^{-1}\}(a^h(t)^2 + a(t)^2)(||\nabla\eta||^2 + ||\eta\cdot\hat{\tau}_j||_\Gamma^2) + \\
&\left(\frac{1}{2} + C\max\{Re, \beta(\delta, Re)^{-1}\}||\nabla\mathbf{w}||^4\right)||\phi^h||^2 + \frac{1}{2}||\eta_t||^2 + \\
&2\sqrt{3}(\bar{C}r)^{3/2}(\underline{C}C_s)^{-1/2}\delta^2||\nabla\eta||_{L^3}^{3/2}.
\end{aligned}$$

Since, by assumption, $||\nabla\mathbf{w}||^4 \in L^1(0,T)$, Gronwall's inequality now implies:

$$\begin{aligned}
&\max_{0\le t\le T}||\phi^h(t)||^2 + \int_0^T \beta(\delta, Re)||\phi^h\cdot\hat{\tau}_j||_\Gamma^2 + Re^{-1}||\nabla\phi^h||^2 + \underline{C}C_s\delta^2||\nabla\phi^h||_{L^3}^3\ dt' \le \\
&C^*(T)||\phi^h(0)||^2 + C^*(T)\int_0^T ||\eta_t||^2 + (\bar{C}r)^{3/2}(\underline{C}C_s)^{-1/2}\delta^2||\nabla\eta||_{L^3}^{3/2}dt' + \\
&C^*(T)\max\{Re, \beta(\delta, Re)^{-1}\}(||a^h(t)||_{L^4(0,T)}^2 + ||a(t)||_{L^4(0,T)}^2)\cdot \\
&\qquad\qquad (||\nabla\eta||_{L^4(0,T;L^2(\Omega))}^2 + ||\eta\cdot\hat{\tau}_j||_{L^4(0,T;L^2(\Omega))}^2),
\end{aligned}$$

where $C^*(T) = C\exp[C\max\{Re, \beta(\delta, Re)^{-1}\}||\mathbf{w}||_{L^4(0,T,H^1(\Omega))}^2]$.
The theorem now follows by the triangle inequality.

■

Chapter 5
Numerical Results

5.1 Objectives

The purpose of this chapter is to provide a careful numerical assessment and comparison of the classical LES model:

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} - \mathrm{Re}^{-1}\Delta \mathbf{w} + \nabla \cdot (\mathbf{w}\mathbf{w}) + \nabla q + \nabla \cdot \left(\dfrac{\delta^2}{2\gamma}\nabla \mathbf{w}\nabla \mathbf{w}\right) & \\ \qquad\qquad - \nabla \cdot \left(C_s \delta^2 |\nabla \mathbf{w}| \nabla \mathbf{w}\right) = \bar{\mathbf{f}} & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{w} = 0 & \text{in} \quad \Omega, \\ \text{plus Initial and Boundary Conditions,} & \end{cases}$$

and the Galdi-Layton LES model:

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} - \mathrm{Re}^{-1}\Delta \mathbf{w} + \nabla \cdot (\mathbf{w}\mathbf{w}) + \nabla q + \nabla \cdot \left[\left(-\dfrac{\delta^2}{4\gamma}\Delta + I\right)^{-1}\left(\dfrac{\delta^2}{2\gamma}\nabla \mathbf{w}\nabla \mathbf{w}\right)\right] & \\ \qquad\qquad - \nabla \cdot \left(C_s \delta^2 |\nabla \mathbf{w}| \nabla \mathbf{w}\right) = \bar{\mathbf{f}}, & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{w} = 0, & \text{in} \quad \Omega, \\ \text{plus Initial and Boundary Conditions.} & \end{cases}$$

as described in Section 2.2. Actually, two Galdi-Layton models will be considered :

1. Galdi-Layton with convolution, where the smoothing operator $\left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1}$ is replaced by smoothing by direct convolution with the Gaussian filter, g_δ;

2. Galdi-Layton with auxiliary problem, where the inverse operator is calculated directly, solving a discrete Poisson problem.

In both the classical and the Galdi-Layton LES model, we are modeling the turbulent fluctuations by a Smagorinsky term $-\nabla \cdot (C_s \delta^2 |\nabla \mathbf{w}| \nabla \mathbf{w})$.

We are focusing herein on global, quantitative properties of the above models vis à vis those of $(\bar{\mathbf{u}}, \bar{p})$. Direct numerical simulation (DNS), the classical LES model, the two variants of the Galdi-Layton LES model and the Smagorinsky model are compared using the two–dimensional driven cavity problem. As noticed in Section 2.2, Lemma 2.2.1 describes an exact cancellation property of the kinetic energy contribution to the large eddies by their interaction with small eddies in the classical model for two – dimensional smooth periodic in space solutions of the classical model. Thus, the clear picture developed of the classical model as (incorrectly) stimulating the kinetic energy in high frequencies of $\mathbf{w}$ is tested numerically in this chapter.

The simplest test problem which fits Proposition 2.2.3's assumption in every respect (except the boundary conditions) is the 2D driven cavity. We thus choose this as being most favorable to the classical model and would anticipate the failure of the classical model to be more severe in 3D.

5.2 Numerical Setting

This problem is given by $\Omega = (0,1)^2$, the boundary conditions $\mathbf{w} = (1,0)$ for $0 < x < 1, y = 1$, homogeneous Dirichlet boundary conditions at the other boundaries, and $\mathbf{f} = \mathbf{0}$ (see Figure 5.1). The driven cavity problem is a classical test example used in a number of papers, e.g. [10], [16], [34]. The incompatibility of the boundary conditions at the corners $(0,1)$ and $(1,1)$ leads to a non–smooth solution such that in general analytical results, such as Proposition 2.2.3, do not hold. Also, an appropriate initial condition for the time dependent problem is not agreed upon in the literature. We have used a, so–called, impulsive start in our computations, i.e. the velocity inside the domain is chosen zero at the beginning. It is well known that such an impulsive start is inconsistent with the physical behaviour of the fluid, see [36]. But, since we are interested in the long time behaviour of the flow, the choice of the initial condition does not play an important role as long as the computations converge.

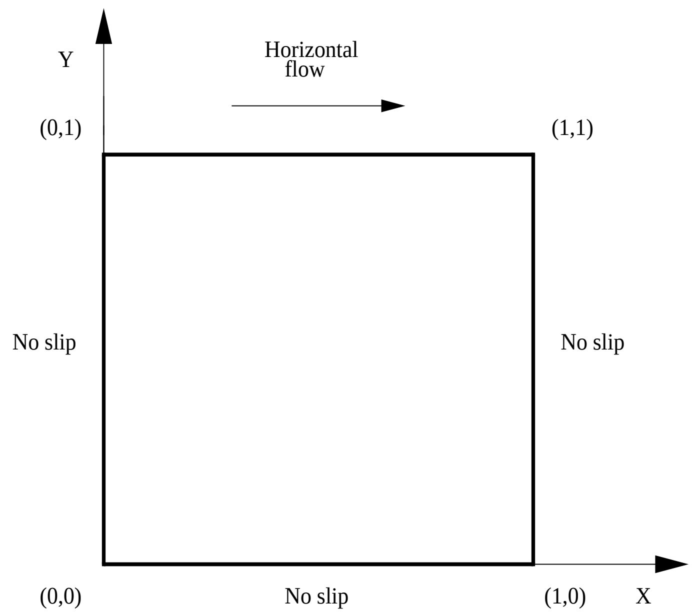

Figure 5.1: Driven Cavity Test Problem.

We did computations for a low Reynolds number flow ($Re = 400$) and a high Reynolds number flow ($Re = 10000$). Results for comparison are available for both Reynolds numbers in [34]. All LES models are applied without turbulent viscosity for $Re = 400$ and with the accepted and common Smagorinsky subgrid scale model for the $\nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'})$ term

$$\nu_T = 0.01\, \delta^2 |\mathbb{D}(\mathbf{w})| \mathbb{D}(\mathbf{w}) \tag{5.2.1}$$

in the tests for $Re = 10000$. The pure Smagorinsky discretization uses also (5.2.1) as turbulent viscosity.

All computations were carried out on equidistant grids with squares of size $h \times h$. First, the Navier–Stokes equations are discretized in time using the fractional–step–θ–scheme which is analyzed for the time dependent Navier–Stokes equations in [56]. This implicit scheme is of second order accuracy, more stable than the Crank–Nicolson scheme and is a popular scheme in the temporal discretization of the Navier–Stokes equations, e.g. see [87]. Unless mentioned otherwise, all computations were carried out with the equidistant time step $\Delta t = 0.01$.

The equations in each time step are discretized by the Q_2/P_1^{disc} finite element discretization, Q_2/P_{-1} in [37], i.e. the velocity is approximated by continuous piecewise biquadratics and the pressure by discontinuous linears. This conforming pair of finite element spaces satisfies the inf–sup or Ladyzhenskaya–Babuška–Brezzi stability condition. It is one of the most popular finite element discretizations for Navier–Stokes equations, e.g. see [25], [37], and has been proven to be superior to other discretizations in a recent study for a benchmark problem for laminar flows [53].

The fractional–step–θ–scheme divides each time step in three sub time steps. In each sub time step, a nonlinear saddle point problem has to be solved. The DNS and the Smagorinsky discretizations are treated always fully nonlinear. In the LES models, the convective term and the turbulent viscosity term are also treated implicitly whereas the additional term coming from the LES models is computed in each sub time step before the nonlinear iteration and it is not changed within this iteration. The linear problems are solved by a coupled multigrid method with Vanka–type smoothers as studied numerically e.g. in [55], [53]. These algorithmic choices are currently considered best in terms of reliability, stability and accuracy in finite element CFD.

We present results for the mesh widths $h = 1/32$ (8 450 velocity degrees of freedom, 3 072 pressure d.o.f.) and $h = 1/64$ (33 282 velocity d.o.f., 12 288 pressure d.o.f.). If not mentioned otherwise, the averaging radius δ is chosen to be the diameter of the mesh cell, i.e. the longest distance of two of its vertices, which is $\delta = \sqrt{2}h$. All computations were carried out with the code MooNMD2.4 [8].

5.3 The low Reynolds number flow

One property of a good turbulence model is that its use in a laminar or low Reynolds number flow results in a solution which is very close to the solution obtained with the DNS. All models in our study show a very similar behaviour with respect to the kinetic energy and the H^1–seminorm of the computed solutions, see Figures 5.2, 5.3 and Table 5.1. This result contradicts the observations in [16] that the classical LES with the P_2/P_1 spatial finite element discretization (Taylor/Hood) on a triangular mesh does not give satisfying results for the driven cavity problem with $Re = 400$. Since the essential difference in our study and the study presented in [16] is the finite element discretization, the Q_2/P_1^{disc} finite element discretization seems to have a better stabilization effect on the classical LES than the P_2/P_1 finite element discretization.

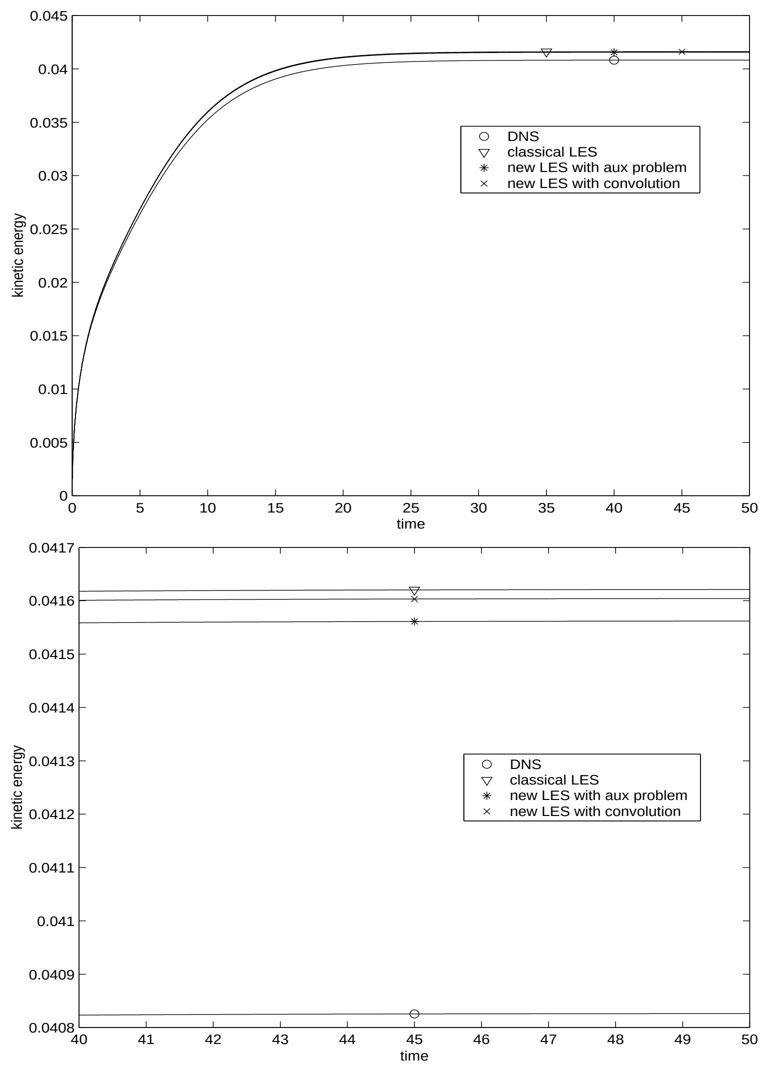

Figure 5.2: Kinetic energy, $Re = 400$, $h = 1/64$, $\delta = \sqrt{2}/64$.

5.4 The high Reynolds number flow

The simulations with the classical LES and Smagorinsky type turbulent viscosity of form (5.2.1) blow up in finite time for $RE = 10000$, see Figures 5.6 - 5.9. Table 5.2 shows that a refinement of the spatial mesh does not prevent the blow up. A reduction of the time step on the same level prevents the blow up, but on the next finer level, the simulations with the classical model blow up again. Even more important, the simulations with the classical LES model blew up also for initial conditions different from the impulsive start, e.g. we used the solutions of the other models for $h = 1/64$ an at $T = 1000$, see Figure 5.10, as initial conditions. Even with these fully developed flows as initial conditions, the simulations with the classical

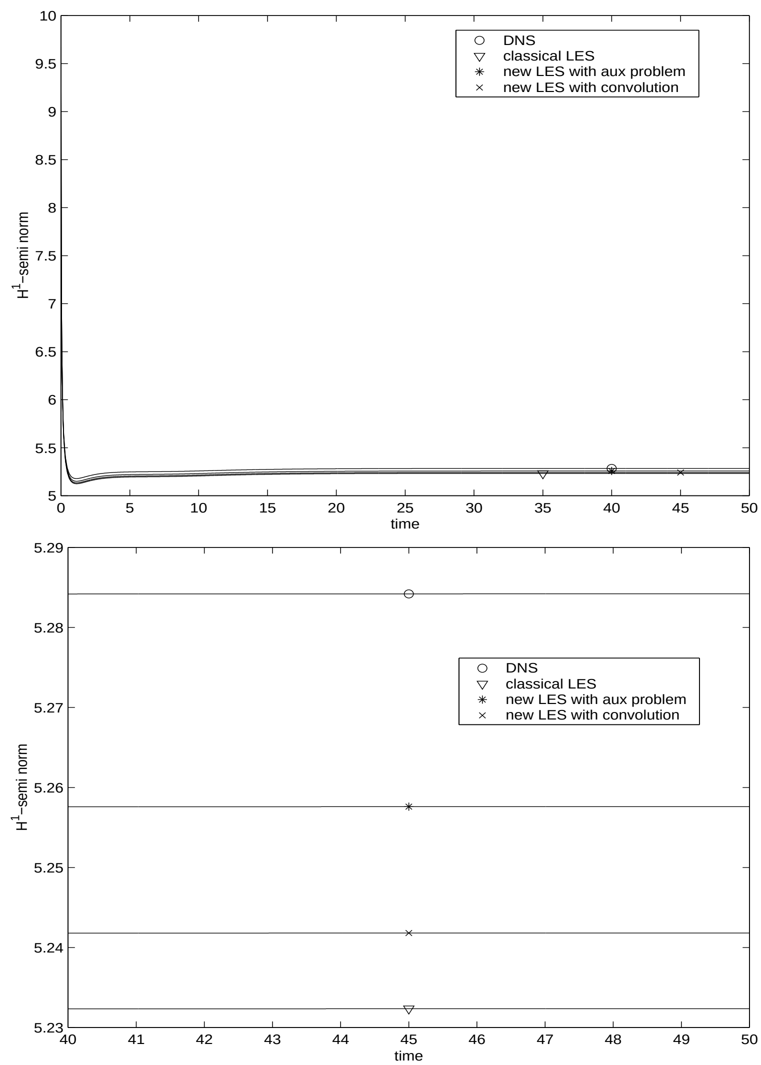

Figure 5.3: H^1–semi norm, $Re = 400$, $h = 1/64$, $\delta = \sqrt{2}/64$.

LES blew up within only few time steps, see Figures 5.4 and 5.5! Thus, the impulsive start is not responsible for the failing of the classical LES.

The failure of the simulations with the classical LES is consistent with results reported by Coletti in [16]. He reports blow up of the simulations with this model for $RE = 10000$ and the P_2/P_1 finite element discretization. In other calculations we have observed the same for the Taylor/Hood finite element on a quadrilateral grid (Q_2/Q_1).

The above observations suggest the blow up of the kinetic energy in the classical model, the reason being, in our opinion, that the classical model does indeed stimulate rather than attenuate small eddies.

Table 5.1: Result obtained for stationary state, $Re = 400$, $h = 1/64$, $\delta = \sqrt{2}/64$.

model	kinetic energy	H^1-semi norm
Galerkin	4.082648e-02	5.284203
classical LES	4.162151e-02	5.232359
Galdi/Layton with auxiliary problem	4.156234e-02	5.257608
Galdi/Layton with convolution	4.160454e-02	5.241819

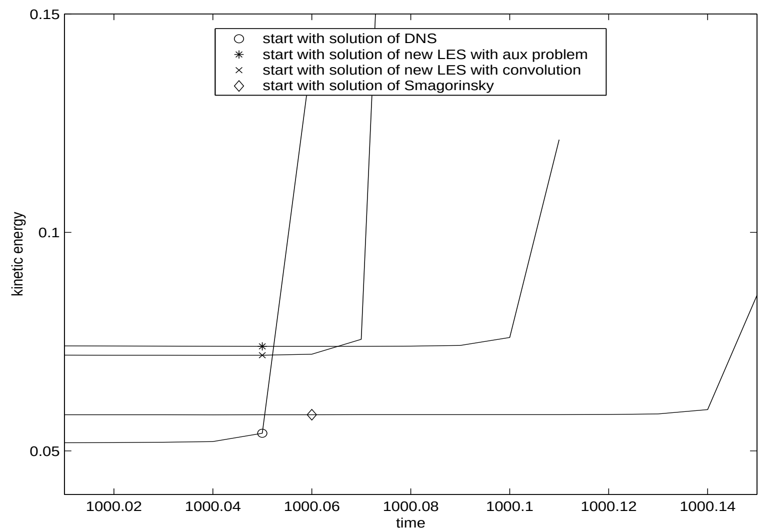

Figure 5.4: Blow up of classical LES starting with various fully developed flow fields.

The simulations with the Galdi/Layton model with the auxiliary problem and with the Galdi/Layton model with convolution, both with turbulent viscosity of type (5.2.1), do not blow up in contrast to the classical LES, see Figures 5.6 - 5.9. Both approaches give similar results for the global energy balance. From the computational point of view, we found that the regularization using the auxiliary problem is much cheaper than performing a convolution.

A closer look on the results presented in Figures 5.6 - 5.10 gives some remarkable observations. The solutions obtained with all models (except the DNS for $h = 1/32$) achieve a stationary or quasi–stationary (with very small oscillations) state. The energy balance of the solutions obtained with the DNS is considerably different for $h = 1/32$ and $h = 1/64$. That indicates that the discretization error in space is still relative large. Also the streamline plot, Figure 5.10, left upper corner, coincides with the results in [34] reasonably well but not perfeclyt. The kinetic energy of all computed solutions for $h = 1/64$ is smaller than for $h = 1/32$. For $h = 1/32$,

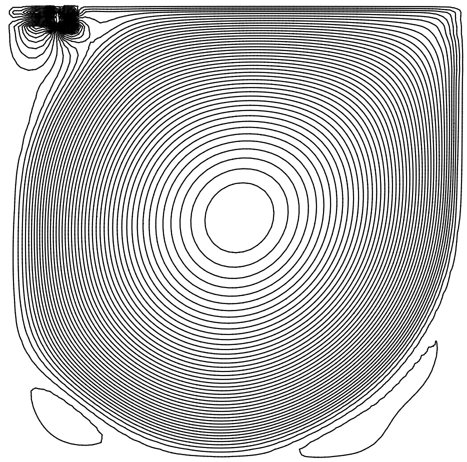

Figure 5.5: Streamlines of classical LES solution just before blow up ($T = 1000.15$ and starting with Smagorinsky solution).

Table 5.2: Blow up times for classical model, $Re = 10000$, $\delta = \sqrt{2}/64$.

$\Delta t \backslash h$	1/64	1/128
10^{-1}	$2\,10^{-1}$	10^{-1}
10^{-2}	$19\,10^{-2}$	$2\,10^{-2}$
10^{-3}	no blow up	$8\,10^{-3}$
10^{-4}	no blow up	$239\,10^{-4}$

the kinetic energy of the DNS solution is larger than for the Smagorinsky solution whereas the situation for $h = 1/64$ is vice versa. The kinetic energy for the solution of both Galdi/Layton models is somewhat larger than for the DNS and Smagorinsky solution.

The Galdi/Layton models as well as the Smagorinsky model show a smoothing effect on the discrete solution. This can be seen form the fact that the H^1–semi norms of the discrete solutions is considerably smaller than for the discrete solution obtained with the DNS, Figures 5.7 and 5.9. This can be expected since the solution of the LES models approximates $g_\delta * \mathbf{u}$ which is $C^\infty(\mathbb{R}^d), d = 2, 3$ for any $\mathbf{u}$ having bounded kinetic energy.

We used values reported in [34] to plot the pictures of the streamlines presented in Figure 5.10. Our plotting program is not able to plot streamlines with absolute value less than 10^{-5} such that we could not plot all the values from [34]. As mentioned above, the streamfunction for the DNS solution coincides reasonable well

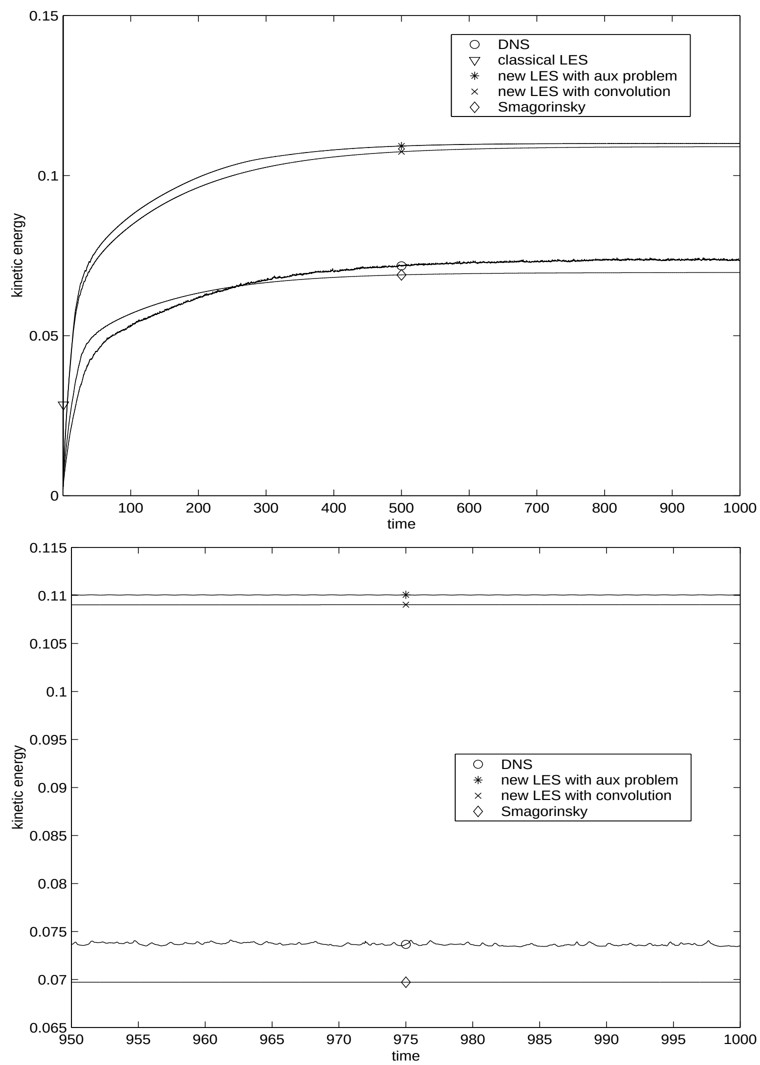

Figure 5.6: Kinetic energy, $Re = 10000$, $h = 1/32$, $\delta = \sqrt{2}/32$.

with the results given in [34]. Of course, the streamfunctions of the solutions obtained with the LES models must be different. These models are expected to suppress small eddies. Because of the conservation of mass, in turn the larger eddies must become even larger. This is achieved very well by both variants of the Galdi/Layton model, Figure 5.10, upper right and lower left corner. The main vortex is in both cases much larger than for the DNS solution and the small vortices are smaller. However, the suppression of the small eddies results in quite different shapes of the various secondary vortices, e.g. compare the lower left corner of the solutions.

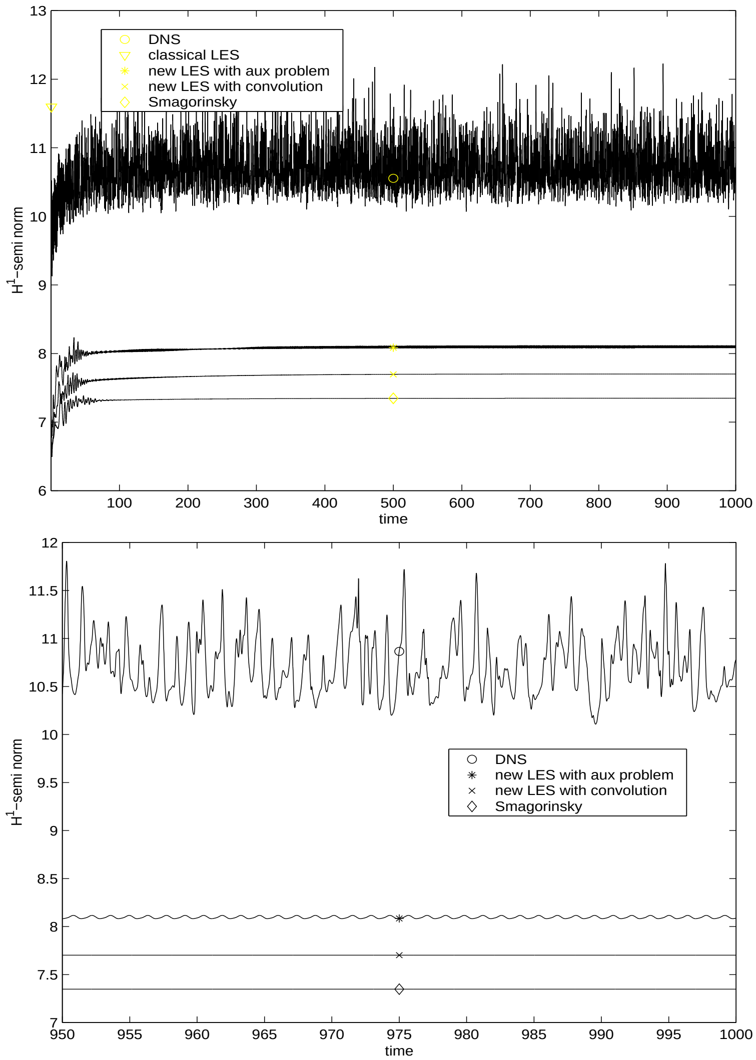

Figure 5.7: H^1–semi norm, $Re = 10000$, $h = 1/32$, $\delta = \sqrt{2}/32$.

5.5 The discretization of the viscous term

A result of our numerical studies which is not directly related to turbulence models but we think is also of interest is presented in Figure 5.11. Often the gradient formulation

$$(\nabla \mathbf{w}, \nabla \mathbf{v}) \tag{5.5.2}$$

is used as variational form of the viscous term instead of the deformation tensor formulation

$$2(\mathbb{D}(\mathbf{w}), \mathbb{D}(\mathbf{v})). \tag{5.5.3}$$

For smooth, divergence-free functions, the two formulations are equivalent. The form (5.5.2) is easier to implement, less matrices have to be stored, and the discrete systems

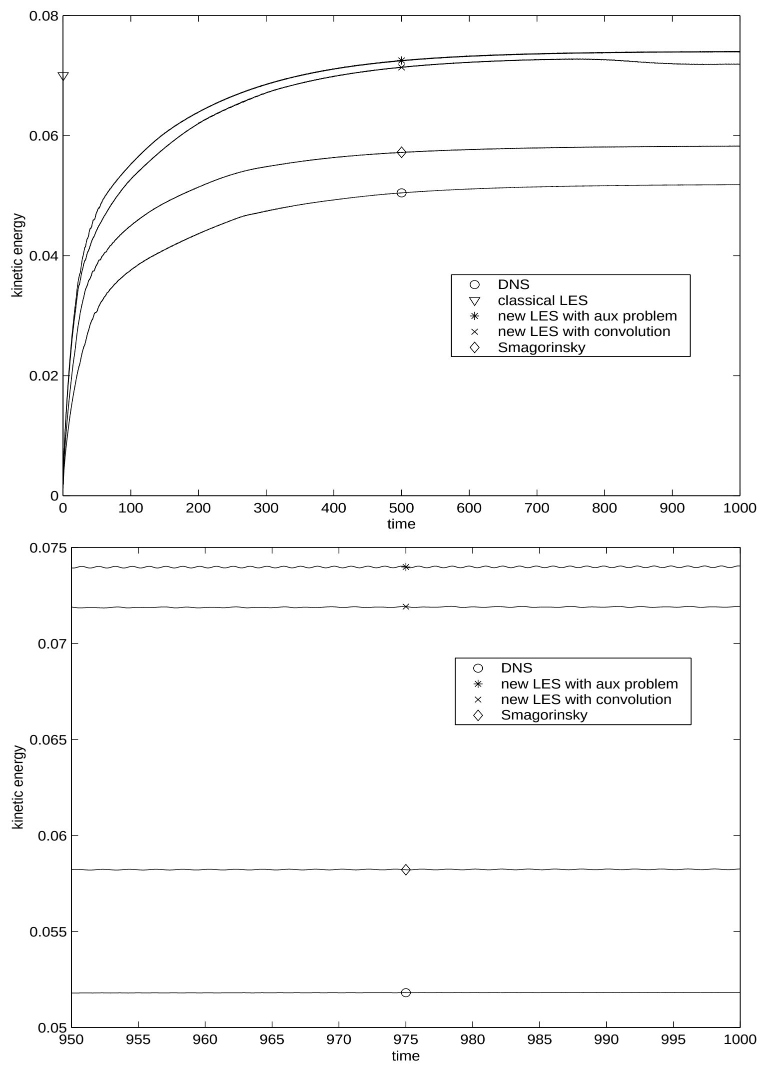

Figure 5.8: Kinetic energy, $Re = 10000$, $h = 1/64$, $\delta = \sqrt{2}/64$.

are easier to solve. However, the numerical results obtained with both discretizations may differ considerably. Whereas the solution obtained with (5.5.3) shows only small and less oscillations in the H^1–semi norm, the solution obtained with (5.5.2) oscillates largely and with a much higher frequence. Thus, the flow computed with (5.5.2) has a much more turbulent character than the flow with (5.5.3). Taking the Reynolds number and the mesh size into consideration, the turbulent behaviour of (5.5.2) seems to be exaggerated such that we consider the solution obtained with (5.5.3) to be better. The computational superiority of the deformation tensor to the gradient formulation of the viscous terms and turbulent diffusion terms is consistent with an early observation of Schultz cited in Ames [4] of the superiority of tensor artificial viscosity in multidimensional finite difference calculations. It is interesting that nonlinear artifi-

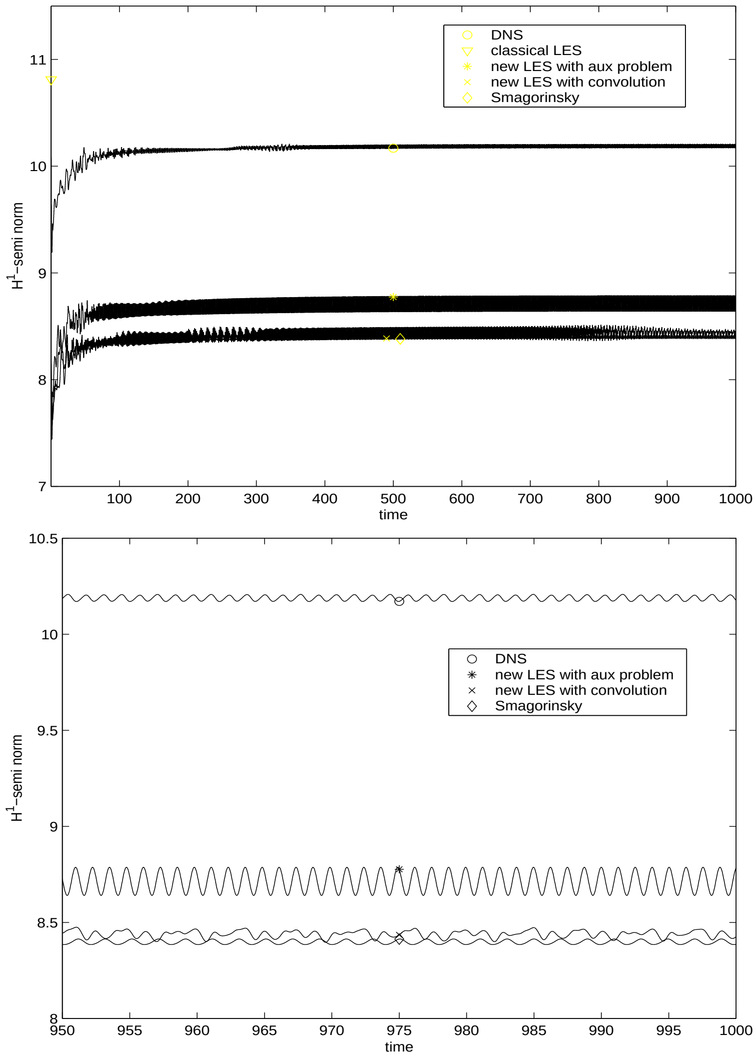

Figure 5.9: H^1–semi norm, $Re = 10000$, $h = 1/64$, $\delta = \sqrt{2}/64$.

cial viscosities were used already in the 1950's by von Neumann for compressible flow with shocks, see e.g. Ames [4], Richtmyer and Morton [80].

5.6 Summary

We want to summarize the most important results of our numerical study:

- The Galdi/Layton LES models did not blow up in the numerical simulation of a high Reynolds number flow for which the classical LES failed.

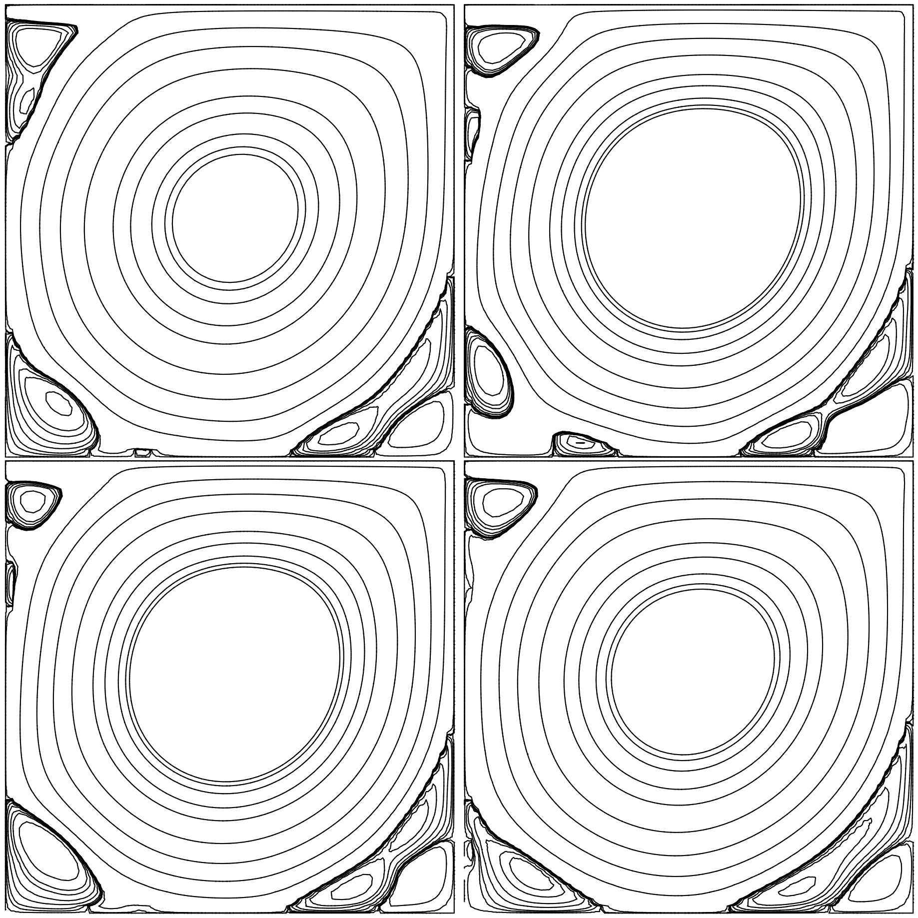

Figure 5.10: Streamlines, upper left DNS, upper right Galdi/Layton with auxiliary problem, lower left Galdi/Layton with convolution, lower right Smagorinsky, $Re = 10000$, $h = 1/64$, $\delta = \sqrt{2}/64$, $t = 200s$.

- The qualitative study of the computed solutions show that the Galdi/Layton LES models behave as expected: small eddies were suppressed and large eddies became larger.

A quantitative investigation of the solutions obtained by the Galdi/Layton LES models, also for flows in three dimensional domains, is subject of a forthcoming study.

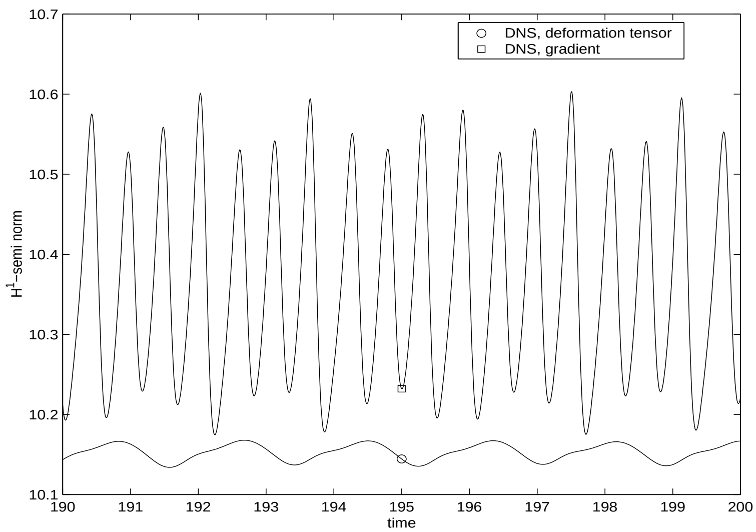

Figure 5.11: H^1–semi norm for DNS, Laplacian with deformation tensor formulation (5.5.3) and with gradient formulation (5.5.2), $Re = 10000$, $\delta = \sqrt{2}/64$.

Chapter 6

Conclusions and Future Resarch

In this thesis, we have analyzed the following LES models:

- the Galdi-Layton LES model, introduced in [30], modeling the cross terms $\overline{\overline{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\overline{\mathbf{u}}}$;
- three new LES models for the turbulent fluctuations $\overline{\mathbf{u}'\mathbf{u}'}$, introduced in [51].

In Chapter 2, we sketched the derivation of the Galdi-Layton model, and then we presented the existence, uniquenes and stability of weak solutions for this LES model with a classical Smagorinsky term modeling the turbulent fluctuations. The mathematical challenges in analyzing this LES model are those for the Navier-Stokes equations plus the nonlinearity added by the two extra terms:

$$-\nabla \cdot \left[\left(-\frac{\delta^2}{4\gamma}\Delta + I \right)^{-1} \left(\frac{\delta^2}{2\gamma}\nabla\mathbf{w}\nabla\mathbf{w} \right) \right], \text{ modeling the cross terms} \tag{6.0.1}$$

$$-\nabla \cdot \left[C_s\delta^2 |\nabla\mathbf{w}|^{2\mu}\nabla\mathbf{w} \right], \text{ modeling the turbulent fluctuations} \tag{6.0.2}$$

Mathematically, the Smagorinsky term (6.0.2) is strongly monotone, and thus helps in the analysis. The difficult term is (6.0.1). In the classical (traditional) model ([14]), the corresponding term is $-\nabla \cdot \left[\left(\frac{\delta^2}{2\gamma}\nabla\mathbf{w}\nabla\mathbf{w} \right) \right]$. It is interesting to note that the inverse operator in (6.0.1) not only that appears naturally in the derivation of the Galdi-Layton model, but is also helpful in the mathematical analysis. Specifically, in Lemma 2.2.1, we use the <u>elliptic regularity</u> introduced by $\left(-\frac{\delta^2}{4\gamma}\Delta + I \right)^{-1}$ to prove the first a priori estimate on $\mathbf{w}$ for $\mu \geq 0.1$. This is a better result compared with the corresponding one obtained by Coletti in [16] (i.e. $\mu \geq 0.5$), since we are not forced to introduce too much extra viscosity in the system. However, we have to note that whereas our result is for small data, the corresponding one in [16] is for large data.

Furthermore, due to the fact that $\left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1}$ is a global operator, we have to use te smallness of data in order to replicate the result in [16] on the monotonicity of the sum of operators (6.0.1) and (6.0.2).

It is the author's belief that the mathematical analysis of the Galdi-Layton LES model can be improved in this respect. Specifically, we should try to find new approaches to avoid imposing the smallness of data. For example, following the remark at the end of Chapter 2, for μ small enough, we can obtain an a priori bound on $\Delta\mathbf{w}$; since we cannot use the Faedo-Galerkin method anymore, we can try to use a modified version of the approach in [38].

Another research direction is to investigate how low the power μ can be in order to guarantee existence of weak solutions. It seems that for the approach we have used in this thesis, $\mu \geq 0.1$ is sharp; thus, we should try to find alternative approaches to decrease the lower bound for μ, an important result not only theoretically, but also practically.

But probably the most important research direction concerning the Galdi-Layton LES model is the model itself. Specfically, a very natural question we can ask ourselves is "Why should we expand everything in terms of δ^2 only? Why not go farther, to δ^4, say?" The advantage of such an approach would be that we could hope to get a more accurate LES model. For example, instead of chopping off the turbulent fluctuations term $\overline{\mathbf{u}'\mathbf{u}'}$ (of order δ^4) and then modeling it, we could just take it into account through our approximation! This approach is, however, beset with difficulties, such as an appropriate treatment of the higher order derivatives resulting from the higher order approximations used in the derivation of the model. These higher order derivatives have to be defined and treated numerically in a correct and careful manner. However, it is the author's opinion that such an approach would increase the amount of mathematical consistency in the corresponding LES models and, a result, decrease the amount of physical intuition involved.

In Chapter 3 we presented three LES models for the turbulent fluctuations $\overline{\mathbf{u}'\mathbf{u}'}$. Introduced in [51], and motivated by Boussinesq assumption $-\nabla \cdot \overline{(\mathbf{u}'\mathbf{u}')} \sim -\nabla \cdot (\nu_T(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^t))$, these models use new approximations for the turbulent viscosity coefficient ν_T in terms of the mean flow based on approximation for the distribution of kinetic energy in $\mathbf{u}'$ in terms of the mean flow $\overline{\mathbf{u}}$. We also prove existence of weak solutions for one such model.

These new models are proposed as an alternative to the popular Smagorinsky model:

$$\nu_T \sim C_s \delta^2 |\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^t|. \tag{6.0.3}$$

To motivate the need for different subgrid-scale models, note that (6.0.3) has at least two intuitive shortcomings. First, for flows with linear velocity profiles (6.0.3) would still introduce significant amounts of turbulent diffusion even though the flow field is laminar. Second, accepting the reasoning of Boussinesq, the amount of turbulent diffusion by small eddies should depend on the kinetic energy in those small eddies so that $\nu_T = \nu_T\left(\frac{1}{2}\rho_0|\mathbf{u}'|^2\right)$ or $\nu_T = \nu_T\left(\overline{\frac{1}{2}\rho_0|\mathbf{u}'|^2}\right)$. The three new models presented in Chapter 3 overcome both shortcomings.

As future research directions, we should mention finding new models faithful to the physics of the flow. Also, the mathematical analysis for these models should give us more insight and additional criteria in our search. But probably the most important research direction is the numerical validation of our models (we will come back to this in more details later).

Two fundamental issues in LES are the assessment of "modeling errors" and "numerical errors". The modeling error refers to the question of how close $\mathbf{w}(\mathbf{x}, t)$ is to the true flow averages: $|||\mathbf{w} - \bar{\mathbf{u}}|||$ for some norm $|||\cdot|||$. To our knowledge, there are no analytical results to date on this question, but there are experimental results comparing various averages of w to those same averages of $\bar{\mathbf{u}}$ (i.e. averages of averages of $\mathbf{u}$). Accepting $\mathbf{w}(\mathbf{x}, t)$ as an interesting model for $\bar{\mathbf{u}}$, "numerical errors" describe how close an approximation $\mathbf{w}^h$ is to $\mathbf{w}$. This leads to classical questions of stability, consistency and convergence for approximations of (4.1.1).

Chapter 4 considers precisely this question for finite element approximations of (4.1.1). In Theorem 4.3.1 we show that the usual, continuous in time, finite element approximation to (4.1.1), $\mathbf{w}^h$, converges to $\mathbf{w}$ as the meshwidth $h \to 0$ for the Reynolds number Re and averaging radius δ fixed.

The long term analytical goals in the numerical analysis of large eddy simulation are then to sharpen this result especially with respect to error dependence on δ and Re, where possible, and complement it with analysis of the modeling error. Preliminary steps in this last direction have recently been made in [52] for a different class of LES models.

Chapter 5 provided a careful numerical assessment and comparison of the classical LES model ([14]) and the Galdi-Layton LES model. Direct numerical simulation (DNS), the classical LES model, two natural variants of the Galdi-Layton LES model and the Smagorinsky model are compared using the two–dimensional driven cavity problem. The numerical experiments clearly suggest that the classical LES model is catastrophically structurally unstable: small perturbations in the model result in blow-up of the kinetic energy. On the contrary, the kinetic energy of the two variants of the Galdi-Layton LES model does not blow-up.

It is the author's belief that the numerical validation and testing of the LES models are two of the most important challenges in the LES of turbulent flows. It is imperative that we find reliable, robust benchmark test problems, so that we can use the numerical simulations as a criterion in selecting the best LES models.

In particular, for our LES models, we plan to run numerical simulations for the channel flow and flow past a cylinder test problems. The next step is to make the transition to 3D numerical calculations. The corresponding numerical results should be compared with DNS on finer meshes, as well as with actual physical experiments to give us an honest numerical evaluation of our LES models.

LES has a huge potential toward the prediction and understanding of turbulent flows. Mathematicians have to keep up with the outstanding increase in available computational power, and advance the LES through better models, better mathematical analysis, better numerical algorithms, and better numerical validation and testing.

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