ON IRREDUCIBLE NO-HOLE $L(2,1)$-COLORING OF TREES

RENU C. LASKAR, GRETCHEN L. MATTHEWS, BETH NOVICK
DEPARTMENT OF MATHEMATICAL SCIENCES
CLEMSON UNIVERSITY
CLEMSON, SC 29634-0975
U.S.A.

JOHN VILLALPANDO
DEPARTMENT OF MATHEMATICS
GONZAGA UNIVERSITY
SPOKANE, WA 99258-2615
U.S.A.

E-MAIL: RCLSK@CLEMSON.EDU, GMATTHE@CLEMSON.EDU, NBETH@CLEMSON.EDU, VILLALPANDO@GONZAGA.EDU

Abstract. We consider a variant of the channel assignment problem in which frequencies are assigned to transmitters in a way that avoids interference while ensuring that all frequencies within the bandwidth are used. This is modeled as an $L(2,1)$-coloring of a graph which is no-hole and irreducible in the sense that no color can be replaced with a smaller one. In this paper, we show that if the network is any tree other than a star, then frequencies may be assigned in this fashion without increasing the bandwidth; that is, we show that for any such tree T, the inh-span of T is equal to its span.

Keywords: channel assignment problem, $L(2,1)$-coloring, irreducible coloring, tree

1. Introduction

In this paper, we consider a variant of the channel assignment problem. The channel assignment problem [11] is to assign frequencies, or channels, to transmitters at various locations in a way that permits interference-free communication. Frequencies assigned to neighboring transmitters are required to be substantially different, and frequencies assigned to pairs of transmitters which are slightly farther away are required to be different so as to prevent interference. It was once the case that the number of usable channels was far greater than the number of transmitters. However, this is no longer true. Hale points this out in his seminal paper in 1980 [11], and, with the ever increasing array of digital communication devices, the need to make wise use of frequencies is even more important today.

The channel assignment problem can be modeled as a vertex coloring problem for graphs. A vertex is assigned to each transmitter. Based on the proximity of the transmitters and the power of the transmissions, edges are placed between vertices to represent possible interference. The frequencies are denoted by nonnegative integers $0, \ldots, \lambda$. In the context described above, the channel assignment problem becomes that of prescribing integer labels for vertices so that neighboring vertices receive labels that differ by at least two while vertices with a common neighbor have different labels. Such a coloring is called an $L(2,1)$-coloring and has been studied extensively in the literature (see [3] and [25] for two recent surveys). The *span* of a graph G, denoted $\lambda(G)$, is the smallest number λ such that there is an $L(2,1)$-coloring of G using the integers $0, \ldots, \lambda$. A *span coloring* of G is an $L(2,1)$-coloring with largest label $\lambda(G)$. Such colorings were first studied by Griggs and Yeh where they show that

$$\Delta(T) + 1 \leq \lambda(T) \leq \Delta(T) + 2$$

for all trees T with maximum degree $\Delta(T)$ [10, Theorem 4.1]. A polynomial time algorithm to determine the span of a tree was developed by Chang and Kuo shortly thereafter [4]. However, no simple characterization of trees with span $\Delta(T)+1$ is known (it is worth mentioning that there has been recent work in this direction, such as [5, 16]).

Since their introduction, $L(2,1)$-colorings of many other classes of graphs have been studied, including three connected graphs and the hypercube [9, 10]; Cartesian products of graphs [9, 12, 13, 15, 18]; chordal graphs and interval graphs [23]; generalized Petersen graphs [1]; and direct and strong products of graphs [14, 17]. Weighted colorings also arise in this context [20]. Algorithms to efficiently assign channels are a related topic of study (see, for example, [2], where the focus is on trees and interval graphs).

Other constraints may be added to the notion of $L(2,1)$-coloring to simulate practical difficulties in the channel assignment problem. Because frequencies are typically purchased in a block, one may wish to use all available frequencies within that block. Making use of the full spectrum of labels available in a span coloring is akin to full coloring, a concept defined by Fishburn and Roberts [8]. Inspired by [21] and [22], Fishburn and Roberts [7] introduced a relaxation of a full coloring in which all colors $0,\ldots,k$ for some integer k (possibly different than the span) are used. Such a coloring is said to be no-hole. Here, we consider a more restrictive type of coloring, an *inh-coloring* (which is short for irreducible no-hole coloring). An *inh-coloring* is a no-hole coloring in which no color can be reduced without violating the distance constraints. An inh-coloring ensures that no spectrum is wasted and that each vertex is labelled with the least available frequency.

The *inh-span* of a graph G, denoted $\lambda_{inh}(G)$, is the smallest number k such that there is an irreducible no-hole $L(2,1)$-coloring of G using the integers $0,\ldots,k$. Fishburn, Laskar, Roberts, and Villalpando [6] proved that if T is a tree that is not a star, then T is inh-colorable and

$$\Delta(T)+1 \leq \lambda_{inh}(T) \leq \Delta(T)+2.$$

This is reminiscent of the earlier mentioned result of Griggs and Yeh on the span of trees. In fact, it has been conjectured that $\lambda_{inh}(T)=\lambda(T)$ for all trees other than stars [24, Conjecture 8]. In this paper, we prove this conjecture. This provides an alternate proof of the full colorability of trees (cf. [8]) and shows that it is possible to minimize bandwidth while guaranteeing that no band is wasted and no frequency can be lowered.

This paper is organized as follows. Section 2 includes notation and preliminaries on inh-coloring. Section 3 contains lemmas used in the proof of the main result which is featured in Section 4. The paper concludes with open problems given in Section 5.

2. Preliminaries

Given two vertices $u, v \in V(G)$ of a graph G, $d(u,v)$ denotes the distance between them, meaning the number of edges in a shortest path between u and v. The edge joining two adjacent vertices u and v is denoted by $\{u,v\}$ or simply uv. The path on n vertices is denoted by P_n. The closed neighborhood (resp., open neighborhood) of a vertex u is denoted $N[u]$ (resp., $N(u)$).

Definition 2.1. *An $L(2,1)$-coloring of a graph G is a vertex coloring $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$ such that*

(1) $|f(u) - f(v)| \geq 2$ *for all* $uv \in E(G)$, *and*
(2) $|f(u) - f(v)| \geq 1$ *if* $d(u,v) = 2$.

The *span* of an $L(2,1)$-coloring f is $span f := \max\{f(v) : v \in V(G)\}$. The *span* of a graph G is

$$\lambda(G) := \min\{span f : f \text{ is an } L(2,1)\text{-coloring of } G\}.$$

An $L(2,1)$-coloring of G whose span is equal to the span of G is called a *span coloring.*

Definition 2.2. *An $L(2,1)$-coloring $f : V(G) \to \{0, 1, 2, \ldots, k\}$ of a graph G is a no-hole coloring provided f is surjective for some integer k.*

The notion of no-hole coloring is related to that of a full coloring.

Definition 2.3. *A no-hole coloring f is called a full coloring if and only if $span f = \lambda(G)$.*

A graph may have a no-hole coloring but not be full-colorable. For example, consider C_6, the cycle on six vertices. Note that the span of C_6 is $\lambda(C_6) = 4$ [10, Proposition 3.1]. Because C_6 has six vertices, a full coloring of C_6 would require that two vertices have the same label, say a. However, every other vertex is adjacent to one of those two vertices. As a result, no vertex may be labelled $a \pm 1$ which creates a hole at at least one of $a \pm 1$. Thus, C_6 is not full-colorable. However, $0, 3, 1, 4, 2, 5$ will give a no-hole coloring as shown in Figure 1.

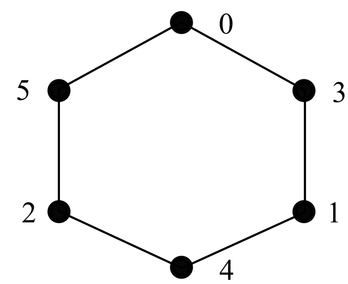

FIGURE 1. A no-hole coloring of C_6.

Next, we define the notion of irreducibility.

Definition 2.4. *An $L(2,1)$-coloring f of a graph G is called reducible if there exists another $L(2,1)$-coloring g of G such that $g(u) \leq f(u)$ for all vertices $u \in V(G)$ and there exists a vertex $v \in V(G)$ such that $g(v) < f(v)$. Otherwise, f is said to be irreducible.*

Every graph has an irreducible $L(2,1)$-coloring. In fact, the following is true.

Lemma 2.5. [19, Theorem 1] *Given any graph G, there is an irreducible $L(2,1)$-coloring f of G such that the span of f equals $\lambda(G)$.*

An *inh-coloring* of a graph G is an $L(2,1)$-coloring of G which is both no-hole and irreducible. A graph G is said to be *inh-colorable* provided G has an inh-coloring. While every graph has an irreducible $L(2,1)$-coloring, not all graphs are inh-colorable. For example, the star $K_{1,n}$ fails to have a no-hole coloring and thus is certainly not inh-colorable. There are graphs which fail to be inh-colorable for other reasons. For example, the graph shown in Figure 2 is not inh-colorable because every no-hole coloring is reducible. In [6], a number of graphs are shown to be inh-colorable, including paths other than P_2 and P_3, cycles of length greater than four, and all trees other than stars. Most graphs with a single cycle and hex graphs with at least five rows and at least five columns are also inh-colorable [19].

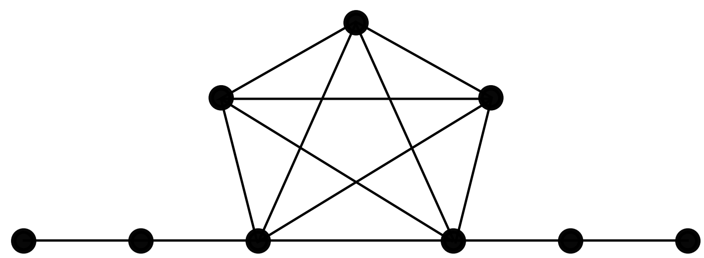

FIGURE 2. An example of a graph in which every no-hole coloring is reducible.

3. Lemmas

This section consists of lemmas used in the proof of the main result. Some of these results are recalled from the literature, while others are proved here.

According to [6], all trees other than stars are inh-colorable. The proof of this result shows that greedily coloring a tree $T \neq K_{1,n}$ provides an inh-coloring with $\Delta(T)+1$ or $\Delta(T)+2$ colors; that is,

$$\Delta(T)+1 \leq \lambda_{inh}(T) \leq \Delta(T)+2.$$

Combining the results of [6] and [10, Proposition 3.1], we immediately see that $\lambda_{inh}(P_n) = \lambda(P_n)$ for all $n \neq 2, 3$. Hence, in the following we restrict our attention to trees with maximum degree at least 3.

Let T be a tree of maximum degree $\Delta \geq 3$ that is not a star. We will show that $\lambda_{inh}(T) = \lambda(T)$. Note that if $\lambda(T) = \Delta + 2$, then the result follows immediately. Note that in any span coloring of T the only possible colors for a maximum degree vertex v are 0 and $\Delta + 1$. Moreover, the only possible colorings of the closed neighborhood of v are

$$f(v) = 0 \text{ and } \{f(u) : u \in N(v)\} = [2, \Delta + 1]$$

and

$$f(v) = \Delta + 1 \text{ and } \{f(u) : u \in N(v)\} = [0, \Delta - 1] \, .$$

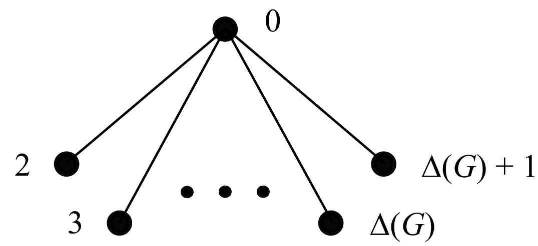

FIGURE 3. The standard star coloring.

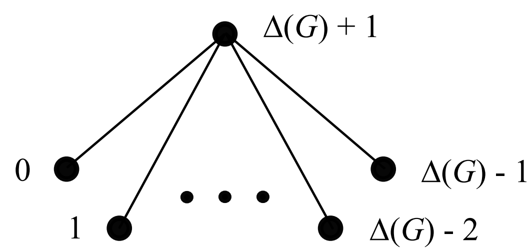

FIGURE 4. The dual star coloring.

A coloring of the former type is called the (standard) star coloring and is illustrated in Figure 3. A coloring of the latter type is called the dual star coloring and is illustrated in Figure 4.

The next result shows that if both 0 and $\Delta(G)+1$ are used to color maximum degree vertices in a span coloring, then the inh-span and the span of G are equal. As a result, if G has two maximum degree vertices which are adjacent or within distance two of one another, then $\lambda_{inh}(G) = \lambda(G)$. This allows us to restrict our attention to trees T in which the distance between any two maximum degree vertices is at least three.

Lemma 3.1. [19, Proposition 1] *For any graph G of maximum degree Δ with $\lambda(G) = \Delta+1$, if there exists a span coloring f and two vertices u and v of degree Δ such that $f(u) = 0$ and $f(v) = \Delta + 1$, then $\lambda_{inh}(G) = \lambda(G)$.*

We next prove a similar result for vertices of degree $\Delta(G)-1$ that will be useful in the proof of the main result.

Lemma 3.2. *Let G be a graph of maximum degree Δ satisfying $\lambda(G) = \Delta+1$. Suppose that f is a span coloring of G with $f(v)=0$ for some maximum degree vertex v. If G has a vertex u of degree $d(u)=\Delta-1$ such that $f(u)\in[3,\Delta]$, then $\lambda_{inh}(G)=\lambda(G)$.*

Proof. Suppose G is a graph of maximum degree Δ satisfying $\lambda(G) = \Delta+1$ with a span coloring f such that $f(v)=0$ for some maximum degree vertex v and $f(u)\in[3,\Delta]$ for some vertex u of degree $d(u) = \Delta-1$. The neighbors of u are colored

$$[0,\Delta+1]\setminus\{f(u), f(u)\pm 1\}.$$

In particular, there is a neighbor v' of u such that $f(v')=1$.

Suppose that f is reducible. Then there exists an $L(2,1)$-coloring f' such that $f'(x)\le f(x)$ for all vertices $x\in V(G)$. The coloring f' may be chosen so that it is irreducible. Note that $f'(N[v]) = [0,\Delta+1]\setminus\{0\}$. In addition, $f'(v')=1$ since $f'(v')\le f(v')=1$ and v' is distance two from u which is colored $f'(u)\le f(u)=0$. Therefore, f' is an irreducible no-hole coloring of G with span $\Delta+1$. □

Notice that if f is an irreducible span coloring of a graph G with $\lambda(G)=\Delta(G)+1$, then the irreducibility of f places strong restrictions on the colors of vertices throughout G. In particular, the following fact (which is used repeatedly in the proof of Theorem 4) follows immediately from the irreducibility of f.

Lemma 3.3. *Suppose that f is an irreducible span coloring of a graph G with $\lambda(G)=\Delta(G)+1$ and $f(v)=0$ for some maximum degree vertex v. Then f is no-hole or each vertex u of G satisfies the following:*

- *If $f(u)=2$ then u is adjacent to a vertex colored 0.*
- *If $f(u)\ge 3$ then either u is adjacent to a vertex colored 0 or u is adjacent to a vertex colored 2 (or both).*

The next result enables us to restrict our consideration to trees in which every leaf is adjacent to a maximum degree vertex.

Lemma 3.4. *Let G be an inh-colorable graph of maximum degree Δ with $\lambda_{inh}(G) = \Delta + 1$. Let $v \in V(G)$ be a vertex of degree at most $\Delta - 2$. Let G^+ denote the graph with vertex set $V(G^+) := V(G) \cup \{u\}$ and edge set $E(G^+) := E(G) \cup \{uv\}$. Then*

$$\lambda_{inh}(G^+) = \lambda_{inh}(G).$$

In particular, if the graph G^- obtained from G by deleting every leaf that is not adjacent to a vertex of maximum degree is inh-colorable with $\lambda_{inh}(G^-) = \Delta + 1$, then $\lambda_{inh}(G) = \Delta + 1$.

Proof. Suppose that f is an inh-coloring of G of span $\Delta + 1$. We claim that f can be extended to an inh-coloring of G^+ using no additional colors. The set of colors available for u among $[0, \Delta + 1]$ is

$$A_u := [0, \Delta + 1] \setminus (\{f(v), f(v) \pm 1\} \cup \{f(x) : x \in V(G), d(x, v) = 1\})$$

which has cardinality

$$|A_u| \geq \Delta + 2 - (3 + d(v)) \geq 1.$$

Define a coloring f^+ of G^+ by $f^+(x) := f(x)$ for all $x \in V(G)$ and $f^+(u) := \min\{a : a \in A_u\}$. Then f^+ is an inh-coloring of G^+ of smallest possible span, namely $\Delta + 1$. □

As a corollary, we see that we may restrict our attention to trees with at least two vertices of maximum degree.

Corollary 3.5. *Let T be a tree of maximum degree Δ that is not a star. If T has only one maximum degree vertex then $\lambda_{inh}(T) = \Delta + 1$.*

Proof. Suppose that T has a unique maximum degree vertex u. Root T at u. Since T is not a star, u has at least one neighbor, say v_1, that has at least one descendant, say w. The subtree induced by $N[u] \cup \{w\}$ has an inh-coloring of span $\Delta + 1$: color $N[u]$ with the standard star coloring, being sure to color v_1 as $\Delta + 1$, and color w as 1. Repeated application of Lemma 3.4 allows adding and coloring of vertices one by one to obtain an inh-coloring of T having span $\Delta + 1$. □

4. Main result

In this section, we build upon the material in the previous sections to prove that the inh-span of a tree is equal to its span, provided the tree is not a star.

Consider a tree T that is not a star with $\Delta := \Delta(T) \geq 3$. Assume $\lambda := \lambda(T) = \Delta+1$. According to Lemma 2.5, T has an irreducible span coloring. Suppose that each irreducible span coloring has a hole, and let f be such a coloring. Without loss of generality, we may assume that $f(v) \neq 1$ for all vertices $v \in V(T)$; that is, we may assume that f has a hole at 1. (Otherwise, take the dual of f and reduce if necessary to obtain an irreducible span coloring.) We will show that f can be altered slightly to produce an inh-coloring of T, yielding a contradiction.

By Corollary 3.5 and Lemma 3.1, we may assume there are at least two maximum degree vertices and that any two distinct maximum degree vertices u and v of T have $d(u, v) \geq 3$. If every leaf that is not adjacent to a vertex of maximum degree were removed, then we obtain a graph that is still inh-colorable (in particular it would contain $N[u]$, $N[v]$ and the path connecting u and v). If we can show that this new graph has inh-span $\Delta+1$ then by Lemma 3.4 the desired result follows. Hence we may assume that every leaf of T is adjacent to a vertex of maximum degree.

We are now ready to prove the main result.

Theorem 4.1. *For any tree T that is not a star,*

$$\lambda(T) = \lambda_{inh}(T).$$

Proof. Root T at a vertex u of maximum degree. Given a vertex $r \in V(T)$, let T_r denote the subtree of T rooted at r. Suppose that v is a maximum degree vertex with $d(u, v)$ as large as possible. Then $f(v) = 0$, and all children of v are leaves. For convenience, set $y := p(v)$, $x := p(y)$, and $w := p(x)$. Since y is not of maximum degree, any child v' of y has a child, and this child must be a leaf, making v' maximum degree. However, it is impossible for v' to be of maximum degree because $d(v', v) = 2$. Therefore v is the only child of y, and so $d(y) = 2$.

If $f(y) \notin \{\Delta, \Delta + 1\}$ and $f(x) \neq \Delta + 1$, then we can modify f to be irreducible and no-hole by maintaining the coloring f for all $s \in V(T) \setminus V(T_v)$ and dual star coloring $N[v]$. Hence, either

(i) $f(y) \in \{\Delta, \Delta + 1\}$ or
(ii) $f(x) = \Delta + 1$.

Note that in case (i), $f(x) = 2$ since otherwise T_y could be recolored so that $f(y) = 1$ and v could be dual star colored, producing an inh-coloring of T with $\Delta + 1$ colors.

We will make repeated use of the fact that if it were possible to recolor x with an integer $a \in [3, \Delta]$, then T_x could be recolored as follows. Available colors for the children of x would be

$$[0, \Delta + 1] \setminus \{f(w), a, a \pm 1\}.$$

This allows y to be colored as 1, $N[v]$ to be dual star colored, and a maximum degree child of x to be star colored (since 0 fails to be available only if $f(w) = 0$ which does not happen if x has such a child), producing an inh-coloring with $\lambda + 1$ colors.

We claim that if w has a child c and $f(c) \notin \{2, \Delta + 1\}$ then $f(c) = 0$ and c is of maximum degree. To see this, assume that w has a child c with $f(c) \in [3, \Delta]$. Since c has no maximum degree grandchild, either c is a leaf or it has a single maximum degree child and is of degree 2. If c is a leaf then we can interchange the colors of x and c and recolor T_x as indicated in the paragraph above. Hence c must have a maximum degree child, say g, and the degree of c is 2. Furthermore, $f(w) \in \{2, \Delta + 1\}$; otherwise, we can recolor c with 1 and dual star color $N[g]$. In either situation we may interchange the colors of x and c, star coloring $N[g]$, and recoloring T_x as indicated above. This establishes our claim.

Recall that if there is an integer $a \in [3, \Delta]$ available for recoloring x, then T_x can be recolored in the manner previously described. Thus,

$$[3, \Delta] \subseteq \{f(w), f(w) \pm 1, f(p(w))\}. \tag{1}$$

If $f(w) = 0$, then (1) implies $\Delta = 3$ and $f(p(w)) = 3$. Therefore $d(p(w)) = \Delta - 1$ and Lemma 3.2 now applies. Hence, $f(w) \neq 0$. If $f(w) = 2$, then (1) implies $\Delta = 4 = f(p(w))$ and $f(x) = 5$. Note that by Lemma 3.3, w must have a maximum degree child. Here, an inh-coloring of T with $\Delta + 1$ colors may be obtained by recoloring T_w as follows: set $f(w) = 1$, $f(x) = 3$, $f(y) = 5$, dual star color the maximum degree child of w and its neighborhood, and star color $N[v]$. Thus $f(w) \neq 2$. If $f(w) = \Delta + 1$, then $f(x) = 2$, $f(y) = \Delta = 4$, and $f(p(w)) = 3$. Here, an inh-coloring of T with $\Delta + 1$ colors may be obtained by recoloring T_w as follows: set $f(w) = 1$, $f(x) = 4$, $f(y) = 2$, and dual star color $N[v]$ as well as the neighborhood of a maximum degree child of w if there is one. Thus, $f(w) \neq \Delta + 1$.

We conclude that $f(w) \in [3, \Delta]$. Reset $f(x)$ to 1. The children of x can now be recolored using colors from $[3, \Delta + 1] \setminus \{f(w)\}$ and each grandchild of x given either the dual star or star coloring. If x has a maximum degree child, it can be dual star colored. If all of the grandchildren of x are dual star colored, then (after possibly reducing $f(x)$ to 0) f is an inh-coloring. If at least one of grandchild of x is star colored, then $f(x)$ cannot be reduced and f is an inh-coloring (as $f(x) = 1$). Therefore, T has an inh-coloring with $\Delta + 1$ colors. □

5. Open problems

In this paper, we have considered inh-colorings of trees. It was previously known that all trees other than stars are inh-colorable [6]. However, there are many other classes of graphs for which the inh-colorability is not known. For instance, while many grid graphs are inh-colorable as are hypercubes of small dimensions, it is not known whether all such graphs are have an inh-coloring. More generally, are all bipartite graphs inh-colorable? It remains an open problem to characterize all inh-colorable graphs.

Acknowledgements

The authors thank the referees for their comments which improved the manuscript as a whole and the proof of the main theorem in particular.

References

[1] S. S. Adams, J. Cass, and D. S. Troxell, An extension of the channel-assignment problem: $L(2,1)$-labelings of generalized Petersen graphs, IEEE Trans. Circuits and Systems I: Regular Papers 53 (2006), 1101–1107.

[2] A. A. Bertossi and C. M. Pinotti, Approximate $L(\delta_1, \delta_2, \ldots, \delta_t)$-coloring of trees and interval graphs, Networks 49 (2007), 204–216.

[3] T. Calamoneri, The $L(h,k)$-labelling problem: A survey and annotated bibliography, Comput J 49 (2006), 585–608.

[4] G. J. Chang and D. Kuo, The $L(2,1)$-labeling problem on graphs, SIAM J Discr Math 9 (1996), 309–316.

[5] G. J. Chang and C. Lu, Distance two labelings of graphs, European J Combin 24 (2003), 53–58.

[6] P. C. Fishburn, R. C. Laskar, F. S. Roberts, and J. Villalpando, Parameters of $L(2,1)$-colorings, preprint.

[7] P. C. Fishburn and F. S. Roberts, No-hole $L(2,1)$-colorings. Discrete Appl Math 130 (2003), 513–519.

[8] P. C. Fishburn and F. S. Roberts, Full color theorems for $L(2,1)$-colorings, SIAM J Discrete Math 20 (2006), 428–443.

[9] J. P. Georges, D. W. Mauro, and M. I. Stein, Labeling products of complete graphs with a condition at distance two, SIAM J Discr Math 14 (2000), 28–35.

[10] J. R. Griggs and R. K. Yeh, Labelling graphs with a condition at distance 2, SIAM J Discr Math 5 (1992), 586–595.

[11] W. K. Hale, Frequency assignment: Theory and applications, Proc IEEE 68 (1980), 1497–1514.

[12] R. E. Jamison, G. L. Matthews, and J. Villalpando, Acyclic colorings of products of trees, Inform Process Lett 99 (2006), 7–12.

[13] P. K. Jha, Optimal $L(2,1)$-labeling of Cartesian products of cycles with an application to independent domination, IEEE Trans Circuits and Systems I: Fundamental Theory and Appl 47 (2000), 1531–1534.

[14] P. K. Jha, Optimal $L(2,1)$-labeling of strong product cycles, IEEE Trans Circuits and Systems I: Fundamental Theory and Appl 48 (2001), 498–500.

[15] P. K. Jha, A. Narayanan, P. Sood, K. Sundaran, and V. Sunder, On $L(2,1)$-labeling of the Cartesian product of a cycle and a path, Ars Combin 55 (2000), 81–89.

[16] E. Jonek, J. H. Hattingh, and C. J. Ras, $\lambda(d,1)$-minimal trees, preprint.

[17] S. Klavzar and S. Spacapan, The Δ^2-conjecture for $L(2,1)$-labeliings is true for direct and strong products of graphs, IEEE Trans Circuits and Systems II: Express Briefs 53 (2006), 274–277.

[18] D. Kuo and J.-H. Yan, On $L(2,1)$-labeling of Cartesian products of paths and cycles, Discrete Math 283 (2004), 137–144.

[19] R. C. Laskar and J. J. Villalpando, Irreducibility of $L(2,1)$-colorings and the inh-colorability of unicyclic and hex graphs, Utilitas Mathematica 69 (2006), 65–83.

[20] C. McDiarmid and B. Reed, Channel assignment and weighted coloring, Networks 36 (2000), 114–117.

[21] F. S. Roberts, No-hole 2-distant colorings, Math Comput Modelling 17 (1993), 139–144.

[22] D. Sakai and C. Wang, No-hole $(r+1)$-distant colorings, Discrete Math. 119 (1993), 175–189.

[23] D. S. Troxell, Labeling chordal graphs: Distance two condition, SIAM J Discr Math 7 (1994), 133–140.

[24] J. Villalpando, Graph parameters: Channel assignment as related to $L(2,1)$-coloring, Ph.D. dissertation, Dept. of Mathematical Sciences, Clemson University, 2002.

[25] R. K. Yeh, A survey on labeling graphs with a condition at distance two, Discrete Math 306 (2006), 1217–1231.