

# A critically finite map, and the associated map on Teichmüller space

Dedicated to the memory of Adrien Douady

J. Cannon<sup>1</sup>   W. Floyd<sup>2</sup>   W. Parry<sup>3</sup>   K. Pilgrim<sup>4</sup>

<sup>1</sup>Department of Mathematics  
Brigham Young University

<sup>2</sup>Department of Mathematics  
Virginia Tech

<sup>3</sup>Department of Mathematics  
Eastern Michigan University

<sup>4</sup>Department of Mathematics  
Indiana University

Spring Topology and Dynamics Conference 2007



# Critically finite branched maps

- An orientation-preserving branched map  $f: S^2 \rightarrow S^2$  is *critically finite* if  $P_f$ , the set of post-critical points, is finite.
- Two such maps  $f$  and  $g$  are *equivalent* if there is a homeomorphism  $h: S^2 \rightarrow S^2$  such that  $h(P_f) = P_g$ ,  $(h \circ f)|_{P_f} = (g \circ h)|_{P_g}$ , and  $h \circ f$  is isotopic, rel  $P_f$ , to  $g \circ h$ .

When is a critically finite map  $f: S^2 \rightarrow S^2$  equivalent to a rational map?

- Following Thurston, we put an orbifold structure  $\mathcal{O}_f$  on  $S^2$  by setting  $\nu_x = \text{lcm}\{D_g(y) : g(y) = x \text{ and } g = f^{\circ n} \text{ for some } n\}$ .
- Let  $\mathcal{T}(\mathcal{O}_f)$  be the Teichmüller space of  $\mathcal{O}_f$ .
- $f$  induces a pullback map  $\tau_f: \mathcal{T}(\mathcal{O}_f) \rightarrow \mathcal{T}(\mathcal{O}_f)$ .

# Thurston's characterization theorem

**Theorem (Thurston):** A critically finite map  $f$  is equivalent to a rational map if and only if  $\tau_f$  has a fixed point.

- *multicurve*  $\Gamma$ : components are nontrivial, non-peripheral, and pairwise non-isotopic
- *invariant* multicurve  $\Gamma$ : each component of  $f^{-1}(\Gamma)$  is trivial, peripheral, or isotopic to a component of  $\Gamma$
- *Thurston matrix*  $A^\Gamma$  for an invariant multicurve
$$A_{\gamma\delta}^\Gamma = \sum_{\alpha \sim \gamma} \frac{1}{\deg(f: \alpha \rightarrow \delta)}$$
- *Thurston obstruction*: an invariant multicurve whose associated matrix has spectral radius (Thurston multiplier) at least 1

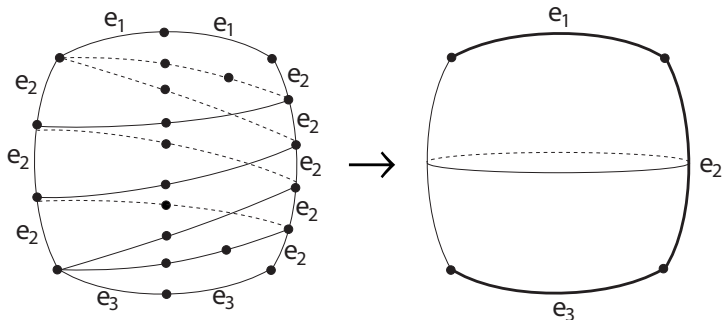
**Theorem (Thurston):** If  $\mathcal{O}_f$  is hyperbolic, then  $f$  is equivalent to a rational map if and only if there are no Thurston obstructions.

# Thurston's characterization theorem

- Thurston presented his characterization theorem in a CBMS lecture series in Duluth in 1983, but he didn't publish his proof.
- Fortunately (for us and many others), Douady and Hubbard published a proof of Thurston's theorem in their paper *A proof of Thurston's topological characterization of rational functions*, Acta Math. **171** (1993), 263–297.
- The proof doesn't make use of a compactification of Teichmüller space. Why?

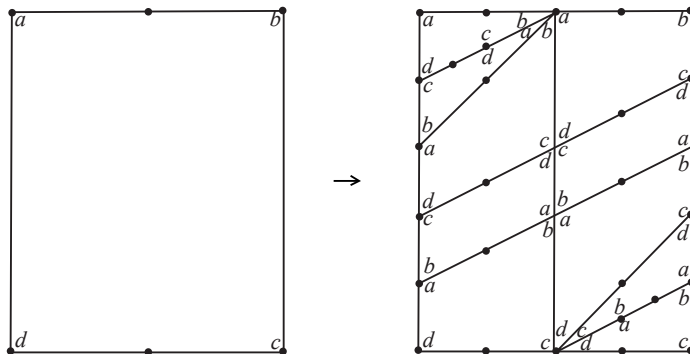
# The map $f$

Here is a critically finite map  $f: S^2 \rightarrow S^2$ .  $f$  is cellular and preserves labels of edges (though many of the labels aren't drawn).



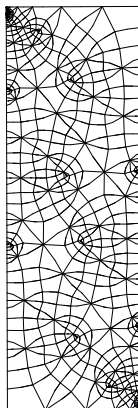
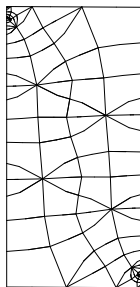
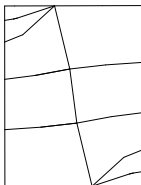
# The map $f$ , viewed as a finite subdivision rule

The map can also be viewed as the subdivision map of a finite subdivision rule. The single tile type is a hexagon. Here is the tile type and its subdivision. Some of the vertices are labeled to indicate the identifications.



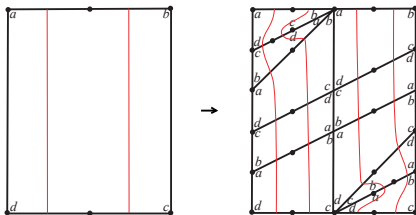
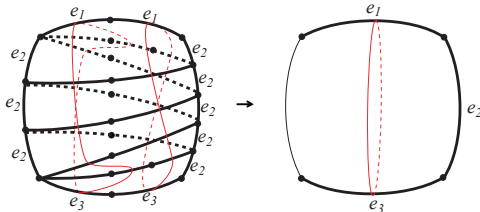
# The first three subdivisions

Here are the first three subdivisions of the tile type, drawn with Ken Stephenson's program CirclePack. They suggest that this example won't be rational, but this example was designed to be difficult to work with.



# An invariant multicurve

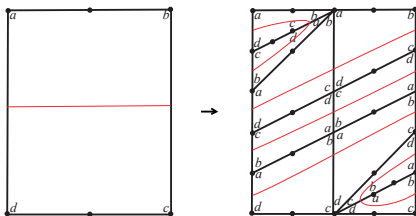
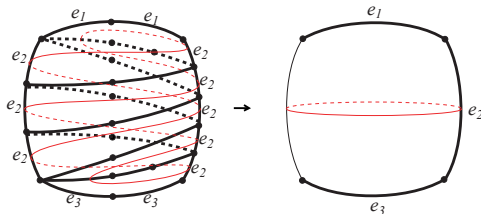
A vertical curve is an invariant multicurve with Thurston multiplier  $\frac{2}{5}$ .





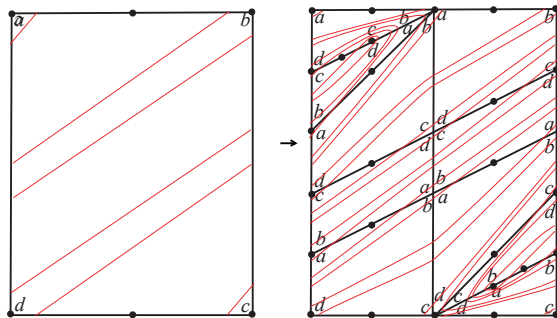
# Another invariant multicurve

A horizontal curve is an invariant multicurve with Thurston multiplier  $\frac{1}{10}$ .



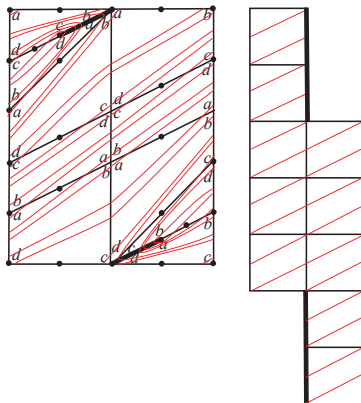
# Is this multicurve invariant?

- For this function  $f$ , does every essential (nontrivial, non-peripheral) curve have a component in its preimage which is essential?
- Are there any other invariant multicurves?
- In general, it isn't obvious whether or not a multicurve is invariant.



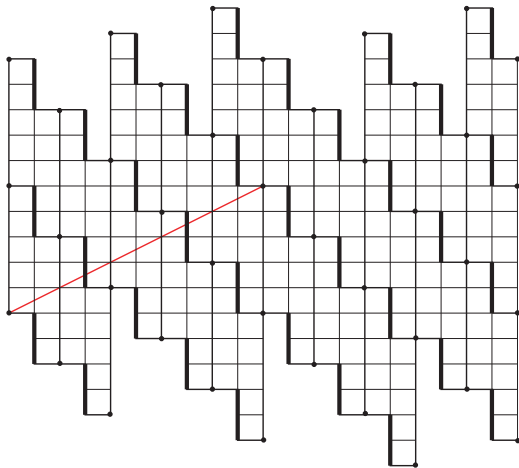
# Twisting mirrors

To more easily figure out the curves in the preimage, one can split the subdivided tile type along the bold arcs. The new subdivision can be realized as a tiling by squares. Also, one can draw a core arc for the curve instead of the curve itself.



# Twisting mirrors

An arc in the preimage now lifts to an arc in the plane. The slope of its image for the example can be computed in terms of how the arc intersects the “twisting mirrors”.



# Defining $\tau_S$ on rational points on the boundary

We begin by defining a function  $\tau_S: \mathbb{Q} \cup \{\infty\} \rightarrow \mathbb{Q} \cup \{\infty\}$ . For convenience, we will often denote  $\infty$  by  $a/0$ , where  $a \in \mathbb{Z}$  and  $a \neq 0$ . The function  $\tau_S$  is defined at  $0/1$  and  $1/0$  by  $\tau_S(0) = 0$  and  $\tau_S(1/0) = 1/0$ . Now suppose that  $p \in \mathbb{Q} \setminus \{0\}$ . We can write  $p$  as  $p = \frac{a}{2b}$  for some integers  $a, b$  with  $\gcd(a, b) = 1$  and  $b > 0$ . Define integers  $A$  and  $B$  as follows:

- I.  $b \equiv 0 \pmod{2}$ 
  - i.  $a + b \equiv 0 \pmod{5}$       Set  $(A, B) = (a, b)$ .
  - ii.  $a + b \not\equiv 0 \pmod{5}$       Set  $(A, B) = (5a, 5b)$ .
- II.  $b \equiv 1 \pmod{2}$ 
  - i.  $a + b \equiv \pm 1 \pmod{5}$       Set  $(A, B) = (a, b)$ .
  - ii.  $a + b \equiv \pm 2 \pmod{5}$       Set  $(A, B) = (3a, 3b)$ .
  - iii.  $a + b \equiv 0 \pmod{5}$       Set  $(A, B) = (2a, 1 + 2b)$ .

# Defining $\tau_S$ on rational points on the boundary

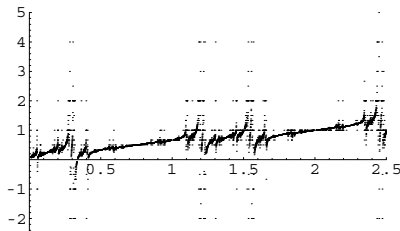
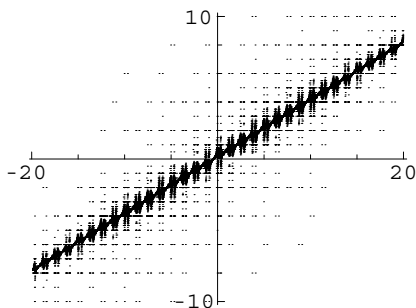
- Suppose we are in one of the first four cases. For each integer  $x$  with  $0 \leq x \leq B$ , let  $q_x \in \mathbb{Z}$  and  $r_x \in (-\frac{5}{2}, \frac{5}{2}]$  such that  $(\frac{a}{b} + 1)x = q_x + r_x$ . Let  $0 < x_1 < \dots < x_n < B$  be those odd integers such that  $|r_x| < 1$ . Let  $x_0 = 0$ , and let  $x_{n+1} = B$ .
- For case II.iii), define  $x_i$  by  $1 < x_1 < \dots < x_n < B$  and let  $x_0 = 1$  and  $x_{n+1} = B$ . For each  $x$ , let  $(\frac{a}{b} + 1)x - \frac{a}{b} = q_x + r_x$ .
- Let
$$N = \sum_{k=0}^n (-1)^k (q_{k+1} - q_k) \quad \text{and} \quad D = \sum_{k=0}^n (-1)^k (x_{k+1} - x_k).$$
- One can show that  $N$  and  $D$  cannot both be 0. (Except in the last case, they can't both be even.)
- $\tau_S(\frac{a}{2b}) = \frac{N}{D}$ .

# The map $\tau_S$

- $\tau_S$  is the “extension” of  $\tau_f$  to rational points in  $\partial\mathcal{T}(\mathcal{O}_f)$ .
- In addition to the fixed points 0 and  $\infty$ ,  $\tau_S$  has a period-3 cycle:  $\frac{2}{3} \mapsto \frac{4}{9} \mapsto \frac{4}{15} \mapsto \frac{2}{3}$ .
- From the definition of  $\tau_S$ , one can show that the only fixed points of  $\tau_S$  are 0 and  $\infty$ . By Thurston’s theorem, this implies that  $f$  is equivalent to a rational map.
- It appears experimentally that under iteration of  $\tau_S$  every point maps to one of the two fixed points or the period-3 cycle given above.

# The graph of $\tau_s$

Here is part of the graph of  $\tau_s$ , and a blow-up of the graph near the origin.





# Functional equations

- Suppose that  $\tau_s(p/q) = r/s$ , and the pre-image of a  $p/q$  curve has  $c$  essential components, each mapping by degree  $d$ .
- Let  $\sigma_1(z) = \frac{(1-pq)z+p^2}{-q^2z+1+pq}$  and let  $\sigma_2(z) = \frac{(1-rs)z+r^2}{-s^2z+1+rs}$ . By considering Dehn twists, one sees the following.
  - If  $q$  is even, then  $\tau_s(\sigma_1^d(z)) = \sigma_2^c(\tau_s(z))$ .
  - If  $q$  is odd, then  $\tau_s(\sigma_1^{2d}(z)) = \sigma_2^{2c}(\tau_s(z))$ .
- Letting  $p/q = 1/0$  gives  $\tau_s(z+5) = \tau_s(z) + 2$ .
- Letting  $p/q = 0/1$  gives  $\tau_s(-\frac{z}{20z+1}) = \frac{\tau_s(z)}{-2\tau_s(z)+1}$ .
- Using the functional equations, one can show that  $\tau_s$  isn't continuous.

# Equivalence to a rational map

- Here is another argument that  $f$  is equivalent to a rational map. We can identify  $\mathcal{T}(\mathcal{O}_f)$  with the upper-half plane model of  $\mathbb{H}^2$ .
- Suppose  $\tau_s(p/q) = p'/q'$  and the Thurston multiplier of  $p/q$  is  $\delta$ . Given  $m > 0$ ,  $B_m(p/q) = \{z: \text{mod}_\tau(p/q) > m\}$  is a horoball tangent to  $-q/p$  with Euclidean radius  $1/(2p^2m)$ .  $\tau(B_m(p/q)) \subset B_{\delta m}(p'/q')$ .
- For  $m$  sufficiently small,  $H = \{z: d(z, \overline{B_m(p/q)}) < d(z, \overline{B_{\delta m}(p'/q')})\}$  is a hyperbolic half-space. It doesn't depend on  $m$ , and its boundary doesn't contain the negative reciprocals of any fixed points of  $\tau_s$  which are Thurston obstructions.
- The boundaries of these half-planes coming from  $p/q = 1/8, 1/4, -1/4, 3/4, 4/9, 5/12$ , and  $7/22$  cover the boundary of  $\mathcal{T}(\mathcal{O}_f)$ , so there are no Thurston obstructions and  $f$  is realizable by a rational map.