Minimal generating sets of Weierstrass semigroups of certain m-tuples on the norm-trace function field

Gretchen L. Matthews and Justin D. Peachey

Abstract. The norm-trace function field is a generalization of the Hermitian function field which is of importance in coding theory. In this paper, we determine the minimal generating set of the Weierstrass semigroup of the m-tuple $(P_\infty, P_{0b_2}, \ldots, P_{0b_m})$ of places on the norm-trace function field.

1. Introduction

Let q be a power of a prime and r be an integer with $r \geq 2$. Consider the function field $\mathbb{F}_{q^r}(x,y)/\mathbb{F}_{q^r}$ where

$$N_{\mathbb{F}_{q^r}/\mathbb{F}_q}(x) = Tr_{\mathbb{F}_{q^r}/\mathbb{F}_q}(y),$$

meaning the norm of x with respect to the extension $\mathbb{F}_{q^r}/\mathbb{F}_q$ is equal to the trace of y with respect to the extension $\mathbb{F}_{q^r}/\mathbb{F}_q$. This function field is called the norm-trace function field. If $r = 2$, then the norm-trace function field coincides with the well-studied Hermitian function field. The norm-trace function field was first studied by Geil in [**G**] where he considered evaluation codes and one-point algebraic geometry codes constructed from this function field. More recently, Munuera, Tizziotti, and Torres [**MTT**] examined two-point algebraic geometry codes and associated Weierstrass semigroups on the norm-trace function field.

Given an algebraic function field $F/\mathbb{F}$, where $\mathbb{F}$ is a finite field, and distinct places $P_1, \ldots, P_m$ of F of degree one, the Weierstrass semigroup of the m-tuple $(P_1, \ldots, P_m)$ is

$$H(P_1, \ldots, P_m) = \left\{ (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^m : \exists f \in F \text{ with } (f)_\infty = \sum_{i=1}^{r} \alpha_i P_i \right\},$$

where $(f)_\infty$ denotes the divisor of poles of f and $\mathbb{N}$ denotes the set of nonnegative integers. The Weierstrass gap set $G(P_1, \ldots, P_m)$ of the m-tuple $(P_1, \ldots, P_m)$ is defined by

$$G(P_1, \ldots, P_m) = \mathbb{N}^m \setminus H(P_1, \ldots P_m).$$

Key words and phrases. Weierstrass semigroup, norm-trace function field, Hermitian function field.

The first author was supported in part by NSF DMS-0901693 and NSA H-98230-06-1-0008.

In this paper, we determine the minimal generating set of the Weierstrass semigroup $H\left(P_\infty, P_{0b_2}, \ldots, P_{0b_m}\right)$ on the norm-trace function field for any m, $2 \leq m \leq q^{r-1}+1$.

This paper is organized as follows. This section concludes with notation utilized in the paper. Section 2 contains relevant background on the norm-trace function field. The main result is found in Section 3. This paper concludes with examples given in Section 4

Notation. The set of integers is denoted $\mathbb{Z}$, and $\mathbb{Z}_+$ denotes the set of positive integers. As usual, given $v \in \mathbb{Z}^m$ where $m \in \mathbb{Z}_+$, the i^{th} coordinate of v is denoted by v_i. Define a partial order $\preceq$ on $\mathbb{Z}^m$ by $(n_1, \ldots, n_m) \preceq (p_1, \ldots, p_m)$ if and only if $n_i \leq p_i$ for all i, $1 \leq i \leq m$. When comparing elements of $\mathbb{Z}^m$, we will always do so with respect to the partial order $\preceq$. We use the notation $n \prec p$ to mean $n \preceq p$ and $n \neq p$.

Given a prime power q, $\mathbb{F}_q$ denotes the field with q elements. Let $F/\mathbb{F}_q$ be an algebraic function field. The divisor of a function $f \in F \setminus \{0\}$ is denoted by (f).

2. Preliminaries on the norm-trace function field

In this section, we review the necessary background on the norm-trace function field; additional details may be found in [**G**].

Consider the norm-trace function field $F := \mathbb{F}_{q^r}(x, y)/\mathbb{F}_{q^r}$ which has defining equation

$$y^{q^{r-1}} + y^{q^{r-2}} + \cdots + y = x^{a+1}$$

where $a := \frac{q^r-1}{q-1} - 1$, q is a power of a prime, and $r \geq 2$ is an integer. The genus of $F/\mathbb{F}_{q^r}$ is $g = \frac{a\left(q^{r-1}-1\right)}{2}$. For each $\alpha \in \mathbb{F}_{q^r}$, there are q^{r-1} elements $\beta \in \mathbb{F}_{q^r}$ such that

$$N_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\alpha) = Tr_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\beta). \tag{2.1}$$

For every pair $(\alpha, \beta) \in \mathbb{F}_{q^r}^2$ satisfying Equation (2.1), there is a place $P_{\alpha\beta}$ of F of degree one which is the common zero of $x - \alpha$ and $y - \beta$. In fact, the places of F of degree one are precisely these $P_{\alpha\beta}$ and P_∞, the common pole of x and y. In particular, there are q^{r-1} places P_{0b} with $b \in \mathcal{B}$ where

$$\mathcal{B} := \left\{b \in \mathbb{F}_{q^r} : Tr_{\mathbb{F}_{q^r}/\mathbb{F}_q}(b) = 0\right\}.$$

In determining the Weierstrass semigroups $H(P_\infty)$ and $H(P_{0b})$, for $b \in \mathcal{B}$, on the norm-trace function field, the following principal divisors are quite useful:

$$(x) = \sum_{b \in \mathcal{B}} P_{0b} - q^{r-1} P_\infty$$

and for any $b \in \mathcal{B}$,

$$(y-b) = (a+1) P_{0b} - (a+1) P_\infty.$$

Combining these with the fact that $|G(P)| = g$ for any place P of degree one, it can be shown that gap set of the infinite place is

$$G(P_\infty) = \left\{ \left(q^{r-1} - i + j - 1\right)(a+1) - jq^{r-1} : \begin{array}{c} 1 \leq j \leq i \leq a - s \text{ and} \\ (s-1)(q-1) \leq i - j < s(q-1) \\ \text{where } 1 \leq s \leq a + 1 - q^{r-1} \end{array} \right\}$$

and the gap set of any place P_{0b} where $b \in \mathcal{B}$ is

$$G(P_{0b}) = \left\{ (i-j)(a+1)+j : \begin{array}{c} 1 \le j \le i \le a-s \text{ and} \\ (s-1)(q-1) \le i-j < s(q-1) \\ \text{where } 1 \le s \le a+1-q^{r-1} \end{array} \right\}.$$

Moreover, each element of the gap set $G(P_\infty)$ has a unique representation of the form above; specifically if

$$(q^{r-1}-i+j-1)(a+1) - jq^{r-1} = (q^{r-1}-i'+j'-1)(a+1) - j'q^{r-1},$$

where $1 \le j, j' \le a-1$, then

$$i' = i \text{ and } j' = j.$$

A similar fact holds for elements of the gap set $G(P_{0b})$ where $b \in \mathcal{B}$. Additional details may be found in [**G**], [**MTT**], and [**M09**].

3. Weierstrass semigroups on the norm-trace function field

In this section, we determine the minimal generating set of the Weierstrass semigroup $H(P_\infty, P_{0b_2}, \ldots, P_{0b_m})$ on the norm-trace function field for any m, $2 \le m \le q^{r-1}$, and any distinct $b_i \in \mathcal{B}$.

Definition 3.1. Let $P_1, \ldots, P_m$ be m distinct places of degree one of an algebraic function field of $F/\mathbb{F}$. Set $\Gamma(P_1) := H(P_1)$; for $m \ge 2$, set

$$\Gamma(P_1, \ldots, P_m) := \left\{ \mathbf{n} \in \mathbb{Z}_+^m : \begin{array}{l} \mathbf{n} \text{ is minimal in } \{\mathbf{p} \in H(P_1, \ldots, P_m) : p_i = n_i\} \\ \text{for some } i, 1 \le i \le m \end{array} \right\}.$$

In [**M04**] it is shown that if $2 \le m \le |\mathbb{F}|$, then $H(P_1, \ldots, P_m) =$

$$\left\{ \operatorname{lub}\{\mathbf{u_1}, \ldots, \mathbf{u_m}\} : \begin{array}{l} \mathbf{u_i} \in \mathbf{\Gamma}(\mathbf{P_1}, \ldots, \mathbf{P_m}) \text{ or } (\mathbf{u_{i_1}}, \ldots, \mathbf{u_{i_k}}) \in \mathbf{\Gamma}(\mathbf{P_{i_1}}, \ldots, \mathbf{P_{i_k}}) \\ \text{for some } \{i_1, \ldots, i_m\} = \{1, \ldots, m\} \text{ such that } i_1 < \cdots < i_k \\ \text{and } u_{i_{k+1}} = \cdots = u_{i_m} = 0 \text{ for some } 1 \le k < m \end{array} \right\}$$

where

$$\operatorname{lub}\{\mathbf{u_1}, \ldots, \mathbf{u_m}\} = (\max\{\mathbf{u_{1_1}}, \ldots, \mathbf{u_{m_1}}\}, \ldots, \max\{\mathbf{u_{1_m}}, \ldots, \mathbf{u_{m_m}}\}) \in \mathbb{N}^{\mathbf{m}}$$

is least upper bound of the vectors $\mathbf{u_1}, \ldots, \mathbf{u_m} \in \mathbb{N}^{\mathbf{m}}$. The set $\Gamma(P_1, \ldots, P_m)$ is called the minimal generating set of the Weierstrass semigroup $H(P_1, \ldots, P_m)$. Hence, to determine the entire Weierstrass semigroup $H(P_1, \ldots, P_m)$, one only needs to determine the minimal generating sets $\Gamma(P_{i_1}, \ldots, P_{i_k})$. The next lemma aids in finding such sets.

Lemma 3.2. [**M04**] *Let $F/\mathbb{F}$ be an algebraic function field where $\mathbb{F}$ is a finite field. Suppose $P_1, \ldots, P_m$ are distinct places of $F/\mathbb{F}$ of degree one and $2 \le m \le |\mathbb{F}|$. Then*

(1) $\Gamma(P_1, \ldots, P_m) \subseteq G(P_1) \times \cdots \times G(P_m)$.

(2) $$\Gamma(P_1, \ldots, P_m) = \left\{ \mathbf{n} \in \mathbb{Z}_+^m : \begin{array}{c} \mathbf{n} \text{ is minimal in} \\ \{\mathbf{p} \in H(P_1, \ldots, P_m) : p_i = n_i\} \\ \text{for all } i, 1 \le i \le m \end{array} \right\}.$$

We aim to find $\Gamma(P_\infty, P_{0b_2}, \ldots, P_{0b_m})$ on the norm-trace function field. The case $m = 2$ appears in [**MTT**] and is recorded here as the next lemma.

Lemma 3.3. [**MTT**] *Let $b \in \mathcal{B}$. The minimal generating set of the Weierstrass semigroup of the pair (P_∞, P_{0b}) of places on the norm-trace function field over $\mathbb{F}_{q^r}$ is*

$$\Gamma\left(P_\infty, P_{0b}\right) = \left\{ v_{ij} : \begin{array}{c} 1 \le j \le i \le a-s, \\ (s-1)(q-1) \le i-j \le s(q-1)-1 \\ \textit{for some } 1 \le s \le a+1-q^{r-1} \end{array} \right\}$$

where

$$v_{ij} := \left((a+1)\left(q^{r-1}-i+j-1\right) - jq^{r-1}, (a+1)(i-j)+j\right).$$

Utilizing the two lemmas above, we next prove the main result.

Theorem 3.4. *Suppose $2 \le m \le q^{r-1}+1$. The minimal generating set of the Weierstrass semigroup of the m-tuple $(P_\infty, P_{0b_2}, \ldots, P_{0b_m})$ of places of the norm-trace function field over $\mathbb{F}_{q^r}$ is*

$$\Gamma(P_\infty, P_{0b_2}, \ldots, P_{0b_m}) = \left\{ \gamma_{j,\mathbf{t}} : \begin{array}{l} \sum_{k=2}^{m} t_k = i-j+1, t_k \in \mathbb{Z}_+, 1 \le j \le i \le a-s, \\ (s-1)(q-1) \le i-j \le s(q-1)-1 \\ \textit{where } 1 \le s \le a+1-q^{r-1} \end{array} \right\}$$

where

$$\gamma_{j,\mathbf{t}} = \left(\left(q^{r-1} - \sum_{k=2}^{m} t_k\right)(a+1) - jq^{r-1}, (t_2-1)(a+1)+j, \ldots, (t_m-1)(a+1)+j \right).$$

Proof. For $2 \le m \le q^{r-1}+1$, set

$$S_m := \left\{ \gamma_{j,\mathbf{t}} : \begin{array}{l} \sum_{k=2}^{m} t_k = i-j+1, t_i \in \mathbb{Z}_+, 1 \le j \le i \le a-s, \\ (s-1)(q-1) \le i-j \le s(q-1)-1 \\ \text{where } 1 \le s \le a+1-q^{r-1} \end{array} \right\}.$$

When convenient, we write H_m to mean $H(P_\infty, P_{0b_2}, \ldots, P_{0b_m})$ and Γ_m to mean $\Gamma(P_\infty, P_{0b_2}, \ldots, P_{0b_m})$, $m \ge 2$. We prove that $S_m = \Gamma_m$ by induction on m. By Lemma 3.3, $S_2 = \Gamma_2$. Assume that $\Gamma_l = S_l$ for $2 \le l \le m-1$. First, we show that $S_m \subseteq \Gamma_m$.

Let $s := \gamma_{j,\mathbf{t}} \in S_m$. Hence,

$$s_1 = \left(q^{r-1} - \sum_{i=2}^{m} t_i\right)(a+1) - jq^{r-1},$$

and for $2 \le i \le m$,

$$s_i = (t_i-1)(a+1)+j.$$

Then $s \in H_m$, since $\left(\frac{x^{a+1-j}}{\prod_{i=2}^{m}(y-b_i)^{t_i}} \right)_\infty =$

$$\left(\left(q^{r-1} - \sum_{i=2}^{m} t_i\right)(a+1) - jq^{r-1} \right) P_\infty + \sum_{i=2}^{m}((t_i-1)(a+1)+j)P_{0b_i}.$$

It remains to show that $s \in \Gamma_m$.

Let $Q_1 := \{p \in H_m : p_1 = s_1\}$. Then $s \in Q_1$. We claim that s is minimal in Q_1. Suppose not; that is, suppose there exists $w \in Q_1$ such that

$$w \prec s.$$

Then there exists $f \in F$ with divisor

$$(f) = A - (w_1 P_\infty + w_2 P_{0b_2} + \cdots + w_m P_{0b_m})$$

where A is effective. Clearly, $w_i \leq s_i$ for $1 \leq i \leq m$ and $w_i < s_i$ for some $2 \leq i \leq m$. We may assume $w_2 < s_2$ as a similar argument holds for any other i. Then

$$w_2 = (t_2 - 1)(a + 1) + j - k$$

for some $k \in \mathbb{Z}^+$.

Suppose that $j \leq k$. Notice that

$$\left(f\,(y - b_2)^{t_2 - 1}\right) = A' - (w_1 + (t_2 - 1)(a + 1))P_\infty - (j - k)\,P_{0b_2} - \sum_{k=3}^{m} w_k P_{0b_k}$$

where A' is an effective divisor. Then

$$v := (w_1 + (t_2 - 1)(a + 1), w_3, \ldots, w_m) \in H\left(P_\infty, P_{0b_3}, \ldots, P_{0b_m}\right)$$

since $j - k \leq 0$. Now, since

$$w_1 + (t_2 - 1)(a + 1) = \left(q^{r-1} - \left(1 + \sum_{i=3}^{m} t_i\right)\right)(a + 1) - jq^{r-1},$$

we obtain that

$$\begin{aligned} v &\preceq \left(\left(q^{r-1} - \sum_{i=3}^{m} t_i'\right)(a + 1) - jq^{r-1}, (t_3' - 2)(a + 1) + j, \ldots, (t_m' - 1)(a + 1) + j\right) \\ &\prec \gamma_{j,(t_3',\ldots,t_m')}, \end{aligned}$$

where $t_3' = t_3 + 1$ and $t_i' = t_i$ for $4 \leq i \leq m$. We claim that

$$\gamma_{j,(t_3',\ldots,t_m')} \in \Gamma\left(P_\infty, P_{0b_3}, \ldots, P_{0b_m}\right).$$

To see this, let $i' = \sum_{i=3}^{m} t_i' + j - 1$. Then, $i' - j + 1 = \sum_{i=3}^{m} t_i'$. First, note that $\sum_{i=3}^{m} t_i' \leq \sum_{i=2}^{m} t_i$. Thus, $i' - j + 1 \leq i - j + 1$. Hence, $i' \leq i \leq a - s$ and $i' - j \leq i - j$. Thus, we can find an s_l such that $1 \leq s_l \leq s \leq a + 1 - q^{r-1}$ and $(s_l - 1)(q - 1) \leq i - j \leq s_l(q - 1) - 1$. Furthermore, $i' \leq a - s \leq a - s_l$. Also, $i' + 1 = \sum_{i=3}^{m} t_i' + j$ implies $i' > j$. Thus, we have that

$$v \prec \gamma_{j,(t_3',\ldots,t_m')}$$

and

$$\gamma_{j,(t_3',\ldots,t_m')} \in \Gamma\left(P_\infty, P_{0b_3}, \ldots, P_{0b_m}\right)$$

which is a contradiction. Hence, it must be that $j > k$.

Now, note that $\left(fx^{j-k}(y - b_2)^{t_2 - 1}\right) =$

$$A'' - (w_1 + (t_2 - 1)(a + 1) + (j - k)q^{r-1})P_\infty - \sum_{i=3}^{m}(w_i - (j - k))P_{0b_i}.$$

where A'' is an effective divisor. Set

$$v := \left(\left(q^{r-1} - \sum_{i=3}^{m} t_i - 1\right)(a + 1) - kq^{r-1}, w_3 - (j - k), \ldots, w_m - (j - k)\right).$$

Then $v \in H_m$. An argument similar to that above shows

$$v \prec \gamma_{k,(t_3',\ldots,t_m')},$$

where $t_3' = t_3 + 1$ and $t_i' = t_i$ for $4 \le i \le m$, and

$$\gamma_{k,(t_3',\ldots,t_m')} \in \Gamma\left(P_\infty, P_{0b_3}, \ldots, P_{0b_m}\right),$$

which is a contradiction. This proves that s is minimal in Q_1. Hence, $s \in \Gamma_m$, and it follows that $S_m \subseteq \Gamma_m$.

Next, we show that $\Gamma_m \subseteq S_m$. Let $n \in \Gamma_m$. By Lemma 3.2(1),

$$n \in G(P_\infty) \times G(P_{0b_2}) \times \cdots \times G(P_{0b_m}).$$

According to Lemma 3.3, this implies

$$\begin{array}{rcl} n_1 & = & (a+1)(q^{r-1} - i_1 + j_1 - 1) - j_1 q^{r-1}, \text{ and} \\ n_l & = & (a+1)(i_l - j_l) + j_l, \text{ for } 2 \le l \le m, \end{array}$$

where for all l, $2 \le l \le m$,

$$\begin{array}{l} 1 \le j_l \le i_l \le a - s_l, \\ (s_l - 1)(q-1) \le i_l - j_l \le s_l(q-1) - 1, \text{ for some } s_l, \\ 1 \le s_l \le a + 1 - q^{r-1}. \end{array}$$

We may assume without loss of generality that

$$j_2 = \min\{j_l : 2 \le l \le m\}$$

since the argument is similar for any j_l where $j_l = \min\{j_l : 2 \le l \le m\}$. Then there exists $h \in F$ with

$$(h)_\infty = n_1 P_\infty + \sum_{k=2}^{m} n_k P_{0b_k}.$$

This implies $\left(h \prod_{k=3}^{m} (y - b_k)^{i_k - j_k + 1}\right)_\infty =$

$$\left(n_1 + (a+1)\sum_{k=3}^{m}(i_k - j_k) + (a+1)(m-2)\right) P_\infty - n_2 P_{0b_2},$$

and

$$v := \left(n_1 + (a+1)\sum_{k=3}^{m}(i_k - j_k + 1), n_2\right) \in H\left(P_\infty, P_{0b_2}\right).$$

By Lemma 3.2(2), there exists $u \in \Gamma_2$ such that $u \preceq v$ and $u_2 = n_2$. Lemma 3.3 implies

$$u_1 = (a+1)(q^{r-1} - i_2 + j_2 - 1) - j_2 q^{r-1}.$$

Furthermore, $u_1 > n_1$; otherwise, $(u_1, u_2, 0, \ldots, 0) \prec n$, which contradicts the minimality of n in $\{p \in H_m : p_2 = n_2\}$. Thus, $n_1 < u_1 \le n_1 + (a+1) \sum_{k=3}^{m} (i_k - j_k + 1)$. Now, let

$$w := (w_1, (i_2 - j_2)(a+1) + j_2, (i_3 - j_3)(a+1) + j_2, \ldots, (i_m - j_m)(a+1) + j_2),$$

where

$$w_1 = \max\left\{0, u_1 - (a+1)\sum_{k=3}^{m}(i_k - j_k + 1)\right\},$$

and let $h = \frac{\prod_{b \in \mathcal{B} \setminus \{b_2, \ldots, b_m\}} (y-b)}{\prod_{k=2}^{m} (y - b_k)^{i_k - j_k} x^{j_2}}$. Then $(h)_\infty = w_1 P_\infty + \sum_{k=2}^{m} w_k P_{0b_k}$. Thus, $w \in H_m$ and $w \preceq n$. Hence,

$$w = n.$$

As a result $w_1 = u_1 - (a+1)\sum_{k=3}^{m}(i_k - j_k + 1) > 0$ and $j_l = j_2$ for all $3 \le l \le m$. Moreover,

$$i_2 + \sum_{k=3}^{m}(i_k - j_k) + (m-2) = i_1 \text{ and } j_2 = j_1$$

by the uniqueness of representation of elements of the gap sets $G(P_\infty)$ and $G(P_{0b})$. Therefore,

$$n = \gamma_{j_2,(i_2-j_2+1, i_3-j_3+1, \ldots, i_m-j_m+1)}.$$

Finally, we must check that $\gamma_{j_2,(i_2-j_2+1,i_3-j_3+1,\ldots,i_m-j_m+1)} \in \Gamma_m$. To do this, we check that $\gamma_{j_2,(i_2-j_2+1,i_3-j_3+1,\ldots,i_m-j_m+1)} \in S_m$. Note that

$$\sum_{k=2}^{m}(i_k - j_k + 1) = i_1 - j_2 + 1,$$

$$1 \le j_2 = j_1 \le i_1 \le a - s, \text{ and}$$

which means

$$(s-1)(q-1) \le i_1 - j_2 \le s(q-1) - 1$$

where $1 \le s \le a + 1 - q^{r-1}$. Therefore, $\Gamma_m \subseteq S_m$. Thus, $\Gamma_m = S_m$ proving the desired description of $\Gamma(P_\infty, P_{0b_2}, \ldots, P_{0b_m})$. □

4. Examples

In this section, we consider two examples.

EXAMPLE 4.1. Consider the norm-trace function field F/F_{q^r} with $r = 2$. Then $a = q$ and $F/\mathbb{F}_{q^2}$ is the Hermitian function field which has defining equation

$$y^q + y = x^{q+1}.$$

Taking $m = 2$ in Theorem 3.4 gives the minimal generating set of $\Gamma(P_\infty, P_{0b_2})$. Because the automorphism group of F is doubly-transitive,

$$\Gamma(P_1, P_2) = \Gamma(P_\infty, P_{0b_2})$$

for any pair (P_1, P_2) of distinct degree one places of the Hermitian function field. This result first appeared as [**M01**, Theorem 3.4].

More generally, the minimal generating set of the Weierstrass semigroup of the m-tuple $(P_\infty, P_{0b_2}, \ldots, P_{0b_m})$ of places of degree one of the Hermitian function field over $\mathbb{F}_{q^2}$ is

$$\Gamma_m = \left\{ \gamma_{j,\mathbf{t}} : \begin{array}{l} \sum_{k=2}^{m} t_k = i - j + 1, t_i \in \mathbb{Z}_+, 1 \le j < i \le q-1, \\ 0 \le i - j \le q - 2 \end{array} \right\}$$

where

$$\gamma_{j,\mathbf{t}} = \left(\left(q - \sum_{i=k}^{m} t_k \right)(q+1) - jq, (t_2 - 1)(q+1) + j, \ldots, (t_m - 1)(q+1) + j \right).$$

This result first appeared as [**M04**, Theorem 10]. We also note that [**MMP**] contains some results related to m-tuples on the Hermitian function field.

Example 4.2. Let $\mathbb{F}_{27} = \mathbb{F}_3(\omega)$ where $\omega^3 - \omega + 1 = 0$. The norm-trace function field with $q = 3$ and $r = 3$ is $\mathbb{F}_{27}(x, y)/\mathbb{F}_{27}$ where

$$y^9 + y^3 + y - x^{13}.$$

The genus of $\mathbb{F}_{27}(x, y)/\mathbb{F}_{27}$ is 48, and there are exactly 9 places of $\mathbb{F}_{27}(x, y)/\mathbb{F}_{27}$ of the form P_{0b}:

$$P_{00}, P_{01}, P_{02}, P_{0\omega}, P_{0\omega^3}, P_{0\omega^9}, P_{0\omega^{14}}, P_{0\omega^{16}}, P_{0\omega^{22}}.$$

Then

$$G(P_\infty) = \left\{ \begin{array}{l} 1,2,3,4,5,6,7,8,10,11,12,14,15,16,17,19,20,21,23,24,25,28, \\ 29,30,32,33,34,37,38,41,42,43,46,47,50,51,55,56,59,60,64, \\ 68,69,73,77,82,86,95 \end{array} \right\},$$

and for all m, $2 \leq m \leq 10$,

$$G(P_{0b_m}) = \left\{ \begin{array}{l} 1,2,3,4,5,6,7,8,9,10,11,14,15,16,17,18,19,20,21,22,23,27, \\ 28,29,30,31,32,33,34,40,41,42,43,44,45,46,53,54,55,56,57, \\ 66,67,68,69,79,80,92 \end{array} \right\}.$$

Taking $m = 2$ in Theorem 3.4 yields $\Gamma(P_\infty, P_{0b_2}) =$

$$\left\{ \begin{array}{l} (1,23),(2,46),(3,69),(4,92),(5,11),(6,34),(7,57),(8,80),(10,22), \\ (11,45),(12,68),(14,10),(15,33),(16,56),(17,79),(19,21),(20,44), \\ (21,67),(23,9),(24,32),(25,55),(28,20),(29,43),(30,66),(32,8),(33,31), \\ (34,54),(37,19),(38,42),(41,7),(42,30),(43,53),(46,18),(47,41),(50,6), \\ (51,29),(55,17),(56,40),(59,5),(60,28),(64,16),(68,4),(69,27), \\ (73,15),(77,3),(82,14),(86,2),(95,1) \end{array} \right\};$$

this also follows from Lemma 3.3. Figure 1 illustrates how the minimal generating set $\Gamma(P_\infty, P_{0b_2})$ is related to the semigroup $H(P_\infty, P_{0b_2})$. In particular, the elements of $\Gamma(P_\infty, P_{0b_2})$ are shown in bold as are the elements of $\Gamma(P_\infty) \cap [0, 2g]$ and $\Gamma(P_{0b_2}) \cap [0, 2g]$.

Taking $m = 3$ in Theorem 3.4 gives $\Gamma(P_\infty, P_{0b_2}, P_{0b_3}) =$

$$\left\{ \begin{array}{l} (1,10,10),(2,7,33),(2,20,20),(2,33,7),(3,4,56),(3,17,43), \\ (3,30,30),(3,43,17),(3,56,4),(4,1,79),(4,14,66),(4,27,53), \\ (4,40,40),(4,53,27),(4,66,14),(4,79,1),(6,8,21),(6,21,8), \\ (7,5,44),(7,18,31),(7,31,18),(7,44,5),(8,2,67),(8,15,54), \\ (8,28,41),(8,41,28),(8,54,15),(8,67,2),(10,9,9),(11,6,32), \\ (11,19,19),(11,32,6),(12,3,55),(12,16,42),(12,29,29),(12,42,16), \\ (12,55,3),(15,7,20),(15,20,7),(16,4,43),(16,17,30),(16,30,17), \\ (16,43,4),(17,1,66),(17,14,53),(17,27,40),(17,40,27),(17,53,14), \\ (17,66,1),(19,8,8),(20,5,31),(20,18,18),(20,31,5),(21,2,54), \\ (21,15,41),(21,28,28),(21,41,15),(21,54,2),(24,6,19),(24,19,6), \\ (25,3,42),(25,16,29),(25,29,16),(25,42,3),(28,7,7),(29,4,30), \\ (29,17,17),(29,30,4),(30,1,53),(30,14,40),(30,27,27),(30,40,14), \\ (30,53,1),(33,5,18),(33,18,5),(34,2,41),(34,15,28),(34,28,15), \\ (34,41,2),(37,6,6),(38,3,29),(38,16,16),(38,29,3),(42,4,17), \\ (42,17,4),(43,1,40),(43,14,27),(43,27,14),(43,40,1),(46,5,5), \\ (47,2,28),(47,15,15),(47,28,2),(51,3,16),(51,16,3),(55,4,4), \\ (56,1,27),(56,14,14),(56,27,1),(60,2,15),(60,15,2),(64,3,3), \\ (69,1,14),(69,14,1),(73,2,2),(82,1,1) \end{array} \right\},$$

as shown in [**M09**].

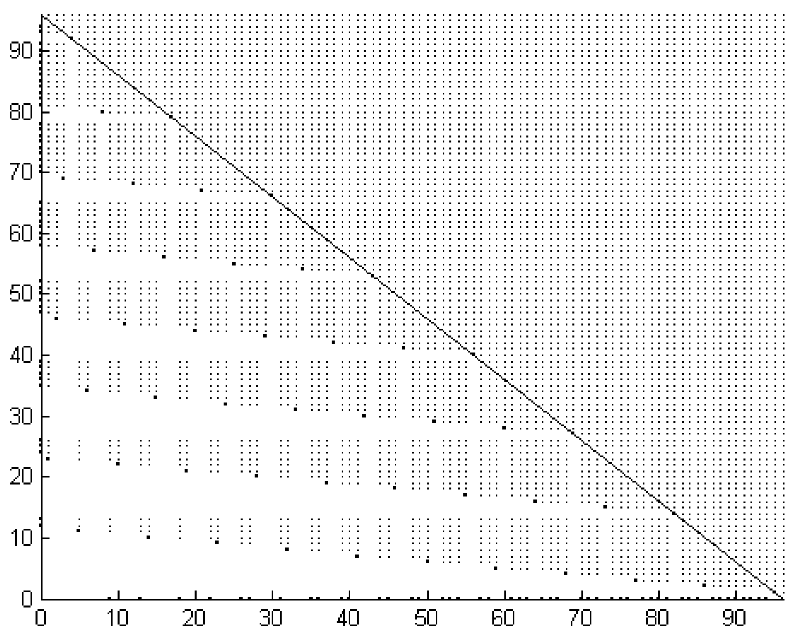

FIGURE 1. $H\left(P_\infty, P_{0b_2}\right) \cap [0, 2g]^2$

Considering $4 \leq m \leq 10$ in Theorem 3.4 gives $\Gamma\left(P_\infty, P_{0b_2}, P_{0b_3}, P_{0b_4}\right) =$

$$\left\{\begin{array}{l}
(69,1,1,1),(56,1,1,14),(56,1,14,1),(56,14,1,1),(60,2,2,2),(47,2,2,15),\\
(47,2,15,2),(47,15,2,2),(51,3,3,3),(38,3,3,16),(38,3,16,3),(38,16,3,3),\\
(42,4,4,4),(29,4,4,17),(29,4,17,4),(29,17,4,4),(33,5,5,5),(20,5,5,18),\\
(20,5,18,5),(20,18,5,5),(24,6,6,6),(11,6,6,19),(11,6,19,6),(11,19,6,6),\\
(15,7,7,7),(2,7,7,20),(2,7,20,7),(2,20,7,7),(6,8,8,8),(43,1,1,27),\\
(43,1,14,14),(43,1,27,1),(43,14,1,14),(43,14,14,1),(43,27,1,1),(30,1,1,40),\\
(30,1,14,27),(30,1,27,14),(30,1,40,1),(30,14,1,27),(30,14,14,14),(30,14,27,1),\\
(30,27,1,14),(30,27,14,1),(30,40,1,1),(34,2,2,28),(34,2,15,15),(34,2,28,2),\\
(34,15,2,15),(34,15,15,2),(34,28,2,2),(21,2,2,41),(21,2,15,28),(21,2,28,15),\\
(21,2,41,2),(21,15,2,28),(21,15,15,15),(21,15,28,2),(21,28,2,15),(21,28,15,2),\\
(21,41,2,2),(25,3,3,29),(25,3,16,16),(25,3,29,3),(25,16,3,16),(25,16,16,3),\\
(25,29,3,3),(12,3,3,42),(12,3,16,29),(12,3,29,16),(12,3,42,3),(12,16,3,29),\\
(12,16,16,16),(12,16,29,3),(12,29,3,16),(12,29,16,3),(12,42,3,3),(16,4,4,30),\\
(16,4,17,17),(16,4,30,4),(16,17,4,17),(16,17,17,4),(16,30,4,4),(3,4,4,43),\\
(3,4,17,30),(3,4,30,17),(3,4,43,4),(3,17,4,30),(3,17,17,17),(3,17,30,4),\\
(3,30,4,17),(3,30,17,4),(3,43,4,4),(7,5,5,31),(7,5,18,18),(7,5,31,5),\\
(7,18,5,18),(7,18,18,5),(7,31,5,5),(17,1,1,53),(17,1,14,40),(17,1,27,27),\\
(17,1,40,14),(17,1,53,1),(17,14,1,40),(17,14,14,27),(17,14,27,14),(17,14,40,1),\\
(17,27,1,27),(17,27,14,14),(17,27,27,1),(17,40,1,14),(17,40,14,1),(17,53,1,1),\\
(4,1,1,66),(4,1,14,53),(4,1,27,40),(4,1,40,27),(4,1,53,14),(4,1,66,1),\\
(4,14,1,53),(4,14,14,40),(4,14,27,27),(4,14,40,14),(4,14,53,1),(4,27,1,40),\\
(4,27,14,27),(4,27,27,14),(4,27,40,1),(4,40,1,27),(4,40,14,14),(4,40,27,1),\\
(4,53,1,14),(4,53,14,1),(4,66,1,1),(8,2,2,54),(8,2,15,41),(8,2,28,28),\\
(8,2,41,15),(8,2,54,2),(8,15,2,41),(8,15,15,28),(8,15,28,15),(8,15,41,2),\\
(8,28,2,28),(8,28,15,15),(8,28,28,2),(8,41,2,15),(8,41,15,2),(8,54,2,2)
\end{array}\right\},$$

$\Gamma\left(P_{\infty}, P_{0b_2}, P_{0b_3}, P_{0b_4}, P_{0b_5}\right) =$

$$\left\{\begin{array}{l}
(56,1,1,1,1),(47,2,2,2,2),(38,3,3,3,3),(29,4,4,4,4),(20,5,5,5,5),(11,6,6,6,6),\\
(2,7,7,7,7),(43,1,1,1,14),(43,1,1,14,1),(43,1,14,1,1),(43,14,1,1,1),(30,1,1,1,27),\\
(30,1,1,14,14),(30,1,1,27,1),(30,1,14,1,14),(30,1,14,14,1),(30,1,27,1,1),\\
(30,14,1,1,14),(30,14,1,14,1),(30,14,14,1,1),(30,27,1,1,1),(34,2,2,2,15),\\
(34,2,2,15,2),(34,2,15,2,2),(34,15,2,2,2),(21,2,2,2,28),(21,2,2,15,15),\\
(21,2,2,28,2),(21,2,15,2,15),(21,2,15,15,2),(21,2,28,2,2),(21,15,2,2,15),\\
(21,15,2,15,2),(21,15,15,2,2),(21,28,2,2,2),(25,3,3,3,16),(25,3,3,16,3),\\
(25,3,16,3,3),(25,16,3,3,3),(12,3,3,3,29),(12,3,3,16,16),(12,3,3,29,3),\\
(12,3,16,3,16),(12,3,16,16,3),(12,3,29,3,3),(12,16,3,3,16),(12,16,3,16,3),\\
(12,16,16,3,3),(12,29,3,3,3),(16,4,4,4,17),(16,4,4,17,4),(16,4,17,4,4),(16,17,4,4,4),\\
(3,4,4,4,30),(3,4,4,17,17),(3,4,4,30,4),(3,4,17,4,17),(3,4,17,17,4),\\
(3,4,30,4,4),(3,17,4,4,17),(3,17,4,17,4),(3,17,17,4,4),(3,30,4,4,4),\\
(7,5,5,5,18),(7,5,5,18,5),(7,5,18,5,5),(7,18,5,5,5),(17,1,1,1,40),(17,1,1,14,27),\\
(17,1,1,27,14),(17,1,1,40,1),(17,1,14,1,27),(17,1,14,14,14),(17,1,14,27,1),\\
(17,1,27,1,14),(17,1,27,14,1),(17,1,40,1,1),(17,14,1,1,27),(17,14,1,14,14),\\
(17,14,1,27,1),(17,14,14,1,14),(17,14,14,14,1),(17,14,27,1,1),(17,27,1,1,14),\\
(17,27,1,14,1),(17,27,14,1,1),(17,40,1,1,1),(4,1,1,1,53),(4,1,1,14,40),(4,1,1,27,27),\\
(4,1,1,40,14),(4,1,1,53,1),(4,1,14,1,40),(4,1,14,14,27),(4,1,14,27,14),(4,1,14,40,1),\\
(4,1,27,1,27),(4,1,27,14,14),(4,1,27,27,1),(4,1,40,1,14),(4,1,40,14,1),\\
(4,1,53,1,1),(4,14,1,1,40),(4,14,1,14,27),(4,14,1,27,14),(4,14,1,40,1),\\
(4,14,14,1,27),(4,14,14,14,14),(4,14,14,27,1),(4,14,27,1,14),(4,14,27,14,1),\\
(4,14,40,1,1),(4,27,1,1,27),(4,27,1,14,14),(4,27,1,27,1),(4,27,14,1,14),\\
(4,27,14,14,1),(4,27,27,1,1),(4,40,1,1,14),(4,40,1,14,1),(4,40,14,1,1),\\
(4,53,1,1,1),(8,2,2,2,41),(8,2,2,15,28),(8,2,2,28,15),(8,2,2,41,2),(8,2,15,2,28),\\
(8,2,15,15,15),(8,2,15,28,2),(8,2,28,2,15),(8,2,28,15,2),(8,2,41,2,2),\\
(8,15,2,2,28),(8,15,2,15,15),(8,15,2,28,2),(8,15,15,2,15),(8,15,15,15,2),\\
(8,15,28,2,2),(8,28,2,2,15),(8,28,2,15,2),(8,28,15,2,2),(8,41,2,2,2)
\end{array}\right\},$$

$\Gamma\left(P_{\infty}, P_{0b_2}, P_{0b_3}, P_{0b_4}, P_{0b_5}, P_{0b_6}\right) =$

$$\left\{\begin{array}{l}
(43,1,1,1,1,1),(30,1,1,1,1,14),(30,1,1,1,14,1),(30,1,1,14,1,1),\\
(30,1,14,1,1,1),(30,14,1,1,1,1),(34,2,2,2,2,2),(21,2,2,2,2,15),\\
(21,2,2,2,15,2),(21,2,2,15,2,2),(21,2,15,2,2,2),(21,15,2,2,2,2),\\
(25,3,3,3,3,3),(12,3,3,3,3,16),(12,3,3,3,16,3),(12,3,3,16,3,3),\\
(12,3,16,3,3,3),(12,16,3,3,3,3),(16,4,4,4,4,4),(3,4,4,4,4,17),\\
(3,4,4,4,17,4),(3,4,4,17,4,4),(3,4,17,4,4,4),(3,17,4,4,4,4),\\
(7,5,5,5,5,5),(17,1,1,1,1,27),(17,1,1,1,14,14),(17,1,1,1,27,1),\\
(17,1,1,14,1,14),(17,1,1,14,14,1),(17,1,1,27,1,1),(17,1,14,1,1,14),\\
(17,1,14,1,14,1),(17,1,14,14,1,1),(17,1,27,1,1,1),(17,14,1,1,1,14),\\
(17,14,1,1,14,1),(17,14,1,14,1,1),(17,14,14,1,1,1),(17,27,1,1,1,1),\\
(4,1,1,1,1,40),(4,1,1,1,14,27),(4,1,1,1,27,14),(4,1,1,1,40,1),\\
(4,1,1,14,1,27),(4,1,1,14,14,14),(4,1,1,14,27,1),(4,1,1,27,1,14),\\
(4,1,1,27,14,1),(4,1,1,40,1,1),(4,1,14,1,1,27),(4,1,14,1,14,14),\\
(4,1,14,1,27,1),(4,1,14,14,1,14),(4,1,14,14,14,1),(4,1,14,27,1,1),\\
(4,1,27,1,1,14),(4,1,27,1,14,1),(4,1,27,14,1,1),(4,1,40,1,1,1),\\
(4,14,1,1,1,27),(4,14,1,1,14,14),(4,14,1,1,27,1),(4,14,1,14,1,14),\\
(4,14,1,14,14,1),(4,14,1,27,1,1),(4,14,14,1,1,14),(4,14,14,1,14,1),\\
(4,14,14,14,1,1),(4,14,27,1,1,1),(4,27,1,1,1,14),(4,27,1,1,14,1),\\
(4,27,1,14,1,1),(4,27,14,1,1,1),(4,40,1,1,1,1),(8,2,2,2,2,28),\\
(8,2,2,2,15,15),(8,2,2,2,28,2),(8,2,2,15,2,15),(8,2,2,15,15,2),\\
(8,2,2,28,2,2),(8,2,15,2,2,15),(8,2,15,2,15,2),(8,2,15,15,2,2),\\
(8,2,28,2,2,2),(8,15,2,2,2,15),(8,15,2,2,15,2),(8,15,2,15,2,2),\\
(8,15,15,2,2,2),(8,28,2,2,2,2)
\end{array}\right\},$$

$$\Gamma\left(P_\infty, P_{0b_2}, P_{0b_3}, P_{0b_4}, P_{0b_5}, P_{0b_6}, P_{0b_7}\right) =$$

$$\left\{\begin{array}{l}
(30,1,1,1,1,1,1),(21,2,2,2,2,2,2),(12,3,3,3,3,3,3),\\
(3,4,4,4,4,4,4),(17,1,1,1,1,1,14),(17,1,1,1,1,14,1),\\
(17,1,1,1,14,1,1),(17,1,1,14,1,1,1),(17,1,14,1,1,1,1),\\
(17,14,1,1,1,1,1),(4,1,1,1,1,1,27),(4,1,1,1,1,14,14),\\
(4,1,1,1,1,27,1),(4,1,1,1,14,1,14),(4,1,1,1,14,14,1),\\
(4,1,1,1,27,1,1),(4,1,1,14,1,1,14),(4,1,1,14,1,14,1),\\
(4,1,1,14,14,1,1),(4,1,1,27,1,1,1),(4,1,14,1,1,1,14),\\
(4,1,14,1,1,14,1),(4,1,14,1,14,1,1),(4,1,14,14,1,1,1),\\
(4,1,27,1,1,1,1),(4,14,1,1,1,1,14),(4,14,1,1,1,14,1),\\
(4,14,1,1,14,1,1),(4,14,1,14,1,1,1),(4,14,14,1,1,1,1),\\
(4,27,1,1,1,1,1),(8,2,2,2,2,2,15),(8,2,2,2,2,15,2),\\
(8,2,2,2,15,2,2),(8,2,2,15,2,2,2),(8,2,15,2,2,2,2),\\
(8,15,2,2,2,2,2)
\end{array}\right\},$$

$$\Gamma\left(P_\infty, P_{0b_2}, P_{0b_3}, P_{0b_4}, P_{0b_5}, P_{0b_6}, P_{0b_7}, P_{0b_8}\right) =$$

$$\left\{\begin{array}{l}
(17,1,1,1,1,1,1,1),(4,1,1,1,1,1,1,14),(4,1,1,1,1,1,14,1),\\
(4,1,1,1,1,14,1,1),(4,1,1,1,14,1,1,1),(4,1,1,14,1,1,1,1),\\
(4,1,14,1,1,1,1,1),(4,14,1,1,1,1,1,1),(8,2,2,2,2,2,2,2)
\end{array}\right\},$$

$$\Gamma\left(P_\infty, P_{0b_2}, P_{0b_3}, P_{0b_4}, P_{0b_5}, P_{0b_6}, P_{0b_7}, P_{0b_8}, P_{0b_9}\right) =$$

$$\left\{\ (4,1,1,1,1,1,1,1,1)\ \right\},$$

and

$$\Gamma\left(P_\infty, P_{0b_2}, P_{0b_3}, P_{0b_4}, P_{0b_5}, P_{0b_6}, P_{0b_7}, P_{0b_8}, P_{0b_9}, P_{0b_{10}}\right) = \emptyset.$$

References

[G] O. Geil, On codes from norm-trace curves. Finite Fields Appl. **9** (2003), no. 3, 351–371.

[HK] M. Homma and S. J. Kim, Goppa codes with Weierstrass pairs. J. Pure Appl. Algebra **162** (2001), no. 2-3, 273–290.

[K] S. J. Kim, On the index of the Weierstrass semigroup of a pair of points on a curve, Arch. Math. **62** (1994), 73–82.

[MMP] H. Maharaj, G. L. Matthews, and G. Pirsic, Riemann-Roch spaces of the Hermitian function field with applications to algebraic geometry codes and low-discrepancy sequences, J. Pure Appl. Algebra. **195** (2005), 261–280.

[M01] G. L. Matthews, Weierstrass pairs and minimum distance of Goppa codes, Des. Codes and Cryptog. **22** (2001), 107–121.

[M04] G. L. Matthews, The Weierstrass semigroup of an m-tuple of collinear points on a Hermitian curve. Finite fields and applications, 12–24, Lecture Notes in Comput. Sci., **2948**, Springer, Berlin, 2004.

[M09] G. L. Matthews, On Weierstrass semigroups of some triples on norm-trace curves, Lecture Notes in Comput. Sci. **5557** (2009), 146-156.

[MTT] C. Munuera, G. C. Tizziotti, and F. Torres, Two-point codes on norm-trace curves, Coding Theory and Applications, Second International Castle Meeting, ICMCTA 2008 (A. Barbero Ed.), 128–136, Lecture Notes in Comput. Sci. **5228**, Springer-Verlag Berlin Heidelberg 2008.

[S93] H. Stichtenoth, Algebraic Function Fields and Codes, Springer-Verlag, 1993.

[S73] H. Stichtenoth, Ober die automorphismengruppe eines algebraischen funktionenkorpers von primzahlcharackteristik, teil 2, Arch. Math. **24** (1973), 615–631.

[S88] H. Stichtenoth, A note on Hermitian codes over $GF\left(q^2\right)$, IEEE Trans. Inform. Theory **34** (1998), no. 5, 1345–1348.

Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975
E-mail address: gmatthe@clemson.edu

Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975
E-mail address: jpeache@clemson.edu