ON A QUESTION OF WILF CONCERNING NUMERICAL SEMIGROUPS

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Abstract. Let S be a numerical semigroup with embedding dimension $e(S)$, Frobenius number $g(S)$, and type $t(S)$. Put $n(S) := \text{Card}(S \cap \{0, 1, \ldots, g(S)\})$. A question of Wilf is shown to be equivalent to the statement that $e(S)n(S) \geq g(S)+1$. This question is answered affirmatively if S is symmetric, pseudo-symmetric, or of maximal embedding dimension. The question is also answered affirmatively in the following cases: $e(S) \leq 3$, $g(S) \leq 20$, $n(S) \leq 4$, $\frac{g(S)+1}{4} \leq n(S)$.

1. Introduction

Let S be a *numerical semigroup*, that is, an additive submonoid of the monoid $\mathbb{N}$ of all non-negative integers. It is well known that any such S is finitely generated (cf. [7, Theorem 2.4(2)]). We assume throughout that any numerical semigroup S under consideration has the property that its set of elements has greatest common divisor 1. (Note that, even if S does not have this property, S is isomorphic to a numerical semigroup with this property.) In this case, it is well known (cf. [7, Theorem 2.4(1)]) that there exists a least integer $g(S) \geq -1$ such that $\{m \in \mathbb{N} : m > g(S)\} \subseteq S$; it is customary to call $g(S)$ the *Frobenius number of* S. An upper bound for $g(S)$ is known in terms of the irredundant generating set $\{a_1, \ldots, a_{e(S)}\}$ of S; that is, the set consisting of $a_1 < \cdots < a_{e(S)}$ in $\mathbb{N}$ such that $S = \left\langle a_1, \ldots, a_{e(S)} \right\rangle := \{\sum_{i=1}^{e(S)} m_i a_i : m_i \in \mathbb{N} \text{ for each } i\}$, $\gcd(a_1, \ldots, a_{e(S)}) = 1$, and $a_i \notin \left\langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{e(S)} \right\rangle$ for all i. Indeed, a result of Schur leads to the fact that $g(S) \leq a_1 a_{e(S)} - a_1 - a_{e(S)}$. (See [3, Theorem B, p. 215] and [8, p. 390].) This inequality is best possible if $e(S) = 2$, for then Sylvester [9] (cf. [2]) has shown that $g(S) = a_1 a_2 - a_1 - a_2$. Our interest here is in another conjectured upper bound for $g(S)$, namely, $e(S)n(S) - 1$. In this expression, $e(S)$ is as above and is called the *embedding dimension of* S; and $n(S) := \mathrm{Card}(S \cap \{0, 1, \ldots, g(S)\})$. Notice that $n(S)$ plays a role in an evident lower bound for $g(S)$. Indeed, $2n(S) - 1 \leq g(S)$, as a consequence of the fact that the assignment $x \mapsto g(S) - x$ establishes an injection $S \cap \{0, 1, \ldots, g(S)\} \to \{0, 1, \ldots, g(S)\} \setminus S$. As explained in Proposition 2.1 (cf. [6, Remark, p. 81]), the conjecture that $e(S)n(S) - 1 \geq g(S)$ is equivalent to a question posed by Wilf [10] in a study of the so-called *presentable* integers obtained as non-negative integral linear combinations of a finite set $\{a_1, \ldots, a_{e(S)}\}$ of relatively prime positive integers. For this reason, we say that S *affirmatively answers the Wilf Question* if $e(S)n(S) \geq g(S) + 1$.

We show first that the Wilf Question is answered affirmatively for numerical semigroups S that are "large" in the following sense. Let $g \in \mathbb{N}$. If g is odd (resp., even), then S is maximal with respect to the property that $g(S) = g$ if and only if S is symmetric (resp., pseudo-symmetric); that is, if and only if $n(S) = \frac{g(S)+1}{2}$ (resp., $n(S) = \frac{g(S)}{2}$). (Cf. [5, Lemmas 1 and 3], [1, Lemmas I.1.8 and I.1.9].) Proposition 2.2 establishes that S affirmatively answers the Wilf Question if S is either symmetric or pseudo-symmetric.

Corollary 2.4 includes the fact that S affirmatively answers the Wilf Question if S is *of maximal embedding dimension*, in the sense that $e(S)$ coincides with a_1 (the minimal positive element of S). To prove this result, we consider the *maximal ideal of* S, given by $M(S) := S \setminus \{0\}$; the semigroup $S(1) := \{m \in \mathbb{N} : m + M(S) \subseteq S\}$; and the *type of* S, given by $t(S) := \mathrm{Card}(S(1) \setminus S)$. It is well known that S is symmetric if and only if either $S = \mathbb{N}$ (in which case, $t(\mathbb{N}) := 0$) or $t(S) = 1$; and, if S is pseudo-symmetric, then $t(S) = 2$ [1, p. 3]. The above-mentioned Corollary 2.4 is a consequence of Proposition 2.3: if $t(S) + 1 \leq e(S)$, then S affirmatively answers the Wilf Question. Another consequence is given in Corollary 2.6: if $e(S) \leq 3$, then S affirmatively answers the Wilf Question. The supporting fact, that $e(S) \leq 3$ implies $t(S) + 1 \leq e(S)$, is known [5, Theorem 11], but we provide a new proof of it in Theorem 2.5. As an upshot, we obtain in Corollary 2.7 that the Wilf Question is answered affirmatively if S is "large" in another sense, namely, that $n(S) \geq \frac{g(S)+1}{4}$. One finds the same conclusion in Corollary 2.12 in case S is "small" in the sense that $g(S) \leq 20$. This follows from Theorem 2.11, where we affirmatively answer the Wilf Question for numerical semigroups that are "small" in another sense, namely, that $n(S) \leq 4$. To prove this result, we consider the ideal $S_i := \{s \in S : s \geq s_i\}$, where $s_0 := 0$ and s_i denotes the i^{th} positive element of S for $1 \leq i \leq n(S)$; the *relative ideal* $S(i) := \{m \in \mathbb{N} : m + S_i \subseteq S_i\}$ for $0 \leq i \leq n(S)$; and the *type sequence* $(t_i(S) : 1 \leq i \leq n(S))$ of S where $t_i(S) := \mathrm{Card}(S(i) \setminus S(i-1))$. Using this notation, we often find it convenient to write $S = \{0, s_1, s_2, \ldots, s_{n(S)-1}, s_{n(S)} = g(S)+1, \rightarrow\}$ where the symbol "$\rightarrow$" means that all subsequent natural numbers belong to S.

The Wilf Question remains unanswered (though we believe it has an affirmative answer) in the following cases: $e(S) \geq 4$; $n(S) \geq 5$; $n(S) < \frac{g(S)+1}{4}$.

For background on numerical semigroups, see [5], [1].

2. Results

We begin by showing that what we have called the Wilf Question is equivalent to a question posed by Wilf [10]. Let S be a numerical semigroup with irredundant generating set $\{a_1, \ldots, a_{e(S)}\}$, as in the Introduction. Wilf lets Ω denote the cardinality of the set of non-presentable non-negative integers; thus, $\Omega = \mathrm{Card}(\{0, 1, \ldots, g(S)\} \setminus S) = g(S) + 1 - n(S)$. Wilf lets χ denote $g(S)+1$; and he lets k denote $e(S)$. The specific question of Wilf concerns $\frac{\Omega}{\chi}$, the ratio of the number of non-presentable non-negative integers to the number of non-negative integers $\leq g$. On [10, page 565], Wilf asks if $\frac{\Omega}{\chi} \leq 1 - \frac{1}{k}$. (As $\chi(\mathbb{N}) = g(\mathbb{N}) + 1 = 0$, we tacitly assume that $S \neq \mathbb{N}$ below.) In studying the Wilf Question, we also tacitly assume that $S \neq 0$ since $e(0)n(0) = 0 \cdot 0 = 0 = g(0) + 1$.

Proposition 2.1. *The question of Wilf is equivalent to the Wilf Question. In other words,* $\frac{\Omega}{\chi} \leq 1 - \frac{1}{k}$ *if and only if* $e(S)n(S) \geq g(S) + 1$.

Proof. $\frac{\Omega}{\chi} \leq 1 - \frac{1}{k} \Leftrightarrow \frac{g(S)+1-n(S)}{g(S)+1} \leq 1 - \frac{1}{e(S)} \Leftrightarrow \frac{-n(S)}{g(S)+1} \leq \frac{-1}{e(S)} \Leftrightarrow \frac{n(S)}{g(S)+1} \geq \frac{1}{e(S)} \Leftrightarrow e(S)n(S) \geq g(S) + 1$. $\square$

We next show that S affirmatively answers the Wilf Question if S is maximal with a given Frobenius number.

Proposition 2.2. *If a numerical semigroup S is either symmetric or pseudo-symmetric, then S affirmatively answers the Wilf Question.*

Proof. Suppose first that S is symmetric. If $S = \mathbb{N}$, then $e(S)n(S) = 1 \cdot 0 = 0 \geq 0 = g(\mathbb{N})+1$. If $S \neq \mathbb{N}$, then $e(S) \geq 2$, and so $e(S)n(S) \geq 2n(S) = 2 \cdot \frac{g(S)+1}{2} = g(S) + 1$.

Suppose next that S is pseudo-symmetric. Then $e(S) \geq 3$, since Sylvester [9] (cf. [2]) has shown that any 2-generated numerical semigroup is symmetric. Therefore, $e(S)n(S) \geq 3 \cdot \frac{g(S)}{2} \geq g(S) + 1$ (since $g(S) \geq 2$). □

Proposition 2.3. *If a numerical semigroup S satisfies $t(S)+1 \leq e(S)$, then S affirmatively answers the Wilf Question.*

Proof. The assertion follows immediately from the fact ([5, Theorem 22], [1, Proposition I.1.11(c)]) that $g(S) + 1 \leq n(S)(t(S) + 1)$. □

The next result refers to Arf semigroups, in the sense of [1]. See [1, Theorem I.3.4] for fifteen characterizations of Arf semigroups.

Corollary 2.4. *(a) Each numerical semigroup of maximal embedding dimension affirmatively answers the Wilf Question.*
(b) Each (numerical) Arf semigroup affirmatively answers the Wilf Question.

Proof.(a) Let S be a numerical semigroup of maximal embedding dimension. Then $e(S) = a_1$, the minimal positive element of S, also known as $\mu(S)$, the so-called multiplicity of S. A general fact about numerical semigroups T (for proofs, see [1, Remarks I.2.7(a), (b) or I.6.3(d)]) states that $t(T) \leq \mu(T) - 1$. In particular, $t(S) + 1 \leq \mu(S) = a_1 = e(S)$. Apply Proposition 2.3.

(b) Each Arf semigroup is of maximal embedding dimension [1, Theorem I.3.4 or page 18]. Apply (a). □

The next two results contain the deepest applications of Proposition 2.3.

Theorem 2.5. *If S is a nonzero numerical semigroup such that $e(S) \leq 3$, then $t(S) + 1 \leq e(S)$.*

Proof. Without loss of generality, $S \neq \mathbb{N}$ (since $t(\mathbb{N}) = 0$ and $e(\mathbb{N}) = 1$). Thus, $e(S)$ is either 2 or 3. Suppose first that $e(S) = 2$. Then, as noted above via [2], S is symmetric, whence $t(S) = 1$ and the assertion holds.

In the remaining case, $e(S) = 3$ and our task is to show that $t(S) \leq 2$. This is known: see [5, Theorem 11] for two proofs of this fact. We next indicate, for the sake of completeness and possible interest, how to modify the methods of Johnson [8] to obtain a third proof that $e(S) = 3$ implies $t(S) \leq 2$.

Let $S = \langle a_1, a_2, a_3 \rangle$. By [6, Proposition 8], we may restrict ourselves to the case where a_1, a_2, a_3 are pairwise relatively prime. Suppose $N \in S(1) \setminus S$. To verify the assertion, it suffices to show that there are at most two possibilities for N. By definition of $S(1)$, N can be expressed as $N = y_{ij}a_j + y_{ik}a_k - a_i$ with $y_{ij}, y_{ik} \in \mathbb{N}$ for $\{i, j, k\} = \{1, 2, 3\}$. As in [8], let L_i be the minimum positive integer K_i such that $K_i a_i \in \langle a_j, a_k \rangle$ for $\{i, j, k\} = \{1, 2, 3\}$. Then we may write $L_i a_i = x_{ij}a_j + x_{ik}a_k$ with $x_{ij}, x_{ik} \in \mathbb{N}$. By [8, Theorem 3], x_{ij} and x_{ik} are uniquely determined and $x_{ij}, x_{ik} > 0$.

We claim that $y_{ij} \leq L_j - 1$. Suppose that $y_{ij} = L_j + d_j$ with $d_j \geq 0$. Then

$$N = (L_j + d_j)a_j + y_{ik}a_k - a_i = (x_{ji}a_i + x_{jk}a_k) + d_j a_j + y_{ik}a_k - a_i$$
$$= (x_{ji} - 1)a_i + (x_{jk} + y_{ik})a_k + d_j a_j \in S$$

since $x_{ji} > 0$. This is a contradiction as $N \notin S$. Hence, the claim holds.

Next, we show that the representations of N of the form $N = y_{ij}a_j + y_{ik}a_k - a_i$ with $y_{ij}, y_{ik} \in \mathbb{N}$ are unique. Suppose that $N = y_{ij}a_j + y_{ik}a_k - a_i = z_{ij}a_j + z_{ik}a_k - a_i$ with $y_{ij}, y_{ik}, z_{ij}, z_{ik} \in \mathbb{N}$. If $y_{ij} = z_{ij}$, then we are done. Otherwise, without loss of generality, we may assume $y_{ij} > z_{ij}$. Then $(y_{ij} - z_{ij})a_j + y_{ik}a_k = z_{ik}a_k$. This leads to $z_{ik} \geq L_k$, which contradicts the fact that $z_{ik} \leq L_k - 1$. Thus, N has unique representations

$$N = y_{31}a_1 + y_{32}a_2 - a_3 = y_{21}a_1 + y_{23}a_3 - a_2 = y_{12}a_2 + y_{13}a_3 - a_1.$$

Next, we show that $y_{31} \neq y_{21}$. If $y_{31} = y_{21}$, then $(y_{32} + 1)a_2 = (y_{23} + 1)a_3$. This leads to $y_{32} + 1 = ma_3$ for some $m \geq 1$ since $(a_2, a_3) = 1$. In particular, $y_{32} + 1 \geq a_3$. By the proof of [8, Theorem 3], $a_3 > L_2$. Thus, $y_{32} + 1 > L_2$, contradicting the fact that $y_{32} \leq L_2 - 1$. Therefore, either $y_{31} < y_{21}$ or $y_{21} < y_{31}$.

We first consider the case $y_{31} < y_{21}$. Here, $(y_{32}+1)a_2 = (y_{21}-y_{31})a_1 + (y_{23}+1)a_3$, whence $y_{32}+1 \geq L_2$. It follows that $y_{32} = L_2 - 1$. Now we have

$$N = y_{31}a_1 + (L_2-1)a_2 - a_3 = y_{12}a_2 + y_{13}a_3 - a_1.$$

This implies $(L_2 - 1 - y_{12})a_2 + (y_{31}+1)a_1 = (y_{13}+1)a_3$. Thus, $y_{13}+1 \geq L_3$ which forces $y_{13} = L_3 - 1$. Now we have $y_{21}a_1 + y_{23}a_3 - a_2 = N = y_{12}a_2 + (L_3-1)a_3 - a_1$. This leads to $(y_{21}+1)a_1 = (y_{12}+1)a_2 + (L_3 - 1 - y_{23})a_3$. As before, this forces $y_{21} = L_1 - 1$. Since $y_{32} = L_2 - 1$,

$$\begin{aligned} N &= y_{31}a_1 + (L_2-1)a_2 - a_3 = y_{31}a_1 + (x_{21}a_1 + x_{23}a_3) - a_2 - a_3 \\ &= (y_{31} + x_{21})a_1 + (x_{23}-1)a_3 - a_2. \end{aligned}$$

By the uniqueness of the representation of N, $L_1 - 1 = y_{31} + x_{21}$ and $x_{23} - 1 = y_{23}$ as $x_{23} > 0$. Similarly, one can show that $y_{31} = x_{31} - 1$. Now we may write

$$\begin{aligned} N &= (L_1-1)a_1 + y_{23}a_3 - a_2 = (y_{31}+x_{21})a_1 + (x_{23}-1)a_3 - a_2 \\ &= (x_{21}a_1 + x_{23}a_3) + y_{31}a_1 - a_3 - a_2 = (L_2-1)a_2 + y_{31}a_1 - a_3 \\ &= (L_2-1)a_2 + (x_{31}-1)a_1 - a_3. \end{aligned}$$

In the remaining case, $y_{21} < y_{31}$. By interchanging subscripts in the above proof, we see that

$$N = (L_3-1)a_3 + (x_{21}-1)a_1 - a_2.$$

This shows that there are at most two possibilities for N, namely, $(L_2-1)a_2 + (x_{31}-1)a_1 - a_3$ and $(L_3-1)a_3 + (x_{21}-1)a_1 - a_2$. Therefore, $t(S) = \mathrm{Card}(S(1) \setminus S) \leq 2$. □

Corollary 2.6. *If S is a numerical semigroup such that $e(S) \leq 3$, then S affirmatively answers the Wilf Question.*

Proof. We observed earlier that 0 affirmatively answers the Wilf Question. On the other hand, if $S \neq 0$, then the assertion follows by combining Theorem 2.5 and Proposition 2.3. □

As noted in the Introduction, each numerical semigroup S satisfies $2n(S) - 1 \leq g(S)$ or, equivalently, $n(S) \leq \frac{g(S)+1}{2}$. We next show that, in a sense, the "upper half" of cases affirmatively answer the Wilf Question.

Corollary 2.7. *Let S be a numerical semigroup such that $n(S) \geq \frac{g(S)+1}{4}$. Then S affirmatively answers the Wilf Question.*

Proof. By Corollary 2.6, we may suppose that $e(S) \geq 4$. Put $g := g(S)$. Since $n(S) < \infty$, there exists a numerical semigroup $T \supseteq S$ such that T is maximal with the property that $g(T) = g$. Suppose that g is odd (resp., even). Then T is symmetric (resp., pseudo-symmetric), by [1, Lemma I.1.8] (resp., [1, Lemma I.1.9]). Let $k := \text{Card}(T \setminus S)$. Then $n(T) = n(S) + k$, since $g(T) = g(S)$. Thus, $n(S) = \frac{g+1}{2} - k$ (resp., $\frac{g}{2} - k$). Accordingly, S affirmatively answers the Wilf Question if and only if $e(S)(\frac{g+1}{2} - k) \geq g + 1$ (resp., $e(S)(\frac{g}{2} - k) \geq g + 1$); that is, if and only if

$$e(S) \geq \frac{g+1}{\frac{g+1}{2} - k} = 2 + \frac{4k}{g+1-2k} \quad \left(\text{resp.}, e(S) \geq \frac{g+1}{\frac{g}{2} - k} = 2 + \frac{4k+2}{g-2k}\right).$$

As $e(S) \geq 4$, it follows that S affirmatively answers the Wilf Question if

$$4 \geq 2 + \frac{4k}{g+1-2k} \quad \left(\text{resp.}, 4 \geq 2 + \frac{4k+2}{g-2k}\right);$$

that is, if $\frac{g+1}{4} \geq k$ (resp., $\frac{g-1}{4} \geq k$); that is, if

$$n(S) = \frac{g+1}{2} - k \geq \frac{g+1}{2} - \frac{g+1}{4} = \frac{g+1}{4}$$

$$\left(\text{resp.}, n(S) = \frac{g}{2} - k \geq \frac{g}{2} - \frac{g-1}{4} = \frac{g+1}{4}\right).$$

Thus, the assertion has been proved in all cases. □

In Theorem 2.11, we settle the Wilf Question for all S with "small" $n(S)$. First, it is convenient to collect some results from [1] and [4] that will be used frequently.

Proposition 2.8. [1, (I.1.10) and Proposition I.1.11 (b)] *Let S be a numerical semigroup. Then:*
(a) $1 \le t_i(S) \le t(S)$ *for all* $1 \le i \le n(S)$.
(b) $g(S) + 1 - n(S) = \sum_{i=1}^{n(S)} t_i(S)$.

Proposition 2.9. [4, Theorem 2.1] *Let S be a semigroup with $n(S) = 3$ and $t_i := t_i(S)$ for each $i = 1, 2, 3$. Then*

$$S = \{0, s_1, t_1 + t_2 + 2, t_1 + t_2 + t_3 + 3, \rightarrow\}, \text{ where}$$

$$s_1 = \begin{cases} t_1 + 2, & \Leftrightarrow t_2 = s_2 - s_1 \le g - s_2 = t_3; \\ t_1 + 1, & \Leftrightarrow t_2 + 1 = s_2 - s_1 > g - s_2 = t_3. \end{cases}$$

Proposition 2.10. [4, Theorem 2.2] *Let S be a semigroup with $n(S) = 4$ and $t_i := t_i(S)$ for each $i = 1, 2, 3, 4$. Then*

$$S = \{0, s_1, s_2, t_1 + t_2 + t_3 + 3, t_1 + t_2 + t_3 + t_4 + 4, \rightarrow\}, \text{ where}$$

$$s_2 = \begin{cases} t_1 + t_2 + 3 & \Leftrightarrow t_3 = s_3 - s_2 \le g - s_3 = t_4; \\ t_1 + t_2 + 2 & \Leftrightarrow t_3 + 1 = s_3 - s_2 > g - s_3 = t_4; \end{cases} \quad \text{and}$$

$$s_1 = \begin{cases} t_1 + 3 & \Leftrightarrow \left\{ \begin{array}{c} \begin{pmatrix} s_2 = t_1 + t_2 + 3 \\ t_2 + t_3 = s_3 - s_1 \le g - s_2 = t_3 + t_4 \end{pmatrix} \\ \text{or} \\ \begin{pmatrix} s_2 = t_1 + t_2 + 2 \\ t_2 + t_3 = s_3 - s_1 \le g - s_2 = t_3 + t_4 + 1 \end{pmatrix} \end{array} \right. \\ t_1 + 2 & \Leftrightarrow \left\{ \begin{array}{c} \begin{pmatrix} s_2 = t_1 + t_2 + 3 \\ t_2 + t_3 + 1 = s_3 - s_1 > g - s_2 = t_3 + t_4 \\ t_2 + 1 = s_2 - s_1 \le g - s_2 = t_3 + t_4 \end{pmatrix} \\ \text{or} \\ \begin{pmatrix} s_2 = t_1 + t_2 + 2 \\ t_2 + t_3 + 1 = s_3 - s_1 > g - s_2 = t_3 + t_4 + 1 \\ t_2 = s_2 - s_1 \le g - s_2 = t_3 + t_4 + 1 \\ t_2 = s_2 - s_1 \ne s_3 - s_2 = t_3 + 1 \end{pmatrix} \end{array} \right. \\ t_1 + 1 & \Leftrightarrow \left\{ \begin{array}{c} \begin{pmatrix} s_2 = t_1 + t_2 + 3 \\ t_2 + 2 = s_2 - s_1 > g - s_2 = t_3 + t_4 \end{pmatrix} \\ \text{or} \\ \begin{pmatrix} s_2 = t_1 + t_2 + 2 \\ t_2 + 1 = s_2 - s_1 > g - s_2 = t_3 + t_4 + 1 \end{pmatrix} \\ \text{or} \\ \begin{pmatrix} s_2 = t_1 + t_2 + 2 \\ t_2 + 1 = s_2 - s_1 = s_3 - s_2 = t_3 + 1 \end{pmatrix} \end{array} \right. \end{cases} .$$

Theorem 2.11. *If S is a numerical semigroup such that $n(S) \leq 4$, then S affirmatively answers the Wilf Question.*

Proof. Without loss of generality, $S \neq \mathbb{N}$. In general, $n(S) \geq 1$. The only numerical semigroups S such that $n(S) = 1$ take the form $S = \langle a, a+1, a+2, \ldots, 2a-1 \rangle$, and any such S satisfies $e(S)n(S) = a \cdot 1 = a = g(S) + 1$. If $n(S) = 2$, then S is an Arf semigroup by [1, Remark I.3.6(b)], and so the assertion follows from Corollary 2.4(b).

Suppose next that $n(S) = 3$. Then S need not be Arf (or even of maximal embedding dimension) [1, Remark I.3.6(c)], but the assertion can be established by the following case analysis.

Let (t_1, t_2, t_3) denote the type sequence of S. By Proposition 2.9,

$$S = \{0, s_1, t_1 + t_2 + 2, t_1 + t_2 + t_3 + 3, \rightarrow\},$$

where either $s_1 = t_1 + 1$ or $s_1 = t_1 + 2$. Given S as above, let

$$J = [t_1 + t_2 + t_3 + 3, (t_1 + t_2 + t_3 + 3) + (s_1 - 1)]$$

and

$$I = J \cap \langle s_1, t_1 + t_2 + 2 \rangle .$$

Let $E(S)$ denote the minimal generating set of S. Then $e(S) = |E(S)|$. To verify the assertion, it suffices by Proposition 2.3 to establish the following claim: $e(S) = t_1 + 1$.

We first consider the case $s_1 = t_1 + 2$; that is,

$$S = \{0, t_1 + 2, t_1 + t_2 + 2, t_1 + t_2 + t_3 + 3, \rightarrow\}.$$

In this case, $t_2 \leq t_3$ by Proposition 2.9. Of course, $t_1 + 2 = \mu(S) \in E(S)$. By Proposition 2.8, $t_2 \leq t_1$. This implies $t_1 + t_2 + 2 < 2(t_1 + 2)$, and so $t_1 + t_2 + 2 \in E(S)$. Therefore, $E(S) = \{t_1 + 2, t_1 + t_2 + 2\} \cup (J \setminus I)$. Hence, $e(S) = |E(S)| = 2 + |J| - |I| = 2 + (t_1 + 2) - |I| = t_1 + 4 - |I|$. Thus, it suffices to show that $|I| = 3$.

Notice that $2(t_1 + 2) \in I$ as $2(t_1 + 2) \in S$ and $t_1 + t_2 + 2 < 2(t_1 + 2)$ imply $t_1 + t_2 + t_3 + 3 \leq 2(t_1 + 2) \leq (t_1 + t_2 + t_3 + 3) + (t_1 + 1)$. Similarly, $(t_1 + 2) + (t_1 + t_2 + 2) \in I$. Also, one can verify that $2(t_1 + t_2 + 2) \in I$ using the fact that $t_2 \leq t_3$. As a result, $|I| \geq 3$.

Suppose $s \in I$. Then $s = u(t_1+2) + v(t_1+t_2+2)$ for some $u, v \in \mathbb{N}$. If $u + v > 2$, then

$$(u(t_1+2) + v(t_1+t_2+2)) - 2(t_1+2) > t_1 + 1.$$

Since $2(t_1+2) \in I \subseteq J$ and J is an interval of length $s_1 - 1 = t_1+1$, $u(t_1+2)+v(t_1+t_2+2) \notin J$. Hence, $u(t_1+2)+v(t_1+t_2+2) \notin I$. Clearly, $t_1+2, t_1+t_2+2 \notin I$ as $t_1+2, t_1+t_2+2 < t_1+t_2+t_3+3$. Therefore, $u+v = 2$. It follows that $|I| \leq 3$, as claimed.

In the remaining case, $s_1 = t_1 + 1$; i.e.,

$$S = \{0, t_1+1, t_1+t_2+2, t_1+t_2+t_3+3, \rightarrow\}.$$

Here, $t_2 + 1 > t_3$ by Proposition 2.9. As above, $t_1 + 1 = \mu(S) \in E(S)$. According to Proposition 2.8, $t_2 \leq t_1$. Hence there are two subcases to consider: $t_2 = t_1$ and $t_2 < t_1$.

Suppose first that $t_1 = t_2$. Then $t_1+t_2+2 = 2(t_1+1) \notin E(S)$, and so $E(S) = \{t_1+1\} \cup (J \setminus I)$. Thus $e(S) = |E(S)| = 1+|J|-|I| = 1+(t_1+1)-|I| = t_1+2-|I|$. To establish the claim, we must show that $|I| = 1$. In this subcase, we have $I = [2t_1+t_3+3, 3t_1+t_3+3] \cap \langle t_1+1 \rangle$. Note that $3(t+1) \in I$ since $t_3 \leq t_2 \leq t_1$. It follows that $u(t_1+1) \notin I$ for $u \neq 3$, as J is an interval of length $s_1 - 1 = t_1$. Hence, $I = \{3(t_1+1)\}$.

In the remaining subcase, $t_2 < t_1$. Here, $t_1+t_2+2 \in E(S)$ since $t_1+t_2+2 < 2(t_1+1)$. Thus, $E(S) = \{t_1+1, t_1+t_2+2\} \cup (J \setminus I)$, and so $e(S) = |E(S)| = 2+|J|-|I| = 2+(t_1+1)-|I| = t_1+3-|I|$. It suffices to show that $|I| = 2$. Notice that $2(t_1+1) \in I$ as $2(t_1+1) \in S$ and $t_1+t_2+2 < 2(t_1+1)$ imply $t_1+t_2+t_3+3 \leq 2(t_1+1) \leq (t_1+t_2+t_3+3)+t_1$. Similarly, $(t_1+1)+(t_1+t_2+2) \in I$. Hence, $\{2(t_1+1), (t_1+1)+(t_1+t_2+2)\} \subseteq I$. However, $2(t_1+t_2+2) > (t_1+t_2+t_3+3)+t_1$ as $t_2+1 > t_3$. As a result, $2(t_1+t_2+2) \notin I$. Since J is an interval of length $s_1 - 1 = t_1$, it follows that $|I| = 2$. This completes the proof for the case $n = 3$.

Finally, suppose that $n(S) = 4$. Let (t_1, t_2, t_3, t_4) denote the type sequence of S. By Proposition 2.10,

$$S = \{0, s_1, s_2, s_3 = t_1+t_2+t_3+3, s_4 = t_1+t_2+t_3+t_4+4, \rightarrow\}$$

where $s_1 \in \{t_1+1, t_1+2, t_1+3\}$ and $s_2 \in \{t_1+t_2+2, t_1+t_2+3\}$. Given such a description of S, let

$$J = [s_4, s_4 + s_1 - 1]$$

and

$$I = J \cap \langle s_1, s_2, s_3 \rangle .$$

Let $E(S)$ denote the minimal generating set of S. Then $e(S) = |E(S)|$. By Proposition 2.3, it suffices to prove the claim that $e(S) \geq t_1+1$, except in the case $s_2 = t_1+t_2+3$, $s_1 = t_1+3$, $s_3 \neq 2s_1$, and $2s_3 \leq s_1 + s_4 - 1$. In this exceptional case, we also show that S affirmatively answers the Wilf Question.

We begin by considering the case $s_2 = t_1 + t_2 + 2$. In this case, $t_3 \geq t_4$ by Proposition 2.10. There are three subcases to consider: $s_1 = t_1 + 1$, $s_1 = t_1 + 2$, and $s_1 = t_1 + 3$.

We begin with the subcase $s_1 = t_1 + 1$. In this subcase, either $t_2 > t_3 + t_4$ or $t_2 = t_3$ by Proposition 2.10. Suppose $s_2, s_3 \notin \langle s_1 \rangle$. Then $s_1, s_2 \in E(S)$. Note that $s_3 \in E(S)$ if $s_3 \neq s_1 + s_2$. Moreover, $s_3 = s_1 + s_2$ implies that $t_1 = t_3$. Since either $t_2 > t_3 + t_4$ or $t_2 = t_3$, it follows from Proposition 2.8 that $t_2 = t_3$. Hence, $t_1 = t_2 = t_3$, and so $s_2 = t_1+t_2+2 = 2(t_1+1) = 2s_1$ which is a contradiction. This shows that $s_1, s_2, s_3 \in E(S)$. As in the proof for the case $n = 3$, $e(S) = 3 + |J| - |I| = 3 + t_1 + 1 - |I| = t_1 + 4 - |I|$. It suffices to show $|I| \leq 3$. Note that $2s_1 \in S$ and $s_2, s_3 \notin \langle s_1 \rangle$ imply that $s_4 \leq 2s_1 \leq s_4+s_1-1$. Hence, $2s_1 \in I$. It follows that $3s_1 > s_4 + s_1 - 1$ since J is an interval of length $s_1 - 1$. This leads to $I \subseteq \{2s_1, s_1+s_2, s_1+s_3, 2s_2, s_2+s_3, 2s_3\}$. Note that $s_2+s_3 > s_4+s_1-1$ as $t_2 \geq t_4$. As a consequence, $s_2 + s_3, 2s_3 \notin I$. If $t_2 > t_3 + t_4$, then $2s_2 > s_4 + s_1 - 1$ and so $2s_2 \notin I$. If $t_2 = t_3$, then $2s_2 = s_1 + s_2$. Therefore, $I \subseteq \{2s_1, s_1 + s_2, s_1 + s_3\}$, as desired.

Next, suppose $s_2 \in \langle s_1 \rangle$ or $s_3 \in \langle s_1 \rangle$. Note that this implies that $s_2 = 2s_1$ or $s_3 = 2s_1$ as $2s_1 \in S$, $2s_1 < 3s_1$, and $s_2 < s_3$. First, assume $s_2 = 2s_1$; that is, assume $t_1 = t_2$. If $t_2 = t_3$, then $s_3 = 3s_1$ and $I = [s_4, s_4+s_1-1] \cap \langle s_1 \rangle = \{4s_1\}$. Hence, $e(S) = 1+|J|-|I| = 1+t_1+1-1 = t_1 + 1$. Otherwise, $t_2 > t_3 + t_4$. Here, $s_1, s_3 \in E(S)$ since $s_3 = 3s_1$ implies $t_1 = t_2 = t_3$ contradicting the fact that $t_2 > t_3 + t_4$ (since $t_4 \geq 1$ by Proposition 2.8). This gives $e(S) = 2+|J|-|I| = 2+t_1+1-|I|$. Note that $I = [2s_1+t_3+t_4+2, 3s_1+t_3+t_4+1] \cap \langle s_1, s_3 \rangle$. Clearly, $3s_1 \in I$ and $s_1 + s_3 \in I$ by Proposition 2.8. As a consequence, $I = \{3s_1, s_1 + s_3\}$, as every element of I is of the form $us_1 + vs_3$, $u, v \in \mathbb{N}$, and J is an interval of length $s_1 - 1$. Therefore, $|I| \le 2$ and $e(S) = t_1 + 3 - |I| \ge t_1 + 3 - 2 = t_1 + 1$.

Finally, suppose $s_3 = 2s_1$. Then $s_1, s_2 \in E(S)$ and $e(S) = 2 + |J| - |I| = 2 + t_1 + 1 - |I| = t_1 + 3 - |I|$, where $I = [2s_1 + t_4 + 1, 3s_1 + t_4] \cap \langle s_1, s_2\rangle$. Clearly, $3s_1 \in I$, as $3s_1 \in S$ and $s_3 = 2s_1$ imply that $s_4 \le 3s_1 \le 3s_1 + t_4$. Since $2s_1 + t_4 + 1 \le 2s_1 + t_3 + 1 \le 2s_1 + t_2 + 1 \le s_1 + s_2 \le s_1 + s_3 + t_4 = 3s_1 + t_4$, we have that $s_1 + s_2 \in I$. If $t_2 = t_3$, then $2s_2 = 3s_1$. If $t_2 > t_3 + t_4$, then $2s_2 > 3s_1 + t_4$ and so $2s_1 + s_2 \notin J$. Then $|I| \le 2$ follows from the facts that $3s_1, s_1 + s_2 \in I$ and J is an interval of length $s_1 - 1$. Hence, $e(S) = t_1 + 3 - |I| \ge t_1 + 3 - 2 = t_1 + 1$. This concludes the proof in the subcase $s_2 = t_1 + t_2 + 2$ and $s_1 = t_1 + 1$.

Next, we consider the subcase $s_1 = t_1 + 2$. In this subcase, $t_3 + t_4 + 1 \ge t_2 > t_4$ and $t_2 \ne t_3 + 1$ by Proposition 2.10. Notice that $s_2 < 2s_1$ as $t_2 \le t_1$ by Proposition 2.8. Thus, $s_1, s_2 \in E(S)$. It follows that $s_3 \in E(S)$ or $s_3 = 2s_1$. Suppose first that $s_3 \in E(S)$; that is, assume $s_3 \notin \langle s_1, s_2\rangle$. As in the previous subcase, $e(S) = 3 + |J| - |I| = 3 + t_1 + 2 - |I| = t_1 + 5 - |I|$, where $I = [s_4, s_4 + t_1 + 1] \cap \langle s_1, s_2, s_3\rangle$. It suffices to show $|I| \le 4$. Note that $2s_1, s_1 + s_2, s_1 + s_3 \in S$ and $s_3 \notin \langle s_1, s_2\rangle$ imply $s_4 \le 2s_1, s_1 + s_3$. Clearly, $2s_1, s_1 + s_2, s_1 + s_3 \le s_4 + s_1 - 1$. Thus, $2s_1, s_1 + s_2, s_1 + s_3 \in I$. As before, by definition of I and J, it follows that $|I| \le 4$, as desired.

Suppose now that $s_3 = 2s_1$. Then $e(S) = 2 + |J| - |I| = 2 + t_1 + 2 - |I| = t_1 + 4 - |I|$, where $I = [2s_1 + t_4 + 1, 3s_1 + t_4] \cap \langle s_1, s_2\rangle$. Clearly, $3s_1 \in I$. Using Proposition 2.8 and the fact that $t_2 > t_4$, one can check that $s_1 + s_2 \in I$. By definition of I and J, $|I| \le 3$. Hence, $e(S) = t_1 + 4 - |I| \ge t_1 + 4 - 3 = t_1 + 1$. This concludes the proof in the subcase $s_2 = t_1 + t_2 + 2$ and $s_1 = t_1 + 2$.

Finally, we consider the subcase $s_1 = t_1 + 3$. Here, $t_2 \le t_4 + 1$ by Proposition 2.10. As in the previous subcase, $s_2 < 2s_1$, whence $s_1, s_2 \in E(S)$ and either $s_3 \in E(S)$ or $s_3 = 2s_1$. Suppose first that $s_3 \in E(S)$. Then $e(S) = 3 + |J| - |I| = 3 + t_1 + 3 - |I| = t_1 + 6 - |I|$, where $I = [s_4, s_4 + s_1 - 1] \cap \langle s_1, s_2, s_3\rangle$. It suffices to show $|I| \le 5$. Note that $2s_1 \in I$, since $2s_1 \in S$ and $s_3 \notin \langle s_1, s_2\rangle$ imply that $s_4 \le 2s_1 \le s_4 + s_1 - 1$. This leads to $I \subseteq \{2s_1, s_1 + s_2, s_1 + s_3, 2s_2, s_2 + s_3, 2s_3\}$ since J is an interval of length $s_1 - 1$. However, $2s_3 > s_4 + s_1 - 1$ as $t_2 + t_3 > t_3 \ge t_4$, whence $2s_3 \notin I$. Therefore, $I \subseteq \{2s_1, s_1 + s_2, s_1 + s_3, 2s_2, s_2 + s_3\}$, as desired.

Suppose now that $s_3 = 2s_1$. Then $e(S) = 2 + |J| - |I| = 2 + t_1 + 3 - |I| = t_1 + 5 - |I|$, where $I = [2s_1 + t_4 + 1, 3s_1 + t_4] \cap \langle s_1, s_2 \rangle$. It suffices to show that $|I| \leq 4$. Note that $I \subseteq \{s_1 + s_2, 2s_2, 3s_1, 2s_1 + s_2, s_1 + 2s_2, 3s_2\}$. Clearly, $s_1 + s_2 \in I$. This leads to $s_1 + 2s_2 = (s_1 + s_2) + s_2 \geq 2s_1 + t_4 + 1 + s_1 > 3s_1 + t_4$, whence $s_1 + 2s_2 \notin I$ and $3s_2 \notin I$. Therefore, $|I| \leq 4$ and so $e(S) \geq t_1 + 1$. This concludes the proof in the case $s_2 = t_1 + t_2 + 2$.

Arguments similar to those above may be used to show that $e(S) \geq t_1 + 1$ in the case $s_2 = t_1 + t_2 + 3$, except in the subcase $s_1 = t_1 + 3$, $s_3 \neq 2s_1$, and $2s_3 \leq s_1 + s_4 - 1$. We now show that the Wilf Question can be answered affirmatively in this exceptional subcase.

In this subcase, $s_1, s_2, s_3 \in E(S)$. This leads to $e(S) = 3+|J|-|I| = 3+t_1+3-|I| = t_1+6-|I|$, where $I \subseteq \{2s_1, s_1+s_2, s_1+s_3, 2s_2, s_2+s_3, 2s_3\}$. Thus, $e(S) = t_1+6-|I| \geq t_1+6-6 = t_1$. Notice that $t_1 + 2 \geq t_2 + t_3 + t_4$ since $2s_1 \geq s_4$. By Proposition 2.3, we may assume that $t_1 \geq 3$. It follows that $g+1 = s_4 = t_1+t_2+t_3+t_4+4 \leq t_1+t_1+2+4 \leq 2t_1+6 \leq 4t_1 \leq 4e(S)$, thus completing the proof for the case $n = 4$. □

It is perhaps a matter of taste whether numerical semigroups S with "small" Frobenius number should be considered as "small" semigroups. In any event, we next show that such S affirmatively answer the Wilf Question.

Corollary 2.12. *If S is a numerical semigroup such that $g(S) \leq 20$, then S affirmatively answers the Wilf Question.*

Proof. Set $n := n(S)$. Let T, k be as in the proof of Corollary 2.7. Suppose that $g := g(S)$ is odd (resp., even). By the proof of Corollary 2.7, the assertion holds if $k \leq \frac{g+1}{4}$ (resp., $k \leq \frac{g-1}{4}$). As $k = n(T) - n = \frac{g+1}{2} - n$ (resp., $\frac{g}{2} - n$), the assertion holds if $n \geq \frac{g+1}{4}$ (resp., $n \geq \frac{g-1}{4}$). By Theorem 2.11, we may suppose that $n \geq 5$. Therefore, the assertion holds if $5 \geq \frac{g+1}{4}$ (resp., $5 \geq \frac{g-1}{4}$); that is, if $g \leq 20$. □

Remark 2.13. (a) Suppose that one had a sharpening of Corollary 2.6 in which there is an integer N such that the Wilf Question were answered affirmatively for all S such that $e(S) \leq N$. Now, let S be a numerical semigroup for which $g := g(S)$ is odd (resp., even).

By the proof of Corollary 2.7, S affirmatively answers the Wilf Question if

$$N+1 \geq 2+\frac{4k}{g+1-2k} \quad \left(\text{resp.}, N+1 \geq 2+\frac{4k+2}{g-2k}\right)$$

where k is as in the proof of Corollary 2.7. Thus, S affirmatively answers the Wilf Question if

$$k \leq \left(\frac{N-1}{N+1}\right)\frac{g+1}{2} \quad \left(\text{resp.}, k \leq \left(\frac{N-1}{N+1}\right)\frac{g}{2}-\frac{1}{N+1}\right);$$

that is, if $n := n(S) = \frac{g+1}{2} - k$ (resp., $\frac{g}{2} - k$) satisfies $n \geq \frac{g+1}{N+1}$. (This agrees with the result in Corollary 2.7, where we used $N = 3$.) The above reasoning quantifies the sense in which sharpenings of Corollary 2.6 would lead to an affirmative resolution of the Wilf Question. To see how a sharpening of Theorem 2.11 would lead to affirmative answers for all S for which $g(S)$ is correspondingly bounded above, we invite the reader to (re)work the proof of Corollary 2.12.

(b) Theorem 2.5 is best possible, in the sense that Backelin [5, pages 15-16] has shown that for each odd number $t \geq 7$, there exists a numerical semigroup S such that $e(S) = 4$ and $t(S) = t$. In particular, $t(S)+1 > e(S)$. Thus if one is to proceed as suggested in (a) for $N = 4$, it would be essential to develop methods that are different from those used above.

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