Cannon's conjecture, subdivision rules, and expansion complexes

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Motivation from the 1970's

Mostow's Rigidity Theorem (special case): If two closed hyperbolic n-manifolds, $n \geq 3$, have isomorphic fundamental groups, then they are isometric.

Thurston's Hyperbolization Conjecture: If M is a closed 3-manifold such that $\pi_1(M)$ is infinite, is not a free group, and does not contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$, then M has a hyperbolic structure.

- A key ingredient of the proof of Mostow's theorem is the action of the fundamental group on the "boundary" of hyperbolic space.
- Can you define the boundary of a group?
- Can you tell when the boundary of a group is a 2-sphere?

Cannon's Conjecture

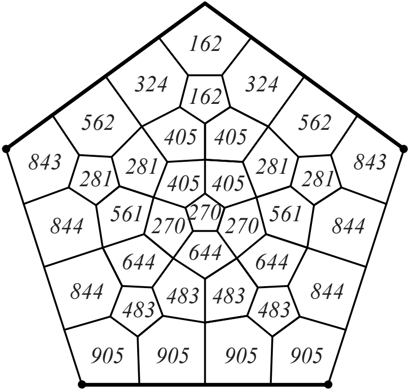
Cannon's Conjecture: If G is a Gromov-hyperbolic discrete group whose space at infinity is S^2, then G acts properly discontinuously, cocompactly, and isometrically on $\mathbb{H}^3$.

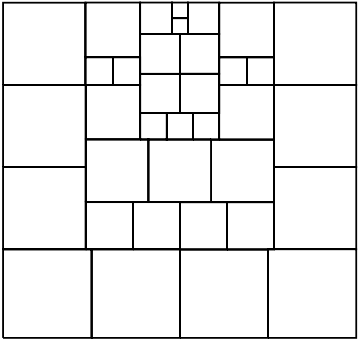
- While a primary motivation for this was Thurston's Hyperbolization Conjecture, even after Perelman's proof of the Geometrization Conjecture this conjecture is still open.
- How do you proceed from combinatorial/topological hypotheses to an analytic conclusion?
- Given a sequence of subdivisions of a tiling, how do you understand/control the shapes of tiles?
- When can you realize the subdivisions so that the subtiles stay almost round?

Weight functions, combinatorial moduli

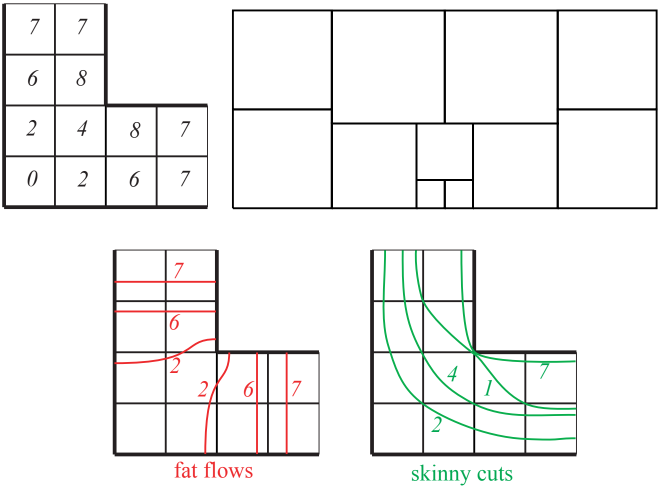
- *shingling* (locally-finite covering by compact, connected sets) $\mathcal{T}$ on a surface S, ring (or quadrilateral) $R \subset S$
- *weight function* ρ on $\mathcal{T}$: $\rho\colon \mathcal{T} \to \mathbb{R}_{\geq 0}$
- ρ-*length* of a curve, ρ-*height* H_ρ of R, ρ-area A_ρ of R, ρ-*circumference* C_ρ of R
- *moduli* $M_\rho = H_\rho^2/A_\rho$ and $m_\rho = A_\rho/C_\rho^2$
- *moduli* $M(R) = \sup_\rho H_\rho^2/A_\rho$ and $m(R) = \inf_\rho A_\rho/C_\rho^2$
- The sup and inf exist, and are unique up to scaling. (This follows from compactness and convexity.)

Optimal weight functions - an example





Optimal weight functions - another example



Combinatorial Riemann Mapping Theorem

- Now consider a sequence of shinglings of S.
- **Axiom 1.** Nondegeneration, comparability of asymptotic combinatorial moduli
- **Axiom 2.** Existence of local rings with large moduli
- *conformal sequence* of shinglings: Axioms 1 and 2, plus mesh locally approaching 0.

Theorem (C): If $\{\mathcal{S}_i\}$ is a conformal sequence of shinglings on a topological surface S and R is a ring in S, then R has a metric which makes it a right-circular annulus such that analytic moduli and asymptotic combinatorial moduli on R are uniformly comparable.

- J. W. Cannon, The combinatorial Riemann mapping theorem, *Acta Math.* **173** (1994), 155–234.

The Cannon-Swenson Theorem

Theorem (C-Swenson): In the setting of Cannon's conjecture, it suffices to prove that the sequence $\{\mathcal{D}(n)\}_{n\in\mathbb{N}}$ of disks at infinity is conformal. Furthermore, the $\mathcal{D}(n)$'s satisfy a linear recursion.

- J. W. Cannon, E. L. Swenson, Recognizing constant curvature groups in dimension 3, *Trans. Amer. Math. Soc.* **350** (1998), 809–849.

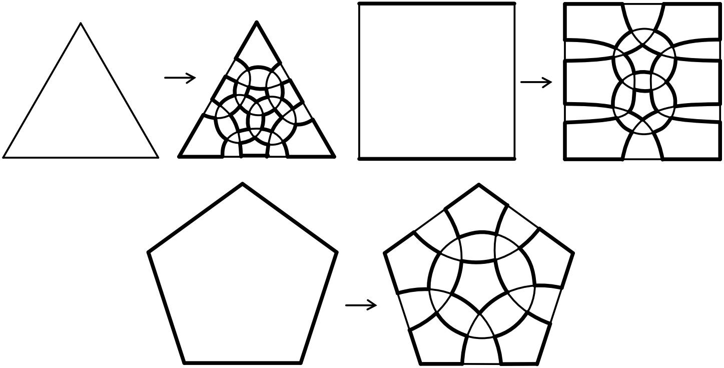
- Finite subdivision rules were created as toy models for the sequences of covers by disks at infinity.

Definition of a finite subdivision rule $\mathcal{R}$

- C-F-P, Finite subdivision rules, *Conform. Geom. Dyn.* **5** (2001), 153–196 (electronic).
- finite subdivision complex $S_{\mathcal{R}}$
- $S_{\mathcal{R}}$ is the union of its closed 2-cells. Each 2-cell is modeled on an n-gon (called a *tile type*) with $n \geq 3$.
- subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$
- subdivision map $\sigma_{\mathcal{R}} \colon \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$
- $\sigma_{\mathcal{R}}$ is cellular and takes each open cell homeomorphically onto an open cell.
- *$\mathcal{R}$-complex*: a 2-complex X which is the closure of its 2-cells, together with a structure map $h \colon X \to S_{\mathcal{R}}$ which takes each open cell homeomorphically onto an open cell
- One can use a finite subdivision rule to recursively subdivide $\mathcal{R}$-complexes. $\mathcal{R}(X)$ is the subdivision of X.

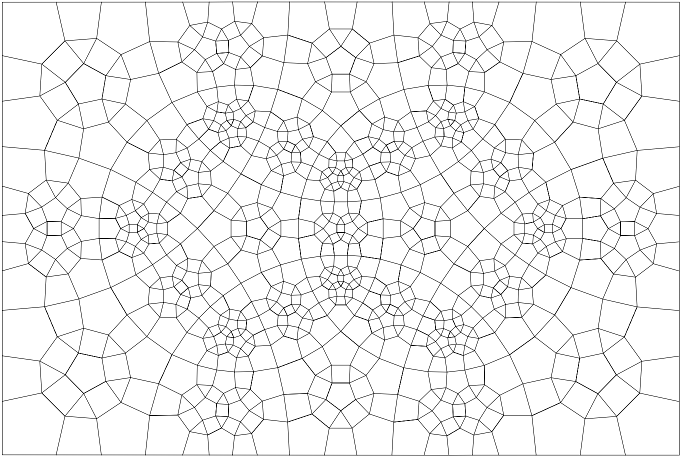
Example 1. The dodecahedral subdivision rule

The subdivisions of the three tile types. Note that there are two edge types.

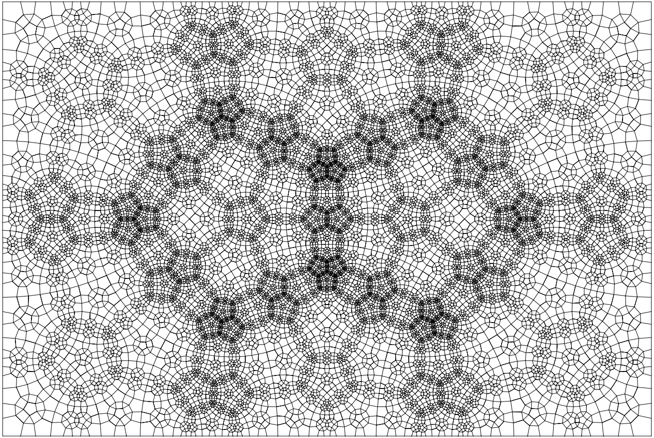


The second subdivision of the quadrilateral tile type

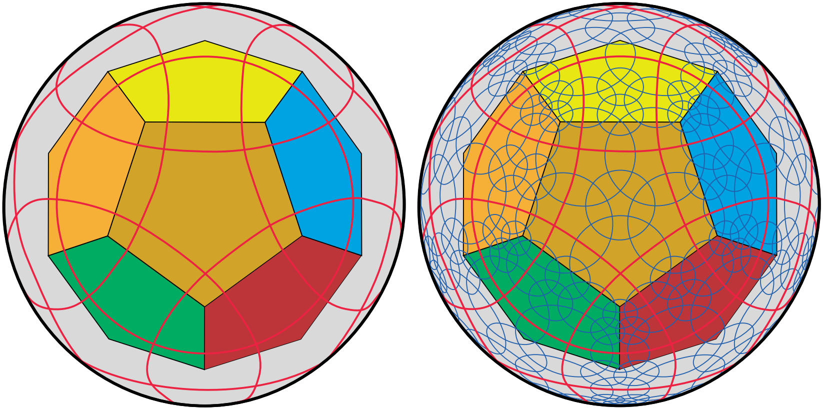
(The subdivision is drawn using Stephenson's CirclePack).



The third subdivision of the quadrilateral tile type

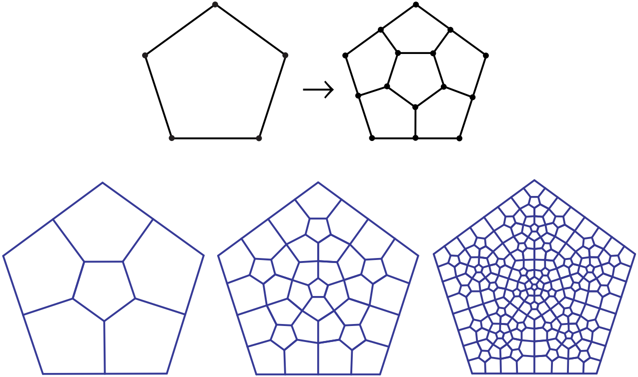


The dodecahedral subdivision rules comes from the recursion at infinity for a Kleinian group.



Example 2. Pentagonal subdivision rule

The subdivision of the tile type and the first three subdivisions.



Postcritically finite branched maps

- An orientation-preserving branched map $f\colon S^2 \to S^2$ is *postcritically finite* if the set P_f of postcritical points is finite.
- Two such maps f and g are *equivalent* if there is a homeomorphism $h\colon S^2 \to S^2$ such that $h(P_f) = P_g$, $(h \circ f)\big|_{P_f} = (g \circ h)\big|_{P_f}$, and $h \circ f$ is isotopic, rel P_f, to $g \circ h$.
- If a finite subdivision rule $\mathcal{R}$ is orientation-preserving and has subdivision complex a 2-sphere, then the subdivision map σ is postcritically finite. (The postcritical points are vertices of $S_{\mathcal{R}}$.)

Realizing subdivision maps by rational maps

- When can the subdivision map of a finite subdivision rule be realized by a rational map?
- **Theorem (C-F-Kenyon-P):** Suppose $\mathcal{R}$ is an orientation-preserving finite subdivision rule which has bounded valence, mesh approaching zero, and subdivision complex a 2-sphere. If $\mathcal{R}$ is (combinatorially) conformal, then it is equivalent to a rational map.
- C-F-K-P, Constructing rational maps from finite subdivision rules, *Conform. Geom. Dyn.* **7** (2003), 76–102 (electronic).
- The converse follows from the expansion-complex machinery of C-F-P.

Realizing rational maps by fsr's

Theorem (Bonk-Meyer, C-F-P-Pilgrim): If *f* is an expanding postcritically finite branched map, then every sufficiently large iterate of *f* is equivalent to the subdivision map of a fsr with mesh approaching zero.

- Idea: pick a simple closed curve containing the postcritical points. For a sufficiently large iterate, that curve can be approximated by a curve in its preimage. Now use the expansion-complex machinery.
- Do you need to pass to an iterate of the map?
- In both proofs, one contructs a fsr with 1-skeleton a circle.

Thurston pullback map

- Suppose $f\colon S^2 \to S^2$ is a postcritically finite branched map. When is it equivalent to a rational map?
- Let $\mathcal{T}(\mathcal{O}_f)$ be the Teichmüller space of $\mathcal{O}_f = S^2 \setminus P_f$.
- f induces a pullback map $\tau_f\colon \mathcal{T}(\mathcal{O}_f) \to \mathcal{T}(\mathcal{O}_f)$.

Theorem (Thurston): If τ_f has a fixed point, then f is equivalent to a rational map.

Thurston obstructions

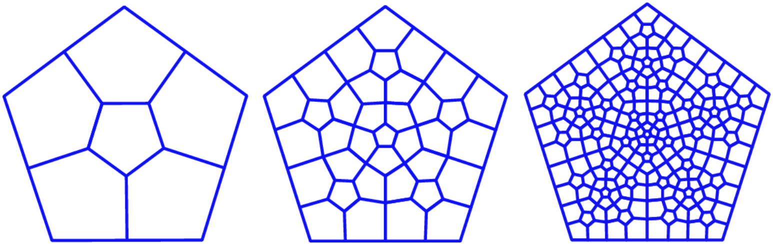
- *multicurve* Γ: components are nontrivial, non-peripheral, and pairwise non-isotopic
- *invariant* multicurve Γ: each component of $f^{-1}(\Gamma)$ is trivial, peripheral, or isotopic to a component of Γ
- *Thurston matrix* A^Γ for an invariant multicurve: $A^\Gamma_{\gamma\delta} = \sum_\alpha \frac{1}{\deg(f\colon \alpha \to \delta)}$
- *Thurston obstruction*: an invariant multicurve with spectral radius at least 1.

Theorem (Thurston): If $\mathcal{O}_f$ is hyperbolic, then f is equivalent to a rational map if and only if there are no Thurston obstructions.

- Unfortunately, the pullback map doesn't extend continuously to Thurston's compactification of Teichmüller space.

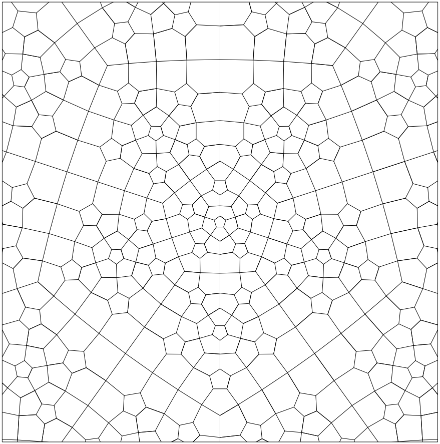
Expansion complexes

- An expansion $\mathcal{R}$-complex is an $\mathcal{R}$-complex X with structure map $f\colon X \to S_{\mathcal{R}}$ such that X is homeomorphic to $\mathbb{R}^2$ and there is an orientation-preserving homeomorphism $\varphi\colon X \to X$ with $\sigma_{\mathcal{R}} \circ f = f \circ \varphi$.
- Expansion complexes arise as direct limits of sequences of subdivisions.

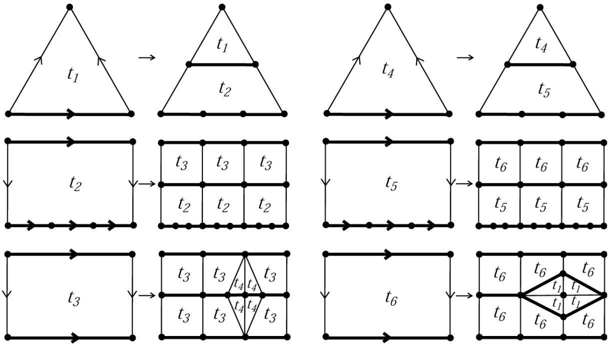


The pentagonal expansion complex

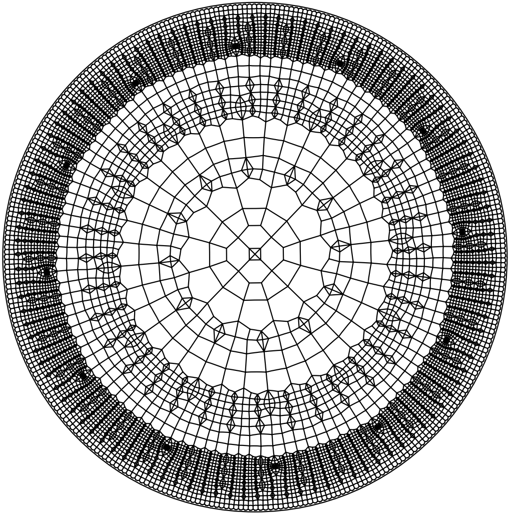
- P.L. Bowers and K. Stephenson, A “regular” pentagonal tiling of the plane, *Conform. Geom. Dyn.* **1** (1997), 58–68 (electronic.)



Example 3. An example with parabolic and hyperbolic expansion complexes



A hyperbolic expansion complex



Another expansion complex