ACYCLIC COLORINGS OF PRODUCTS OF TREES

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Abstract. We obtain bounds for the coloring numbers of products of trees for three closely related types of colorings: acyclic, distance 2, and $L(2,1)$.

1. Introduction

A *k-coloring* of a graph G with vertex set $V(G)$ is a labeling $f : V(G) \to \{1,\ldots,k\}$. A k-coloring of a graph is a *proper coloring* provided any two adjacent vertices have distinct colors. The chromatic number of G, denoted $\chi(G)$, is the minimum k such that G has a proper k-coloring. An *acyclic coloring* of a graph G is a proper coloring of G such that the subgraph of G induced by any two color classes of G contains no cycles. The *acyclic chromatic number* of a graph G, denoted $AC(G)$, is the minimum number k such that G has an acyclic k-coloring.

Acyclic colorings were introduced by Grünbaum in [8]. The study of acyclic colorings for planar graphs was carried on by Berman and Albertson [1] and Borodin [5]. This was followed by work on the acyclic chromatic number for graphs on certain surfaces [3]. In addition, acyclic colorings have been studied by Alon, McDiarmid, and Reed [2] and Mohar [11]. Nowakowski and Rall have investigated the behavior of several graph parameters with respect to an array of different graph products [12].

Here we address a natural extension of the very nice work by Ferrin, Godard, and Raspaud [6] in which acyclic colorings of products of paths were studied. We consider acyclic colorings of products of trees, obtaining bounds for the acyclic chromatic number. The product we are taking is the usual *Cartesian* (or *box*) product. The vertex set of $G \Box H$ is the Cartesian product $V(G) \times V(H)$ of the vertex sets of G and H. There is an edge between two vertices (a,b) and (x,y) of the product if and only if they are adjacent in exactly one coordinate and agree in the other.

In addition, we also investigate certain close but more restrictive relatives of acyclic colorings. A *distance* 2 *coloring* of G is a proper coloring in which any three vertices lying on a path of length two in G have distinct colors. The *distance* 2 *chromatic number* of G, denoted $\chi(G^2)$, is the minimum number k such that G has

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a distance 2 coloring with k colors. Recall that the *square* G^2 of a graph G has the same vertex set as G but has two vertices adjacent if and only if they are at most distance two apart in G. By definition, the distance 2 chromatic number is just the chromatic number of the square, hence the notation $\chi(G^2)$.

An even more restrictive coloring notion is that of $L(2,1)$-coloring. An $L(2,1)$ *coloring* is a coloring in which adjacent vertices are assigned color labels which differ by at least 2 and vertices at distance two apart get color labels which differ by at least one. The smallest k such that G has an $L(2,1)$-coloring with k colors is the $L(2,1)$-*chromatic number* of G and is denoted by $\lambda(G)$. Whittlesey, Georges, and Mauro investigated $L(2,1)$-colorings of products of paths [13]. More recently, Kuo and Yan considered the $L(2,1)$-chromatic numbers of products of paths and cycles [10].

Since these three types of colorings are progressively more restrictive, it is evident that for any graph G we have

$$\chi(G) \leq AC(G) \leq \chi(G^2) \leq \lambda(G)$$

It is also evident that each of these colorings are hereditary in the sense that the restriction of a coloring of one of the four types - proper, acyclic, distance 2, or $L(2,1)$ - to a subgraph is again of the same type. Thus, the corresponding chromatic numbers are nondecreasing from subgraph to supergraph.

All of the graphs we consider will be simple (no loops or multiple edges). The path on n vertices will be denoted by P_n, and Q_d will denote the (1-skeleton of the) d-dimensional cube; that is, the d-cube Q_d is the d-fold Cartesian/box product of a single edge P_2 with itself.

2. Acyclic Coloring

In [6, Theorem 2], it is shown that the acyclic chromatic number of a product of paths is at most $d+1$. Here, we generalize this to a product of trees.

Theorem 2.1. *Let $G = T_1 \square T_2 \square \cdots \square T_d$ be a product of trees $T_1, T_2, \ldots, T_d$. Then $\lceil \frac{d+3}{2} \rceil \leq AC(G) \leq d+1$.*

Proof. Let $G = T_1 \square T_2 \square \cdots \square T_d$, where $T_1, T_2, \ldots, T_d$ are trees. To obtain the lower bound on $AC(G)$, note that G contains $Q_d = P_2 \square \cdots \square P_2$, the d-dimensional cube. Then, according to [6, Theorem 4], $\lceil \frac{d+3}{2} \rceil \leq AC(Q_d) \leq AC(G)$.

To obtain the upper bound on $AC(G)$, root each tree T_i at some vertex r_i. Then, for each tree T_i, direct all edges of T_i away from the root r_i. Given a vertex $u = (u_1, \ldots, u_d) \in V(G)$, define $f(u) \in \{1, \ldots, d+1\}$ by

$$f(u) \equiv \sum_{k=1}^{d} k \cdot \operatorname{dist}(u_k, r_k) \pmod{d+1},$$

where $\operatorname{dist}(u_k, v_k)$ denotes the distance between two vertices u_k and v_k of the tree T_k. Given $1 \leq m < n \leq d+1$, consider the two color classes $L_m := \{v \in V(G) : f(v) = m\}$ and $L_n := \{v \in V(G) : f(v) = n\}$. Let H be the subgraph of G induced by the color classes L_m and L_n. We claim that H contains no cycles.

First note that G contains no oriented cycle. It follows that any cycle in H must contain a vertex $v = (v_1, \ldots, v_d)$ with both arrows going into v. Let $v' = (v'_1, \ldots, v'_d)$ and $v'' = (v''_1, \ldots, v''_d)$ be the neighbors of v in H. Since v and v' are adjacent in G, there exists i, $1 \leq i \leq d$, such that v_i and v'_i are adjacent in the tree T_i and $v_k = v'_k$

for all $k \neq i$. Note that v'_i is the parent of v_i in T_i since the edge $v'v$ is directed from v' to v. Similarly, since v and v'' are adjacent in G, there exists j, $1 \leq j \leq d$, such that v_j and v''_j are adjacent in the tree T_j and $v_l = v''_l$ for all $l \neq j$. Again, v''_j is the parent of v_j in T_j. Since any vertex in a tree has at most one parent, if $i = j$, then $v'_i = v''_i$ which implies $v' = v''$. Since this cannot be the case, $i \neq j$. Then

$$f(v') \equiv \sum_{k=1}^{d} k \operatorname{dist}(v'_k, r_k) \equiv f(v) - i(\operatorname{dist}(v_i, r_i) - \operatorname{dist}(v'_i, r_i)) \equiv f(v) - i \pmod{d+1}$$

since $\operatorname{dist}(v_i, r_i) = \operatorname{dist}(v'_i, r_i) + 1$. Similarly,

$$f(v'') \equiv f(v) - j \pmod{d+1}.$$

To prove the claim, we will show that $|\{f(v), f(v'), f(v'')\}| = 3$.

Since $1 \leq i, j \leq d$, $f(v) \neq f(v')$ and $f(v) \neq f(v'')$ implying that f is a proper coloring. It remains to show that $f(v') \neq f(v'')$. Suppose $f(v') = f(v'')$. From the above, this implies that

$$f(v) - i \equiv f(v) - j \pmod{d+1}.$$

Thus, $(d + 1) \mid (i - j)$. However, $-d \leq i - j \leq d$ as $1 \leq i, j \leq d$. This contradicts the fact that $i \neq j$. Hence, $f(v') \neq f(v'')$. Therefore, $|\{f(v), f(v'), f(v'')\}| = 3$. It follows that H cannot contain a cycle, proving the claim. Since G contains no bichromatic cycles, $f : V(G) \to \{1, \ldots, d + 1\}$ is an acyclic coloring. Consequently, $AC(G) \leq d + 1$. ■

Corollary 2.2. *The acyclic chromatic number of the product of two trees is 3.*

Proof. Any graph containing a cycle must have acyclic chromatic number at least 3. The above result constructs an acyclic 3-coloring for a product of two trees. ■

Theorem 2.3. *The acyclic chromatic number of the product of three trees is 4.*

Proof. The product of three (nontrivial) trees necessarily contains Q_3, the product of three edges. Hence there is no acyclic coloring of a product of three trees with 3 colors as the acyclic chromatic number of Q_3 is 4 [6, Table 1]. Thus by Theorem 2.1, the acyclic chromatic number of the product of three trees is 4. ■

The lower bound in Theorem 2.1 can be improved in some cases using a modification of a technique in [6]. This will be shown in a general context in [9].

3. Distance 2 coloring

3.1. **Distance 2 colorings of general products of trees.** In any graph G the closed neighborhood $N[v]$ of any vertex v is a clique in the square G^2. As such its order is a lower bound on the distance 2 chromatic number. The order of $N[v]$ is just the degree of v plus one (to count v itself). Thus,

$$\chi(G^2) \geq \Delta(G) + 1,$$

where $\Delta(G)$ denotes the maximum degree of a vertex of G.

Now the degree of a vertex in a product graph is just the sum of the degrees of its coordinates. In particular, choosing each coordinate to have maximal degree in its factor produces a vertex of maximal degree in the product. In particular, this argument yields the following bounds:

Theorem 3.1.
(1) *If G is a product of trees T_i $(i = 1, \ldots, d)$ with maximum degrees Δ_i, then $\chi(G^2) \geq (\sum_{i=1}^{d} \Delta_i) + 1$.*
(2) *If G is a product of d paths of lengths at least 2, then $\chi(G^2) \geq 2d + 1$.*
(3) *If G is the d-dimensional cube, then $\chi(G^2) \geq d + 1$.*

The construction of distance 2 colorings giving an upper bound on $\chi(G^2)$ is more involved and it will be helpful to standardize notation before embarking on the construction. Let $G = T_1 \square T_2 \square \cdots \square T_d$, where $T_1, T_2, \ldots, T_d$ are non-trivial trees, and let Δ_i denote the maximum degree of T_i. Without loss of generality, we can assume the factor trees are ordered by increasing maximum degree: $\Delta_i \leq \Delta_{i+1}$ for $i = 1, 2, \ldots d-1$. Let n denote the number of factor trees T_i with $\Delta_i = 1$. That is, n is the number of factors which are just single edges. Let each T_i be rooted at a leaf $r_i \in V(T_i)$.

For our current purposes define a *weighting* to be a map

$$\mathrm{wt} : E(T_1) \cup E(T_2) \cup \cdots \cup E(T_d) \longrightarrow \mathbb{Z}^+$$

which assigns a positive integer to each edge in each factor tree T_i. A weighting is *admissible* if and only if it satisfies the following two conditions:

(AW1) Edges from different factor trees get different weights;
(AW2) For all v, the mapping $\{v, u_i\} \rightarrow \mathrm{wt}(\{v, u_i\})$ is one-to-one on the children $u_1, u_2, \ldots u_j$ of v.

Given a positive integer m, the *m-coloring of G based on a weighting* wt is defined as follows. First for each factor tree T_i, define a labeling $g_i : V(T_i) \rightarrow \mathbb{Z}^+$ of each of the vertices $v \in V(T_i)$ of T_i by

$$g_i(v) = \sum_{k=1}^{j} \mathrm{wt}(\{x_k, x_{k-1}\})$$

where $r_i = x_0, x_1, \ldots, x_{j-1}, x_j = v$ is the unique path in T_i from the root r_i to the vertex v; that is, $g_i(v)$ is the weight of the path in T_i from the root r_i to the vertex v. Now given a vertex $u = (u_1, \ldots, u_d) \in V(G)$, define $f(u) \in \{0, 1, 2, \ldots, m-1\}$ by

$$f(u) \equiv \sum_{k=1}^{d} g_k(u_k) \pmod{m}.$$

Note that we are deviating slightly from the definition of coloring given in the first section. Namely, since we are now working modulo m, it is more convenient to let the color set be the system of residues $\{0, 1, 2, \ldots, m-1\}$ rather than the first m positive integers.

Note also that since each edge of G corresponds to moving some component along some edge in a factor tree, each edge of G is associated with an edge in a factor tree. Thus we can lift the weighting from the edges of the factor trees to the edges of the full product G.

Lemma 3.2. *Let* wt *be an admissible weighting on a product of trees G. If $m = 2\max_{\{u,v\} \in E(G)}(\mathrm{wt}(\{u, v\})) + 1$, then the m-coloring f based on the weighting* wt *is a distance 2 coloring of G.*

Proof. First, we will show that f is a proper coloring of G. Consider

$$v = (v_1, \ldots, v_d), v' = (v'_1, \ldots, v'_d) \in V(G)$$

where $\{v, v'\} \in E(G)$. Then there exists an i with $1 \le i \le d$ such that v_i and v'_i are adjacent in T_i and $v_k = v'_k$ for all $k \neq i$. Relabeling if necessary, we may assume that v'_i is the parent of v_i in T_i. Then

$$f(v) - f(v') \equiv \sum_{k=1}^{d} g_k(v_k) - \sum_{k=1}^{d} g_k(v'_k) \equiv g_i(v_i) - g_i(v'_i) \equiv \mathrm{wt}(\{v'_i, v_i\}) \pmod{m}.$$

From the definition of m and the fact that the weights are positive, we have $1 \le \mathrm{wt}(\{v'_i, v_i\}) \le \frac{m-1}{2}$. So $\mathrm{wt}(\{v'_i, v_i\}) \neq 0$ and $m \nmid \pm\mathrm{wt}(\{v'_i, v_i\})$. Thus, $f(v) \neq f(v')$. Therefore, f is a proper coloring of G.

Now we must show that if $\mathrm{dist}(v, v'') = 2$, then $f(v) \neq f(v'')$ and hence f is a distance 2 coloring of G. Suppose that $\mathrm{dist}(v, v'') = 2$ for some vertices $v = (v_1, \ldots, v_d) \in V(G)$ and $v'' = (v''_1, \ldots, v''_d) \in V(G)$. Then there exists $v' = (v'_1, \ldots, v'_d) \in V(G)$ such that v and v' are adjacent and v' and v'' are adjacent. Since v and v' are adjacent, there exists i with $1 \le i \le d$, such that v_i and v'_i are adjacent in the tree T_i and $v_k = v'_k$ for all $k \neq i$. Since v' and v'' are adjacent, there exists j with $1 \le j \le d$ such that v'_j and v''_j are adjacent in the tree T_j and $v'_k = v''_k$ for all $k \neq j$.

Now we have

$$\begin{aligned} f(v) - f(v'') &\equiv (f(v) - f(v')) + (f(v') - f(v'')) \\ &\equiv \pm\mathrm{wt}(\{v_i, v'_i\}) \pm \mathrm{wt}(\{v'_j, v''_j\}) \pmod{m}. \end{aligned} \tag{1}$$

By choice of m, each of the two weights above are at most $\frac{m-1}{2}$, so adding or subtracting them will give a value between $-(m-1)$ and $m-1$. Thus if $f(v) = f(v'')$, then the two weights must cancel out. To show this cannot happen, there are three cases.

First, if $i \neq j$, then the edges $\{v_i, v'_i\}$ and $\{v'_j, v''_j\}$ are in different factor trees. Condition (AW1) says they are assigned different weights, so they cannot cancel out.

Next suppose $i = j$ and v_i, v'_i, and v''_i lie on a directed path. Then the weights in Equation (1) will occur with the same signs: positive if $v_i - v'_i - v''_i$ is going toward the root and negative if it is going away.

Finally suppose $i = j$ and v' is the parent of both v and v''. Then by condition (AW2) the edge weights from v' to v and to v'' will be different, so that these weights cannot cancel out. ■

Theorem 3.3. *Let $G = T_1 \square T_2 \square \cdots \square T_d$ be a product of trees $T_1, T_2, \ldots, T_d$. Let $n = |\{i : 1 \le i \le d, T_i = P_2\}|$ and $s = \sum_{i=1}^{d} \Delta_i$, where Δ_i denotes the degree of a vertex of T_i of maximum degree. Then $\chi(G^2) \le 2(s - d + n) + 1$.*

Proof. Let $G = T_1 \square T_2 \square \cdots \square T_d$, where $T_1, T_2, \ldots, T_d$ are trees. Let $n = |\{i : 1 \le i \le d, T_i = P_2\}|$ be the number of trees $T_i = P_2$, a path on two vertices. Define $\Delta_0 = 1$. For $1 \le i \le d$, let Δ_i denote the degree of a vertex of T_i of maximum degree. Set $s = \sum_{i=1}^{d} \Delta_i$. Then, if $n \ge 1$, we may assume that $T_1 = \cdots = T_n = P_2$ and that $\Delta_i \ge 2$ for all i, $n + 1 \le i \le d$. If $n = 0$, then $\Delta_i \ge 2$ for all i, $1 \le i \le d$.

Now we create a weighting on G. First, root each tree T_i at a leaf $r_i \in V(T_i)$. Then direct all edges of T_i away from the root r_i. Recall that each T_i, $1 \le i \le n$, has exactly one edge. Assign the weight i to such an edge. Now consider a tree T_i where $n + 1 \le i \le d$; that is, consider a tree T_i with $\Delta_i \ge 2$. Given a vertex

$v_i \in V(T_i)$, v_i has degree at most Δ_i and so has at most $\Delta_i - 1$ children. If $v_i \in V(T_i)$ has children $u_1, \ldots, u_t$, where $1 \leq t \leq \Delta_i - 1$, then define the weight of the edge $\{v_i, u_j\} \in E(T_i)$ by

$$\text{wt}(\{v_i, u_j\}) = n + \left(\sum_{l=n}^{i-1} (\Delta_l - 1)\right) + j.$$

Since $1 \leq j \leq \Delta_i - 1$,

$$\left(\sum_{k=n}^{i-1} (\Delta_k - 1)\right) + 1 \leq \text{wt}(\{v_i, u_j\}) \leq n + \sum_{k=n}^{i} (\Delta_k - 1)$$

for all edges $\{v_i, u_j\} \in E(T_i)$. In addition, $\text{wt}(\{v_i, v_i'\}) \neq wt(\{v_j, v_j'\})$ for two any edges $\{v_i, v_i'\} \in E(T_i)$, $\{v_j, v_j'\} \in E(T_j)$ in distinct trees T_i and T_j. This proves that wt is an admissible weighting on G.

Thus, by Lemma 3.2, $f : V(G) \to \{1, 2, \ldots, m\}$ is a distance 2 coloring of G. ■

We conclude this section with a summary of our findings for products of paths.

Corollary 3.4. (1) *If G is a product of paths T_i $(i = 1, \ldots, d)$ and n denotes the number of paths of length 1, then $2d + 1 - n \leq \chi(G^2) \leq 2d + 1$.*
(2) *If G is a product of d paths of lengths at least 2, then $\chi(G^2) = 2d + 1$.*
(3) *If G is the d-dimensional cube, then $d + 1 \leq \chi(G^2) \leq 2d + 1$.*

3.2. **Distance 2 colorings of hypercubes.** Our Corollary 3.4(2) is already contained in [6], where they show that if G is a product of paths of lengths 2 or more, then $\chi(G^2) = 2d+1$. Although the hypothesis that the paths have lengths greater than 2 is not included in the statement of their theorem, its omission is surely an oversight since it is mentioned in the proof and, as we now show, the result is false without it. We will look at lower bounds for $\chi(Q_d^2)$ where Q_d^2 is the square of the cube of dimension d; that is, lower bounds for the distance 2 chromatic number of a product of d paths of length 1.

Theorem 3.5. *If $d = 2^r - 1$ for some integer $r > 1$, then $\chi(Q_d^2) = d + 1$.*

Proof. Let $\mathcal{H}_r$ denote the Hamming code of length $2^r - 1$. As is well known, $\mathcal{H}_r$ is a perfect linear code of weight 3 for all $r \geq 2$ [4]. This applies to our situation as follows. Let $d = 2^r - 1$ and consider the d-cube Q_d. The vertices of Q_d may be regarded as binary vectors of length d. The Hamming distance for vectors is in this case the same as the graph distance in Q_d. The fact that the code is linear and the weight is 3 implies that the distance between any two vertices (= vectors) in the code is at least 3. That is, the code $\mathcal{H}$ is an independent set in the square Q_d^2 of the d-cube. The fact the code is perfect implies that it is an independent set of *maximum* size. Now $\mathcal{H}$ has dimension $d - r$ and so contains 2^{d-r} vectors. The cube has 2^d vertices, so the minimum number of independent sets required to partition the vertex set (i.e., the minimum number of colors in a proper coloring) of Q_d is $2^d/2^{d-r} = 2^r = d + 1$. The Hamming code $\mathcal{H}_r$ achieves this minimum, so $\chi(Q_d^2) = d + 1$. ■

Recall that Corollary 3.4(c) shows $\chi(Q_d^2) \geq d + 1$. The above result shows that equality is obtained if $d+1$ is a power of 2. Next, we see that this is the only time this occurs. To do this, we derive another lower bound on the distance 2 chromatic number of a hypercube.

Lemma 3.6. *The distance* 2 *chromatic number of the d-dimensional cube Q_d satisfies*

$$\chi(Q_d^2) \geq \left\lceil \frac{2^d}{\left\lfloor \frac{2^d}{d+1} \right\rfloor} \right\rceil.$$

Proof. Clearly, $\alpha(Q_d^2)\chi(Q_d^2) \geq 2^d$ where $\alpha(Q_d^2)$ denotes the independence number of the square of the d-dimensional cube Q_d. By the Sphere Packing Bound, $\alpha(Q_d^2) \leq \frac{2^d}{d+1}$. Since $\alpha(Q_d^2)$ is an integer, this gives $\alpha(Q_d^2) \leq \left\lfloor \frac{2^d}{d+1} \right\rfloor$. It follows that $\chi(Q_d^2) \geq \frac{2^d}{\left\lfloor \frac{2^d}{d+1} \right\rfloor}$. Since $\chi(Q_d^2)$ is an integer, the desired bound is obtained. ■

Theorem 3.7. *The distance* 2 *chromatic number of the d-dimensional cube satisfies $\chi(Q_d^2) \geq d+1$ with equality if and only if $d+1$ is a power of* 2*.*

Proof. Let $z := \frac{2^r}{d+1}$. Suppose that $z \notin \mathbb{Z}$. Then $\lfloor z \rfloor < z$ and so $\frac{2^r}{\lfloor z \rfloor} > \frac{2^r}{z} = d+1$. By Lemma 3.6, this yields

$$\chi(Q_d^2) \geq \left\lceil \frac{2^r}{\lfloor z \rfloor} \right\rceil \geq \frac{2^r}{\lfloor z \rfloor} > d+1.$$

■

Corollary 3.8. *If $d = 2^r - 2$ for some integer $r > 1$, then $\chi(Q_d^2) = d+2$.*

Proof. Let $d = 2^r - 2$ where $r > 1$. By Theorem 3.7, $2^r \leq \chi(Q_d^2)$. Since Q_{d+1} contains Q_d, we have that $\chi(Q_d^2) \leq \chi(Q_{d+1}^2)$. Now Theorem 3.5 implies $2^r \leq \chi(Q_d^2) \leq \chi(Q_{d+1}^2) = 2^r$. ■

Finding the distance 2 chromatic number of the d-dimensional cube in the case $d \notin \{2^r - 2, 2^r - 1 : r > 1\}$ remains an interesting open question.

4. $L(2,1)$-coloring

An $L(2,1)$-coloring of a graph G, introduced by J. Griggs and R. Yeh [7], is a labeling f of the vertex set onto the set of non-negative integers such that

(1) $|f(u) - f(v)| \geq 2$ if $\{u, v\} \in E(G)$
(2) $|f(u) - f(v)| \geq 1$ if $\text{dist}(u, v) = 2$.

For an $L(2,1)$-coloring f on a graph G, let $k = max_{u \in V(G)} f(u)$. Then the $L(2,1)$-chromatic number of a graph G, denoted by $\lambda(G)$, is defined to be the integer $\min\{k : f \text{ is an } L(2,1)\text{−coloring of } G\}$.

We again consider G to be the product of trees. We develop a bound for the $L(2,1)$-chromatic number of G.

Lemma 4.1. *Let $G = T_1 \square T_2 \square \cdots \square T_d$ be a product of nontrivial trees $T_1, T_2, \ldots, T_d$. Let* wt *be an admissible weighting on G where $\text{wt}(\{u,v\}) \neq 1$ for all edges $\{u,v\}$ of G. If $m = 2\max_{\{u,v\} \in E(G)} (\text{wt}(\{u,v\})) + 1$, then the m-coloring based on the weighting* wt *of G is an $L(2,1)$-coloring.*

Proof. Let $G = T_1 \Box T_2 \Box \cdots \Box T_d$ be a product of non-trivial trees $T_1, T_2, \ldots, T_d$ and $m = 2\max_{\{u,v\}\in E(G)}(\mathrm{wt}(\{u,v\}))+1$. Let wt be an admissible weighting on G and f the m-coloring of G based on wt. Assume that $\mathrm{wt}(\{u,v\}) \neq 1$ for all edges $\{u,v\}$ of G.

We begin by showing that adjacent vertices differ in color by at least two. Consider $v = (v_1, \ldots, v_d), v' = (v'_1, \ldots, v'_d) \in V(G)$ where $\{v, v'\} \in E(G)$. Then there exists i with $1 \leq i \leq d$ such that v_i and v'_i are adjacent in T_i and $v_k = v'_k$ for all $k \neq i$. Relabeling if necessary, we may assume that v'_i is the parent of v_i in T_i. Then

$$f(v) - f(v') \equiv \sum_{k=1}^{d} g_k(v_k) - \sum_{k=1}^{d} g_k(v'_k) \equiv g_i(v_i) - g_i(v'_i) \equiv \mathrm{wt}(\{v'_i, v_i\}) \pmod{m}.$$

By definition of wt and m, $2 \leq \mathrm{wt}(\{v'_i, v_i\}) \leq \frac{m-1}{2} \leq m-1$. Thus, $|f(v)-f(v')| \geq 2$.

Now note that vertices of distance 2 apart differ in color as f is a distance 2 coloring by Lemma 3.2. This completes the proof that f is an L(2,1) coloring. ■

Theorem 4.2. *Let $G = T_1 \Box T_2 \Box \cdots \Box T_d$ be a product of trees $T_1, T_2, \ldots, T_d$. Let $n = |\{i : 1 \leq i \leq d, T_i = P_2\}|$ and $s = \sum_{i=1}^{d} \Delta_i$, where Δ_i denotes the degree of a vertex of T_i of maximum degree. Then $\lambda(G) \leq 2(s - d + n) + 3$.*

Proof. Let $G = T_1 \Box T_2 \Box \cdots \Box T_d$, where $T_1, T_2, \ldots, T_d$ are trees. Let $n = |\{i : 1 \leq i \leq d, T_i = P_2\}|$ be the number of trees $T_i = P_2$, a path on two vertices. Define $\Delta_0 = 1$. For $1 \leq i \leq d$, let Δ_i denote the degree of a vertex of T_i of maximum degree. Set $s = \sum_{i=1}^{d} \Delta_i$. Then, if $n \geq 1$, we may assume that $T_1 = \cdots = T_n = P_2$ and that $\Delta_i \geq 2$ for all i, $n+1 \leq i \leq d$. If $n = 0$, then $\Delta_i \geq 2$ for all i, $1 \leq i \leq d$.

To prove this we create a weighting on G. First, root each tree T_i at a leaf $r_i \in V(T_i)$. Then direct all edges of T_i away from the root r_i. Recall that each T_i, $1 \leq i \leq n$, has exactly one edge. Assign the weight $i+1$ to such an edge. Now consider a tree T_i where $n+1 \leq i \leq d$; that is, consider a tree T_i with $\Delta_i \geq 2$. Given a vertex $v_i \in V(T_i)$, v_i has degree at most Δ_i and so has at most $\Delta_i - 1$ children. If $v_i \in V(T_i)$ has children $u_1, \ldots, u_t$, where $1 \leq t \leq \Delta_i - 1$, then define the weight of the edge $\{v_i, u_j\} \in E(T_i)$ by

$$\mathrm{wt}(\{v_i, u_j\}) = n + \left(\sum_{k=n}^{i-1} (\Delta_k - 1)\right) + j.$$

Since $1 \leq j \leq \Delta_i - 1$,

$$\left(\sum_{k=n}^{i-1} (\Delta_k - 1)\right) + 1 \quad \leq \quad \mathrm{wt}(\{v_i, u_j\}) \quad \leq \quad n + \sum_{k=n}^{i} (\Delta_k - 1)$$

for all edges $\{v_i, u_j\} \in E(T_i)$. In addition, $\mathrm{wt}(\{v_i, v'_i\}) \neq \mathrm{wt}(\{v_j, v'_j\})$ for two any edges $\{v_i, v'_i\} \in E(T_i)$, $\{v_j, v'_j\} \in E(T_j)$ in distinct trees T_i and T_j. This proves that wt is an admissible weighting on G.

Note that $\mathrm{wt}(\{u,v\}) \neq 1$ if $\{u,v\} \in E(G)$. Thus Lemma 4.1 applies and $f : V(G) \to \{1, 2, \ldots, m\}$ is an $L(2,1)$-coloring of G. ■

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