Cannon's conjecture, finite subdivision rules, and rational maps

J. Cannon[1] W. Floyd[2] W. Parry[3]

[1]Department of Mathematics
Brigham Young University

[2]Department of Mathematics
Virginia Tech

[3]Department of Mathematics
Eastern Michigan University

Groups007

Cannon's Conjecture

Conjecture: If G is a Gromov-hyperbolic discrete group whose space at infinity is S^2, then G acts properly discontinuously, cocompactly, and isometrically on $\mathbb{H}^3$.

- Suppose G is a group and Γ is a locally finite Cayley graph. G is *Gromov-hyperbolic* if Γ has *thin triangles*.
- Points in the space at infinity are equivalence classes of geodesic rays; $R \sim S$ if $\sup\{d(R(t), S(t)) : t \geq 0\} < \infty$.

How do you proceed from combinatorial/topological information to analytic information?

Combinatorial structure

Theorem (C): If G is a cocompact, discrete group of isometries of hyperbolic space, then G has a linear recursion.

- J. W. Cannon, The combinatorial structure of cocompact discrete hyperbolic groups, *Geom. Dedicata* **16** (1984), 123–148.
- The proof shows that the cone type of a vertex depends only on the order type of a finite ball around the vertex in the Cayley graph, and hence there are only finitely many cone types.
- The proof depends on hyperbolic space having thin triangles. Once Gromov-hyperbolic spaces are defined, the proof applies to Gromov-hyperbolic groups.

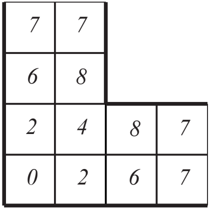
Weight functions

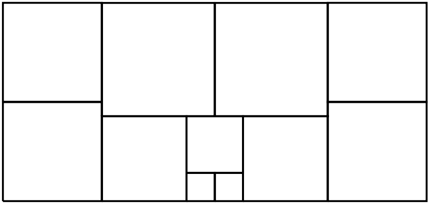
- *shingling* (locally-finite covering by compact, connected sets) $\mathcal{T}$ on a surface S, ring (or quadrilateral) $R \subset S$
- *weight function* ρ on $\mathcal{T}$: $\rho \colon \mathcal{T} \to \mathbb{R}_{\geq 0}$
- ρ*-length* of a curve α in S: $\Sigma_{\{t\colon\, t\cap\alpha\neq\emptyset\}}\rho(t)$
- ρ*-height* H_ρ of R: infimum of the ρ-lengths of the height curves
- ρ*-area* A_ρ of R: $\Sigma_{\{t\colon\, t\cap A\neq\emptyset\}}\rho(t)^2$
- ρ*-circumference* C_ρ of R: infimum of the ρ-lengths of separating curves
- *moduli* $M_\rho = H_\rho^2/A_\rho$ and $m_\rho = A_\rho/C_\rho^2$

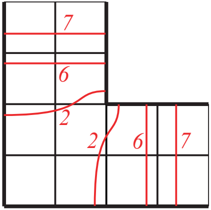
Combinatorial moduli

- *moduli* $M(R) = \sup_\rho H_\rho^2 / A_\rho$ and $m(R) = \inf_\rho A_\rho / C_\rho^2$
- The sup and inf exist, and are unique up to scaling. (This follows from compactness and convexity.)
- Now consider a sequence of shinglings of S.
- **Axiom 1.** Nondegeneration, comparability of asymptotic combinatorial moduli
- **Axiom 2.** Existence of local rings with large moduli
- *conformal sequence* of shinglings: Axioms 1 and 2, plus mesh locally approaching 0.

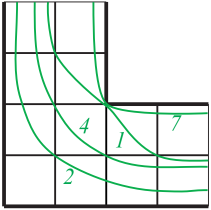
Optimal weight functions - an example







fat flows



skinny cuts

Finite Riemann Mapping Theorem

Theorem (C-F-P, Schramm): For a tiling of a quadrilateral, the optimal weight function determines a squaring of a rectangle.

- C-F-P, Squaring rectangles: the finite Riemann mapping theorem, Contemp. Math., 169 (1994), 133-212.
  O. Schramm, Square tilings with prescribed combinatorics, *Israel J. Math.* **84** (1993), 97–118.
- The optimal weight function for fat flows is also the optimal weight function for skinny cuts.
- This optimal weight function is a weighted sum of fat flows and a weighted sum of fat cuts. The flows and cuts give a grid for the squaring.
- With an eye toward Cannon's Conjecture, Hersonsky has a new proof using harmonic maps.

Combinatorial Riemann Mapping Theorem

Theorem (C): If $\{\mathcal{S}_i\}$ is a conformal sequence of shinglings on a topological surface S and R is a ring in S, then R has a metric which makes it a right-circular annulus such that analytic moduli and asymptotic combinatorial moduli on R are uniformly comparable.

- J. W. Cannon, The combinatorial Riemann mapping theorem, *Acta Math.* **173** (1994), 155–234.

Corollary (C): If $\{\mathcal{S}_i\}$ is a conformal sequence of shinglings on a topological surface S, then there is a quasiconformal structure on S such that the analytic moduli of rings are comparable to the asymptotic combinatorial moduli.

Cannon-Swenson

- G a Gromov-hyperbolic group, Γ a locally finite Cayley graph, base vertex $\mathcal{O}$
- space at infinity Γ_∞: points are equivalence classes of geodesic rays based at $\mathcal{O}$
- *half-space* $H(R,n) = \{x \in \Gamma \colon \; d(x, R([n,\infty)) \leq d(x, R([0,n])\}$
- *disk at infinity* $\mathcal{D}(R,n) = \{[S] \in \Gamma_\infty \colon \; \lim_{t\to\infty} d(S(t), \Gamma \setminus H(R,n)) = \infty\}$
- cover $\mathcal{D}(n) = \{\mathcal{D}(R,n) \colon \; R \text{ is a geodesic ray based at } \mathcal{O}\}$

Cannon-Swenson

Theorem (C-Swenson): In the setting of Cannon's conjecture, it suffices to prove that the sequence $\{\mathcal{D}(n)\}_{n\in\mathbb{N}}$ is conformal. Furthermore, the $\mathcal{D}(n)$'s satisfy a linear recursion.

- J. W. Cannon, E. L. Swenson, Recognizing constant curvature groups in dimension 3, *Trans. Amer. Math. Soc.* **350** (1998), 809–849.
- The disks at infinity give a basis for the topology of Γ_∞.
- The CRMT implies there is a quasiconformal structure on Γ_∞. It is quasiconformally equivalent to an analytic structure. The group action is uniformly quasiconformal so by Sullivan/Tukia it is conjugate to a conformal action.
- The linear recursion follows from finite cone types.

Sufficiently rich families

- **Axiom 1.** Nondegeneration, comparability of asymptotic combinatorial moduli
- **Axiom 2.** Existence of local rings with large moduli
- **Axiom 0.** Existence of local rings whose moduli don't degenerate to 0
- *buffered ring*: made out of three subrings; the outer rings have moduli bounded below (by a fixed constant) and the spanning ring has moduli bounded above
- *buffered ring cover*: bounded valence family of closed disks which cover, have disjoint inner disks, and have complements that are buffered annuli

Theorem (C-F-P): In the setting of Cannon's conjecture, Axioms 1 and 2 can be replaced by Axiom 0 plus the existence of buffered ring covers of arbitrarily small mesh. Furthermore, for Axiom 0 it suffices to check the moduli of finitely many annuli.

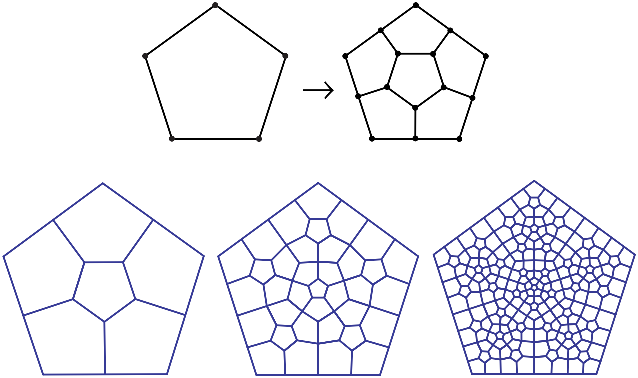
- C-F-P, Sufficiently rich families of planar rings, *Ann. Acad. Sci. Fenn. Math.* **24** (1999), 265–304.
- The proof of CRMT can be adapted so you only need Axiom 2 plus the existence of the buffered ring covers.
- By the finite recursion, finding buffered ring covers reduces to finding them for finitely many disks.
- Axiom 2 follows from Axiom 0 because of the finiteness and the subadditivity of moduli for nested annuli.

Definition of a finite subdivision rule $\mathcal{R}$

- C-F-P, Finite subdivision rules, *Conform. Geom. Dyn.* **5** (2001), 153–196 (electronic).
- subdivision complex $S_{\mathcal{R}}$
- $S_{\mathcal{R}}$ is the union of its closed 2-cells. Each 2-cell is modeled on an n-gon (called a *tile type*) with $n \geq 3$.
- subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$
- subdivision map $\sigma_{\mathcal{R}} \colon \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$
- $\sigma_{\mathcal{R}}$ is cellular and takes each open cell homeomorphically onto an open cell.
- *$\mathcal{R}$-complex*: a 2-complex X which is the closure of its 2-cells, together with a structure map $h \colon X \to S_{\mathcal{R}}$ which takes each open cell homeomorphically onto an open cell
- One can use a finite subdivision rule to recursively subdivide $\mathcal{R}$-complexes. $\mathcal{R}(X)$ is the subdivision of X.

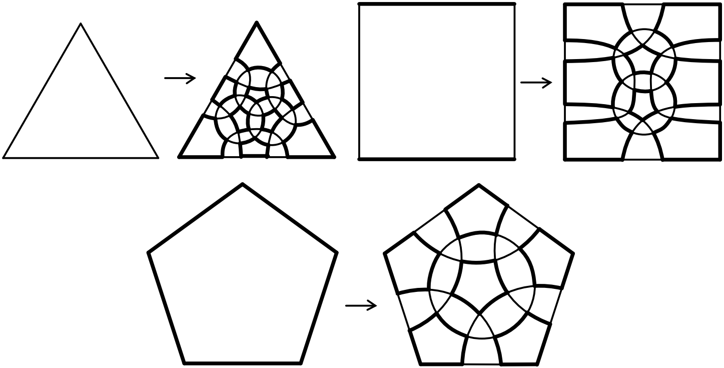
Example 1. Pentagonal subdivision rule

The pentagonal sudivision rule and the first three subdivisions. (The subdivisions are drawn using Stephenson's CirclePack).



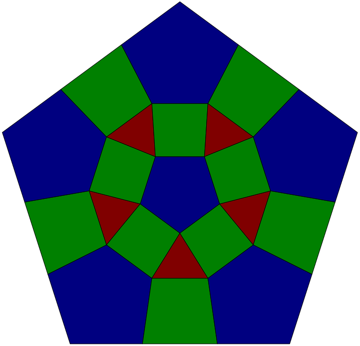
Example 2. The dodecahedral subdivision rule

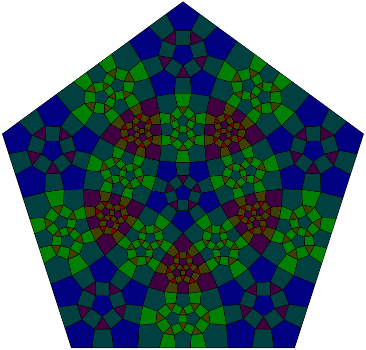
The subdivisions of the three tile types. Note that there are two edge types.



Example 2. The dodecahedral subdivision rule

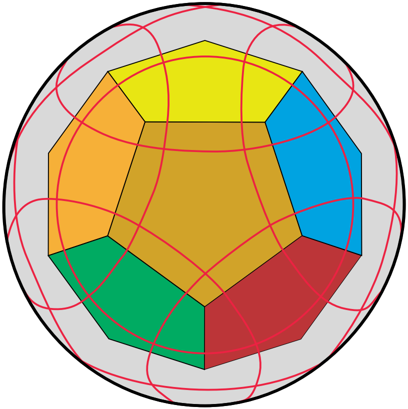
The first two subdivisions of the pentagonal tile type.

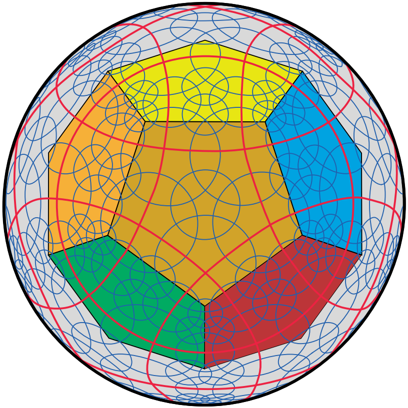




This subdivision rule on the sphere at infinity

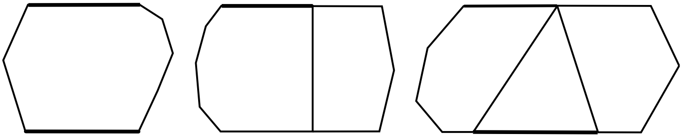
The dodecahedral subdivision rule comes from the recursion at infinity for a Kleinian group.





Conformal finite subdivision rules

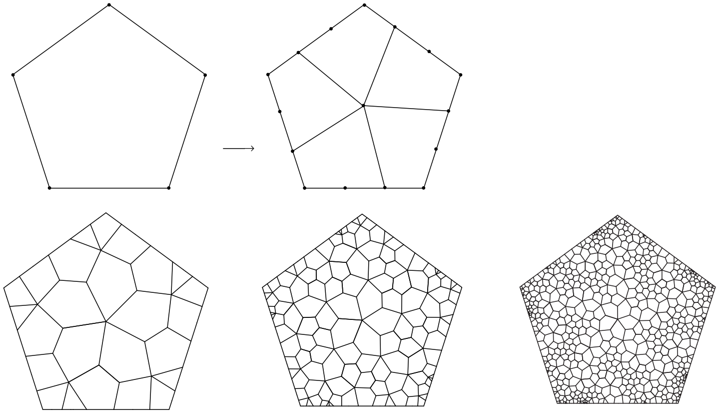
- A fsr $\mathcal{R}$ has *bounded valence* if there is an upper bound to the valences of the vertices in the subdivisions $\mathcal{R}^n(S_{\mathcal{R}})$.
- $\mathcal{R}$ has *mesh approaching* 0 if, for any open cover U of $S_{\mathcal{R}}$, for n sufficiently large each tile of $\mathcal{R}^n(S_{\mathcal{R}})$ is contained in an element of U.
- $\mathcal{R}$ is *conformal* if each $\mathcal{R}$-complex is conformal with respect to the sequence of tiles of $\mathcal{R}^n(X)$.
- For fsr's with bounded valence and mesh approaching 0, Axiom 0 implies conformality.
- 1,2,3-tile criterion. It suffices to consider finitely many $\mathcal{R}$-complexes.



Symmetry

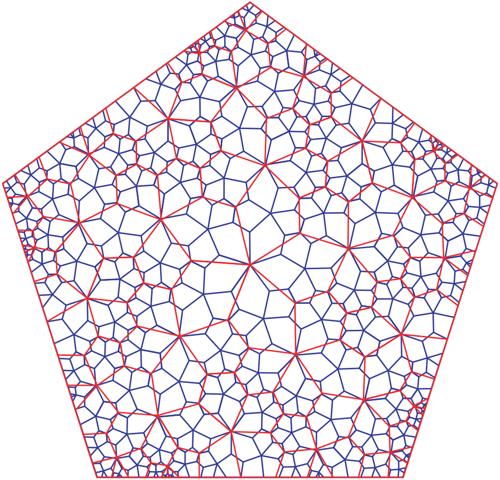
- A fsr with bounded valence, mesh approaching 0, a single tile type and dihedral symmetry is conformal. This uses the 1,2,3-tile criterion.
- A one-tile orientation preserving fsr with rotational symmetry is conformal. The proof uses expansion complexes and conformal structures on the subdivision complex.
- An *expansion complex* for a fsr $\mathcal{R}$ is an $\mathcal{R}$-complex X with $X \simeq \mathbb{R}^2$ and an orientation preserving homeomorphism $\varphi\colon X \to X$ such that $h \circ \varphi = \sigma_{\mathcal{R}} \circ h$, where $h\colon X \to S_{\mathcal{R}}$ is the structure map for X.
- Loosely, an expansion map corresponds to a horoball, and the map φ corresponds to moving along a geodesic to the corresponding point at infinity.

An example with rotational symmetry



Superimposed subdivisions

Here are the third and fourth subdivisions, superimposed. Note the vertices.



The expansion complex

- The direct limit of the subdivisions is an expansion complex X. One can put a piecewise conformal structure on X with regular pentagons, and then use power maps to extend over the vertices. (This is inspired by a Bowers-Stephenson construction.)
- The expansion map agrees with a conformal map on the vertices. One can conjugate to get a new fsr for which this conformal map is the expansion map. The subdivision map is conformal with respect to the induced conformal structure on the subdivsion complex.
- The existence of an invariant conformal structure implies (combinatorial) conformality of the fsr.
- C-F-P, Expansion complexes for finite subdivision rules I, *Conform. Geom. Dyn.* **10** 63–99 (2006) (electronic)
  C-F-P, Expansion complexes for finite subdivision rules II, *Conform. Geom. Dyn.* **10** 326–354 (2006) (electronic)

Critically finite branched maps

- A fsr has an *edge pairing* if the subdivision complex is a surface. In this case, if the subdivision map isn't a covering, the subdivision complex is S^2 and the subdivision map f is a critically finite branched map.
- f is *critically finite* if P_f, the set of post-critical points, is finite.
- Suppose $\mathcal{R}$ is an orientation-preserving fsr with mesh approaching 0 and $S_{\mathcal{R}} \simeq S^2$? When is the subdivision map f equivalent to a rational map?
- Here $f \sim g$ if there is a homeomorphism $h\colon\ S^2 \to S^2$ such that $h(P_f) = P_g$, $(h \circ f)\big|_{P_f} = (g \circ h)\big|_{P_f}$, and $h \circ f$ is isotopic, rel P_f, to $g \circ h$.
- Put an orbifold structure $\mathcal{O}_f$ on $S_{\mathcal{R}}$ by setting $\nu_x = \text{lcm}\{D_g(y)\colon\ g(y) = x \text{ and } g = f^{\circ n} \text{ for some } n\}$.

Thurston obstruction

- Let $\mathcal{T}(\mathcal{O}_f)$ be the Teichmüller space of $\mathcal{O}_f$.
- f induces a pullback map $\tau_f : \mathcal{T}(\mathcal{O}_f) \to \mathcal{T}(\mathcal{O}_f)$.

Theorem (Thurston): If τ_f has a fixed point, then f is equivalent to a rational map.

- *multicurve* Γ: components are nontrivial, non-peripheral, and pairwise non-isotopic
- *invariant* multicurve Γ: each component of $f^{-1}(\Gamma)$ is trivial, peripheral, or isotopic to a component of Γ
- *Thurston matrix* A^Γ for an invariant multicurve $A^\Gamma_{\gamma\delta} = \sum_\alpha \frac{1}{\deg(f:\ \alpha\to\delta)}$
- *Thurston obstruction*: an invariant multicurve with spectral radius at least 1

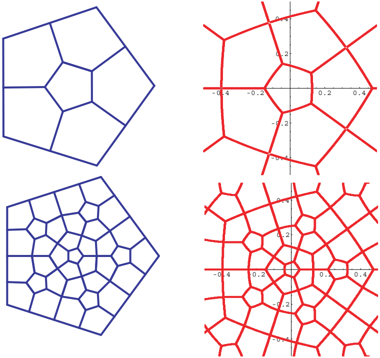
Thurston's characterization theorem

Theorem (Thurston): If $\mathcal{O}_f$ is hyperbolic, then f is equivalent to a rational map if and only if there are no Thurston obstructions.

- If f is the subdivision map of an orientation-preserving fsr with mesh approaching 0 and $S_{\mathcal{R}}$ is a 2-sphere, then f is realizable by a rational map if and only if f is (combinatorially) conformal.
- Most of the proof of Thurston's theorem can be recast in term of fsr's. The chief stumbling block in giving a fsr proof is Mumford's theorem.

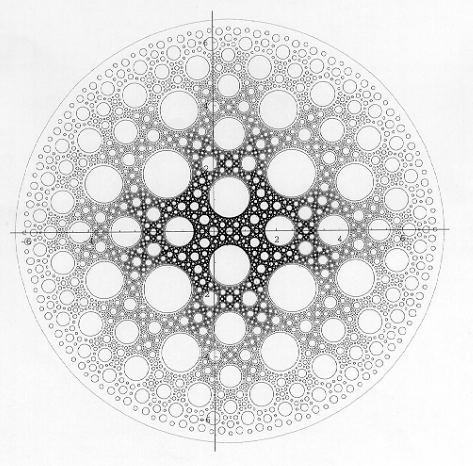
The pentagonal subdivision rule

- The pentagonal subdivision rule is closely associated with a fsr (with triangular tile types) which is realizable by a rational map. Here are subdivisions drawn by CirclePack and by preimages under the rational map (unfolding by $z \mapsto z^5$).



The barycentric subdivision rule

- The barycentric subdivision rule can be realized by the rational map $f(z) = \frac{4(z^2-z+1)^3}{27z^2(z-1)^2}$. Here is the Julia set.



Realizing rational maps by fsr's

Theorem (C-F-P, Bonk-Meyer): If f is a critically finite rational map without periodic critical points, then every sufficiently large iterate of f is equivalent to the subdivision rule of a fsr.

- Our proof: Pick a simple closed curve containing the post-critical points. For a sufficiently large iterate, that curve can be approximated by a curve in its preimage. Now use the expansion complex machinery.
- Do you need to pass to an iterate of the map?
- What about critically finite maps with periodic critical points? This corresponds to fsr's with unbounded valence.