

COMMUNICATION BETWEEN THE MATHEMATICAL AND MATH-EDUCATION COMMUNITIES

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ABSTRACT. Communication between K-12 and college educators is needed to reverse a decline in preparation for study in technical fields. Attempts have been largely unsuccessful and sometimes so unpleasant they are referred to as “wars”. We analyze obstacles and particularly try to separate linguistic differences from conflicts of underlying mindsets and priorities. Annotated lists of sample problems may offer the best approach.

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1. INTRODUCTION

The mathematical community is primarily concerned with developing new mathematics and training in the professional use of mathematics. The US math-education community is concerned with teaching mathematics at least three hundred years old to the general population. There is not much overlap between these primary concerns and so—until recently—little need for systematic communication.

In recent decades the US math-ed community, largely under the leadership of the National Council of Teachers of Mathematics (NCTM), has developed an innovative, coherent and forcefully articulated approach to K-12 math education. At the

Date: June 2006, edited October 2008.

same time students graduating from this system have shown a significant decline in preparation for technical careers. For decades the US has imported technical training to make up the shortfall but demand has now outpaced imports and technical jobs are being exported. This is an impending national disaster. Being coherent and innovative is not the same as being right, and the current K–12 program seems to be seriously flawed in this regard.

Part of the problem is that the K–12 focus has been on the weakest students (No Child Left Behind) at the expense of high-achievers. This focus is so complete that most math educators either deny there is a high-end problem (skills are “different”, not worse) or believe that more of the same will fix it.

The academic mathematical community has to deal with the output of the K–12 system so is much more sensitive to the problem. Many now believe that some sort of action is needed. The most vocal proposals have been simplistic and unconstructive: “game over, return to the 19th century” or “adopt the proven program used in (country name)”. But attempts at thoughtful communication have also been unsuccessful.

This essay is concerned with recognizing barriers to communication between the two communities, and seeking ways to avoid them.

Some of the barriers are linguistic: in the second section we describe the very different interpretations given to words such as “understand”, “know”, “apply” and “recall”. But in the third section we suggest that these differences are adapted to the jobs at hand rather than arbitrary, so they cannot be settled by linguistic negotiation. Our conclusion is that successful communication on an abstract or conceptual level will be so difficult that alternatives must be found.

This requires a sharpening of the problem: exactly what is it that needs to be communicated, and to what end? The first need is to communicate about preparation of students for success in college and technical work. This is investigated in the fourth section where we suggest that it might be done on a very primitive level—essentially annotated lists of sample problems.

2. LANGUAGE DIFFERENCES

Professional communities develop specialized language for precision and clarity. Naturally, different communities will give terms different specialized meanings, and while this is an obvious and well-known source of confusion it almost always takes participants by surprise. For instance a core part of the new math-ed vision is a shift from drill and rote mechanical work to conceptual skills such as “understanding” and “knowing” and conceptual activities such as “creating” and “recognizing”. These goals sound good to everyone but the words have rather different meanings in the math and math-ed communities. In this section we describe some of these differences. The next section suggests reasons for the differences.

2.1. Understanding. In the math-ed community “understand” does not imply “able to work problems with”, while mathematical use includes this and more. We expand on the differences, then consider the problems they invite.

The use of “understand” as something untestable and distinct from “able to use” is clear and consistent in recent math-ed literature and many (probably over 100) standards documents. The Executive Summary of the 2000 NCTM publication *Principles and Standards for School Mathematics* provides a useful and authoritative example. There are 24 occurrences of the word “understand” in the text,

eleven referring to student learning objectives. Ten of these are accompanied by a phrase “and apply”, “as well as use” etc. The intent is to clarify that students should be able to *do* as well as understand, but this also clarifies that “understand” is not taken to include doing. The eleventh appearance is in the sentence “An understanding of numbers allows computational procedures to be learned and recalled with ease.” This is one of the few explicit suggestions of a utility for understanding, but again it is clear that a failure to “learn and recall” is not taken as evidence of a lack of understanding.

In mathematics “understand” means mastery and certainly includes ability to work problems. It is correct usage to say “evidently you *don't* understand since you can't use it effectively.” Non-functional exposure is considered useless and not given a name.

The difference in usage means any statement containing the word “understand” will be misunderstood. Standards documents often include a phrase like “understand the quadratic formula”. College teachers think this means “know cold, be able to recite the formula instantly, apply quickly and accurately, and translate quickly and easily between the various formulations of the outcome.” Math educators are more likely to interpret it as “recognize the name and realize it has something to do with solving equations with a squared term” and expect that if anything more is intended it would be spelled out.

Awareness of language differences would help but not be a complete solution. Consider for instance “I would rather my calculus students have a mathematical-level understanding of algebra alone than a math-ed understanding of algebra, calculus, statistics and probability” (really!). Educators might understand this as an assertion about the importance of precision and “fluency” but would not know what it involves in detail.

2.2. Remembering, recalling, knowing. A central tenant of the new vision of math education that memorization is a superficial approach to learning and should be avoided. A corollary is that teachers should not require a level of precision that might require memorization. “Remember” and “recall” implement this idea: they indicate a “bringing to mind” that demonstrates familiarity but is tolerant of error. This is close to the meaning in common language where words such as “recite” are available to indicate “recall with precision”.

The presumption is that students’ “remembering” will become more precise as their “understanding” grows, and it is better to wait for understanding than to require memorization. Thus when a Standards document states that students should “know properties of numbers”, “recall multiplication facts”, or “remember facts about solutions of quadratic equations” the math-ed interpretation is that they should *not* be required to do so *correctly*. These are not endpoints but processes that in the fullness of time are supposed to lead to accuracy.

Deliberate fault tolerance does not make sense to mathematicians. Mathematics is empowering only if it is used exactly right, and errors in “recalling” almost always render it useless. Most of the work in teaching college mathematics is getting students to find and fix errors in their “recollections”. This is not to say mathematicians are fans of memorization. The author often teaches a calculus class that requires use of trig addition formulas. He can't remember these formulas, but can derive them in a few seconds from properties of the complex exponential function. He would be overjoyed to have a student who could do this too, but the

realistic and honest approach is to say “you need these; memorize them”. Indeed mathematical experience is that understanding in the strong mathematical sense *follows from* accurate recall and extensive practice, and cannot precede them.

Some educators acknowledge the need for better “recollection” and the term “automatic recall” is starting to appear. This does not have an agreed standard meaning so use of it does not indicate a commitment to anything in particular. The fact that “memorize” and “recite” are still being avoided suggests that the outcome is uncertain.

2.3. Applying, evaluating. “Applying” and “evaluating” appear in Standards documents in phrases like “apply math concepts” and “evaluate mathematical statements”. These phrases are also common in mathematics but indicate activities developmentally inappropriate before the second or third year of college. Indeed the shift from problem-solving to concepts and logical evaluation is the main reason so many students who start out as math majors change their minds. In this case it is obvious, at least to mathematicians, that there are profound differences in meaning in the two communities. The nature of the differences may be less clear.

Mathematicians use “apply” as a synonym for “work problems with” while the math-ed meaning is closer to “illustrate”. For instance if the concept is that multiplication is related to stacking blocks then “applying” might mean stacking blocks to model a multiplication problem. If the concept is commutativity of multiplication then “applying” might mean rearranging stacks of blocks to illustrate this. This clarifies that “applying” is an understanding-enhancing activity, not a testable skill. Generally the practice in Standards and the educational literature is that if something does not contain an unambiguous phrase such as “computational fluency” or “work problems” then a non-testable interpretation is acceptable and probably intended. The examples also illustrate a mismatch in the meaning of “concept”. Mathematicians think of commutativity of multiplication of numbers as a property rather than a concept and use “concept” for higher-level abstractions.

The mathematical meaning of “evaluate mathematical statements” is “demonstrate correctness or incorrectness”. This is certainly a testable skill. For example students might be asked “Evaluate the statement ‘addition distributes over multiplication’ ”. and be expected to know that to show the statement is false one should find particular values for which the two sides of the equation give different answers; and second to be able to find such values. In the math-ed interpretation demonstrations are “understanding-building” activities and not expected of students. The most that would appear on a test would be “true or false: addition distributes over multiplication”.

2.4. Creating, discovering. “Creating” and “discovering” may provide the most extreme examples of mismatch between the mathematical and math-ed communities. These activities are very highly valued by mathematicians: the primary requirement for the PhD degree is that the student demonstrate he is capable of creativity and discovery. Undergraduate research projects are fashionable but difficult and extremely time-consuming. How could this possibly be scaled down to K-12? Mathematicians generally find the whole idea disturbing.

Quite a few math educators suggest that students should “discover” their own versions of algorithms for multiplication or division. But the standard algorithms are finely-tuned instruments developed with the difficulty and depth of experience

required by the US Bill of Rights. Would government teachers ask students to discover the Bill of Rights? Probably not unless the plan was to spend half a year explaining why the discovered versions were inadequate. Would carpentry teachers show students screws and nails and ask them to discover screwdrivers and hammers? And if they did, would a shortage of competent carpenters and an epidemic of carpentry-anxiety be a mystery?

The point is that some things are simply out of reach of student discovery. The problem goes beyond that however. Professional experience is that 90% of math discovery is either dysfunctional or outright wrong, and consequently 90% of the effort in *effective* discovery is spent finding and correcting errors. It would be truly wonderful if K-12 students could experience this. However few students are willing to be wrong (and get corrected) 90% of the time and few teachers have the time or training to guide the necessary diagnosis.

The math-ed interpretation of “discovery” is quite different: either a process intended to build “understanding” but so tightly controlled by the teacher that it can’t go wrong, or a less-directed activity that is unevaluated because it lacks the refinement process needed to be effective. The outcome (e.g. an algorithm) can be tested but the discovery process itself is not a testable skill.

2.5. Teaching vs learning. The final terminology problem is much more profound and concerns location of responsibility in the educational process. One view centers on students: learning requires effort and it is their responsibility to put in this effort, or at the very least not disrupt efforts of others. The other view centers on teachers: teachers are providers, students are recipients, and if engagement is required then it is the responsibility of the teacher to develop it. Are teachers “learning facilitators” or are students “teaching customers”? Are grades “given” by the teacher or “earned” by the student? On the slogan level, “you can lead a student to knowledge but you can’t make him think” vs. “if the student hasn’t learned then the teacher hasn’t taught”.

In the US K-12 system responsibility is placed primarily on teachers. It is standard practice for teachers and school system to be punished if students do poorly on state tests. At the college level there is simply no way to avoid placing primary responsibility for learning on students. This is incorporated into the way college teachers think and interpret terminology. As a result even the words “teaching” and “learning” will cause interpretation problems in K-12-college communications¹.

3. MATHEMATICS AND LEARNING

In this section we suggest that the dramatically different word usages described above are adapted to their subjects: there are actually reasons for them. The mismatch is not simply linguistic and cannot be solved by linguistic compromise.

3.1. Mathematics. The demanding nature of mathematics is suggested by the fact that it was an organized subject of study for three thousand years before it really got off the ground. Mathematical conclusions are like legal documents: powerful if fine print is satisfied and loopholes are avoided but you can lose your shirt if you make the smallest mistake. After three thousand years of lost shirts we figured this out and learned to read and write fine print. Mathematics did not become routinely successful as a profession until this was incorporated into

¹For an extended discussion see [Teaching vs. Learning in Mathematics Education](#).

community norms. “Know” and “understand” came to mean “so intimately familiar with the fine print that blunders are minimized”. Rigorous standards made math slow and difficult and were resisted by many mathematicians during the changeover, but they were enforced by mathematics itself. Sloppy people were less effective and ended up marginalizing themselves. It took a century of such reenforcement for rigorous standards to win general acceptance.

The seeming ridiculously high standards of modern mathematics are simply what it takes to be successful, not a conspiracy to shut out non-members.

3.2. Human learning. People see patterns and connect facts quickly and with little effort. This instinctive facility is thought to have developed because it enhances survival in dangerous situations. Inevitably many of these patterns and connections are incorrect, but people do *not* recognize and correct errors either quickly or easily: the persistence of superstition and gullible belief is well known. Apparently error correction does not enhance survival.

People also have difficulty understanding abstract *explanations* of patterns. It is frequently more effective to provide examples and hints and let them find the patterns themselves.

Effective learning requires finding and fixing errors in natural learning. Young students need help with diagnosing errors. Learning to do this oneself is the key to effective learning at higher levels. At the highest level mathematicians need such accurate and reliable understanding that they must learn to vigorously test—almost attack—the impressions coming from natural learning.

3.3. Math education. “Learning” in US math education seems to correspond to the “natural learning” described above. It makes sense to have a term for this because there is quite a change of gears between this and the more disciplined error-correcting phase. It does not make sense to have no terminology for, or even awareness of, the later phase.

Disciplined areas regard natural learning as only a starting point for understanding and knowledge. The US educational community has taken a different approach: redefine “understanding” and “knowledge” in a sufficiently fault-tolerant way that natural learning is nearly sufficient. Some error-correction is still needed but instead of doing it explicitly the US practice is to cycle through the natural-learning process multiple times. This sets some students more firmly into bad habits and is a mind-numbing waste of time for the ones who got it right the first time, but does give some improvement in the middle.

Lack of concern for errors in learning seems to pervade the profession. Elementary math textbooks are packed with distractors and intellectual content is diffused. The distractors are supposed to maintain interest and enrich the learning process. They also increase the error rate. The error-tolerance of the math-ed community is so great that either they cannot see this or they regard it as a good exchange for “enrichment”. Error tolerance makes educators’ job easier and reduces the effort required of students but it also largely cuts students off from areas requiring high-precision knowledge.

How can error tolerance coexist with something as black-and-white as a math test? Not well. US students fare poorly in international comparisons. Statewide high-stakes tests are causing dislocations, though this is softened by the political need to set standards low enough that most students pass. To improve grades

teachers can use simplified problems and standard phrasing in classroom tests. Credit for routine homework with low quality control provides a buffer against low test scores. Valuing “knowledge” etc. provides a loophole: teachers can say “I can give you credit even though you can’t work problems because I see you basically understand it”. Finally calculators have been a godsend: students can be trained to get good numbers via keystrokes without a disciplined grasp of detail.

There is an historical explanation for error-tolerance that long predates the NCTM vision. The old view that mathematics is good training in disciplined logical thinking is explicitly not error-tolerant. About a century ago some US educational leaders asserted that elementary mathematics should focus on and be valued for its applications. This may have been a mistake. There has been a decline in disciplined logical thinking in US society, and elementary math is now personified to many people by bizarre and contrived word problems. In any case math education was released from the constraints of rigor. Moreover this happened so many generations ago that it is deeply ingrained in the mindset, literature, teacher training methods and community standards. Any change will be slow and painful.

4. COMMUNICATING ABOUT STUDENT PREPARATION

We have argued that terminology and mindset differences are too great for abstract statements such as “students should know how to solve quadratic equations” to be successful. Here we outline a more concrete and direct approach. This has been arrived at by a process of elimination—anything else seems likely to fail—but it has a number of other significant benefits.

4.1. Quadratic example.

4.1.1. *Task.* Students should be able to recite the quadratic formula and use it to find exact solutions to quadratic equations with either numerical or symbolic coefficients. For instance:

Solve the equation $2x^2 + ax - 5a^2 = 0$ for a in terms of x . Use this to rewrite the left-hand side in the form $-5(a - R)(a - S)$.

(A real-life treatment would continue with many more examples.)

4.1.2. *Annotation.* many mathematical procedures require solving an equation. (Examples: finding extrema, intersections of curves, solutions of some differential equations.) Quadratics are one of the very few general classes of equations that can easily be solved so they are heavily used in examples and problems. Students who can work with quadratics easily and without much thought will be able to focus on the new material. Students who have difficulty with quadratics will constantly find this a barrier to further learning.

4.2. **The general pattern.** The core of a document intended to communicate goals would be an extensive list of sample problems. These problems would be selected to illustrate key points: the example above illustrates that coefficients might involve symbols; that roots may be irrational and also involve symbols; and that the variable being solved for may not always be called “ x ”. There should be notes explaining why such problems are important; how they may be used; abstract principles underlying them; and so on, but the notes should be clearly subordinate

to the problems. Alternate interpretations of notes does not justify simplifying or discarding problem types.

The web site of the AMS Working Group on Preparation for Technical Careers, <http://amstechnicalcareers.wikidot.com> is an attempt to implement this idea.

4.3. Benefits. The first virtue of this approach is that it avoids the modes of failure identified in earlier sections. This alone would make it worth pursuing, but it seems to have some significant further advantages.

4.3.1. Neutrality. The math-ed community has learned how to teach K–12 math. The mathematical community believes this learning is flawed and needs correction. Error correction may be a routine part of mathematical culture but it is not in math education. Teachers as well as students worry about the difference between “you have made an error” and “you are stupid”; between being offered correction and being disrespected. Linguistic differences exacerbate this.

Problem lists provide a neutral meeting place for the professions. Mathematicians can formulate goals without judgmental overtones or misinterpretations. Educators can see for themselves what the core concerns are, and formulate these conclusions in their own language.

4.3.2. Focus on outcomes. There is always tension and confusion between process and outcome. It is faster and easier to say “do this and you will come to the right place” rather than carefully describe the “right place”. But the fact that it is faster means it is more susceptible to misunderstanding, and the fact that it is easier makes it more susceptible to error. It should bring focus and clarity to the process to undertake describing the “right place” concretely in terms of sample problems.

Focus on problems would also help the mathematical community develop a coherent position. “Mathematicians” are not a coherent group with uniform views: there is a great deal of shared culture and agreement on general principles, but agreement disintegrates quickly as one gets into specific issues. The community lacks mechanisms for developing agreement and in particular lacks the forceful and articulate leadership provided by the NCTM in the math-ed community. However most of the disagreement concerns process or terminology: the right *way* rather than the right outcome. Mathematicians will probably agree that certain sample problems are good even if they disagree on *why* they are good.

4.3.3. Professional autonomy. Mathematicians have no business dictating in detail how K–12 math should be taught. They can and should specify the outcomes they need as a basis for further education. They can further describe interconnections of patterns and flow of mathematical ideas that might make teaching easier. If educators find these ideas useful they will use them but they should not be ordered to use them.

For instance mathematicians might specify—through sample problems—that students be able to multiply and divide polynomials “with facility”. They might further point out that the algorithms used for long division and multiplication of numbers are also needed to divide or multiply extremely large numbers using calculators or computers with limited digit capability, and again for division or multiplication of polynomials. Students who learn the simpler versions early may find the steps up in complexity or abstraction relatively easy, while those who come to the advanced versions without preparation will find them difficult. Educators

may find this a convincing reason to return the algorithms to the early curriculum, or they may prefer to experiment with ways to tackle it later. As long as they accept the final goals and get the job done it shouldn't matter. The process should be up to educators and not dictated by mathematicians.

Those who observe that dictatorial control worked well in Soviet math education should realize this was a matter of great luck in choice of dictators. The same dictators forbade the teaching of evolution and essentially destroyed effective biology outside the biological warfare programs. Dictatorial control of process is considered a failed management model and one successful example does not change this.

The need to be persuasive—rather than imposing solutions by fiat—may also encourage mathematicians to be a bit more clear and coherent.

4.3.4. *Testing.* High-stakes statewide tests are nearly universal and national tests seem increasingly likely. Preparing students for these tests has become a matter of personal survival for many teachers. They need to know what the students will face and what is needed to prepare them. In principle this information is provided in Standards documents. In practice there is such a gap between the abstract goals in Standards and problems on tests that they are useless and old tests are de facto the authoritative guides.

The other side of this coin is the dilemma facing test designers. Ostensibly tests should reflect goals set by Standards documents. However these goals are so vague and inflated that this can't be taken seriously and again old tests serve as the main guides. The Standards document itself is effectively removed from the process and any intellectual content is lost.

A great virtue of a goals document organized around sample problems is that it connects clearly and directly to tests. For teachers the phrasing “students should be able to work problems like the following” becomes “test problems will be like the following”. For test designers, explanations of what the sample problems are supposed to illustrate become instructions on how they can be varied while accomplishing the same goals. The results should be better tests and better student preparation for them.

5. CONCLUSION

We have argued that annotated problem lists would be an effective way for the mathematical community to communicate goals for student preparation in K-12. More than that we have argued that other—more “conceptual”—approaches face such severe obstacles as to be a waste of time. The mathematical community should undertake the development of such lists as soon as possible².

² An attempt can be found at <http://amstechnicalcareers.wikidot.com>.