

EXPLICIT NON-SPECIAL DIVISORS OF SMALL DEGREE AND LCD CODES FROM KUMMER EXTENSIONS

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ABSTRACT. A linear code is linear complementary dual, or LCD, if and only if the intersection between the code and its dual is trivial. Introduced by Massey in 1992, LCD codes have attracted recent attention due to their application. In this vein, Mesnager, Tang, and Qi considered complementary dual algebraic geometric codes, giving several examples from low genus curves as well as instances using places of higher degree from Hermitian curves. In this paper, we consider the hull of an algebraic geometry code, meaning the intersection of the code and its dual. We demonstrate how LCD codes from Kummer extensions (and Hermitian curves in particular) may be defined using only rational points. Our primary tool is the explicit construction of non-special divisors of degrees g and $g - 1$ on certain families of curves with many rational points. As a result, we provide algebraic geometric LCD codes from some maximal curves (such as the Hermitian code and a quotient) along with other curves (such as the norm-trace curve) supported by only rational points by appealing to Weierstrass semigroups.

1. INTRODUCTION

The hull of a linear code C is $Hull(C) := C \cap C^\perp$. Hulls were first studied formally in the early 1990s by Assmus and Key [1], though their use may be tied to earlier notions. For instance, a code is self-orthogonal if and only if $Hull(C) = C$ and self-dual if and only if $Hull(C) = C^\perp$. The dimensions of hulls govern the complexity of some algorithms of interest in cryptography [3], [15] as well as properties of some entanglement-assisted quantum error-correcting codes [8]. In this paper, we see that hulls of certain algebraic geometry codes are again algebraic geometry codes and use this as a launch point for studying codes which are linearly complementary dual, or LCD, meaning their hulls are trivial. The study of LCD codes was initiated by Massey [10] in 1992. This has attracted recent attention due to Carlet and Guilley's work on using LCD codes to protect against side-channel attacks and fault injection attacks [4]. They note that LCD codes with large rate and minimum distance improve the resilience against both types of attacks, prompting researchers to seek constructions for such codes. LCD codes have a number of applications ranging from communications to data storage.

In [12], rational points on elliptic curves and hyperelliptic curves as well as non-rational points on Hermitian curves are used to define algebraic geometry codes which are LCD. Recently, Jin and Xing [9] proved that there are asymptotically good algebraic geometry codes which are LCD. Around the same time, Carlet, Mesnager, Tang, Qi, and Pellikaan [5]

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showed that every linear code over a finite field \mathbb{F}_q with $q > 3$ elements is equivalent to one which is LCD. This completely settles the question of what parameters are obtainable for LCD codes over \mathbb{F}_q , $q > 3$. Even so, explicit constructions of LCD algebraic geometric codes over curves of higher genus using rational points remain elusive. Such constructions are necessary to obtain longer LCD codes, as the length n of an algebraic geometry code constructed from a curve of genus g over \mathbb{F}_q is bounded above according to the Hasse-Weil bound

$$n \leq q + 1 + 2g\sqrt{q}.$$

In this paper, we construct LCD codes using only rational points on certain maximal curves, including the Hermitian curve. More generally, we consider Kummer extensions given by

$$(1) \quad \prod_{i=1}^r (y - \alpha_i) = x^m$$

over \mathbb{F}_q with $(r, m) = 1$. They allow us to consider several families of curves of particular interest in coding theory, including those given by

$$X : y^{q^{r-1}} + y^{q^{r-2}} + \cdots + y = x^u$$

over \mathbb{F}_{q^r} where $u \mid \frac{q^r-1}{q-1}$. Hermitian curves, which are maximal over \mathbb{F}_{q^2} are seen by taking $r = 2$ and $u = \frac{q^r-1}{q-1}$. The first examples of non-classical curves are obtained when $r = 2$ [14] and are also maximal. The norm-trace curves, obtained by setting $u = \frac{q^r-1}{q-1}$, meet the Geil-Matsumoto bound on the number of rational points over \mathbb{F}_{q^r} .

The codes considered are multipoint codes by necessity, as we will see in Section 2; this section also provides the background on Weierstrass semigroups which will be used to address the problem. Our primary tool is the explicit construction of non-special divisors of small degree, as detailed in Section 3. This builds on the work of Ballet and Le Brigand [2] where the existence of non-special divisors of degree $g - 1$ (resp., of degree g) is proven for curves of genus $g \geq 2$ over a field of size $q \geq 2$ (resp., $q \geq 3$) and they note that explicit constructions of such divisors is a challenging task. Here, we provide explicit constructions for such divisors on the Kummer extensions given in (1). In Section 4, we use these divisors to construct LCD codes. Section 5 contains a conclusion.

2. PRELIMINARIES

We begin this section by setting up the notation and terminology to be used throughout the paper. For more details on algebraic geometry codes, we refer the reader to the standard references [16], [18].

Let X be a nonsingular, projective curve of genus g over a finite field \mathbb{F} with algebraic closure $\overline{\mathbb{F}}$. The field of rational functions on X is denoted $\mathbb{F}(X)$, the set of K -rational points on X is written $X(K)$ for an intermediate field K of the extension $\overline{\mathbb{F}}/\mathbb{F}$, and the vector space of Weil differentials is given by Ω_X . Given a divisor A , we may write $A = \sum_{P \in X(\overline{\mathbb{F}})} a_P P$ with $a_P \in \mathbb{Z}$; in this case, we say $v_P(A) := a_P$ and $\deg A = \sum_{P \in X(\overline{\mathbb{F}})} v_P(A)$. There is a partial order on $Div(X)$, the set of divisors on X , given by $A \leq B$ if and only if $v_P(A) \leq v_P(B)$ for all $P \in X(\overline{\mathbb{F}})$. The divisor of a function $f \in \mathbb{F}(X) \setminus \{0\}$ is written $(f) = \sum_{P \in X(\overline{\mathbb{F}})} a_P P$ where P is a zero (resp., pole) of multiplicity a_P (resp., $-a_P$) provided $a_P > 0$ (resp., $a_P < 0$).

The pole divisor of $f \in \mathbb{F}(X) \setminus \{0\}$ is $(f)_\infty = \sum_{\substack{P \in X(\mathbb{F}) \\ v_P(f) < 0}} a_P P$. A divisor A on X defines

a space of functions

$$\mathcal{L}(A) := \{f \in \mathbb{F}(X) : (f) \geq -A\} \cup \{0\}$$

on X as well as a space of differentials

$$\Omega(A) := \{\omega \in \Omega_X : (\omega) \geq A\} \cup \{0\}$$

where $(\omega) \in \text{Div}(X)$ denotes the divisor associated with the differential ω . The dimension of $\mathcal{L}(A)$ is given by $\ell(A)$ and satisfies the Riemann-Roch Theorem, meaning

$$\ell(A) = \deg A + 1 - g + \ell(K - A)$$

where K is a canonical divisor. Moreover, if $\deg A \geq 2g - 1$, then

$$(2) \quad \ell(A) = \deg A + 1 - g.$$

We will use the fact that

$$A \leq B \Rightarrow \mathcal{L}(A) \subseteq \mathcal{L}(B).$$

The support of the divisor A is $\text{Supp} A := \{P \in X(\mathbb{F}) : v_P(A) \neq 0\}$, and we say that A is supported by $P \in X(\mathbb{F})$ if and only if $v_P(A) \neq 0$. A divisor A is linearly equivalent to B , denoted $A \sim B$, if and only if there exists $f \in \mathbb{F}(X)$ such that $A - B = (f)$. In the setting where we consider a curve $X : f(y) = g(x)$, we use P_{ab} to denote a point on X corresponding to $x = a$ and $y = b$. If X has a unique point at infinity, we note it by P_∞ .

We use the standard notation from coding theory. Because this paper only considers linear codes, we use the term code to mean linear code. An $[n, k, d]$ code over a finite field \mathbb{F} is a code of length n , dimension k , and minimum distance d (taken with respect to the Hamming metric). The set of indices of codewords of a code of length n is $[n] := \{1, \dots, n\}$. Given $v \in \mathbb{F}^n$, we denote its i^{th} component by v_i where $i \in [n]$. The dual of an $[n, k, d]$ code C is

$$C^\perp := \{w \in \mathbb{F}^n : w \cdot c = 0 \ \forall c \in C\};$$

that is, the dual is taken with respect to the usual (or Euclidean) inner product. The set of $m \times n$ matrices over a field \mathbb{F} is denoted $\mathbb{F}^{m \times n}$, $\text{Row}_i M$ (resp., $\text{Col}_j M$) denotes the i^{th} row of $M \in \mathbb{F}^{m \times n}$ (resp., the j^{th} column of M). We write \mathbb{N} to mean the set of nonnegative integers and \mathbb{Z}^+ for the set of positive integers. At times, we make use of the partial order \leq on \mathbb{N}^m given by $v \leq w$ if and only if $v_i \leq w_i$ for all $i \in [m]$; we write $v \not\leq w$ to mean there exists $i \in [m]$ with $v_i > w_i$.

Suppose G and $D = Q_1 + \dots + Q_n$ are divisors on X defined over \mathbb{F} , where Q_1, \dots, Q_n are distinct \mathbb{F} -rational points on X , each not belonging to the support of G . The algebraic geometry code $C(D, G)$ is defined by

$$C(D, G) = \{ev(f) : f \in \mathcal{L}(G)\}$$

where

$$ev(f) := (f(Q_1), f(Q_2), \dots, f(Q_n)).$$

Certainly, n is length of the code. For convenience, we suppose $n > \deg G$. Then, by the Riemann-Roch Theorem, the code $C(D, G)$ has dimension $k = \ell(G)$ and minimum Hamming distance $d \geq n - \deg G$. If the support of G consists of m points, then $C(D, G)$ is called an m -point code and said to be a *multipoint code* provided $m \geq 2$.

The dual of $C(D, G)^\perp$ is

$$C(D, G)^\perp = \{(\omega_{Q_1}(1), \dots, \omega_{Q_n}(1)) : \omega \in \Omega(G - D)\}.$$

It is well known that if there is a differential $\eta \in \Omega_X$ which has simple poles at the points in the support of D , then

$$(3) \quad C(D, G)^\perp = \{(\eta_{Q_1}(1)f(Q_1), \eta_{Q_2}(1)f(Q_2), \dots, \eta_{Q_n}(1)f(Q_n)) : f \in \mathcal{L}(D - G + (\eta))\},$$

meaning that the dual of the algebraic geometry code is a generalized algebraic geometry code. This leads to the following result (cf. [16, Proposition 2.2.10]).

Lemma 1. *Consider the algebraic geometry code $C(D, G)$ where $D = Q_1 + \dots + Q_n$ as above.*

(1) *If there exists $\eta \in \Omega_X$ such that $v_{Q_i}(\eta) \geq -1$ and $\eta_{Q_i}(1) = 1$ for all $i \in [n]$, then*

$$C(D, G)^\perp = C(D, D - G + (\eta)).$$

(2) *If, in addition, $G = \sum_{i=1}^m a_i P_i$ so that $C(D, G)$ is an m -point code and $\text{Supp}(\eta) \subseteq \text{supp}G \cup \text{Supp}D$, then $C(D, G)^\perp = C(D, \sum_{i=1}^m a'_i P_i)$ for some $a'_i \in \mathbb{Z}$, meaning that the dual is also an m -point code.*

Consider a one-point code $C(D, \alpha P)$ on a curve X satisfying the hypotheses of Lemma 1(2). Then the dual of the one-point code $C(D, \alpha P)$ on X is also a one-point code, say $C(D, \alpha' P)$. In this case,

$$(4) \quad \begin{aligned} C(D, \alpha P) \cap C(D, \alpha P)^\perp &= C(D, \alpha P) \cap C(D, \alpha' P) \\ &= C(D, \min\{\alpha, \alpha'\} P) \end{aligned}$$

which is trivial if and only if $\alpha < 0$ or $\alpha' < 0$. Thus, there are no nontrivial one-point codes on X which are LCD. Many maximal curves, including the Hermitian curve, satisfy the conditions of Lemma 1(2). Hence, considering nontrivial algebraic geometry codes from such curves that have the potential to be LCD necessarily requires looking at multipoint codes.

To define the desired LCD codes, we turn our attention to the explicit construction of non-special divisors. Recall that the index of speciality of a divisor A is

$$i(A) := \ell(A) - (\deg(A) + 1 - g).$$

A divisor A is called non-special if and only if $i(A) = 0$, meaning $\ell(A) = \deg(A) + 1 - g$, and special otherwise. Mesnager, Tang, and Qi [12] used non-special divisors to construct LCD codes. Based on their contribution and motivated by (4), we define two useful divisors. Given divisors G and H on a projective curve X over a finite field \mathbb{F} , define their greatest common divisor as

$$\gcd(G, H) := \sum_{P \in X(\mathbb{F})} \min\{v_P(G), v_P(H)\} P$$

and their least multiple divisor as

$$\text{lmd}(G, H) := \sum_{P \in X(\mathbb{F})} \max\{v_P(G), v_P(H)\} P.$$

It is immediate that

$$\begin{aligned} \mathcal{L}(G) \cap \mathcal{L}(H) &= \mathcal{L}(\gcd(G, H)), \\ f \in \mathcal{L}(G) \text{ and } h \in \mathcal{L}(H) &\Rightarrow f - h \in \mathcal{L}(\text{lmd}(G, H)), \end{aligned}$$

and

$$G + H = \gcd(G, H) + \text{lmd}(G, H).$$

Using these ideas, we prove the next result which is strongly inspired by [12, Theorem 4]. We will later combine it with other tools to produce LCD codes using only rational points on the Hermitian curve and some of its relatives.

Theorem 2. *Let $G, H,$ and $D = \sum_{i=1}^n Q_i$ be divisors on a curve X such that $Q_i \neq Q_j$ for $i \neq j$ and G and H has disjoint support from D . Assume that $G + H - D$ is a canonical divisor such that*

$$C(D, G)^\perp = C(D, H).$$

If $2g - 2 < \deg G < n$ and $\gcd(G, H)$ is non-special, then

$$\text{Hull}(C(D, G)) = C(D, \gcd(G, H))$$

Proof. It is clear that $C(D, \gcd(G, H)) = \text{ev}(\mathcal{L}(G) \cap \mathcal{L}(H)) \subseteq C(D, G) \cap C(D, H)$. On the other hand, if $c \in \text{Hull}(C(D, G))$, then there exists $f \in \mathcal{L}(G)$ and $h \in \mathcal{L}(H)$ such that

$$\text{ev}(f) = \text{ev}(h) = c$$

and then $f - h \in \mathcal{L}(\text{lmd}(G, H) - D)$. Since $G + H - D$ is canonical, $\ell(\text{lmd}(G, H) - D) = \ell(\gcd(G, H)) = 0$ as $\gcd(G, H)$ is non-special. Then $f = h$ and $f \in \mathcal{L}(\gcd(G, H))$. \square

Corollary 3. *Let G, H, D as in the previous theorem.*

- (1) *The dimension of the Hull of $C(D, G)$ is $\ell(\gcd(G, H))$.*
- (2) *If $\deg \gcd(G, H) = g - 1$, then $C(D, G)$ is LCD.*
- (3) *If $G \leq H$, then $C(D, G)$ is self-orthogonal. If the equality holds, then the code is self-dual.*

In light of Theorem 3, we set out to find explicit non-special divisors of degree $g - 1$. A primary tool in our approach is the Weierstrass semigroup. Given m distinct \mathbb{F} -rational points P_1, \dots, P_m on a curve X , the Weierstrass semigroup of the m -tuple (P_1, \dots, P_m) is

$$H(P_1, \dots, P_m) = \left\{ \alpha \in \mathbb{N}^m : \exists f \in \mathbb{F}(X) \text{ with } (f)_\infty = \sum_{i=1}^m \alpha_i P_i \right\}.$$

Equivalently, $\alpha \in H(P_1, \dots, P_m)$ if and only if

$$\ell \left(\sum_{i=1}^m \alpha_i P_i \right) = \ell \left(\sum_{i=1}^m \alpha_i P_i - P_j \right) + 1$$

for all $j \in [m]$. For short, if the m -tuple of points (P_1, \dots, P_m) is clear from the context, we sometimes write H_m to mean $H(P_1, \dots, P_m)$. The Weierstrass gap set of a t -tuple of points (P_1, \dots, P_m) is

$$G(P_1, \dots, P_m) := \mathbb{N}^m \setminus H(P_1, \dots, P_m).$$

Notice that $\alpha \in H(P_1, \dots, P_m) \setminus \{0\}$ implies $\ell(\sum_{i=1}^m \alpha_i P_i) \geq 2$. Indeed, there exists $j \in [m]$ with $\alpha_j > 0$, and

$$\ell \left(\sum_{i=1}^m \alpha_i P_i \right) > \ell \left(\sum_{i=1}^m \alpha_i P_i - P_j \right) \geq \ell(0) = 1$$

since $\sum_{i=1}^m \alpha_i P_i - P_j \geq 0$. The multiplicity of the semigroup $H(P)$, where $P \in X(\mathbb{F})$, is

$$\gamma(H(P)) := \min \{a : a \in H(P) \setminus \{0\}\}.$$

In the next section, we use features of these semigroups to define non-special divisors of least possible degree.

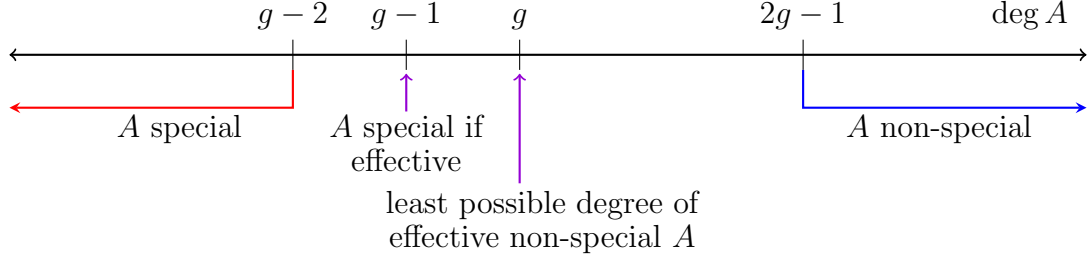


FIGURE 1. Impact of $\deg A$ on the index of speciality of a divisor A on a curve X of genus g

3. EXPLICIT NON-SPECIAL DIVISORS OF SMALL DEGREE

Recall that for a divisor $A \in \text{Div}(X)$ with $\deg A \geq 2g - 1$, $i(A) = 0$ according to (2). Hence, divisors of degree at least $2g - 1$ are non-special. On the other hand, if $\deg A < g - 1$, then $\deg A + 1 - g < 0$ and so $\ell(A) \neq \deg A + 1 - g$. Thus, divisors of degree less than $g - 1$ are special. Consequently, the least possible degree of a non-special divisor is $g - 1$. It is also worth noting that if $A \in \text{Div}(X)$ has degree $\deg A = g - 1$, then A non-special implies A is not effective. To see this, notice that if $A \geq 0$, then $\mathbb{F} \subseteq \mathcal{L}(A)$; hence, $\ell(A) \geq 1 \neq 0 = \deg(A) + 1 - g$. We conclude that the least possible degree of an effective non-special divisor is g . This is captured in Figure 1. Furthermore, if A is non-special and $A \leq B$, then B is also non-special. This follows from the fact that if $\ell(A) = \deg A + 1 - g$, then $\ell(K - A) = 0$ for a canonical divisor K which implies $\ell(K - B) = 0$ and $\ell(B) = \deg B + 1 - g$.

Ballet and Le Brigand [2] prove the existence of non-special divisors of degree $g - 1$ (resp., of degree g) for curves of genus $g \geq 2$ over fields of size $q \geq 2$ (resp., $q \geq 3$). In particular, they show the following.

Lemma 4. *Suppose X is a curve of genus $g \geq 1$ over a field \mathbb{F} with $|\mathbb{F}| \geq 4$.*

- (1) *If $|X(\mathbb{F})| \geq g$, then there exists a non-special divisor $A \geq 0$ with $\deg A = g$ and $\text{Supp} A \subseteq X(\mathbb{F})$.*
- (2) *If $|X(\mathbb{F})| \geq g + 1$, then there exists a non-special divisor A with $\deg A = g - 1$ and $\text{Supp} A \subseteq X(\mathbb{F})$.*

In this section, we give explicit constructions of non-special divisors of degrees $g - 1$ and g for certain families of curves. We first focus our attention on those of degree g . To determine a non-special divisor A of degree g , we seek A with $\ell(A) = 1$, meaning $\mathcal{L}(A) = \mathbb{F}$.

Notice that if A and B are effective divisors with $A \leq B$ and $v_P(A) \in H(P) \setminus \{0\}$, then $\ell(B) \geq \ell(A) \geq 2$. Since we seek $A = \sum_{i=1}^m \alpha_i P_i$ with $\ell(A) = 1$, we may restrict our search to divisors

$$A = \sum_{i=1}^m \alpha_i P_i \text{ with } \alpha_i < \gamma(H(P_i)) \quad \forall i \in [m].$$

This alone, however, is not enough to guarantee $\ell(A) = 1$ as the following example demonstrates.

Example Consider the Hermitian curve X given by $y^q + y = x^{q+1}$ over \mathbb{F}_{q^2} . It is well-known that $H(P) = \langle q, q + 1 \rangle$ for all \mathbb{F}_{q^2} -rational points P on X , so $\gamma(H(P)) = q$ for all

$P \in X(\mathbb{F}_{q^2})$. Furthermore,

$$(x) = \sum_{\substack{b \in \mathbb{F}_{q^2} \\ b^q + b = 0}} P_{0b} - qP_\infty,$$

and

$$(y) = (q+1)P_{00} - (q+1)P_\infty.$$

Consequently,

$$\left(\frac{x^2}{y}\right) = 2 \sum_{\substack{b \in \mathbb{F}_{q^2}^* \\ b^q + b = 0}} P_{0b} - (q-1)P_{00} - (q-1)P_\infty.$$

It follows that $\frac{x^2}{y} \in \mathcal{L}((q-1)P_{00} + (q-1)P_\infty)$ and $\ell((q-1)P_{00} + (q-1)P_\infty) \geq 2$ despite the fact that

$$v_P((q-1)P_{00} + (q-1)P_\infty) = q-1 < \gamma(H(P))$$

for $P \in \{P_{00}, P_\infty\}$.

As we will see, it is necessary to consider divisors $A = \sum_{i=1}^m \alpha_i P_i$ where

$$\beta \in G(P_1, \dots, P_m) \forall (\beta_1, \dots, \beta_m) \leq (\alpha_1, \dots, \alpha_m).$$

This naturally leads one to the minimal generating set of the Weierstrass semigroup. For $i \in [m]$, let $\Gamma^+(P_i) = H(P_i)$, and for $\ell \geq 2$, let

$$\Gamma^+(P_{i_1}, \dots, P_{i_\ell}) = \left\{ v \in \mathbb{Z}^{+\ell} : v \text{ is minimal in } \{w \in H(P_{i_1}, \dots, P_{i_\ell}) : v_i = w_i\} \text{ for some } i \in [\ell] \right\}.$$

The *least upper bound* of $v^{(1)}, \dots, v^{(\ell)} \in \mathbb{N}^n$ is given by

$$\text{lub}(v^{(1)}, \dots, v^{(\ell)}) = (\max\{v_1^{(1)}, \dots, v_1^{(\ell)}\}, \dots, \max\{v_n^{(1)}, \dots, v_n^{(\ell)}\}).$$

For each $I \subseteq [m]$ let ι_I denote the natural inclusion $\mathbb{N}^\ell \rightarrow \mathbb{N}^m$ into the coordinates indexed by I . The *minimal generating set* of $H(P_1, \dots, P_m)$ is

$$\Gamma(P_1, \dots, P_m) = \bigcup_{\ell=1}^m \bigcup_{\substack{I=\{i_1, \dots, i_\ell\} \\ i_1 < \dots < i_\ell}} \iota_I(\Gamma^+(P_{i_1}, \dots, P_{i_\ell})).$$

The Weierstrass semigroup $H(P_1, \dots, P_m)$ is completely determined by the minimal generating set $\Gamma(P_1, \dots, P_m)$ [11, Theorem 7]: If $1 \leq m < |\mathbb{F}|$, then

$$(5) \quad H(P_1, \dots, P_m) = \{\text{lub}\{\mathbf{v}_1, \dots, \mathbf{v}_m\} : \mathbf{v}_1, \dots, \mathbf{v}_m \in \Gamma(P_1, \dots, P_m)\}.$$

The next observation relates the minimal generating set of the semigroup of the m -tuple (P_1, \dots, P_m) to non-special effective divisors of degree g .

Proposition 5. *Consider an effective divisor $A = \sum_{i=1}^m \alpha_i P_i \in \text{Div}(X)$ of degree g . If $\gamma \not\leq \alpha$ for all $\gamma \in \Gamma(P_1, \dots, P_m)$, then A is non-special.*

Proof. Let $A = \sum_{i=1}^m \alpha_i P_i \in \text{Div}(X)$ where $\alpha_i \geq 0$ for all $i, 1 \leq i \leq m$ and $\sum_{i=1}^m \alpha_i = g$. We prove the contrapositive of the statement. Suppose A is special. Then $\ell(A) > \deg A + 1 - g = 1$. Since $\ell(A) \geq 2$, there exists $w \in H(P_1, \dots, P_m)$ such that $w \leq \alpha$. According to (5), there exist $\mathbf{v}_1, \dots, \mathbf{v}_m \in \Gamma(P_1, \dots, P_m)$ satisfying $\mathbf{v}_1 \leq \text{lub}\{\mathbf{v}_1, \dots, \mathbf{v}_m\} = w \leq \alpha$, completing the proof. \square

We are interested in Kummer extensions defined by

$$\prod_{i=1}^r (y - \alpha_i) = x^m$$

where $\alpha_i \in \mathbb{F}_q$ and $(r, m) = 1$, which is a function field of genus $\frac{(m-1)(r-1)}{2}$. They allow us to consider several families of curves of particular interest in coding theory, including those given by

$$X : y^{q^{r-1}} + y^{q^{r-2}} + \cdots + y = x^u$$

over \mathbb{F}_{q^r} where $u \mid \frac{q^r-1}{q-1}$. This family contains some maximal curves, such as the Hermitian curves, seen by taking $r = 2$ and $u = \frac{q^r-1}{q-1}$, as well as their quotients which are given in the case $r = 2$. The norm-trace curves, obtained by setting $u = \frac{q^r-1}{q-1}$, are not maximal unless $r = 2$ but do meet the Geil-Matsumoto bound.

To prove our result, we need an explicit description of $\Gamma^+(P_1, \dots, P_l)$ where $P_i \in \text{Supp}((x)) \setminus P_\infty$. We note that analogous results for the Hermitian and norm-trace curves were obtained previously [11], [13].

Proposition 6 ([6, Theorem 3.2], [17, Theorem 10]). *Let $F/\mathbb{F}_q(y)$ be the Kummer extension defined by*

$$x^m = \prod_{i=1}^r (y - \alpha_i)$$

as above, and let P_i the the place associated with $x - \alpha_i$. Then

$$\Gamma^+(P_1) = \mathbb{N}_0 \setminus \left\{ mk + j \mid 1 \leq j \leq m - 1 - \left\lfloor \frac{m}{r} \right\rfloor, 0 \leq k \leq r - 2 - \left\lfloor \frac{rj}{m} \right\rfloor \right\}$$

and for $2 \leq l \leq r - \lfloor \frac{r}{m} \rfloor$, $\Gamma^+(P_1, \dots, P_l)$ is given by

$$\left\{ (ms_1 + j, \dots, ms_l + j) \mid 1 \leq j \leq m - 1 - \left\lfloor \frac{m}{r} \right\rfloor, s_i \geq 0, \sum_{i=1}^l s_i = r - l - \left\lfloor \frac{rj}{m} \right\rfloor \right\}$$

Before applying this proposition to the construction of non-special divisors, we state facts that will be needed to determine their degrees.

Lemma 7. *Let $r, m \in \mathbb{Z}^+$ be relatively prime.*

(1) *Let $1 \leq j \leq m - 1$ and set $t = r \bmod m$. Then*

$$\left\lfloor \frac{r(j+1)}{m} \right\rfloor - \left\lfloor \frac{rj}{m} \right\rfloor = \begin{cases} \left\lfloor \frac{r}{m} \right\rfloor + 1 & j = \left\lfloor \frac{km}{t} \right\rfloor, 1 \leq k \leq t - 1 \\ \left\lfloor \frac{r}{m} \right\rfloor & \text{otherwise.} \end{cases}$$

(2) *If $t < m$, then*

$$\sum_{k=1}^{t-1} \left\lfloor \frac{km}{t} \right\rfloor = \frac{(m-1)(t-1)}{2}.$$

Proof. (1) First, observe that

$$\left\lfloor \frac{r(j+1)}{m} \right\rfloor - \left\lfloor \frac{rj}{m} \right\rfloor = \left\lfloor \frac{r}{m} \right\rfloor + \left\lfloor \frac{t(j+1)}{m} \right\rfloor - \left\lfloor \frac{tj}{m} \right\rfloor.$$

Let $\lfloor \frac{km}{t} \rfloor < j < j+1 \leq \lfloor \frac{(k+1)m}{t} \rfloor$ for some $1 \leq k \leq t-1$. Since $(r, m) = 1$, $(t, m) = 1$ and we can guarantee that

$$k \leq \frac{tj}{m} < \frac{t(j+1)}{m} < k+1.$$

Thus, $\lfloor \frac{tj}{m} \rfloor = \lfloor \frac{t(j+1)}{m} \rfloor$. On the other hand, if $j = \lfloor \frac{km}{t} \rfloor$ for some $1 \leq k \leq t-1$, then $j+1 = \lceil \frac{km}{t} \rceil$. Since $(t, m) = 1$,

$$j < \frac{km}{t} < j+1 \implies \frac{tj}{m} < k < \frac{t(j+1)}{m},$$

from which $\lfloor \frac{t(j+1)}{m} \rfloor - \lfloor \frac{tj}{m} \rfloor = 1$. Given the observation at the beginning of this proof, we can conclude the result.

(2) From Proposition 6, we know that for the Kummer extension given by $\prod_{i=1}^m (y - \alpha_i) = x^t$, the number of gaps of P_1 is

$$\begin{aligned} |\{tk + j \mid 1 \leq j \leq t-1, 0 \leq k \leq m-2 - \lfloor \frac{mj}{t} \rfloor\}| &= \sum_{j=1}^{t-1} (m-1 - \lfloor \frac{mj}{t} \rfloor) \\ &= \frac{(m-1)(t-1)}{2} \end{aligned}$$

from which the conclusion follows. \square

We are ready to provide explicit effective non-special divisors of degree g over some Kummer extensions.

Theorem 8. *Let $F/\mathbb{F}_q(y)$ be a Kummer extension given by*

$$\prod_{i=1}^r (y - \alpha_i) = x^m$$

where $\alpha_i \in \mathbb{F}_q$ and $(r, m) = 1$. For $1 \leq j \leq m-1 - \lfloor \frac{m}{r} \rfloor$, define the following values:

- $l_j = r - \lfloor \frac{rj}{m} \rfloor$.
- $s_j = l_j - l_{j+1}$ if $j < m-1 - \lfloor \frac{m}{r} \rfloor$ and $s_{m-1 - \lfloor \frac{m}{r} \rfloor} = l_{m-1 - \lfloor \frac{m}{r} \rfloor} - 1 = \max\{1, \lfloor \frac{r}{m} \rfloor\}$.

Then A is an effective non-special divisor of degree g with support contained in $\{P_{0b} : \prod_{i=1}^r (b - \alpha_i) = 0\}$ if and only if

$$A = \sum_{j=1}^{m-1 - \lfloor \frac{m}{r} \rfloor} j \sum_{i=1}^{s_j} P_{0b_{j_i}}.$$

In particular, if $r < m$,

$$A = \sum_{j=1}^{r-1} \left\lfloor \frac{jm}{r} \right\rfloor P_{0b_j}.$$

Proof. First, note that

$$A = \sum_{j=1}^{m-1 - \lfloor \frac{m}{r} \rfloor} j D_j = \sum_{j \in S} j \sum_{i=1}^{s_j} P_{0b_{j_i}}$$

where

$$D_j = \begin{cases} \sum_{i=1}^{s_j} P_{0b_{j_i}} & \text{if } s_j > 0 \\ 0 & \text{if } s_j = 0 \end{cases}$$

and $S := \sum_{j \in S} j \sum_{i=0}^{s_j} P_{0b_{j_i}}$.

We will prove that A is of degree g . Let $t = r \bmod m$, we have:

$$\begin{aligned} \deg A &= \sum_{i=1}^{m-1 - \lfloor \frac{m}{r} \rfloor} j s_j \\ &= \sum_{i=1}^{m-1} j \lfloor \frac{r}{m} \rfloor + \sum_{k=1}^{t-1} \lfloor \frac{km}{t} \rfloor \\ &= \frac{(m-1)m}{2} \lfloor \frac{r}{m} \rfloor + \frac{(m-1)(t-1)}{2} \\ &= \frac{(m-1)}{2} \left(m \lfloor \frac{r}{m} \rfloor + t - 1 \right) \\ &= \frac{(m-1)(r-1)}{2} = g \end{aligned}$$

where the second equality follows from Lemma 7 (1) and the third one from Lemma 7 (2). Therefore, A is effective of degree g . Now, we are going to prove that A is non special using Proposition 5.

Take $v \in \mathbb{N}^{l_1-1}$ such that $A = \sum_{i=1}^{l_1-1} v_i P_i$. Since $v_i \leq m - 1 - \lfloor \frac{m}{r} \rfloor$ for all i , then by Proposition 6 $v_i < w$ for any $w \in \Gamma^+(P_i)$, and so

$$\iota_{\{i\}}(w) \not\leq v$$

for any $w \in \Gamma^+(P_i)$. Take $w \in \Gamma^+(P_{i_j} \mid j \in I \subset [l_1 - 1])$. If for some i , $w_i > m - \lfloor \frac{m}{r} \rfloor$, then $w \not\leq v$, so assume $w = (k, \dots, k)$ for some $1 \leq k \leq m - 1 - \lfloor \frac{m}{r} \rfloor$. By Proposition 6, we know that $|I| = l_k$ and the number of entries of v greater or equal than k are $\sum_{i=k}^{m-1 - \lfloor \frac{m}{r} \rfloor} s_i = l_k - 1$, therefore $\iota_I(w) \not\leq v$ for any I of cardinality l_k . This concludes that $w \not\leq v$ for all $w \in \Gamma(\text{Supp } A)$ and then A is non-special.

Set $\gamma = m - 1 - \lfloor \frac{m}{r} \rfloor$ and choose B an effective non-special divisor of degree g supported on $\text{Supp}((x))$. If $v_P(B) \geq \gamma + 1$ for some P , then $\iota_i(\gamma + 1) \leq B$, contradicting the non-speciality of B .

Write $B = \sum_{j=1}^{\gamma} j D_j$ where D_j is zero or is the sum of distinct rational places of degree 1 and $\text{Supp}(D_j) \cap \text{Supp}(D_h) = \emptyset$ for $j \neq h$.

Observe that $|\text{Supp}(B)| \leq l_1 - 1 < r = |\text{Supp}((x)_0)|$. For D_γ we know

$$\deg D_\gamma \leq l_\gamma - 1 = s_\gamma$$

otherwise it would contradict the non-specialty of B . Take $D'_\gamma \geq D_\gamma$, $\text{Supp}(D'_\gamma) \subset \text{Supp}(D_\gamma) \cup (\text{Supp}((x)_0) \setminus \text{Supp}(B))$, such that

$$\deg D'_\gamma = |\text{Supp} D'_\gamma| = l_\gamma - 1.$$

Similarly, for $1 \leq h < \gamma$, we know

$$\sum_{j=h}^{\gamma} \deg D_j \leq \deg D_h + \sum_{j=h+1}^{\gamma} \deg D'_j \leq l_h - 1,$$

so take $D'_h \geq D_h$, $\text{Supp}(D_h) \subseteq \text{Supp}((x)_0) \setminus \text{Supp}(B + \sum_{j=h+1}^{\gamma} D'_j)$ such that

$$\sum_{j=h}^{\gamma} D'_h = \sum_{j=h}^{\gamma} |\text{Supp} D'_j| = l_h - 1.$$

From this construction, it is clear

$$\deg D'_h = s_h$$

and so

$$g = \deg B \leq \sum_{j=1}^{\gamma} j \deg D'_j = \sum_{j=1}^{\gamma} j s_j = g.$$

Then $D'_h = D_h$ for any $1 \leq h \leq \gamma$ and B has the desired form.

In the case $r < m$, by Lemma 7 we have $s_j = 1$ if $j = \lfloor \frac{km}{r} \rfloor$ and 0 otherwise and then $D_j = P_k$ or $D_j = 0$. \square

Corollary 9. *On the norm-trace curve given by $y^{q^{r-1}} + y^{q^{r-2}} + \cdots + y = x^{\frac{q^r-1}{q-1}}$ over \mathbb{F}_{q^r} , any effective non-special divisor of degree g supported by points P_{0b} is of the form*

$$\sum_{i=1, q \nmid i}^{\frac{q^r-1}{q-1}-2} iP_{0b_i}.$$

Proof. Given

$$\left\lfloor \frac{(r-1)u}{q^{r-1}} \right\rfloor = u - 2,$$

it is enough to prove that $\lfloor \frac{jm}{r} \rfloor$ cannot be divisible by q , since

$$u - 2 - |\{i \in [u-2] \mid q \mid i\}| = u - 2 - \frac{u-1}{q} + 1 = \frac{u-1}{q}(q-1) = q^{r-1} - 1$$

and then A should be $\sum_{j=1}^{q^{r-1}-1} \lfloor \frac{ju}{q^{r-1}} \rfloor P_j$, implying that A is non-special of degree g . By Theorem 8, any other divisor with these characteristics should be of this form.

Thus, we will prove $q \nmid \lfloor \frac{ju}{q^{r-1}} \rfloor$ for any $1 \leq j \leq q^{r-1} - 1$. If j is such that $\lfloor \frac{ju}{q^{r-1}} \rfloor = qk$ for some $1 \leq k \leq \frac{u-1}{q} - 1$, then

$$ju = q^r k + z = u(q-1)k + k + z.$$

This expression implies that $u \mid k + z$, but

$$k + z < \frac{u-2}{q} + q^{r-1} = \frac{uq-1}{q} < u.$$

Then no such k exists, and the conclusion follows. \square

Corollary 10. *On the Hermitian curve $y^q + y = x^{q+1}$ over \mathbb{F}_{q^2} , any effective non-special divisor of degree g with support contained in $\{P_{0b_i} : 1 \leq i \leq q\}$ is of the form $A = \sum_{i=1}^{q-1} iP_{0b_i}$.*

Remark Theorem 8 also gives effective non-special divisors of degree g for curves of the form $X : y^{q^{r-1}} + \cdots + y = x^u$ over \mathbb{F}_{q^r} where $u \mid \frac{q^r-1}{q-1}$.

Now that we have explicit constructions for non-special divisors of degree g , we make use of the following idea to obtain non-special divisors of degree $g - 1$. These divisors of degree $g - 1$ will support the construction of LCD codes in Section 4.

Lemma 11. [2, Lemma 3] *If A is a non-special divisor of degree g on a curve X and there exists $P \in X(\mathbb{F}) \setminus \text{Supp}A$, then $A - P$ is non-special.*

Combining Lemma 11 with Theorem 8 yields the following result.

Theorem 12. *Let F/\mathbb{F}_q be a Kummer extension defined by*

$$\prod_{i=1}^r (y - \alpha_i) = x^m$$

with $\alpha_i \in \mathbb{F}_q$ and $(r, m) = 1$. Then

$$A = \left(\sum_{j=1}^{m-1-\lfloor \frac{m}{r} \rfloor} j \sum_{i=1}^{s_j} P_{0b_{ji}} \right) - P$$

is a non-special divisor of degree $g - 1$ for all $P \in \{P_{ab} \mid a \neq 0 \text{ or } b \neq b_{ji}\} \cup \{P_\infty\}$. In particular, there exist non-special divisors of degree $g - 1$ supported on $\text{Supp}((x)_0) \cup \{P_{ab}\}$ for any $a \neq 0$.

Proof. Note that $A + P_{ab}$ is non-special of degree g by Theorem 8 and, by Lemma 11, we have A is non-special too. Given

$$|\text{Supp}(A)| = r - \left\lfloor \frac{r}{m} \right\rfloor - 1 \leq r - 1,$$

we can take $P \in \text{Supp}(x) \setminus \text{Supp}(A) \neq \emptyset$. □

4. LCD CODES FROM SOME KUMMER EXTENSIONS

In this section, we described LCD codes utilizing the Kummer extensions studied in the previous sections. In all cases, we consider a non-special divisor A of degree $g - 1$ and a divisor $B \geq A$ such that $A + B - D = (\eta)$ is canonical, with simple poles at D and $\text{res}_P(\eta) = 1$ for $P \in \text{Supp}(D)$. With this, we would have that any two divisors G, H of degree less than $\deg D$, such that $\gcd(G, H) = A$ and $\text{lcm}(G, H) = B$, satisfy

$$C(D, G)^\perp = C(D, H) \quad \text{and} \quad C(D, G) \cap C(D, H) = \{0\},$$

meaning that $C(D, G)$ is LCD.

Theorem 13. *Let $F/\mathbb{F}_{q^2}(y)$ be a maximal Kummer extension of genus g defined by $F(y) = x^{q+1}$, where F is a separable and linearized polynomial of degree $1 < \rho \leq q$. Take $D = \sum_{a \in \mathbb{F}_{q^2} \setminus \{0\}} P_{ab}$ ($\deg D = n$) and $A + P$ an effective non-special divisor of degree g such that $\text{Supp}(A + P) \subset \{P_{0b_i} \mid 1 \leq i \leq \rho\}$ and $P \notin \text{Supp}(D)$. Set P_∞ the only pole of x . If G, H are divisors such that $2g - 2 < \deg G, \deg H < n$ and*

$$\gcd(G, H) = A,$$

$$G + H = (2g + \rho - 2)P_\infty + \sum_{i=1}^{\rho} (q^2 - 2)P_{0b_i},$$

then $C(D, G)$ is LCD.

Proof. By Theorem 8, we know that for any $R \in \text{Supp}(A)$,

$$v_P(A) \leq q - \left\lfloor \frac{q+1}{\rho} \right\rfloor \leq q - 1.$$

Then $v_{P_{0b_i}}(2A) \leq 2q - 2 \leq q^2 - 2$ for any $1 \leq i \leq \rho$ and so

$$A \leq (2g + \rho - 2)P_\infty + \sum_{i=1}^{\rho} (q^2 - 2)P_{0b_i} - A.$$

Therefore, G, H are well defined.

Now, take $H' = H - (x^{q^2-1})$. Since $ev_D(x^{q^2-1}) = (1, \dots, 1)$, then $C(D, H) = C(D, H')$. We have:

$$\begin{aligned} G + H' - D &= G + H - D - (x^{q^2-1}) \\ &= (2g + \rho - 2)P_\infty + \sum_{i=1}^{\rho} (q^2 - 2)P_{0b_i} - D - (x^{q^2-1}) \\ &= (n + 2g - 2)P_\infty - D + (x^{q^2-2}/x^{q^2-1}) \\ &= (n + 2g - 2)P_\infty - D - (x) \\ &= (2g - 2)P_\infty - (x^{16} - x) \end{aligned}$$

This is a canonical divisor of simple poles at D and residues -1 at any point of D , then

$$C(D, G)^\perp = C(D, H') = C(D, H).$$

By Corollary 3, we have the conclusion. □

Now let us describe some LCD codes for specific curves.

Proposition 14. *Let $4|q$ and consider the curve $X : y^{\frac{q}{2}} + y^{\frac{q}{4}} + \dots + y = x^{q+1}$ over \mathbb{F}_{q^2} . Let $(x) = \sum_{i=1}^{\frac{q}{2}} P_{0b_i} - \frac{q}{2}P_\infty$ and take $D = \sum_{a \in \mathbb{F}_q \setminus \{0\}} P_{ab}$ and $G = (q^2 - 1)P_{0b_{\frac{q}{2}}} + \sum_{j=1}^{\frac{q-2}{2}} 2jP_{0b_j}$. Then $C(D, G)$ is a LCD code of dimension q^2 .*

Proof. Take $A = \sum_{j=1}^{\frac{q-2}{2}} 2jP_{0b_j}$ and $n = \deg D$. By Theorem 8, A is non-special of degree g , the genus of the curve. Take $H = \frac{q^2 - q - 4}{2}P_\infty + \sum_{i=1}^{\frac{q-2}{2}} (q^2 - 2 - 2j)P_{0b_i} - P_{0b_{\frac{q}{2}}}$. Then

$$\gcd(G, H) = A - P_{0b_{\frac{q}{2}}}$$

$$\text{lcm}(G, H) = \frac{q^2 - q - 4}{2}P_\infty + \sum_{i=1}^{\frac{q}{2}} (q^2 - 2)P_{0b_i} - A + P_{0b_{\frac{q}{2}}}.$$

By Theorem 13, $C(D, G)$ is LCD. Since $A \leq G$ and A is non-special, G is non-special and given $\deg G = g + q^2 - 1 < n$, we have

$$\dim C(D, G) = \ell(G) = \deg G - g + 1 = q^2$$

□

Proposition 15. Let q a power of a prime and r odd. Consider the curve $X : y^q + y = x^{q^r+1}$ on $\mathbb{F}_{q^{2r}}$. Let $D = \sum_{a \in \mathbb{F}_q^{2r} \setminus \{0\}} P_{ab}$ and $(x) = \sum_{i=1}^q P_{0b_i} - qP_\infty$. Let

$$G = (q^r + 1)(q - 1)P_\infty + \sum_{i=1}^{q-1} q^{r-1} j P_{0b_j}.$$

Then $C(D, G)$ is a LCD code of dimension $(q^r + 1)(q - 1) + 1$.

Proof. By Theorem 8, $A = \sum_{i=1}^{q-1} q^{r-1} j P_{0b_j}$ is non-special of degree g . Take $H = \sum_{i=1}^q (q^{2r} - 2)P_{0b_i} - A - P_\infty$. We have $\gcd(G, H) = A - P_\infty$ and

$$G + H = ((q^r + 1)(q - 1) - 1)P_\infty + \sum_{i=1}^q (q^{2r} - 2)P_{0b_i}.$$

Then, by Theorem 13, $C(D, G)$ is LCD. Since $\deg G = g + (q^r + 1)(q - 1) < n$ and G is non-special, then $\dim C(D, G) = \ell(G) = (q^r + 1)(q - 1) + 1$. \square

Corollary 16. Consider the Hermitian curve $X : y^q + y = x^{q+1}$ on \mathbb{F}_{q^2} and take $D = \sum_{a \in \mathbb{F}_{q^2} \setminus \{0\}} P_{ab}$ and $(x) = \sum_{i=1}^q P_{0b_i} - qP_\infty$. For $G = \sum_{i=1}^{q-1} j P_{0b_j} + (q^2 - 1)P_{0b_q}$ and $H = \sum_{i=1}^{q-1} j P_{0b_j} + (q^2 - 1)P_\infty$, $C(D, G)$ is a LCD code of dimension q^2 .

The following are examples of LCD codes over maximal curves.

Example Let $q = 4$ and consider $X : y^2 + y = x^5$ over \mathbb{F}_{q^2} . This curve has 33 rational points and its genus is $g = 2$. Take P_∞ the point at infinity and $D = \sum_{a \in \mathbb{F}_q \setminus \{0\}} P_{ab}$. We have $\deg D = 30$ and for $G = 4P_\infty + 12P_{00} - P_{01}$ we have that $C(D, G)$ is of dimension 14 and it is generated by the evaluations of functions in the set

$$\mathcal{B} := \left\{ x, x^2, \frac{x}{y}, \frac{x^2}{y}, \frac{x^3}{y}, \frac{x^4}{y}, \frac{x}{y^2}, \frac{x^2}{y^2}, \frac{x^3}{y^2}, \frac{x^4}{y^2}, \frac{x^5}{y^2}, \frac{x^3}{y^3}, \frac{x^4}{y^3}, \frac{x^5}{y^3} \right\}.$$

On the other hand, for $H = 2P_{00} + 15P_{01}$, $C(D, H)$ is of dimensions 16 and it is generated by the evaluations of

$$\left\{ 1, \frac{1}{x^2}, \frac{1}{x}, \frac{1}{x^2(y+1)}, \frac{1}{x(y+1)}, \frac{1}{y+1}, \frac{x}{y+1}, \frac{x^2}{y+1}, \frac{1}{x^2(y+1)^2}, \frac{1}{x(y+1)^2}, \frac{1}{(y+1)^2}, \frac{x}{(y+1)^2}, \frac{x^2}{(y+1)^2}, \frac{1}{(y+1)^3}, \frac{x}{(y+1)^3}, \frac{x^2}{(y+1)^3} \right\}.$$

By Proposition 14, $C(D, G)^\perp = C(D, H)$ and $C(D, G) + C(D, H) = \mathbb{F}_{16}^{30}$.

Example Let $q = 2$ and consider the Hermitian curve defined by $y^2 + y = x^3$ over \mathbb{F}_4 . This curve has 9 rational points and its genus is $g = 1$. Take P_∞ the point at infinity and $D = \sum_{a \in \mathbb{F}_4 \setminus \{0\}} P_{ab}$. We have $\deg D = 6$ and for $G = 3P_\infty + P_{00}$, $C(D, G)$ is of dimension 4. Its generator matrix is

	(a, a)	(a^2, a)	$(1, a)$	(a, a^2)	(a^2, a^2)	$(1, a^2)$
$\frac{x^2}{y}$	a	1	a^2	1	a^2	a
y	a	a	a	a^2	a^2	a^2
x	a	a^2	1	a	a^2	1
1	1	1	1	1	1	1

Now, for $H = P_{00} + 2P_{01} - P_{\infty}$, $C(D, H)$ is of dimension 2 and its generator matrix is

$$\begin{array}{c|cccccc} & (a, a) & (a^2, a) & (1, a) & (a, a^2) & (a^2, a^2) & (1, a^2) \\ \hline \frac{1}{x} & a^2 & a & 1 & a^2 & a & 1 \\ \frac{x}{y+1} & a^2 & 1 & a & 1 & a & a^2 \end{array}$$

We can easily check that $C(D, G)^\perp = C(D, H)$ and $C(D, G) \cap C(D, H) = \{0\}$.

5. CONCLUSION

In this paper, we determined explicit non-special divisors of degree $g - 1$ and g on certain Kummer extensions. As a consequence, we obtained the expressions for non-special divisors of the smallest degree as well as for effective non-special divisors of least degree on families of Hermitian and norm-trace curves and some of their quotients. These divisors support the construction of LCD codes on some families of maximal curves which are relevant to coding theory. All of the work done in this paper considers codes relative to the Hamming metric and the usual Euclidean inner product.

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