

On Weierstrass semigroups arising from finite graphs

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Abstract

We consider numerical semigroups that arise from finite graphs in a manner similar to that of Weierstrass semigroups from curves. Specifically, we study the collection $H_f(P)$ of nonnegative integers α that arise as coefficients of pole divisors of functions supported by the single vertex P of a finite, simple graph G . Our work is motivated by the classical Weierstrass semigroup of a rational point on a curve whose properties are tied to the Riemann-Roch Theorem as well as the analogue of the Riemann-Roch Theorem for finite graphs demonstrated by Baker and Norine. Here, we show that $H_f(P)$ is a numerical semigroup with at most g gaps, meaning nonnegative integers which are not elements of $H_f(P)$, where g denotes the cyclomatic number of the graph G . In addition, we determine $H_f(P)$ for vertices P of trees, unicyclic graphs, and bicyclic graphs as well as complete graphs and complete bipartite graphs.

1 Introduction

In this paper, we consider numerical semigroups that arise from finite graphs in a manner similar to that of Weierstrass semigroups from curves. Weierstrass semigroups are objects of classical study, dating back to the work of Hurwitz [11]. Given a nonsingular curve X of genus g over a finite field \mathbb{F} and an \mathbb{F} -rational point P on X , the Weierstrass semigroup of P is the collection of nonnegative integers α such that there exists a function f on X with pole divisor exactly αP ; that is,

$$H(P) := \{ \alpha \in \mathbb{N} : \exists f \text{ with } (f) = A - \alpha P, A \geq 0, \text{ and } P \notin \text{supp } A \}$$

where \mathbb{N} denotes the set of nonnegative integers, (f) denotes the divisor of the function f , and $\text{supp } A$ denotes the support of the divisor A . It is well known that $H(P)$ satisfies the following properties:

- (i) $H(P)$ is a numerical semigroup, meaning a subset of \mathbb{N} which is closed under addition, contains the zero element, and has a finite complement in \mathbb{N} ;

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(ii) $\mathbb{N} \setminus H(P) \subseteq [0, 2g - 1]$;

(iii) $|\mathbb{N} \setminus H(P)| = g$; and

(iv) $\alpha \in H(P)$ if and only if $\ell(\alpha P) = \ell((\alpha - 1)P) + 1$ where $\ell(\alpha P)$ denotes the dimension of the space of functions on X poles only at P and of order at most α .

For more on Weierstrass semigroups, please see [1, 6]. In this paper, we consider a similar notion that arises in the context of finite, simple graphs.

Given a vertex P of a finite, simple graph G , we consider the collection $H_f(P)$ of non-negative integers α that arise as coefficients of pole divisors of functions supported by the single vertex P . We prove that $H_f(P)$ is a numerical semigroup satisfying

$$\mathbb{N} \setminus H_f(P) \subseteq [0, 2g - 1]$$

and

$$|\mathbb{N} \setminus H_f(P)| \leq g$$

where $g := |E(G)| - |V(G)| + 1$ denotes the cyclomatic number of G . In [3], Baker and Norine define the rank of a divisor D on a graph G , denoted by $r(D)$ and prove that rank satisfies a Riemann-Roch Theorem; this study is carried on in [2, 4]. We show that if $r(\alpha P) = r((\alpha - 1)P) + 1$, then $\alpha \in H_f(P)$. However, the equality given in (iii) and the equivalence given in (iv) above may fail. We determine $H_f(P)$ for vertices P of several important families of graphs (trees, unicyclic graphs, bicyclic graphs, complete graphs, and complete bipartite graphs) and demonstrate that $H_f(P) \setminus \{\alpha \in \mathbb{N} : r(\alpha P) = r((\alpha - 1)P) + 1\}$ can be arbitrarily large.

This section concludes with a description of the notation used throughout the paper. Section 2 contains the necessary background on divisors of functions on graphs. Section 3 contains the main results. Examples are given in Section 4. The paper concludes with further discussion and open questions in Section 5.

Notation. The set of nonnegative integers is denoted by \mathbb{N} , and \mathbb{Z}^+ denotes the set of positive integers. Furthermore, given $a_1, \dots, a_k \in \mathbb{Z}^+$ with $\gcd(a_1, \dots, a_k) = 1$, the numerical semigroup generated by a_1, \dots, a_k is

$$\langle a_1, \dots, a_k \rangle := \left\{ \sum_{i=1}^k c_i a_i : c_i \in \mathbb{N} \right\}.$$

We say that α is a gap of $\langle a_1, \dots, a_k \rangle$ (equivalently, of a numerical semigroup S) if and only if $\alpha \in \mathbb{N} \setminus \langle a_1, \dots, a_k \rangle$ (equivalently, $\alpha \in \mathbb{N} \setminus S$). A general reference for numerical semigroups is [9].

All graphs in this paper are undirected, unweighted, finite, connected, with no loops, and with no multiple edges. Hence, we say a graph to mean a graph satisfying these conditions. Given a graph G , $V(G)$ denotes the vertex set of G , and $E(G)$ denotes the edge set of G . The neighborhood $Nbhd(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v .

Given a matrix A and $i, j \in \mathbb{Z}^+$, A_{ij} denotes the entry of A in the i^{th} row and j^{th} column. The transpose of A is denoted A^T .

2 Principal divisors on graphs

Let $V(G) = \{P_1, P_2, \dots, P_n\}$ be the set of vertices of a graph G . The group of divisors on G is the free abelian group on $V(G)$, denoted $Div(G)$. Elements of $Div(G)$ are called divisors, so a divisor D on G is of the form

$$D = \sum_{i=1}^n a_i P_i,$$

where $a_i \in \mathbb{Z}$. A divisor D is said to be effective if and only if $a_i \geq 0$ for all $P_i \in V(G)$, and the degree of D is $\deg(D) = \sum_{i=1}^n a_i$. The support of the divisor D is

$$\text{supp } D := \{P \in V(G) : a_i \neq 0\}.$$

The subset of all effective divisors of degree k is denoted by

$$Div_+^k(G) = \{D \in Div(G) : D \geq 0 \text{ and } \deg(D) = k\}.$$

There is a partial order \leq on $Div(G)$: given two divisors $A = \sum_{i=1}^n a_i P_i$ and $B = \sum_{i=1}^n b_i P_i$ on G , we say that $A \leq B$ if and only if $a_i \leq b_i$ for all $i, 1 \leq i \leq n$.

Consider the abelian group of integer-valued functions on the vertices of G ,

$$\mathcal{M}(G) = Hom\{V(G), \mathbb{Z}\}.$$

Associated with each function $f \in \mathcal{M}(G)$, there is a divisor on G defined as follows.

Definition 2.1. *Given a function $f \in \mathcal{M}(G)$, the divisor of f is*

$$\Delta(f) = \sum_{v \in V(G)} \left(\sum_{w \in N\text{bhd}(v)} (f(v) - f(w)) \right) v.$$

Notice that

$$\Delta(f) = \sum_{v \in V(G)} \left(f(v) \deg v - \sum_{w \in N\text{bhd}(v)} f(w) \right) v.$$

Definition 2.2. *Given a vertex P of a graph G and a function $f \in \mathcal{M}(G)$, set*

$$\Delta_P(f) := f(P) \deg P - \sum_{w \in N\text{bhd}(P)} f(w)$$

so that

$$\Delta(f) = \sum_{P \in V(G)} \Delta_P(f) P.$$

We say that f has a pole at P (resp., zero at P) if and only if $\Delta_P(f) < 0$ (resp., $\Delta_P(f) < 0$).

A divisor D is called principal provided $D = \Delta(f)$ for some $f \in \mathcal{M}(G)$. One may verify that every principal divisor has degree 0.

Fact 2.3. 1. Given $f, h \in \mathcal{M}(G)$, $f + h \in \mathcal{M}(G)$ and

$$\Delta(f + h) = \Delta(f) + \Delta(h). \quad (1)$$

As a consequence, for every $a \in \mathbb{Z}$, $\Delta(f + a) = \Delta(f)$.

2. Given $f \in \mathcal{M}(G)$, there exists $a \in \mathbb{Z}$ so that $h := f + a$ has

$$h(v) < 0 \quad \forall v \in V(G)$$

and

$$\Delta(h) = \Delta(f).$$

To verify Fact 2.3, observe that

$$\begin{aligned} \Delta(f) + \Delta(h) &= \sum_{v \in G} \left(f(v) \deg v - \sum_{w \in \text{Nbhd}(v)} f(w) \right) v \\ &\quad + \sum_{v \in G} \left(h(v) \deg v - \sum_{w \in \text{Nbhd}(v)} h(w) \right) v \\ &= \sum_{v \in G} \left((f(v) + h(v)) \deg v - \sum_{w \in \text{Nbhd}(v)} (f(w) + h(w)) \right) v \\ &= \sum_{v \in G} \left((f + h)(v) \deg v - \sum_{w \in \text{Nbhd}(v)} (f + h)(w) \right) v. \end{aligned}$$

Certainly, given $a \in \mathbb{Z}$, the function h defined by $h(v) = a$ for all $v \in V(G)$ has $\Delta(h) = 0$. Thus, the fact that $\Delta(f + a) = \Delta(f)$ follows immediately as a special case of the above. Given $f \in \mathcal{M}(G)$, take $a := \max \{f(v) : v \in V(G)\}$. Then $(f - a)(P) = f(P) - a \leq 0$.

Let A be the adjacency matrix of the graph G , so that $A \in \mathbb{Z}^{n \times n}$ with A_{ij} is the number of edges between P_i and P_j . Let $D \in \mathbb{Z}^{n \times n}$ be the matrix with entries $D_{ij} = 0$ if $i \neq j$ and $D_{ii} = \deg(P_i)$. The Laplacian of the graph G is the matrix $Q = D - A$. Hence, $Q \in \mathbb{Z}^{n \times n}$, and

$$Q_{ij} = \begin{cases} -|\{\text{edges between } P_i \text{ and } P_j\}| & \text{if } i \neq j \\ \deg(P_i) & \text{if } i = j. \end{cases}$$

Let (P_1, P_2, \dots, P_n) be a list consisting of all vertices of a graph G . If $f \in \mathcal{M}(G)$, we define $[f] \in \mathbb{Z}^n$ by $[f]_i = f(P_i)$. Similarly, if $D \in \text{Div}(G)$ where $D = \sum_{i=1}^n a_i P_i$, $[D]_i = a_i$. If Q is the Laplacian of G ,

$$[\Delta(f)] = Q[f].$$

Example 2.4. Consider the graph G given in Figure 2.1. Then

$$\mathcal{M}(G) = \{f : f(P_i) \in \mathbb{Z} \text{ for } i = 1, 2, 3, 4\}.$$

Consider $f \in \mathcal{M}(G)$ such that $f(P_1) = -1$, $f(P_3) = -1$, $f(P_2) = 0$, and $f(P_4) = 0$.

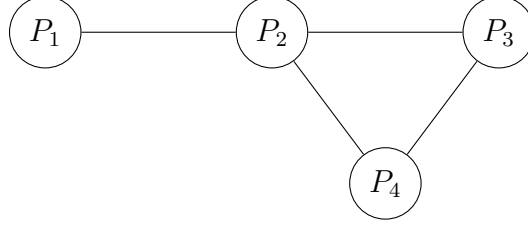


Figure 2.1: Graph G in Example 2.4

Then

$$\Delta_{P_1}(f) = -1 - 0 = -1$$

$$\Delta_{P_2}(f) = (0 - (-1)) + (0 - (-1)) = 2,$$

$$\Delta_{P_3}(f) = (-1 - 0) + (-1 - 0) = -2,$$

and

$$\Delta_{P_4}(f) = 0 - (-1) = 1.$$

Thus, the divisor of the function f is

$$\Delta(f) = 2P_2 + P_4 - P_1 - 2P_3.$$

The Laplacian of G is

$$Q = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

and $[f] = [-1, 0, -1, 0]^T$. One may verify that

$$Q[f] = [-1, 2, -2, -1]^T,$$

confirming the observation above that $\Delta(f) = 2P_2 + P_4 - P_1 - 2P_3$.

The Jacobian of G is the quotient group

$$Jac(G) = Div^0(G) / Prin(G)$$

where $Div^0(G)$ denotes the set of all divisors of degree zero on G , and $Prin(G)$ denotes the set of all principal divisors on G . We will make use of the Abel-Jacobi map

$$S_P^{(k)} : \begin{array}{ccc} Div_+^k(G) & \rightarrow & Jac(G) \\ D & \mapsto & D - kP \end{array}$$

defined in [3].

3 Weierstrass semigroups on graphs

Let G be an undirected, unweighted, finite, connected, with no loops, and with no multiple edges. Given a vertex P of G , consider the set

$$H_f(P) = \{\alpha \in \mathbb{N} : \exists f \in \mathcal{M}(G) \text{ with } \Delta(f) = A - \alpha P, A \geq 0 \text{ and } P \notin \text{supp} A\}$$

where $\mathcal{M}(G)$ denotes the set of integer valued functions on the vertex set $V(G)$ of G and $\Delta(f)$ is the divisor of the function f . The set $H_f(P)$, called the Weierstrass semigroup of the vertex P , is the primary object of study in this section; in Theorem 3.3, we confirm that $H_f(P)$ is indeed a numerical semigroup.

To aid in our understanding of $H_f(P)$, we consider another set which depends on ranks of divisors as defined in [3]. Let

$$H_r(P) := \{\alpha \in \mathbb{N} : r(\alpha P) = r((\alpha - 1)P) + 1\}$$

where

$$r(D) = \max \{k \in \mathbb{N} : \exists f \in \mathcal{M}(G) \text{ such that } \Delta(f) \geq E - D \text{ for all } E \in \text{Div}_+^k(G)\}$$

if such a nonnegative integer exists and $r(D) := -1$ otherwise. In [3, Theorem 1.12], Baker and Norine prove an analogue of the Riemann-Roch Theorem for finite graphs:

$$r(D) = \deg(D) + 1 - g + r(K - D)$$

where K is a canonical divisor on G . Notice that $r(D) = -1$ if $\deg D < 0$, and if $A \leq B$, then $r(B) - r(A) \leq \deg(B - A)$.

We will see that $H_r(P) \subseteq H_f(P)$, $\mathbb{N} \setminus H_r(P) \subseteq [0, 2g - 1]$, and $|\mathbb{N} \setminus H_r(P)| = g$ in addition to the fact that $H_r(P) \setminus H_f(P)$ can be arbitrarily large. Let

$$G_f(P) = \mathbb{N} \setminus H_f(P)$$

and

$$G_r(P) = \mathbb{N} \setminus H_r(P).$$

Lemma 3.1. *Let G be a graph, and let $P \in V(G)$. If $\alpha \geq 2g$, $\alpha \in H_r(P)$. Hence, $G_r(P)$ is finite. In fact,*

$$|G_r(P)| = g$$

and $G_r(P) \subseteq [0, 2g - 1]$.

Proof. If $\alpha \geq 2g$, then

$$r(\alpha P) = \alpha + 1 - g + (-1) = \alpha + g > (\alpha - 1) + 1 - g + (-1) = r((\alpha - 1)P).$$

Hence, $\alpha \in H_r(P)$ and $G_r(P) \subseteq [0, 2g - 1]$. To see that $|G_r(P)| = g$, notice that

$$0 = r(0P) \leq r(P) \leq r(2P) \leq r((2g - 1)P) = g - 1.$$

□

The following result establishes the relationship between $H_r(P)$ and $H_f(P)$.

Theorem 3.2. *Let G be a graph and P be a vertex of G . Then*

$$H_r(P) \subseteq H_f(P).$$

Thus, $|G_f(P)| \leq g$.

Proof. Let $\alpha \in H_r(P)$. Then $r((\alpha - 1)P) = k$ and $r(\alpha P) = k + 1$ for some $k \in \mathbb{Z}$. It follows that there exists $E_0 \in \text{Div}_+^{k+1}(G)$ so that for all $f \in \mathcal{M}(G)$,

$$(\alpha - 1)P - E_0 + \Delta(f) \not\geq 0.$$

By definition of $r(\alpha P)$, for all $E \in \text{Div}_+^{k+1}(G)$, there exists $f \in \mathcal{M}(G)$ so that

$$\alpha P - E + \Delta(f) \geq 0.$$

Thus, there exists $h \in \mathcal{M}(G)$ so that

$$\alpha P - E_0 + \Delta(h) \geq 0.$$

Thus, it must be that $\Delta_P(h) = \alpha$ and $\Delta_v(h) \geq 0$ for all $v \in V(G) \setminus \{P\}$. As a result, $\alpha \in H_f(P)$. \square

Next, we see that $H_f(P)$ is a numerical semigroup.

Proposition 3.3. *Let G be a graph and P be a vertex of G . If $\alpha, \beta \in H_f(P)$, $\alpha + \beta \in H_f(P)$.*

Proof. If $\alpha, \beta \in H_f(P)$, there exist $f, h \in \mathcal{M}(G)$ such that $\Delta(f) = A - \alpha P$ and $\Delta(h) = B - \beta P$ where $A, B \geq 0$ and $P \notin \text{supp } A \cup \text{supp } B$. According to (1),

$$\Delta(f + h) = A + B - (\alpha + \beta)P$$

which demonstrates that $\alpha + \beta \in H_f(P)$. \square

Next, we define a family of functions that will be useful in determining certain elements of numerical semigroups.

Definition 3.4. *Let G be a graph with vertex set $V(G) = \{P_1, P_2, \dots, P_n\}$. The indicator function f_{P_i} is defined by*

$$f_{P_i}(P_j) = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Notice that

$$\Delta(f_P) = \sum_{Q \in \text{Nbhd}(P)} Q - \deg P P.$$

Hence,

$$\deg P \in H_f(P). \tag{2}$$

Next, in Proposition 3.5, we see that $\deg(P)$ is the smallest nonzero element of $H_f(P)$ provided that the graph G' obtained from G by deleting the vertex P and its incident edges is connected.

Recall that the edge connectivity of a graph G , denoted $\lambda(G)$ is the minimum number of edges of G whose deletion results in a graph G' which is not connected. We say that G is $(k+1)$ -edge connected if and only if for every subset $E \subseteq E(G)$ with $|E| = k$, the deletion of the edges in E gives rise to a connected graph G' .

Proposition 3.5. *Given a vertex P of a graph G ,*

$$\lambda(G) \leq \min \{ \alpha \in H_f(P) : \alpha \neq 0 \}.$$

If G' is connected, where G' denotes the graph obtained from G by deleting P and its incident edges, then

$$\min \{ \alpha \in H_f(P) : \alpha \neq 0 \} = \deg P.$$

Proof. Let $\lambda := \lambda(G)$. Consider $0 < \gamma \leq \lambda - 1$. We claim that $\gamma \in G_f(P)$. Observe that G is $(\gamma + 1)$ -edge connected, because the graph resulting from the deletion of any γ edges of G is connected.

Suppose there exists $f \in \mathcal{M}(G)$ such that $\Delta(f) = A - \gamma P$ with $A \geq 0$ and $P \notin \text{supp } A$. Notice that $A \in \text{Div}_+^{\gamma} G$, since principal divisors have degree zero. According to [3, Theorem 1.8], S_P^{γ} is injective. Since

$$S_P^{\gamma}(A) = 0 \in \text{Jac}(G)$$

and

$$S_P^{\gamma}(\gamma P) = 0 \in \text{Jac}(G)$$

it must be that $A = \gamma P$ which is a contradiction. Thus, $\gamma \in G_f(P)$ for all γ , $0 < \gamma \leq \lambda - 1$.

Now assume that G' is connected. From Proposition 3.5, $\deg P \in H_f(P)$. We claim that $0 < c < \deg P$ implies $c \in G_f(P)$. Suppose $c \in H_f(P) \setminus \{0\}$ and $c < \deg P$. Then there exists $f \in \mathcal{M}(G)$ such that $\Delta(f) = A - cP$ for some $A \geq 0$ with $P \notin \text{supp } A$. For convenience, we may assume that $f(v) < 0$ for all $v \in V(G)$ as in Fact 2.3. Let $Q \in V(G) \setminus \{P\}$ be such that

$$f(Q) = \min \{ f(v) : v \in V(G) \setminus \{P\} \}.$$

If $Q \notin \text{Nbhd}(P)$, then

$$\Delta_Q(f) = \deg Q f(Q) - \sum_{v \in \text{Nbhd}(Q)} f(v) = \sum_{v \in \text{Nbhd}(Q)} f(Q) - f(v) \leq 0$$

which implies $f(v) = f(Q)$ for all $v \in \text{Nbhd}(Q)$; otherwise f has a pole at Q which is a contradiction. Since every vertex $v \in V(G) \setminus \{P\}$ has a neighbor $w \notin \text{Nbhd}(P)$,

$$f(v) = f(Q) \quad \forall v \in V(G) \setminus \{P\}.$$

It follows that

$$-c = \Delta_P(f) = \deg P (f(P) - f(Q)) \leq -\deg P,$$

meaning $c \geq \deg P$, which is a contradiction. Thus, no such c exists and

$$\deg P = \min \{ \alpha \in H_f(P) : \alpha \neq 0 \}.$$

□

4 Weierstrass semigroups of families of graphs

In this section, we determine the Weierstrass semigroups of vertices of trees, unicyclic graphs, bicyclic graphs, complete graphs, and complete bipartite graphs.

We begin by considering graphs G with small cyclomatic number $g = |E(G)| - |V(G)| + 1$. Loosely speaking, one may think of the cyclomatic number as the number of independent cycles in the graph. Observe that if G has cyclomatic number $g = 0$, then G is a tree.

Proposition 4.1. *If G is a tree, then $H_f(P) = \mathbb{N}$ for all vertices P of G .*

Proof. This follows immediately from Lemma 3.1 as G has cyclomatic number $g = 0$, meaning that $|G_r(P)| = 0$ and

$$\mathbb{N} = H_r(P) \subseteq H_f(P).$$

□

Next, we consider those graphs with cyclomatic number $g = 1$. Such a graph is called unicyclic.

Proposition 4.2. *If G is a unicyclic graph and $P \in V(G)$ where G' is connected, then*

$$H_f(P) = \begin{cases} \mathbb{N} & \text{if } \deg P = 1 \\ \langle 2, 3 \rangle & \text{otherwise.} \end{cases}$$

Proof. Assume that G has cyclomatic number $g = 1$ and G' is connected. According to Proposition 3.5, if $\deg P = 1$, then $H_f(P) = \mathbb{N}$. Suppose that $\deg P \geq 2$. Then, according to Proposition 3.5, $\deg P$ is the least nonzero element of $H_f(P)$. This, combined with Theorem 3.2 gives $|G_f(P)| = 1$. Since the only numerical semigroup with a single gap is $\langle 2, 3 \rangle$, $H_f(P) = \langle 2, 3 \rangle$. □

Next, we consider those graphs with cyclomatic number $g = 2$. Such a graph is called bicyclic.

Proposition 4.3. *If G is a bicyclic graph and $P \in V(G)$ where G' is connected, then*

$$H_f(P) = \begin{cases} \mathbb{N} & \text{if } \deg P = 1 \\ \langle 3, 4, 5 \rangle & \text{if } \deg P = 3 \\ \langle 2, 3 \rangle \text{ or } \langle 2, 5 \rangle & \text{otherwise.} \end{cases}$$

Proof. Assume that G is a bicyclic graph, $P \in V(G)$, and G' is connected. Proposition 3.5 handles the case where $\deg P = 1$.

Henceforth, we assume that $\deg P \geq 2$. Notice that $\deg P \leq 3$; otherwise, Proposition 3.5 gives $\{1, \dots, \deg P\} \subseteq G_f(P)$ which is a contradiction to the fact that $|G_f(P)| \leq 2$ by Theorem 3.2. If $\deg P = 3$, then Theorem 3.2 and Proposition 3.5 give $|G_f(P)| = \{1, 2\}$. Thus, $H_f(P) = \langle 3, 4, 5 \rangle$. Similarly, if $\deg P = 2$, then $H_f(P) = \langle 2, 3 \rangle$ or $H_f(P) = \langle 2, 5 \rangle$. □

Next, we consider Weierstrass semigroups of complete graphs and complete bipartite graphs.

Proposition 4.4. *Let P be a vertex of the complete graph K_n on n vertices. Then*

$$H_f(P) = \langle n-1, n \rangle.$$

Proof. First, notice that the indicator function for P has divisor

$$\Delta(f_P) = \sum_{v \in V(G) \setminus \{P\}} v - (n-1)P.$$

Let $Q \in V(K_n) \setminus \{P\}$. Then the function $f = f_P - f_Q$ has divisor

$$\Delta(f) = nQ - nP$$

according to (1). By Theorem 3.3, $\langle n-1, n \rangle \subseteq H_f(P)$.

Suppose $\alpha \in H_f(P) \setminus \langle n-1, n \rangle$. Then

$$\Delta(f) = A - \alpha P$$

for some $f \in \mathcal{M}(G)$ and $A \geq 0$ with $P \notin \text{supp } A$. Moreover,

$$\alpha = i(n-1) + j$$

with $0 \leq i < j \leq n-2$ as α must be element of the following array.

$$\begin{array}{ccccccc} 1 & 2 & 3 & \cdots & n-4 & n-3 & n-2 \\ n+1 & n+2 & n+3 & \cdots & 2n-4 & 2n-3 & \\ 2n+1 & 2n+2 & 2n+3 & \cdots & 3n-4 & & \\ \vdots & & & \ddots & & & \\ \vdots & & & \ddots & & & \\ \vdots & & & \ddots & & & \\ (n-1)n+1 & & & \ddots & & & \end{array}$$

It follows that

$$\sum_{v \in \text{Nbhd}(P)} h(v) - j = (n-1)(h(P) + i)$$

as $\Delta_P(f) = f(P)(n-1) - \sum_{v \in \text{Nbhd}(P)} h(v) = -\alpha = -i(n-1) - j$.

We may assume that $f(v) \leq 0$ for all $v \in V(K_n)$.

Choose $Q \in V(K_n) \setminus \{P\}$ so that $f(Q) = \min \{f(v) : v \in V(K_n) \setminus \{P\}\}$. Suppose that $f(Q) \geq f(P) + i + 1$. Then

$$\sum_{v \in V(K_n) \setminus P} f(v) \geq (n-1)f(Q) \geq (n-1)(f(P) + i + 1)$$

implies

$$j = \sum_{v \in V(K_n) \setminus P} f(v) - (n-1)(f(P) + i + 1) \geq n-1 > j$$

which is a contradiction. Thus, $f(Q) \leq f(P) + i$. It follows that

$$\begin{aligned} \Delta_Q(f) &= f(Q)(n-1) - \sum_{v \in \text{Nbhd}(Q)} f(v) \\ &= f(Q)n - f(P) - \sum_{v \in \text{Nbhd}(P)} f(v) \\ &\leq n(f(P) + i) - f(P) - \sum_{v \in \text{Nbhd}(P)} f(v) \\ &= \Delta_P(f) + ni \\ &= -\alpha + ni = i - j < 0 \end{aligned}$$

which is a contradiction as Q is not a pole of f . Thus, $H_f(P) \setminus \langle n-1, n \rangle = \emptyset$. \square

The next result follows immediately from the fact that K_n has cyclomatic number

$$g = \binom{n}{2} - n + 1 = |\mathbb{N} \setminus \langle n-1, n \rangle|.$$

Corollary 4.5. *Given any vertex P of the complete graph K_n , $H_r(P) = \langle n-1, n \rangle$.*

Proposition 4.6. *Let (U, V) be the natural partition of the vertices of the complete bipartite graph $K_{m,n}$ where $|U| = m$ and $|V| = n$. If $P \in U$, then*

$$H_f(P) = \langle n, (m-1)n+1, (m-1)n+2, \dots, (m-1)n+(n-1) \rangle.$$

Proof. Let $U = \{P, P_1, \dots, P_{m-1}\}$ and $V = \{Q_1, \dots, Q_n\}$. Consider the function

$$h := mf_P + \sum_{i=1}^l f_{Q_i}$$

where $1 \leq l \leq n-1$. Then, according to (1),

$$\begin{aligned} \Delta(h) &= m\Delta(f_P) + \sum_{i=1}^l \Delta(f_{Q_i}) \\ &= m \sum_{v \in \text{Nbhd}(P)} v - mnP + \sum_{i=1}^l \left(\sum_{w \in \text{Nbhd}(Q_i)} w - mQ_i \right) \\ &= m \sum_{v \in V} v + l \sum_{w \in V} w - \sum_{i=1}^l mQ_i - mnP \\ &= \sum_{i=l+1}^n mQ_i + l \sum_{w \in U \setminus \{P\}} w - (mn-l)P. \end{aligned}$$

Since $l \leq n-1 \leq n \leq mn$, h has a pole at P of order $mn-l$, and $mn-l = (m-1)n+n-l \in H_f(P)$. Consequently,

$$\langle n, (m-1)n+1, \dots, (m-1)n+n-1 \rangle \subseteq H_f(P).$$

Now assume there exists $\alpha \in H_f(P) \setminus \langle n, (m-1)n+1, \dots, (m-1)n+n-1 \rangle$. Then $\alpha = in + j$ with $0 \leq i \leq m-2$ and $1 \leq j \leq n-1$. In addition, there exists $f \in \mathcal{M}(G)$ with $\Delta(f) = A - \alpha P$ where $A \geq 0$ and $P \notin \text{supp } A$. As before, we may assume that $f(v) \leq 0$ for all $v \in V(G)$. Then $-\alpha = \Delta_P(f) = nf(P) - \sum_{v \in N_{\text{bhd}}(P)} f(v)$, which implies

$$\sum_{v \in V} f(v) = tn + j$$

with $t := f(P) + i \leq -1$. Let $v \in V(K_{m,n})$ be such that

$$f(v) = \min \{f(w) : w \in V(K_{m,n}) \setminus \{P\}\}.$$

Suppose that $v = P_i \in U$. Then

$$\Delta_v(f) = nf(v) - \sum_{w \in V} f(w) = n(f(v) - t) - j$$

which implies $f(v) > t$ as v is not a pole of f . Then

$$\sum_{i=1}^n f(Q_i) \geq nf(v) \leq n(t+1) > nt + j = \sum_{v \in V} f(v)$$

which is a contradiction. Consequently, $v \notin U$, $v = Q_j \in V$ for some $1 \leq j \leq n$, and $f(Q_j) < f(P_i)$ for all i, j . Then

$$\begin{aligned} \Delta_{Q_j}(f) &= \deg Q_j f(Q_j) - f(P) - \sum_{i=1}^m f(P_i) \\ &= mf(Q_j) - \sum_{i=1}^m f(P_i) - f(P) \\ &\leq -(m-1) + f(Q_j) - f(P) \\ &\leq -(m-1) + i \\ &\leq -1 \end{aligned}$$

which is a contradiction as f does not have a pole at Q_j . Thus,

$$H_f(P) \setminus \langle n, (m-1)n+1, (m-1)n+2, \dots, (m-1)n+(n-1) \rangle = \emptyset.$$

□

The next result follows immediately from the fact that $K_{m,n}$ has cyclomatic number

$$g = mn - (m+n) + 1 = |\mathbb{N} \setminus \langle n, (m-1)n+1, (m-1)n+2, \dots, (m-1)n+(n-1) \rangle|.$$

Corollary 4.7. *Let (U, V) be the natural partition of the vertices of the complete bipartite graph $K_{m,n}$ where $|U| = m$ and $|V| = n$. If $P \in U$, then*

$$H_r(P) = \langle n, (m-1)n+1, (m-1)n+2, \dots, (m-1)n+(n-1) \rangle.$$

Based on the results above for certain graphs G (such as trees, cycles, complete graphs, and complete bipartite graphs), one might be tempted to conjecture that $H_r(P) = H_f(P)$ for all $P \in V(G)$. As the next examples demonstrate, this does not hold in general.

Example 4.8. Consider the graph G in Example 2.4. Note that $g = 1$. By Proposition 3.1, $|G_r(P)| = 1$ and if $\alpha \geq 2$, $\alpha \in H_r(P)$. Thus,

$$H_r(P) = \langle 2, 3 \rangle$$

for all $P \in V(G)$. However, according to Proposition 3.5, $H_f(P_1) = \mathbb{N}$, and $H_r(P_1) \neq H_f(P_1)$.

The next example shows we do not always obtain $H_f(P) = H_r(P)$ even for $P \in V(G)$ for a regular graph G .

Example 4.9. Consider the cube graph Q_3 as shown in Figure 4. Then $\deg(P) = 3$ for all $P \in V(Q_3)$. According to Proposition 3.5, $3 \in H_f(P)$ for all $P \in V(Q_3)$. However, using Sage [17], one can determine that $H_r(P) = \mathbb{N} \setminus \{1, 2, 3, 4, 5\} = \langle 6, 7, 8, 9, 10, 11 \rangle$. Thus,

$$H_r(P) \subsetneq H_f(P).$$

We mention two ways in which one can determine $H_f(P)$ in this example. First, using Theorem 3.2, $G_f(P) \subseteq \{1, 2, 4, 5\}$. It is easy to check that the Smith normal form of the Laplacian of G is $A := \text{Diag}(1, 1, 1, 1, 2, 8, 24, 0) \in \mathbb{Z}^{8 \times 8}$ and that there are no integral solutions to $Ax = [-4, b_2, b_3, b_4, b_5, b_6, b_7, b_8]^T$ with $b_i \in \mathbb{N}$ and $\sum_{i=2}^8 b_i = 4$. It follows that $4, 5 \in G_f(P)$. Thus, $H_f(P) = \langle 3, 7, 8 \rangle$.

Alternatively, $H_f(P)$ can be computed without the assistance of $H_r(P)$ as follows. Suppose $P = P_1$; a similar argument holds for any other vertex of Q_3 . To see that $7 \in H_f(P)$, consider the function $f \in \mathcal{M}(Q_3)$ given by

$$f(v) = \begin{cases} -3 & \text{if } v = P_1 \\ -1 & \text{if } v = P_2, P_4 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\Delta(f) = 2P_3 + 3P_5 + P_6 + P_8 - 7P_1.$$

To see that $8 \in H_f(P)$, consider the function $h \in \mathcal{M}(Q_3)$ given by

$$h(v) = \begin{cases} -3 & \text{if } v = P_1 \\ -1 & \text{if } v = P_2 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\Delta(f) = P_3 + 3P_4 + 3P_5 + P_6 - 8P_1,$$

and $8 \in H_f(P)$. This allows us to see that $\langle 3, 7, 8 \rangle \subseteq H_f(P)$, and it may be checked as above that $4, 5 \notin H_f(P)$.

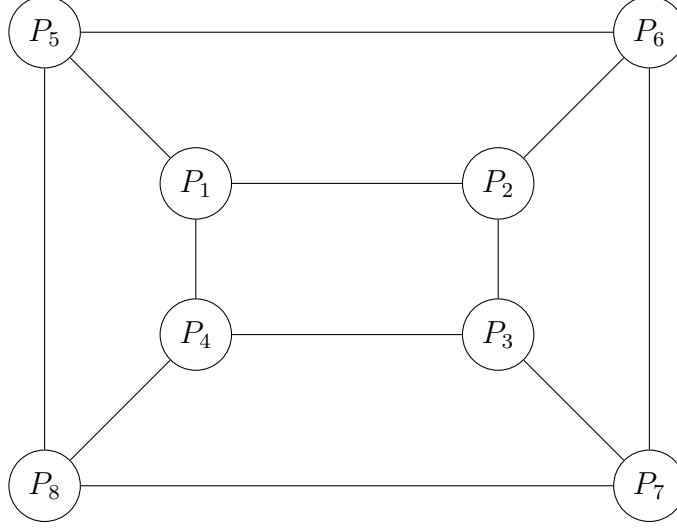


Figure 4.1: The cube graph Q_3 considered in Example 4.9

We conclude with a result demonstrating that $H_f(P) \setminus H_r(P)$ can be arbitrarily large.

Proposition 4.10. *For every $n \in \mathbb{Z}^+$, there exists a graph G with vertex P so that*

$$|H_f(P) \setminus H_r(P)| = n.$$

Proof. To construct such a graph let G_1 be a graph with cyclomatic number g_1 and G_2 be a graph with cyclomatic number g_2 . Consider the graph G formed by introducing a new vertex P of degree two in G which is adjacent to exactly one vertex of G_1 and exactly one vertex of G_2 . To see that $H_f(P) = \mathbb{N}$, consider the function $f \in \mathcal{M}(G)$ given by

$$f(v) = \begin{cases} 1 & \text{if } v \in V(G_1) \\ 0 & \text{otherwise.} \end{cases}$$

which has

$$\Delta(f) = Q - P$$

where $Q \in V(G_1)$ is adjacent to P . However, G has cyclomatic number $g_1 + g_2$, which implies $|G_r(P)| = g_1 + g_2$ while $H_f(P)$ has no gaps. As a result,

$$|H_f(P) \setminus H_r(P)| = g_1 + g_2.$$

□

To demonstrate Proposition 4.10, we provide the following example.

Example 4.11. *Consider the graph G shown in Figure 4.2.*

Note that $g = 2$. Then, $2 = |G_r(P)|$ for all $P \in V(G)$ by Proposition 3.1. Now, consider $f \in \mathcal{M}(G)$ where $f(P_i) = 0$ for $i = 1, 2, 3, 7$ and $f(P_i) = 1$ for $i = 4, 5, 6$. Then, $\Delta(f) = P_5 - P_7$. Thus, $1 \in H_f(P_7)$. By Proposition 3.3, $H_f(P_7) = \mathbb{N}$.

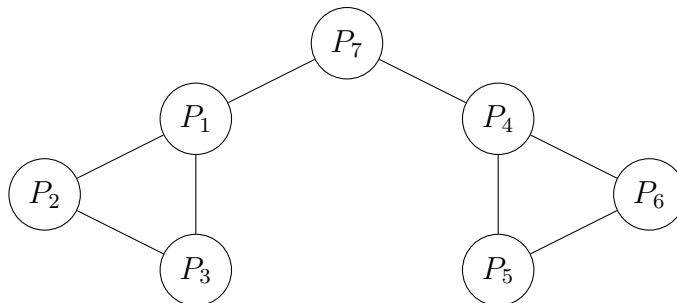


Figure 4.2: The graph G considered in Example 4.11

5 Conclusion

Our work prompts several questions. Which numerical semigroups arise as a Weierstrass semigroup of a vertex of a finite graph? The analogous problem for points on curves is a deep one, first suggested by Hurwitz [11]. Nearly 100 years later, Buchweitz [5] proved that not every numerical semigroup is the Weierstrass semigroup of a point on a curve and defined what is now called the Buchweitz Criterion. This problem was further addressed in [8, 13] (see also references therein) and more recently [12] but remains open. What can be said about the structure of $H_r(P)$? The analogous set for points on curves (defined appropriately) has the property that $H_r(P) = H_f(P)$. However, we see that this fails dramatically for vertices on finite graphs, as demonstrated in Proposition 4.10. This leaves one to consider what can be said about $H_r(P)$. Of course, one may study Weierstrass semigroups of vertices on certain families of graphs. In particular, one may consider covers of graphs and associated semigroups as has been done for coverings of curves [7, 10, 14, 15, 16].

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