Frobenius numbers of generalized Fibonacci semigroups

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Abstract

The numerical semigroup generated by relatively prime positive integers $a_1, \ldots, a_n$ is the set S of all linear combinations of $a_1, \ldots, a_n$ with nonnegative integral coefficients. The largest integer which is not an element of S is called the Frobenius number of S. Recently, J. M. Marín, J. L. Ramírez Alfonsín, and M. P. Revuelta determined the Frobenius number of a Fibonacci semigroup, that is, a numerical semigroup generated by a certain set of Fibonacci numbers. In this paper, we consider numerical semigroups generated by certain generalized Fibonacci numbers. Using a technique of S. M. Johnson, we find the Frobenius numbers of such semigroups obtaining the result of Marín et. al. as a special case. In addition, we determine the duals of such semigroups and relate them to the associated Lipman semigroups.

1. Introduction

Given a set of relatively prime positive integers $a_1, \ldots, a_n$, let S denote the set of linear combinations of $a_1, \ldots, a_n$ with nonnegative integral coefficients. Since $a_1, \ldots, a_n$ are relatively prime, every sufficiently large integer N is an element of S. The largest integer which is not an element of S is called the Frobenius number of S and is denoted by $g(S)$. The Frobenius problem is to determine $g(S)$. An excellent general reference on the Frobenius problem is [11].

In discussing the Frobenius problem, it is convenient to use the terminology of numerical semigroups. The set S defined above is called the numerical semigroup generated by $a_1, \ldots, a_n$ and is denoted by $S = \langle a_1, \ldots, a_n \rangle$; that is,

$$\langle a_1, \ldots, a_n \rangle := \left\{ \sum_{i=1}^{n} c_i a_i : c_i \in \mathbb{N} \right\}$$

where $\mathbb{N}$ denotes the set of nonnegative integers. Typically, we assume that

$$a_i \notin \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \rangle$$

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for all i, $1 \le i \le n$. Then we say that S is a n-generated semigroup. General references on numerical semigroups include [2, 6, 7, 8].

The Frobenius problem takes its name from the fact that Frobenius is said to have mentioned it repeatedly in his lectures [3]. However, the first published work on this problem appears to be due to Sylvester [13] where he determined the number of elements of $\mathbb{N} \setminus \langle a, b\rangle$ where a and b are relatively prime. Though not stated explicitly in [13] (or in the often cited [12]), it is suspected that Sylvester knew that $g(\langle a, b\rangle) = ab - a - b$ and this fact is typically attributed to him. Given such a simple formula for the Frobenius number of a two-generated semigroup, it is natural to try to find the Frobenius number of an n-generated semigroup for other small values of n. However, Curtis proved that such a closed-form expression cannot be given for the Frobenius number of a general n-generated semigroup for $n > 2$ [4]. For this reason, Frobenius problem enthusiasts often consider semigroups whose generators are of a particular form.

In [10], the authors determine the Frobenius numbers of so-called Fibonacci semigroups which are numerical semigroups of the form $\langle F_i, F_{i+2}, F_{i+k}\rangle$ where F_j denotes the j^{th} Fibonacci number. In studying these semigroups, it is useful to recall the convolution property of Fibonacci numbers:

$$F_n = F_m F_{n-m+1} + F_{m-1} F_{n-m}$$

for all $m, n \in \mathbb{Z}^+$ (where $\mathbb{Z}^+$ denotes the set of positive integers). A Fibonacci semigroup is three-generated if and only if $3 \le k < i$ (equivalently, $F_k < F_i$). To see this, we consider two cases depending on the value of k. If $k = i$, then

$$F_{i+k} = F_{2i} = L_i F_i \in \langle F_i, F_{i+2}\rangle$$

where L_i denotes the i^{th} Lucas number. If $k > i$, then

$$F_{i+k} \ge F_{2i+1} = F_i F_{i+2} + F_{i-1} F_{i+1} > g(\langle F_i, F_{i+2}\rangle)$$

and so $F_{i+k} \in \langle F_i, F_{i+2}\rangle$.

In this paper, we consider semigroups of the form $S = \langle a, a+b, aF_{k-1} + bF_k\rangle$ where $a > F_k$ and $gcd(a, b) = 1$. Such a semigroup will be called a generalized Fibonacci semigroup. Notice that if $a = F_i$ and $b = F_{i+1}$, then $a + b = F_{i+2}$ and

$$aF_{k-1} + bF_k = F_i F_{k-1} + F_{i+1} F_k = F_{i+k}$$

and so every Fibonacci semigroup is a generalized Fibonacci semigroup. Using a method of S. M. Johnson [9], we find the dual of a generalized Fibonacci semigroup. Recall that the dual of a numerical semigroup S is defined to be

$$B(S) := \{x \in \mathbb{N} : x + (S \setminus \{0\}) \subseteq S\}.$$

It is immediate that $g(S) \in B(S)$ for any numerical semigroup $S \ne \mathbb{N}$ as $g(S) + s > g(S)$ for all $s \in \mathbb{Z}^+$. Moreover,

$$g(S) = \max\{x \in B(S) : x \notin S\}.$$

provided $S \neq \mathbb{N}$. Hence, we determine the Frobenius number of a generalized Fibonacci semigroup obtaining the result of [10] as a corollary.

This paper is organized as follows. Section 2 outlines Johnson's method and applies it to find the dual of a generalized Fibonacci semigroup. Section 3 contains results relating the dual and Lipman semigroups. The paper concludes with Section 4 where several open problems are posed.

2. Johnson's method

We begin this section with a review of S. M. Johnson's method [9] for determining the dual of a semigroup generated by three relatively prime positive integers.

Let $S := \langle a_1, a_2, a_3 \rangle$ where a_1, a_2, and a_3 are pairwise relatively prime. Suppose $N \in B(S) \setminus S$. Then $N + a_i \in S$ for $i = 1, 2, 3$; that is,

$$N = y_{ij} a_j + y_{ik} a_k - a_i$$

for some $y_{ij}, y_{ik} \in \mathbb{N}$. Since the a_i are relatively prime, the semigroup generated by any two of them has a Frobenius number. Hence, any sufficiently large integer will be contained in such a semigroup. Let

$$L_i := \min \{ c : c a_i \in \langle a_j, a_k \rangle \} .$$

Then there exist $x_{ij}, x_{ik} \in \mathbb{N}$ such that

$$L_i a_i := x_{ij} a_j + x_{ik} a_k .$$

According to [9, Theorem 3], x_{ij} and x_{ik} are positive integers and are unique. Recall that

$$\begin{aligned} N &= y_{21} a_1 + y_{23} a_3 - a_2 \\ &= y_{31} a_1 + y_{32} a_2 - a_3 . \end{aligned}$$

Then

$$N = \begin{cases} (L_2 - 1)\, a_2 + (x_{13} - 1)\, a_3 - a_1 & \text{if } y_{21} < y_{31} \\ (x_{21} - 1)\, a_2 + (L_3 - 1)\, a_3 - a_1 & \text{if } y_{31} < y_{21}. \end{cases}$$

and so

$$B(S) \setminus S = \left\{ \begin{array}{c} (L_2 - 1)\, a_2 + (x_{13} - 1)\, a_3 - a_1, \\ (x_{21} - 1)\, a_2 + (L_3 - 1)\, a_3 - a_1 \end{array} \right\} .$$

Next, we apply this method to a generalized Fibonacci semigroup. Consider

$$S := \langle a, a + b, a F_{k-1} + b F_k \rangle$$

where $a > F_k$ and the generators of S are pairwise relatively prime. Note that

$$F_k (a + b) = F_{k-2} a + (F_{k-1} a + F_k b)$$

From this and the argument [9, p. 395-396], it follows that

$$L_2 = F_k, x_{21} = F_{k-2}, \text{ and } x_{23} = 1.$$

In addition, we find that

$$\begin{array}{lllllllll} L_1 & = & (a+b) - F_{k-2}\left\lfloor \frac{a}{F_k} \right\rfloor & x_{12} & = & a - F_k\left\lfloor \frac{a}{F_k} \right\rfloor & x_{13} & = & \left\lfloor \frac{a}{F_k} \right\rfloor \\ L_2 & = & F_k & x_{21} & = & F_{k-2} & x_{23} & = & 1 \\ L_3 & = & \left\lfloor \frac{a}{F_k} \right\rfloor + 1 & & & & & & \end{array}$$

As a consequence,

$$B(S) \setminus S = \left\{ \left(a + b - F_{k-2}\left\lfloor \frac{a}{F_k} \right\rfloor - 2 \right) a - b, \left(F_{k-2} - 2 \right) a + \left\lfloor \frac{a}{F_k} \right\rfloor \left(aF_{k-1} + bF_k \right) - b \right\}.$$

This proves the next two results.

Proposition 1 *Assume $a > F_k$. If $S = \langle a, a+b, aF_{k-1} + bF_k \rangle$ is generated by three pairwise relatively prime integers, then the dual of S is*

$$B(S) = S \cup \left\{ \left(a + b - F_{k-2}\left\lfloor \frac{a}{F_k} \right\rfloor - 2 \right) a - b, \left(F_{k-2} - 2 \right) a + \left\lfloor \frac{a}{F_k} \right\rfloor \left(aF_{k-1} + bF_k \right) - b \right\}.$$

Theorem 2 *The Frobenius number of $S = \langle a, a+b, aF_{k-1} + bF_k \rangle$ where $a > F_k$ and the generators of S are pairwise relatively prime is*

$$g(S) = \max \left\{ \left(a + b - F_{k-2}\left\lfloor \frac{a}{F_k} \right\rfloor - 2 \right) a - b, \left(F_{k-2} - 2 \right) a + \left\lfloor \frac{a}{F_k} \right\rfloor \left(aF_{k-1} + bF_k \right) - b \right\}.$$

This theorem gives a formula for $g(F_i, F_{i+2}, F_{i+k})$, due to Marin et al. [10], when $a = F_i$ and $b = F_{i+1}$. However, we should point out that the technique used in [10], different from ours, allowed the authors not only to state explicitly when the maximum is obtained in each case but also to give a formula for the genus (meaning $|\mathbb{N} \setminus S|$) of such Fibonacci semigroups.

3. Duals and Lipman semigroups

In this section, we compare two chains of semigroups. One of the chains is based on the dual construction. The other chain arises by taking Lipman semigroups. We first describe the Lipman semigroup. Then we relate it to the dual of S. Finally, we consider these for Fibonacci semigroups.

Suppose $S = \langle a_1, a_2, \ldots, a_n \rangle$ is a numerical semigroup with $a_1 < \cdots < a_n$. The Lipman semigroup of S is defined as

$$L(S) := \langle a_1, a_2 - a_1, \ldots, a_n - a_1 \rangle .$$

Clearly, $S \subseteq L(S)$. Moreover, $B(S) \subseteq L(S)$ since $x \in B(S)$ implies $x + a_1 = \sum_{i=1}^{n} c_i a_i$ for some $c_i \in \mathbb{N}$.

Given a numerical semigroup S, both its dual $B(S)$ and its Lipman semigroup $L(S)$ are numerical semigroups. Hence, one may iterate the B and L constructions to obtain two ascending chains of numerical semigroups

$$B_0(S) := S \subseteq B_1(S) := B(B_0(S)) \subseteq \cdots \subseteq B_{h+1}(S) := B(B_h(S)) \subseteq \cdots$$

and

$$L_0(S) := S \subseteq L_1(S) := L(L_0(S)) \subseteq \cdots \subseteq L_{h+1}(S) := L(L_h(S)) \subseteq \cdots$$

as in [2]. We will refer to these as the B- and L-chains. Notice that for $S \neq \mathbb{N}$, $S \subsetneqq B(S)$ and $S \subsetneqq L(S)$. This together with the fact that $\mathbb{N} \setminus S$ is finite implies that there exist smallest non-negative integers $\beta(S)$ and $\lambda(S)$ such that $B_{\beta(S)}(S) = \mathbb{N}_0 = L_{\lambda(S)}(S)$. Since

$$B_0(S) = S = L_0(S),$$

$$B_1(S) \subseteq L_1(S),$$

and

$$B_{\beta(S)}(S) = \mathbb{N}_0 = L_{\lambda(S)}(S),$$

it is natural to compare the two chains. In [2] the authors suggest that $B_j(S) \subseteq L_j(S)$ for all $0 \leq j \leq \beta(S)$. While true for two-generated semigroups, this containment may fail in general; there are examples of four-generated semigroups T for which $B_2(T) \nsubseteq L_2(T)$; see [5]. This prompts the question of whether or not $B_j(S) \subseteq L_j(S)$ for all $j \geq 0$ for a three-generated semigroup S. Here, we consider this question for Fibonacci semigroups.

Suppose that $S = \langle a, a+b, aF_{k-1} + bF_k \rangle$ where $a = F_i$ and $b = F_{i+1}$; that is, suppose

$$S = \langle F_i, F_{i+2}, F_{i+k} \rangle .$$

Then the Lipman semigroup of S is $L_1(S) = \langle a, b \rangle$. Since $a < b$, $L_2(S) = \langle a, b - a \rangle$. Continuing this process yields the following result.

Proposition 3 *Given a Fibonacci semigroup $S = \langle F_i, F_{i+2}, F_{i+k} \rangle$,*

$$L_j(S) = \langle F_{i-j+1}, F_{i-j+2} \rangle$$

for all j, $1 \leq j \leq i - 3$. In particular, $\lambda(S) = i - 3$.

Of course, one may obtain similar results for generalized Fibonacci semigroups. Because the description depends on sizes of a and b, we omit this here and leave the details to the reader. Since $L_j(S)$ is a two-generated semigroup, $L_j(S)$ is symmetric for all $1 \leq j < \lambda(S)$. Therefore, $L_j(S)$ is maximal in the set of all numerical semigroups with Frobenius number $g\left(L_j(S)\right)$. In particular, we notice that if

$$F_i = \min\left\{x \in B_{F_i - \left\lfloor \frac{F_i - 1}{F_k} \right\rfloor F_{k-2} - 2}(S) : x \neq 0\right\}, \tag{1}$$

then

$$g\left(B_{F_i - \left\lfloor \frac{F_i - 1}{F_k} \right\rfloor F_{k-2} - 1}(S)\right) = g(S) - \left(F_i - \left\lfloor \frac{F_i - 1}{F_k} \right\rfloor F_{k-2} - 2\right) F_i = g\left(L_1(S)\right)$$

[2, Proposition I.1.11]. Hence, if (1) holds and

$$F_{i+1} \in B_{F_i - \left\lfloor \frac{F_i - 1}{F_k} \right\rfloor F_{k-2} - 1}(S), \tag{2}$$

then

$$B_{F_i - \left\lfloor \frac{F_i - 1}{F_k} \right\rfloor F_{k-2} - 1}(S) = L_1(S)$$

and $B_j(S) \subseteq L_j(S)$ for all nonnegative integers j would follow from [5, Theorem 2.6]. Unfortunately, because $B_1(S)$ is not three-generated, $B_2(S)$ may not be computed using the method in Section 2. Instead, one may compute this directly from the definition and obtain the next result.

Proposition 4 *Given a Fibonacci semigroup* $S = \langle F_i, F_{i+2}, F_{i+k} \rangle$

$$B_2(S) = \left\langle \begin{array}{l} F_i, F_{i+2}, F_{i+k}, l - F_{i+2}, h - F_{i+k} \text{ if } x_{32} = 1, \\ l - F_i \text{ if } k > 4 \text{ or } x_{12} = 1, l - F_{i+k} \text{ if } x_{13} \geq 2 \text{ or } x_{13} = x_{31} = 1, \\ h - F_i \text{ if } x_{31} \geq 2 \text{ or } x_{31} = x_{13} = 1, h - F_{i+2} \text{ if } x_{12} \geq 2 \text{ or } x_{12} = x_{21} = 1 \end{array} \right\rangle$$

where

$$h = \left(F_k - 1\right) F_{i+2} + \left(F_{i+2} - F_{k-2}\left(\left\lfloor \frac{F_i}{F_k} \right\rfloor + 1\right) - 1\right) F_i - F_{i+k}$$

and

$$l = \left(F_k - 1\right) F_{i+2} + \left(\left(\left\lfloor \frac{F_i}{F_k} + 1 \right\rfloor\right) F_k - F_i - 1\right) F_{i+k} - F_i.$$

Consequently,

$$B_2(S) \subseteq L_1(S) \subseteq L_2(S).$$

In light of Proposition 4, determining $B_j(S)$ for $j > 3$ will not be an immediate consequence of Johnson's method. We leave this (and settling when (1) and (2) hold) as a problem for further study.

4. Conclusion

In this paper, we determined the Frobenius number of generalized Fibonacci semigroups. In addition, we also obtained the dual of such a semigroup. We leave as an open problems to

1. determine if $B_j(S) \subseteq L_j(S)$ for each $j \geq 0$ for Fibonacci semigroups S (or, more generally, three-generated semigroups), and
2. find the Frobenius number and dual of other semigroups generated by generalized Fibonacci numbers.

We conclude by mentioning another interesting problem relating numerical semigroups and Fibonacci numbers. M. Bras-Amoros conjectures that the number of semigroups with a particular genus g behaves asymptotically as the Fibonacci sequence [1].

References

[1] M. Bras-Amoros, Fibonacci-like behavior of the number of numerical semigroups of a given genus, Semigroup Forum, to appear.

[2] V. Barucci, D. E. Dobbs and M. Fontana, Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains, Memoirs Amer. Math. Soc. **125/598** (1997).

[3] A. Brauer, *On a problem of partitions*, Amer. J. Math., **64** (1942), 299–312.

[4] F. Curtis, On formulas for the Frobenius numer of a numerical semigroup, Math Scand. **67** (1990), 190–192.

[5] D. E. Dobbs and G. L. Matthews, On comparing two chains of numerical semigroups and detecting Arf semigroups, Semigroup Forum **63** (2001), 237-246.

[6] R. Fröberg, C. Gottlieb and R. Häggkvist, On numerical semigroups, Semigroup Forum **35** (1987), no. 1, 63–83.

[7] R. Fröberg, C. Gottlieb and R. Häggkvist, Semigroups, semigroup rings and analytically irreducible rings, Reports Dept. Math. Univ. Stockholm no. 1 (1986).

[8] R. Gilmer, Commutative semigroup rings. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1984.

[9] S. M. Johnson, A linear Diophantine problem, Canad. J. Math. **12** (1960), 390–398.

[10] J. M. Marín, J. L. Ramírez Alfonsín and M. P. Revuelta, On the Frobenius number of Fibonacci numerical semigroups, Integers **7** (2007), A14.

[11] J. L. Ramírez Alfonsín, The Diophantine Frobenius Problem, Oxford Lecture Series in Mathematics and its Applications **30**, Oxford University Press, 2005.

[12] J. J. Sylvester, Mathematical questions with their solutions, Educational Times **41** (1884), 21.

[13] J. J. Sylvester, On Subvariants, i.e. semi-invariants to binary quantics of an unlimited order, Amer. J. Math. 5 (1882), no. 1-4, 79–136.