Analysis of termatiko sets in measurement matrices

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Abstract

Termatiko sets are combinatorial structures that have been shown to hinder the success of the Interval-Passing Algorithm in compressed sensing. In this paper, we show how termatiko sets relate to other combinatorial structures in graphs representing measurement matrices that are also known to cause failure in similar iterative algorithms. We give bounds on the sizes of termatiko sets of measurement matrices based on finite geometries and also investigate the effect of the redundancy of the matrices on the number of these sets.

1 Introduction

Compressed sensing is a novel technique used to recover a sparse signal by sampling at a rate much lower than the Nyquist rate. Researchers have shown that perfect recovery is achievable by effectively exploiting the sparsity of the signal in some basis known only at the receiver. While the signals can be of any nature, the results also apply to recovering sparse discrete-time signals that may be represented as sparse vectors. In particular, to recover a signal that is a k-sparse vector, meaning the number of its nonzero positions is at most k, Candes and Tao showed that using a small number of measurements and a linear programming (LP) algorithm over the ℓ_1-norm was effective in recovering the signal perfectly and provided bounds on the number of measurements needed [1].

While linear programming is a practical technique in many applications, its implementation complexity is polynomial in the length of the signal. More efficient and almost linear-time algorithms are desirable for many practical applications. A close to linear-time algorithm is the Interval-Passing Algorithm (IPA) [2]. Typically, to recover a k-sparse vector $\mathbf{x}$ of length n, an $m \times n$ measurement matrix M is used. The measurement $\mathbf{y}$ is obtained as $M\mathbf{x}$. Random real-valued entries in the measurement matrix have been shown to be effective for the compressed

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sensing problem, and it was shown recently that sparse binary matrices can be as effective [3, 4]. When sparse measurement matrices are used, the IPA complexity is almost linear-time.

In [5], it was shown that there are close connections between LP decoding of low-density parity-check (LDPC) codes over a communication channel and LP reconstruction of sparse signals in compressed sensing. Specifically, using the same sparse binary parity-check matrices for LDPC codes as measurement matrices in compressed sensing, the authors showed that if the LP decoder could recover any k symbol errors, then the LP reconstruction could recover any k-sparse signal. LDPC codes are typically decoded using iterative graph-based message-passing algorithms, such as the Sum-Product Algorithm (SPA), which can be visualized as operating on the incidence graph, called the Tanner graph, of the corresponding parity-check matrix. Similarly, taking the incidence graph of the measurement matrix, one can visualize the IPA as a message-passing algorithm operating on this graph. The sparser the matrix, the lower the complexity of the algorithm.

The efficiency of algorithms like the SPA and IPA makes them preferred in practice, despite the fact that they are suboptimal. Therefore, a main goal is to improve their performance by careful design of the corresponding matrices (equivalently, graphs). Failure of the sum-product algorithm operating on an LDPC code graph has been characterized by combinatorial substructures in the Tanner graph. Among these substructures are stopping sets (in the case of the binary erasure channel), trapping sets, absorbing sets, and pseudocodewords [6, 7, 8, 9]. Similarly, it was recently shown that certain graph structures characterize when the IPA fails to recover the signal [3]. Specifically, the IPA will fail if and only if the k nonzero positions of the signal correspond to the vertices in a so-called *termatiko set*.

In this paper, we consider the relationship between termatiko sets and other combinatorial structures that characterize failure in related iterative decoding algorithms. This includes stopping, trapping, and absorbing sets. We also examine some classes of Tanner graphs including those based on finite geometries and those with degree regularity to obtain bounds on the cardinality of the smallest termatiko sets. These sets are those most likely to lead to failure in IPA.

This paper is organized as follows. In Section 2, we provide the necessary background. In Section 3, we present some results characterizing the relationship between termatiko sets and stopping sets, and we give a lower bound on the size of termatiko sets in left-regular Tanner graphs. We also examine canonical examples of trapping and absorbing sets to determine whether or not they are also termatiko. In Section 4, we share a case study for whether redundancy in the measurement matrix affects the presence of termatiko sets. In Section 5, we give results on the sizes and types of termatiko sets that occur in finite-geometry based measurement matrices. Section 6 concludes the paper.

2 Preliminaries

In this section, we provide any necessary definitions and notation used in this paper, including the background from coding theory as well as compressed sensing.

Given a parity-check matrix H of a binary LDPC code $\mathcal{C}$, the columns correspond to the coordinate positions of a codeword and the rows correspond to the parity-check equations (or more general constraints). The corresponding *Tanner graph* G is a bipartite graph where one set of vertices V corresponds to the columns (called *variable nodes*) and one set of vertices W corresponds to the rows (called *check nodes*). An edge connects the ith variable node to the jth check node if and only if the (j, i)th entry of H is non-zero. We sometimes say the Tanner graph

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

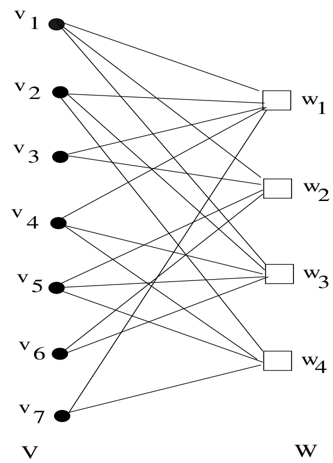


Figure 1: A measurement matrix M on the left, with its corresponding Tanner graph representation on the right.

of $\mathcal{C}$ to mean the Tanner graph corresponding to a particular parity-check matrix of $\mathcal{C}$.

The Tanner graph of an $m \times n$ measurement matrix M in compressed sensing is defined similarly. The columns correspond to the coordinate positions of the signal and the rows correspond to the measurements. In the corresponding Tanner graph, the columns are represented by a set V of variable nodes and the rows are represented by a set W of *measurement nodes*. If the matrix is non-binary, the edges in the Tanner graph are labeled with the corresponding entry. However, for binary matrices, the presence of an edge implies a 1 in the matrix, and no edge label is needed. In this paper, we consider only binary matrices.

Example 2.1. Figure 1 shows an example of a measurement matrix M and its corresponding Tanner graph. The variable nodes are represented by shaded circles, and the measurement nodes are represented by squares.

For a subset X of vertices in a graph G, let $N(X)$ denote the set of vertices that are adjacent to at least one vertex in X. In addition, $N(u) := N(\{u\})$ for a vertex u of G.

It is common in coding theory to draw Tanner graphs in their *left-right representation*, meaning that the variable node set is on the left, and the check/measurement node set is on the right, as in Figure 1. Moreover, when all variable nodes have degree j, the Tanner graph is said to be *j-left regular*, and when additionally, all check/measurement nodes have degree k, the Tanner graph is said to be *(j,k)-regular*. The *girth* of a graph is the length of its shortest cycle. For bipartite simple graphs (meaning those with no multiple edges), the girth is even and at least four. Since small cycles are known to cause problems with iterative decoding [10], it is common to consider graphs with girth at least six.

In [6], it was shown that iterative decoder failure on the Binary Erasure Channel (BEC) is characterized by the presence of stopping sets in the associated Tanner graph.

Definition 2.2. A *stopping set* $S \subseteq V$ is a subset of the variable nodes such that each check node in $N(S)$ is adjacent to at least two vertices in S. Equivalently, S is a stopping set if there is no check node with exactly one edge adjacent to a vertex in S.

The idea is that if all of the vertices in the stopping set correspond to positions in the codeword that are erased, then none of the check nodes will be able to recover any of the erased bits on the stopping set since each check neighbor of the set connects to it at least twice. Thus, the decoder fails exactly when a subset of the erased positions forms a stopping set. Consequently, one wants the smallest stopping sets to be as large as possible to minimize the probability of decoder failure. The size of a smallest nonempty stopping set in a Tanner graph is called the *stopping distance* and is denoted by $s_{\min}$.

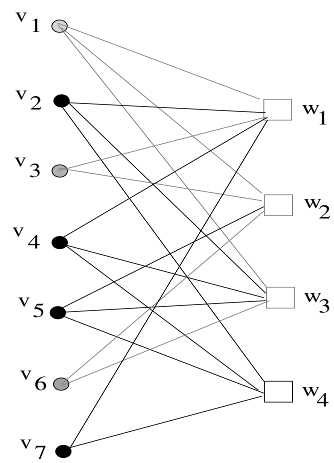


Figure 2: The set $\{v_1, v_3, v_6\}$ is both a stopping set and a termatiko set in the Tanner graph.

Example 2.3. In the Tanner graph in Figure 1, the vertices v_1, v_3, v_6 form a stopping set S, since each vertex in the set of their neighbors $\{w_1, w_2, w_3\}$ connects to the set S at least twice. The set $S = \{v_1, v_3, v_6\}$ is highlighted in Figure 2. On the other hand, the set $\{v_1, v_6\}$ does not form a stopping set since w_1 is only adjacent to v_1. The smallest stopping set in the graph in Figure 2 is $\{v_2, v_4\}$, thus the stopping distance of the graph is $s_{\min} = 2$.

In [3], it was shown that IPA failure is characterized by the presence of termatiko sets in the associated Tanner graph.

Definition 2.4. Let $G = (V, W; E)$ be a Tanner graph corresponding to a measurement matrix M. Let $T \subseteq V$, let $N(T) \subseteq W$ be the set of check nodes neighbors of T and let $U = \{u \in V \backslash T \mid N(u) \subseteq N(T)\}$. Then T is a *termatiko set* if and only if for each $c \in N(T)$ one of the two conditions holds:

I. c is adjacent to a vertex $u \in U$.

II. c is not adjacent to any vertex $u \in U$, but c is adjacent to at least two vertices v_1 and v_2 in T and every check node $c' \in N(v_1) \cup N(v_2)$ is adjacent to at least two vertices in T.

Throughout this paper, when we refer to Condition I and Condition II, we will mean the conditions in Definition 2.4. Moreover, references to U will mean the set U corresponding to a termatiko set T (from context) in Definition 2.4.

Example 2.5. The set $T = \{v_1, v_3, v_6\}$ in Figure 2 is also a termatiko set. To see this, note that $N(T) = \{w_1, w_2, w_3\}$ and none of the vertices v_2, v_4, v_5, v_7 have all of their neighbors in T. Thus, $U = \emptyset$. To be a termatiko set, each measurement node in $N(T)$ must satisfy Condition I or II. It is easy to verify that each of the nodes w_1, w_2, w_3 satisfy Condition II.

Notice that $T = \{v_2, v_4\}$ is another termatiko set. Indeed, in this case, $N(T) = \{w_1, w_3, w_4\}$ and $U = \{v_7\}$. Both w_1 and w_4 are adjacent to $v_7 \in U$; hence, w_1 and w_4 satisfy Condition I. However, w_3 fails to satisfy Condition I as it is not adjacent to v_7, the only element of U. Even so, w_3 is adjacent to both v_2 and v_4 (which are vertices in T). Moreover, investigating the vertices in $N(v_2) \cup N(v_4) = \{w_1, w_3, w_4\}$, we can confirm that each of them is adjacent to both v_2 and v_4. As a result, w_3 satisfies Condition II. This verifies that $T = \{v_2, v_4\}$ is a termatiko set and demonstrates the situation in which some elements of $N(T)$ satisfy Condition I while others satisfy Condition II.

As with stopping sets, the IPA algorithm is improved when the smallest termatiko sets are as large as possible. The size of a smallest nonempty termatiko set is called the *termatiko distance* and is denoted by $t_{\min}$.

3 Connections between termatiko sets and other structures

We start this section by examining the connection between termatiko sets and stopping sets. As noted in [3], every stopping set is a termatiko set. To see this, consider the following. If $S \subseteq V$ is a stopping set, then by Definition 2.2, each $c \in N(S) \subseteq W$ is adjacent to at least two vertices v_1 and v_2 in S. Since every check node $c' \in N(v_1) \cup N(v_2)$ is also adjacent to at least two vertices in S, it follows that S is termatiko; indeed, each node in $N(T)$ satisfies Condition II. Therefore, for any Tanner graph, $t_{\min} \leq s_{\min}$.

However, not every termatiko set is a stopping set.

Example 3.1. For an example of a termatiko set that is not a stopping set, see Figure 3. The set $T = \{v_6, v_7\}$ is termatiko because all check node neighbors satisfy Condition I of the definition. In this case $U = \{v_1, v_2, v_3, v_4, v_5\}$, since $N(T) = \{w_1, w_2, w_3, w_4\}$. T is not a stopping set since (for example) the check node w_1 is adjacent to one vertex in T.

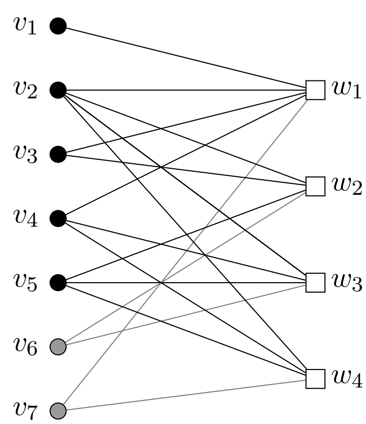


Figure 3: The set $T = \{v_6, v_7\}$ is a termatiko set in which every check node neighbor of vertices in T satisfies Condition I of the termatiko set definition.

In what follows, we will prove some results relating termatiko sets to stopping sets. Throughout we assume the set U is as defined in Definition 2.4.

Although a termatiko set T is not necessarily a stopping set, we can show that $T \cup U$ is one.

Proposition 3.2. *For any termatiko set T in a Tanner graph G, $T \cup U$ is a stopping set in G.*

Proof. Let T be a termatiko set, and U be as defined in Definition 2.4. Let $c \in N(T \cup U) = N(T)$. Suppose $c \in N(U)$. Thus, there exists $u \in U$ that is adjacent to c. Also, since $c \in N(T)$, there exists a $t \in T$ that is adjacent to c. Therefore, c is adjacent to at least two vertices in $T \cup U$.

Now suppose $c \in N(T) \setminus N(U)$. By definition of a termatiko set, c must satisfy Condition II. In particular, this means that c is adjacent to at least two vertices in T and therefore c is adjacent to at least two vertices in $T \cup U$. Since any $c \in N(T \cup U)$ is always adjacent to at least two vertices in $T \cup U$, the set $T \cup U$ is a stopping set. □

The previous result shows that every termatiko set T is contained in a stopping set. In particular, when $U = \emptyset$, T is a stopping set. Note that this is not a necessary condition, since T may be a stopping set with $U \neq \emptyset$.

Proposition 3.3. *Suppose T is a termatiko set that is not a stopping set. Let W' be the set of check nodes that are adjacent to exactly one vertex in T, and let $U' = N(W') \cap U$. Then the set $T \cup U'$ is a stopping set.*

Proof. Suppose T is a termatiko set that is not a stopping set, and assume $W' \subseteq N(T)$ is such that each $c \in W'$ is adjacent to exactly one vertex in T. Since T is termatiko, each $c \in W'$ is adjacent to at least one vertex in U. Let $U' = N(W') \cap U$. Then $T \cup U'$ is a stopping set since each vertex $c \in N(T) \setminus W'$ is adjacent to at least two vertices in T, and each vertex $w \in W'$ is adjacent to one vertex in T and at least one vertex in U'. □

Sparse matrices that have j nonzero positions per column for $j \geq 3$ have been heavily studied in coding theory and are good candidates for measurement matrices. They correspond to j-left regular Tanner graphs. We first look at the cardinality of the smallest termatiko sets that can exist in a j-left regular graph, assuming the girth of the graph is at least six.

Theorem 3.4. *Suppose G is a j-left regular Tanner graph with girth at least 6. Then any termatiko set T in G has size $|T| \geq j$. Moreover, for every integer $j \geq 2$, there exists a j-left regular Tanner graph with girth at least 6 and termatiko distance j.*

Proof. We first treat the case $j = 2$. If G is 2-left regular, then it remains to show there cannot be a termatiko set of size 1. Observe that if $T = \{v_1\}$ is a termatiko set, then without loss of generality, let $N(v_1) = \{c_1, c_2\}$. If $u \in U$, then $N(u) = \{c_1, c_2\}$ since u has degree 2 also. But then G has a 4-cycle which contradicts the assumption. Therefore $U = \emptyset$ and neither c_1 nor c_2 can satisfy Condition I. Observe that c_1 and c_2 also cannot satisfy Condition II since each has only one neighbor in T. Thus, the smallest termatiko set in G has size $j \geq 2$.

Let T be a termatiko set in G, and suppose that $|T| < j$. We first show that no check node in $N(T)$ satisfies Condition II. Suppose there exists a vertex $c \in N(T)$ such that $c \notin N(U)$ and c is adjacent to $x, y \in T$ and every $c' \in N(x) \cup N(y)$ is adjacent to some two vertices in T. Then any $d \in N(x) \setminus \{c\}$ is adjacent to x and at least one other vertex in T. If two vertices in $N(x)$ were adjacent to the same vertex $v \neq x \in T$, then there would be a 4-cycle in G, e.g. (c, x, d, v, c). Thus, each of the $j - 1$ elements in $N(x) \setminus \{c\}$ are adjacent to a different vertex $v \neq x \in T$. So we have $j - 1$ different vertices other than x in T that are adjacent to $d \in N(x) \setminus \{c\}$. This means T has size greater than $j - 1$, which is a contradiction. Thus, Condition II cannot hold for any $c \in N(T)$.

We can therefore assume that for all $c \in N(T)$, Condition I is satisfied, so $c \in N(U)$. Let $c \in N(T)$ and suppose c is adjacent to some $v \in T$. By assumption, c is adjacent to some $u \in U$. Since G is j-left regular, u has $j - 1$ neighbors other than c in $N(T)$. Moreover, no two vertices

$d_1, d_2 \in N(u)$ are adjacent to the same vertex $x \in T$, since this would create a 4-cycle. This means $N(u)$ is adjacent to j distinct vertices in T, which is a contradiction. Thus, $|T| \geq j$.

Given an integer $j \geq 2$, we may construct a j-left regular Tanner graph $G = (V, W; E)$ with girth at least 6 and termatiko distance j as follows. Let $V = T \cup U$, where $T = \{v_1, \ldots, v_j\}$ and $U = \{u_1, \ldots, u_j\}$, and $W = \{c_1, c_2, \ldots, c_{j^2}\}$. For $i = 1, \ldots, j$, let $N(v_i) = \{c_{j(i-1)+1}, c_{j(i-1)+2}, \ldots, c_{ij}\}$, and let $N(u_i) = \{c_i, c_{i+j}, \ldots, c_{i+(j-1)j}\}$. Then G is j-left regular, $N(U) = N(T) = W$, and each $c \in W$ is adjacent to a vertex in U. Hence, Condition I is satisfied for each $c_i \in N(T)$, and T is a termatiko set of size j. It remains to show that there are no 4-cycles. First observe that the neighborhoods of any pair $v_i, v_j \in T, i \neq j$ have empty intersection, as do the neighborhoods of any pair $u_i, u_j \in U, i \neq j$. Further, the neighborhoods of $v_k \in T$ and $u_\ell \in U$ intersect only in check node $c_{j(k-1)+\ell}$, and thus there are no 4-cycles in G. By the previous argument, G does not have any smaller termatiko set, and G has termatiko distance j. □

While stopping sets characterize iterative decoder failure on the binary erasure channel, absorbing sets and trapping sets have been shown to characterize iterative decoder failure on other communication channels. In the rest of the section, we will look at specific types of absorbing sets and trapping sets that have been heavily studied and examine whether they are termatiko.

Let $G_D = (D, N(D); E_D)$ denote the subgraph induced by $D \subseteq V$ and $N(D) \subseteq W$ in a Tanner graph G, where E_D is the set of edges between D and $N(D)$.

Definition 3.5. An (a, b)-*trapping set* is a subset $D \subseteq V$ of variable nodes in the Tanner graph $G = (V, W; E)$ of a code such that $|D| = a$ and $|O(D)| = b$, where $O(D)$ is the subset of check nodes of odd degree in the subgraph G_D.

Definition 3.6. An (a, b)-*absorbing set* is a subset $D \subseteq V$ of variable nodes in the Tanner graph $G = (V, W; E)$ of a code such that $|D| = a$, $|O(D)| = b$, and each variable node in D has strictly fewer neighbors in $O(D)$ than in $W \setminus O(D)$, where $O(D)$ is the subset of check nodes of odd degree in the subgraph G_D.

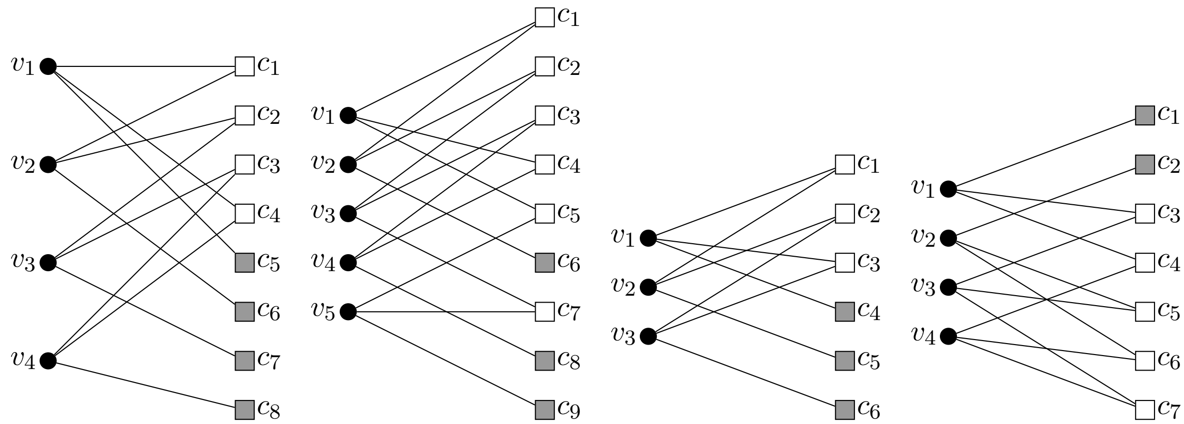
Figure 4 contains examples of the smallest trapping and absorbing sets that occur in 3-left regular Tanner graphs with girth at least 6. These are regarded as the most harmful trapping and absorbing sets with respect to some iterative algorithms. Note that the absorbing sets in Figure 4(c) and Figure 4(d) are the unique absorbing sets of the given sizes.

We now show that in a 3-left regular Tanner graph with girth at least 6, if the $(3, 3)$-absorbing set (shown in Figure 4(c)) occurs as a subgraph, it is not termatiko. A similar statement also holds for the $(4, 2)$-absorbing set (Figure 4(d)).

Proposition 3.7. *Consider a 3-left regular Tanner graph G with girth $g \geq 6$. Then a $(3, 3)$-absorbing set of G is not termatiko, and a $(4, 2)$-absorbing set is not termatiko.*

Proof. Assume T is a (3,3)-absorbing set in a 3-left regular Tanner graph with girth $g \geq 6$. As in Figure 4(c), let v_1, v_2, v_3 denote the vertices in T and $c_1, c_2, \ldots, c_6$ denote the vertices of $N(T)$ where c_4, c_5, and c_6 are each adjacent to exactly one vertex in T. Suppose by way of contradiction that T is termatiko.

We first show that no vertices in $N(T)$ satisfy Condition II. Vertices c_4, c_5, c_6 are not adjacent to two vertices in T so Condition II of the termatiko set definition is not satisfied for these vertices. Each of the vertices c_1, c_2, c_3 are adjacent to exactly two vertices in the set $\{v_1, v_2, v_3\}$.



(a) $(4,4)$-trapping set (b) $(5,3)$-trapping set (c) $(3,3)$-absorbing set (d) $(4,2)$-absorbing set

Figure 4: Subfigures (a) - (d) show examples of trapping sets [11] and absorbing sets [7], where each trapping/absorbing set is defined by the variable nodes shown on the left of the respective bipartite graph. Each of these graphs can be considered as a potential induced subgraph of a larger graph. The check nodes of odd degree are shown in gray.

Since each vertex in $\{v_1, v_2, v_3\}$ is adjacent to one vertex in $\{c_4, c_5, c_6\}$, $N(v_i) \cup N(v_j)$ where $i, j \in \{1, 2, 3\}$, $i \neq j$, includes a vertex that is adjacent to only one vertex in T and therefore none of the vertices c_1, c_2, c_3 satisfy Condition II.

Since T is termatiko, each vertex in $\{c_1, c_2, \ldots, c_6\}$ must satisfy Condition I. This means each is adjacent to at least one vertex in a set $U \subset V$ such that $N(U) = N(T)$. Since the Tanner graph is 3-left regular, each vertex of U has degree 3. For $i, j \in \{1, 2, 3\}$, and $i \neq j$, c_i and c_j cannot be adjacent to the same vertex $u^* \in U$ because otherwise there is a 4-cycle in the graph. Thus, let u_1, u_2, u_3 be vertices in U that are adjacent to c_1, c_2, c_3 respectively. Since the Tanner graph is 3-left regular, each of u_1, u_2, and u_3 must be adjacent to exactly two vertices from $\{c_4, c_5, c_6\}$. Consider u_1 which is adjacent to c_1. If u_1 and c_4 are adjacent, then $(u_1, c_4, v_1, c_1, u_1)$ is a 4-cycle, a contradiction. If u_1 and c_5 are adjacent, then $(u_1, c_5, v_2, c_1, u_1)$ is a 4-cycle, a contradiction. Thus, u_1 can only be adjacent to c_1 and c_6, which means u_1 does not have degree 3 which contradicts the 3-left regular assumption on the graph. Therefore, T is not termatiko.

The proof that a $(4, 2)$-absorbing set, as in Figure 4(d), is not termatiko is similar to the previous result and is omitted. We show that while one measurement node in the absorbing set graph satisfies Condition II (assuming that the set is termatiko), it is not possible to get the rest of the measurement nodes to satisfy Condition I and degree regularity simultaneously. □

Proposition 3.8. *There exists a 3-left regular Tanner graph with girth at least 6 such that a* $(4, 4)$*-trapping set is a termatiko set. Moreover, there exists a 3-left regular Tanner graph with girth at least 6 such that the (5,3)-trapping set is termatiko.*

Proof. Consider the graph in Figure 4(a). Let $T = \{v_1, v_2, v_3, v_4\}$. We will show how to embed the graph into a 3-left regular Tanner graph with girth ≥ 6 so that T is termatiko. First, it is straightforward to show that none of the measurement nodes $c_i \in N(T)$, for $i = 1, \ldots, 8$ satisfy Condition II. Therefore, each c_i must satisfy Condition I and be adjacent to least one element $u \in U$. Observe that it is not possible for two of the nodes in $\{c_1, c_2, c_3, c_4\}$ to be adjacent to the same $u_i \in U$ for the graph to have both girth at least 6 and 3-left regularity. We will construct the set U as follows. Let $U = \{u_1, u_2, u_3, u_4\}$. Define the neighborhoods as follows: $N(u_1) = \{c_1, c_7, c_8\}$, $N(u_2) = \{c_2, c_5, c_8\}$, $N(u_3) = \{c_3, c_5, c_6\}$, $N(u_4) = \{c_4, c_6, c_7\}$. Note that each measurement node in $N(T)$ is adjacent to at least one vertex in U, each vertex in U has degree 3, $N(U) = N(T)$, and the girth of the resulting graph is 6.

Consider the $(5, 3)$-trapping set in Figure 4(b). Let $T = \{v_1, v_2, v_3, v_4, v_5\}$. As in the previous proof, we will show how to embed the graph into a 3-left regular Tanner graph with girth ≥ 6 so that T is termatiko. Since none of the measurement nodes in $N(T)$ satisfy Condition II, we define a set $U = \{u_1, u_2, u_3\}$ as follows. Let $N(u_1) = \{c_1, c_3, c_9\}, N(u_2) = \{c_2, c_5, c_8\}$, and $N(u_3) = \{c_4, c_6, c_7\}$. Note that each measurement node in $N(T)$ is adjacent to a vertex in U, each vertex in U has degree 3, $N(U) = N(T)$, and the girth of the resulting graph is 6. □

This result shows that whenever there is a $(4, 4)$-trapping set, it may coincide with a termatiko set of size 4. More importantly, some of the existing techniques to remove $(4, 4)$-trapping sets [11] can be used to remove these types of termatiko sets. Similarly, it shows that whenever there is a $(5, 3)$-trapping set, it may coincide with a termatiko set of size 5. Thus, one may consider methods for removing them.

4 Redundancy

Redundancy in the parity-check matrix representations of LDPC codes has been shown to improve iterative decoder performance due to the fact that harmful structures like stopping sets, absorbing sets, and trapping sets tend to be removed when additional constraints are present [12]. This gain comes at the expense of iterative decoder complexity, so finding the optimal tradeoff is advantageous. In this section, we examine several measurement matrix representations to see the effect of redundancy on the presence of termatiko sets in the corresponding Tanner graphs. This case study gives insight to design rules for measurement matrices that can improve IPA reconstruction performance.

Consider the measurement matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

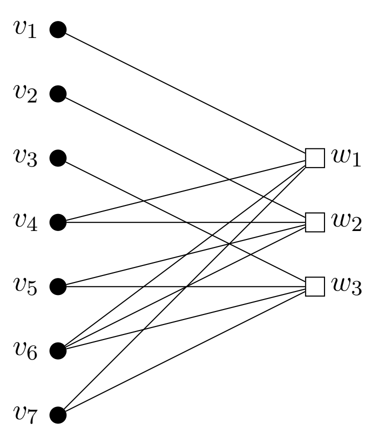


Figure 5: The Tanner graph associated to the measurement matrix M.

The corresponding Tanner graph is shown in Figure 5. The following matrices have additional rows of redundancy.

$$M_A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad M_B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$M_C = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad M_D = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

In particular, if $Row_i S$ denotes the ith row of a matrix S, then $Row_4 M_A = Row_2 M + Row_3 M$; $Row_4 M_B = Row_1 M + Row_2 M + Row_3 M$; $Row_4 M_C = Row_2 M + Row_3 M$ and $Row_5 M_C = Row_1 M + Row_2 M$; and $Row_4 M_D = Row_2 M + Row_3 M$, $Row_5 M_D = Row_1 M + Row_2 M$, $Row_6 M_D = Row_1 M + Row_3 M$, and $Row_7 M_C = Row_1 M + Row_2 M + Row_3 M$. Since the smallest stopping set in each of the representations is 3, $t_{\min} \leq 3$. The following table shows the numbers of termatiko sets of sizes 1 and 2 in the corresponding graphs.

Representation	# Size 1 Termatiko	# Size 2 Termatiko
M	4	15
M_A	3	15
M_B	1	12
M_C	0	7
M_D	0	0

This example demonstrates that redundancy in the parity-check matrix can result in fewer termatiko sets of small size. Moreover, the choice of redundant measurement node matters. Indeed, representations M_A and M_B each have only one redundant row but different number of small cardinality termatiko sets. One may also observe that representation M_D attains the best termatiko distance possible in this scenario with $t_{\min} = 3$.

5 Measurement matrices from finite geometries

In this section, we focus on measurement matrices obtained from incidence structures in finite geometries. LDPC codes based on finite geometries have been shown to have nice properties [13]. Specifically, we consider the incidences of points and lines in the finite Euclidean and projective geometries and analyze the sizes of termatiko sets in the corresponding graphs.

The Euclidean geometry constructions involve defining a subgeometry without the origin point and creating incidence matrices of points and lines for these families of subgeometries. We recall the basic properties of finite projective and Euclidean geometries, starting with the definitions of affine and projective space [14].

A *linear space* is a collection of points and lines such that any line has at least two points, and two points are on precisely one line. A hyperplane of a linear space is a maximal proper subspace. A *projective plane* is a linear space in which any two lines meet, and there exists a set of four points, no three of which are collinear. A projective plane has dimension 2. A *projective space* is a linear space in which any two-dimensional subspace is a projective plane. An *affine space* is a projective space with one hyperplane removed. Throughout, $\mathbb{F}_q$ denotes the field with q elements, where q is a power of a prime.

Like the Euclidean space $\mathbb{R}^m$, the set of points formed by m-tuples with entries from the finite field $\mathbb{F}_q$ forms an affine space, called a finite Euclidean geometry. A finite Euclidean geometry satisfies the axioms of affine space, and comprises one of the families of finite geometries that we will consider in this paper. In the case of $m = 2$, lines are sets of points (x, y) satisfying an equation $y = hx + b$ or $x = a$, where $h, b, a \in \mathbb{F}_q$.

Definition 5.1. The *m-dimensional finite Euclidean geometry* $\mathrm{EG}_0(m, q)$, has the following parameters.

- There are q^m points.
- There are $\dfrac{q^{m-1}(q^m - 1)}{q - 1}$ lines.
- Each line contains q points.
- Each point is on $\dfrac{q^m - 1}{q - 1}$ lines.

Any two points have exactly one line in common, and any two lines either have one point in common or are parallel (i.e., have no points in common).

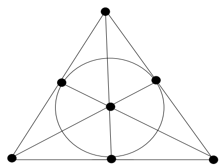
It is common to define a code from a modified version of $\mathrm{EG}_0(m, q)$ in which the origin point is removed and every line containing the origin is also deleted. By convention, the notation $\mathrm{EG}(m, q)$ is used to refer to the Euclidean geometry with the origin removed [13]. We use $\mathrm{EG}_0(m, q)$ to distinguish the case when the origin and all lines containing it are included. The parameters for $\mathrm{EG}(2, q)$ and $\mathrm{EG}_0(2, q)$ are provided in Table 1.

Definition 5.2. The *m-dimensional finite projective geometry* $\mathrm{PG}(m, q)$ has the following parameters.

- There are $\dfrac{q^{m+1} - 1}{q - 1}$ points.
- The number of lines is $\dfrac{(q^m + \cdots + q + 1)(q^{m-1} + \cdots + q + 1)}{(q + 1)}$.
- Each line contains $q + 1$ points.
- Each point is on $\dfrac{q^m - 1}{q - 1}$ lines.

Any two points have exactly one line in common, and each pair of lines has exactly one point in common.

An essential subclass of finite geometries are the finite Euclidean and projective planes. The parameters in these cases are provided in Table 1. Any Tanner graph of a $\mathrm{PG}(m, q)$ or an $\mathrm{EG}(m, q)$-LDPC code with $m > 1$ has girth 6 [13]. One way to obtain a measurement matrix from a finite geometry is to create an incidence matrix from the points and lines of the geometry, and let that matrix be M. In this paper, the columns of the incidence matrix correspond to lines



$$M = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

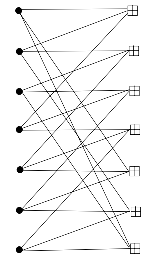


Figure 6: The point-line incidence graph of $\mathrm{PG}(2,2)$ is on the left, its corresponding measurement matrix in the middle, and its corresponding Tanner graph is on the right.

in the geometry, and the rows of the matrix correspond to points in the geometry. This choice allows for less redundancy in the resulting measurement matrix in the Euclidean geometry case than if the role of points and lines were to be reversed. Therefore, the lines in the geometry correspond to variable nodes in the Tanner graph, and the points in the geometry correspond to check nodes. This process of obtaining a measurement matrix M from a finite geometry is illustrated in the next example.

Example 5.3. An incidence matrix for the points and lines of $\mathrm{PG}(2,2)$ is given in Figure 6, along with the corresponding Tanner graph. The geometry is commonly known as the Fano plane. □

	$\mathrm{EG}_0(2,q)$	$\mathrm{EG}(2,q)$	$\mathrm{PG}(2,q)$
Number of points	q^2	q^2-1	q^2+q+1
Number of lines	$q(q+1)$	q^2-1	q^2+q+1
Number of points on each line	q	q	$q+1$
Number of lines that intersect at a point	$q+1$	q	$q+1$

Table 1: Parameters of finite planes.

It is convenient to rephrase the definition of termatiko set in terms of points and lines, where the lines correspond to the variable nodes and the points correspond to the measurement nodes of the measurement matrix based on the underlying finite geometry. Let L be the set of all lines in $\mathrm{EG}(m,q)$. Let $T \subseteq L$, and let $N(T)$ be the set of points that lie on at least one line in T. Define $U = \{\ell \in L \backslash T \mid N(\ell) \subseteq N(T)\}$. Then T is a termatiko set if and only if for each $p \in N(T)$ one of the two conditions holds:

I. p lies on a line in U.

II. p is not on a line in U, and there are at least two lines ℓ_1 and ℓ_2 in T that contain p and every point in $\ell_1 \cup \ell_2$ is on at least two lines from T.

Proposition 5.4. *In the line-point incidence graph of* $\mathrm{EG}(2,q)$*, a maximal set of parallel lines is a termatiko set, all of whose check nodes satisfy Condition I. The complement of a maximal set of parallel lines is also a termatiko set of this type. Furthermore, any set of parallel lines that is not maximal is not a termatiko set.*

Proof. Let T be a maximal set of parallel lines in $\mathrm{EG}(2,q)$. Then T is incident to all points in the geometry. Thus, $V = U \cup T$. Moreover, since each point is on $q+1>1$ lines, each point is incident to a line in U, and therefore each point satisfies Condition I.

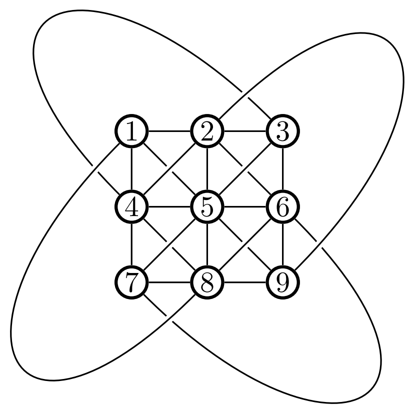


Figure 7: The finite Euclidean plane $\mathrm{EG}(2,3)$.

The set of lines $V \setminus T = U$ is also termatiko, since each point in $\mathrm{EG}(2,q)$ is incident to one line in T, which now plays the role of U in Condition I. Therefore, all points satisfy Condition I.

We now prove that a set of parallel lines that is not maximal is not a termatiko set. Let T be a set of $k \leq q-1$ parallel lines in $\mathrm{EG}(2,q)$. We show that $U = \{\ell \in V \backslash T \mid N(\ell) \subseteq N(T)\} = \emptyset$. Suppose $\ell \in V \backslash T$. From the geometry, it is known that ℓ contains q points which means that in the line-point incidence graph of $\mathrm{EG}(2,q)$, $|N(\ell)| = q$. Since a pair of lines can have at most one point in common, $N(\ell) \cap N(\ell_T) \leq 1$ for each line $\ell_T \in T$. Since there are k lines in T, this means $|N(\ell) \cap N(T)| \leq k \leq q-1$. It follows that there exists at least one point in $N(\ell) \backslash N(T)$ and therefore $N(\ell) \nsubseteq N(T)$.

Since $U = \emptyset$, there are no points in $N(T)$ that satisfy Condition I. Additionally, since T is a set of parallel lines, $N(\ell_1) \cap N(\ell_2) = \emptyset$ for all $\ell_1, \ell_2 \in T$. Therefore, there is no point $p \in N(T)$ that is contained in two lines of T and thus there are no points that satisfy Condition II. Therefore, T is not termatiko. $\square$

Remark 5.5. We state a few other observations for termatiko sets in $\mathrm{EG}(2,q)$.

- If T is the union of a maximal set of parallel lines with one additional line, then U has parallel lines and one more line. Then T is a termatiko set and both Conditions I and II occur.
- If T has q lines, none of which are parallel, then $U = \emptyset$, and T is a termatiko set with each point satisfying Condition II.
- If T has three lines intersecting in a point, then T is not termatiko since there are some points that satisfy neither condition.

Theorem 5.6. *Let T be a subset of the lines in* $\mathrm{EG}(2,q)$. *Then T is a termatiko set of smallest size in the line-point incidence graph of* $\mathrm{EG}(2,q)$ *if and only if T is a maximal set of parallel lines.*

Proof. Let T be a subset of lines in $\mathrm{EG}(2,q)$, and let $t_{\min}$ denote the smallest size of a termatiko set in the line-point incidence graph of $\mathrm{EG}(2,q)$. We first show that if T is a set of q lines and is not a maximal set of parallel lines, then T is not a termatiko set.

Suppose T is a set of q lines, and assume T is not a bundle of parallel lines. Then there are at most $q-1$ parallel lines and in particular, there is a line in T that intersects the $q-1$ (or fewer) parallel lines. Without loss of generality, suppose there are $q-1$ parallel lines in T, and label these such that for $0 \le i \le q-2$, line $\ell_{i+1} \in T$ contains points $N(\ell_{i+1}) = \{p_{iq+1}, \ldots, p_{(i+1)q}\} \subset N(T)$, and for $1 \le i \le q-1$, line ℓ_q has points $N(\ell_q) = \{p_1, p_{q+1}, \ldots, p_{iq+1}, \ldots, p_{(q-1)q+1}\} \subset N(T)$. Thus, ℓ_q intersects the parallel lines $\ell_1, \ldots, \ell_{q-1}$. Recall that $U = \{\ell \in V \backslash T \mid N(\ell) \subseteq N(T)\}$.

Towards a contradiction, suppose one of the intersection points of ℓ_q and a point of T, say p_1, is on line $\ell^* \in U$. Since p_1 is on ℓ^*, it follows that ℓ^* cannot contain points $p_2, \ldots, p_q$ because otherwise, points p_1 and some point of $\{p_2, \ldots, p_q\}$ is on both ℓ_1 and ℓ^*. This means that the other $q-1$ points of ℓ^* are in the sets $N(\ell_2), \ldots, N(\ell_q)$. If ℓ^* has two points in the set $N(\ell_i)$, for some $2 \le i \le q$, then the lines ℓ^* and ℓ_i have two points in common, giving a contradiction. Thus, ℓ^* has a point p_i^* in $N(\ell_i)$ for each $i = 2, \ldots, q$. Then, in particular, ℓ^* and ℓ_q intersect in both points p_1 and p_q^*, resulting in a contradiction. Thus, none of the special intersection points, like p_1, are incident to the lines of U. Thus, they fail Condition I in the definition of a termatiko set. In the case of fewer than $q-1$ parallel lines, even more points in $N(T)$ are these special intersection points, and thus, these points still fail Condition I.

We now need to show that one of these intersection points in $N(T)$ also fails Condition II. Without loss of generality, we once again consider $p_1 \in N(T)$, since from the above argument we know p_1 is not on a line of U. The only lines that contain p_1 in $T \cup U$ are ℓ_1 and ℓ_q. Now the points of $N(\ell_1) \cup N(\ell_q)$ are $\{p_1, p_2, \ldots, p_q\} \cup \{p_1, p_{q+1}, \ldots, p_{(q-1)q+1}\}$, and we need all these points to be on at least two lines in T. But we know for example that p_2 is only on ℓ_1 since the lines $\ell_1, \ldots, \ell_{q-1}$ are parallel. Thus, we have shown that there are points in $N(T)$ that also fail Condition II. In the case of fewer than $q-1$ parallel lines, it is easy to show that there is again a common intersection point that fails to satisfy Condition II, regardless of whether a subset of the lines intersecting the parallel set are themselves intersecting or parallel.

Thus, if T is a set of q lines in $\mathrm{EG}(n,q)$, not all parallel, then T is not a termatiko set. Moreover, by Theorem 3.4, since the graph is q-left regular with girth at least 6, any set of fewer than q lines is not termatiko. □

Corollary 5.7. *The termatiko distance of* $\mathrm{EG}(2,q)$ *is* q.

Proof. This follows immediately from Theorem 5.6. □

Theorem 5.8. *The termatiko distance of* $\mathrm{EG}(m,q)$ *is at most* q.

Proof. A maximal set of parallel lines in $\mathrm{EG}(2,q)$ occurs as a subset of lines in $\mathrm{EG}(m,q)$, and still forms a termatiko set. □

Proposition 5.9. *In* $\mathrm{EG}(m,q)$*, for* $m \ge 2$*, the set of all lines through a given point is not a termatiko set.*

Proof. Let p^* be a point in $\mathrm{EG}(m,q)$, and let $T = \{\ell_1, \ldots, \ell_{n_t}\}$ be the set of $n_t = (q^m-1)/(q-1)$ lines[1] through point p^*. Furthermore, let $N(T)$ be the set of points that lie on at least one line in T, and let $U = \{\ell \in V \backslash T \mid N(\ell) \subseteq N(T)\}$. Then by choice of T, point p^* does not lie on

[1]In $\mathrm{EG}(m,q)$, each point is on $n_t = (q^m-1)/(q-1)$ lines.

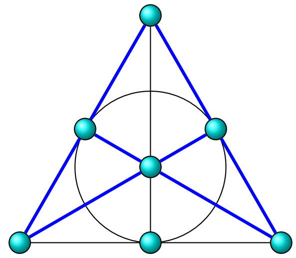


Figure 8: Minimum termatiko set in PG$(2,2)$, shown in blue.

a line in U. Now consider any two lines in T that contain p^*, say ℓ_1 and ℓ_2. Then any point $p \in N(\ell_1) \cup N(\ell_2)$ with $p \neq p^*$ is on only one line in T since all lines of T contain p^* and every two points in EG(m,q) have at most one line in common. Since this holds for any pair of lines of T, it follows that T is not termatiko. □

We will now consider measurement matrices based on finite projective geometries. For shorthand, we use the notation $t_{min}(\mathrm{PG}(m,q))$ to denote the size of a termatiko set of minimum size in the Tanner graph constructed as the incidence graph of lines and points in PG(m,q), where variable nodes correspond to lines in the geometry and measurement nodes correspond to points.

Similar to Theorem 5.7, we now show the relationship between termatiko sets in PG(m,q) and PG$(2,q)$, for $m > 2$.

Theorem 5.10. *For all integers $m \geq 2$, $t_{\min}(\mathrm{PG}(m,q)) \leq t_{\min}(\mathrm{PG}(2,q))$.*

Proof. Let T be a termatiko set in PG$(2,q)$. Recall that PG$(2,q)$ is a subspace of PG(m,q); specifically, there are $\frac{(q^{m+1}-1)(q^{m+1}-q)(q^{m+1}-q^2)}{(q^3-1)(q^3-q)(q^3-q^2)}$ copies of PG$(2,q)$ appearing in PG(m,q). We claim that the termatiko set T of PG$(2,q)$ is also a termatiko set in PG(m,q). Since PG$(2,q)$ is a subspace of PG(m,q), all lines through the points in $N(T)$ are contained in PG$(2,q)$. Therefore, the conditions satisfied by the points in $N(T)$ in PG$(2,q)$ are also satisfied in PG(m,q). Since a minimum size termatiko set in PG$(2,q)$ also exists in PG(m,q), we must have that the termatiko distance of PG(m,q) is no more than the termatiko distance of PG$(2,q)$. □

Conjecture 5.11. For $m \geq 2$, $t_{\min}(\mathrm{PG}(m,q)) = t_{\min}(\mathrm{PG}(2,q))$.

Based on the structure of PG(m,q), Conjecture 5.11 says that it is not possible to obtain a smaller termatiko set in a higher dimensional geometry. Next we give two examples of termatiko sets of minimum size in PG$(2,q)$ for small q.

Example 5.12. In this example we show that $t_{min}(\mathrm{PG}(2,2)) = 4$. The termatiko set in Figure 8 has all lines in T, and $U = \emptyset$. Thus, T is also a stopping set.

Next we will show that no set of three or fewer lines in PG$(2,2)$ can form a termatiko set. It is clear that a set of one or two lines in PG$(2,2)$ cannot be a termatiko set since there would be at least three points on these lines that do not satisfy either of the conditions on points in $N(T)$. For sets of three lines in PG$(2,2)$, there are two cases.

1. Suppose that a set of three lines in PG$(2,2)$ intersect pairwise at distinct points. In this case there would be three points, p_1, p_2, p_3 where two of the lines in T intersect, and three other points, each lying on a single line in T. By inspection in Figure 9, the points p_1, p_2, and p_3 would fail to satisfy either condition on $N(T)$, so T cannot be a termatiko set.

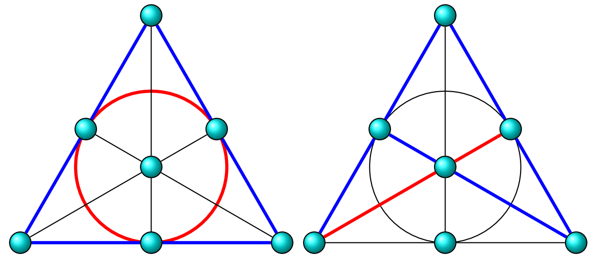


Figure 9: Two sets T of size three in PG$(2,2)$ that are not termatiko. The sets T are shown in blue, and the sets U are shown in red.

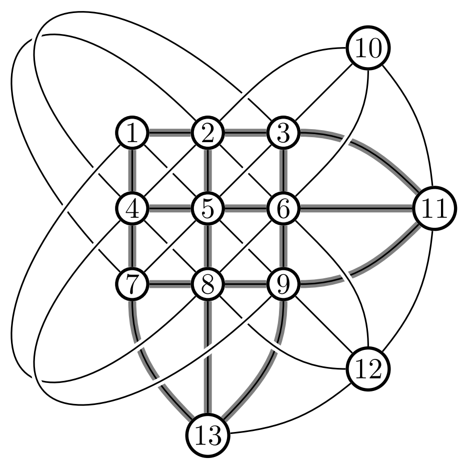


Figure 10: A termatiko set of size 6 in PG$(2,3)$ is shown in gray.

2. Suppose that the set of three lines intersects at a single point, p. Then all other points p' on the lines in T would have the property that p' is on a line in U, since $U = L \setminus T$ in this case. However, the point of intersection p is not in U, and it also fails the other condition on elements of $N(T)$. Thus, the set T is not a termatiko set.

Example 5.13. This example deals with the case of PG$(2,3)$. The set of six lines in Figure 10 is a termatiko set since every point in $N(T)$ is on at least two lines in T, meaning that all points in $N(T)$ satisfy Condition II in the definition of termatiko set. This also makes T a stopping set, and $U = \emptyset$. Therefore, $t_{min}(\mathrm{PG}(2,3)) \leq 6$.

Next we give partial results on the sizes of neighborhoods of T, assuming that T is a termatiko set in a code from PG$(2,q)$.

Proposition 5.14. *Let T be a termatiko set in* PG$(2,q)$*. Then*

$$|T|(q+1) - \binom{|T|}{2} \leq |N(T)| \leq |T|q + 1.$$

Proof. Given a termatiko set T in PG$(2,q)$, we consider the smallest number of points lying on T. Notice that each line in T has $q+1$ points, but each pair of lines must intersect at a point.

We consider two extreme scenarios: if all lines in T intersect at a single point, the number of points lying on T could be as large as $|T|q+1$. An equivalent way of counting this set of point is: $|T|(q+1)-(|T|-1)$. On the other hand, if we assume that each pair of lines in T intersects at a distinct point, the number of points lying on T could be as small as $|T|(q+1)-\binom{|T|}{2}$. $\square$

Remark 5.15. We conclude this section with a discussion about $|U|$ in minimum size termatiko sets in $\mathrm{PG}(m,q)$. First, we note that in $\mathrm{PG}(2,2)$, if $|S|>1$, then $|T|>4$, by case analysis. Therefore, any minimum-size termatiko set in $\mathrm{PG}(2,2)$ has $|U|=\emptyset$. Likewise all examples of termatiko sets in $\mathrm{PG}(2,3)$ with $|U|>0$ that we constructed were not minimum size. This leads to the following conjecture.

Conjecture 5.16. Termatiko sets of minimum size in $\mathrm{PG}(m,q)$ have $U=\emptyset$, and thus are stopping sets.

6 Conclusion

In this paper, we considered the relationship between termatiko sets and stopping, trapping, and absorbing sets. We characterized the relationship between termatiko sets and stopping sets and provided a lower bound on the size of termatiko sets in left-regular Tanner graphs. We determined whether certain trapping and absorbing sets are also termatiko. It would be interesting to pursue a formal study of how redundancy in the measurement matrix affects the presence of termatiko sets, based on the case study provided here. While initial results on types of termatiko sets that occur in finite-geometry based measurement matrices have been described here, a number of questions regarding their sizes remain open.

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