A SURVEY OF TWISTED FACE-PAIRING 3-MANIFOLDS

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Abstract. The twisted face-pairing construction gives an efficient way to generate face-pairing descriptions for many interesting closed 3-manifolds. Our work in this paper is directed toward the goal of determining which closed, connected, orientable 3-manifolds can be generated from this construction. We succeed in proving that all lens spaces, the Heisenberg manifold (Nil geometry), $S^2 \times S^1$, and all connected sums of twisted face-pairing manifolds are twisted face-pairing manifolds. We show how to obtain most closed, connected, orientable, Seifert-fibered manifolds as twisted face-pairing manifolds. It still seems unlikely that all closed, connected, orientable 3-manifolds can be so obtained.

The twisted face-pairing construction of our earlier papers [1], [2], [3] gives an efficient way of generating, mechanically and with little effort, myriads of relatively simple face-pairing descriptions of interesting closed 3-manifolds. Papers [1] and [2] established the basic properties of these manifolds. In [3] we investigated a special subclass of twisted face-pairing manifolds, namely the ample manifolds, and showed that each has Gromov hyperbolic fundamental group with the 2-sphere as the space at infinity. In [4] we showed how to construct Heegaard diagrams for twisted face-pairing 3-manifolds. From that construction it is easy to give framed surgery descriptions for twisted face-pairing 3-manifolds.

Our work in this paper is directed towards determining which closed, connected, orientable 3-manifolds can be realized as twisted face-pairing manifolds. Our methods are powerful enough to show that the class of twisted face-pairing manifolds contains all lens spaces (Corollary 4.2), the Heisenberg manifold (Example 7.4), $S^2 \times S^1$ (Example 6.2.1 of [4]), every orientable torus bundle over a circle with Solv geometry (Theorem 5.1), most closed, connected, orientable, Seifert fibered manifolds (Theorem 8.1), and all connected sums of twisted face-pairing manifolds (Theorem 9.1). It still seems unlikely, however, that all closed, connected, orientable 3-manifolds can be obtained as twisted face-pairing manifolds. Thus the following tantalizing questions remain unanswered:

Questions. Is every closed, connected, orientable 3-manifold a twisted face-pairing manifold? In particular, is there any twisted face-pairing manifold based on Euclidean geometry or on $\mathbf{H}^2 \times \mathbf{R}$? Is the 3-torus a twisted face-pairing manifold?

We have identified twisted face-pairing manifolds from each of Thurston's eight 3-manifold geometries except Euclidean geometry and $\mathbf{H}^2 \times \mathbf{R}$.

Much of the paper is devoted to detailed analyses of the class of twisted face-pairing manifolds arising from two classes of model face-pairings. The first of these is reflection face-pairings, in which the model faceted 3-ball P is the unit ball in $\mathbf{R}^3$ and each face-pairing map is reflection through the xy-plane. These are

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used in showing that every lens space is a twisted face-pairing 3-manifold and that every torus bundle over S^1 with Solv geometry is a twisted face-pairing 3-manifold. The other class of model face-pairing that is studied in detail is the class of lune complexes, in which the faceted 3-ball P has exactly two vertices and each face of P is a digon. Every twisted face-pairing 3-manifold arising from a lune complex is a Seifert fibered manifold, and most closed, connected, orientable, Seifert fibered 3-manifolds are twisted face-pairing manifolds.

Section 1 establishes the terminology for two standard constructions, twist moves and slam-dunks, on framed links in S^3. These constructions are used throughout the paper.

Section 2 gives a connected-sum operation for corridor complex links, which are defined in [4]. Suppose P is a faceted 3-ball, ϵ is an orientation-reserving face-pairing on P, and mul is a multiplier function for (P, ϵ). Let $M(\epsilon, \text{mul})$ be the associated twisted face-pairing manifold. The corridor construction of [4] produces a planar projection of a framed link L in S^3 such that $M(\epsilon, \text{mul})$ is obtained from S^3 by framed surgery on L. The corridor complex link L has two kinds of components, face components and edge components. Each face component has framing 0, and the framing of an edge component is the blackboard framing of its given planar projection plus the reciprocal of the multiplier of the associated edge cycle. In Section 2 we show that the connected sum of two corridor complex links along edge components is itself a corridor complex link as long as one of the edge components has an edge cycle representative with distinct vertices. This is used in Sections 6, 8, and 9.

Section 3 begins the analysis of reflection face-pairings. Given a reflection face-pairing ϵ on a faceted 3-ball P and a multiplier function mul, we prove that the corridor complex link can be replaced by a much simpler link L such that the twisted face-pairing 3-manifold $M(\epsilon, \text{mul})$ is obtained from S^3 by framed surgery on L. The corridor complex link is constructed from a Heegaard diagram for $M(\epsilon, \text{mul})$ where the Heegaard surface is the boundary of a handlebody obtained from P by joining each pair of faces by a solid tube so that the solid tubes follow corridors in the 1-skeleton of P. For reflection face-pairings, one obtains the new link from a Heegaard diagram where the Heegaard surface is the boundary of a handlebody obtained from $S^3 \setminus \text{int}(P)$ by adding a vertical tube in P for each face pair. As for corridor complex links, the link is composed of face components and edge components. The face components all have framing 0, and the framing of an edge component is the reciprocal of the multiplier of its associated edge cycle (each edge component has blackboard framing 0).

In Section 4 we show that every lens space is a twisted face-pairing manifold coming from a reflection face-pairing in which the faceted 3-ball is a scallop. By a slight modification of the construction, in Section 5 we show that every torus bundle over S^1 with Solv geometry is a twisted face-pairing 3-manifold.

Suppose that L is an unframed corridor complex link. Then for each multiplier function mul on the set of edge cycles of the associated faceted 3-ball, one obtains a twisted face-pairing manifold from the framed surgery on L in which each face component has framing 0 and each edge component has framing its blackboard framing plus the reciprocal of the multiplier of its associated edge class. Hence, by varying the multipliers, one obtains infinitely many manifolds by surgery on L. In Section 6 we show that we still get a twisted face-pairing 3-manifold by framed surgery on L if we give each face component framing 0 and either give each edge component its blackboard framing plus an arbitrary positive rational number or give each edge component its blackboard framing minus an arbitrary positive rational number. The proof of this result involves modifying the model faceted 3-ball by "attaching scallops" and "splitting edges". This result is used in Section 8 to show that "most" closed, connected, orientable, Seifert fibered 3-manifolds are twisted face-pairing manifolds.

Following the notation of [7], we describe a closed, connected, orientable , Seifert fibered 3-manifold M by either

$$(Oog|b;(\alpha_1,\beta_1),\dots,(\alpha_s,\beta_s)) \qquad \text{or} \qquad (Onk|b;(\alpha_1,\beta_1),\dots,(\alpha_s,\beta_s)).$$

Here O signifies that M is orientable, og signifies that the base surface is an orientable surface with g handles and nk signifies that the base surface is a nonorientable surface with k crosscaps, b is an arbitrary integer, s is an arbitrary nonnegative integer, and for $i \in \{1,\dots,s\}$ α_i and β_i are relatively prime positive integers with $\alpha_i > \beta_i$. Given the above data, let S be the surface described by og or nk, and let N be the orientable circle bundle over S which has a section. Then M is obtained from N by Dehn surgery on $s+1$ fibers with framings $1/b, \alpha_1/\beta_1,\dots,\alpha_s/\beta_s$.

Twisted face-pairing manifolds arising from lune complexes are analyzed in Section 7. Suppose that P is a lune complex and ϵ is an orientation-reversing face-pairing on P. We can assume that P is the unit 3-ball in $\mathbf{R}^3$, the vertices of P are the north pole and the south pole, and ϵ preserves the equator. Then ϵ restricts to an edge-pairing on the equatorial disk D obtained by intersecting P with the xy-plane. The quotient of D under the edge-pairing is a closed surface S with a vertex corresponding to each edge cycle and an edge corresponding to each face pair. We prove in Theorem 7.5 that for each multiplier function mul, the twisted face-pairing 3-manifold $M(\epsilon,\mathrm{mul})$ is a Seifert fibered manifold with base surface S. If b is the number of edge cycles with multiplier 1 and $m_1,\dots,m_s$ are the multipliers that are greater than 1, then M is either $(Oog|b;(m_1,1),\dots,(m_s,1))$ or $(Onk|b;(m_1,1),\dots,(m_s,1))$, where og or nk describes the topological type of S. Combining this with Theorem 6.1, in Theorem 8.1 we show that a closed, connected, orientable, Seifert fibered 3-manifold M is a twisted face-pairing manifold as long as $b \geq 0$ and either $b > 0$ or $s > 0$. By changing the orientation on M, one can show that a closed, connected, orientable, Seifert fibered 3-manifold is a twisted face-pairing 3-manifold unless either $b = s = 0$ or $s > 0$ and $-s < b < 0$.

The Seifert fibered 3-manifolds described in the previous sentence are not the only Seifert fibered 3-manifolds which are twisted face-pairing manifolds. Example 7.2 of [4] shows that Dehn surgery on the figure eight knot with positive integer framing m yields a twisted face-pairing manifold. These twisted face-pairing manifolds are also mentioned in Section 10 below. According to page 95 of [6], Dehn surgery on the figure eight knot with framing 1 gives the Seifert fibered manifold $(Oo0|-1;(2,1),(3,1),(7,1))$. The framing $m=2$ gives $(Oo0|-1;(2,1),(4,1),(5,1))$, and the framing $m=3$ gives $(Oo0|-1;(3,1),(3,1),(4,1))$. These three twisted face-pairing manifolds are not obtained by the results described in the previous paragraph.

We prove in Section 9 that the connected sum of two twisted face-pairing 3-manifolds is a twisted face-pairing 3-manifold. In Section 10 we briefly describe our computations with SnapPea [12] to construct hyperbolic twisted face-pairing manifolds with small volume. Finally, we conclude the paper in Section 11 with some questions.

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1. Dehn surgery preliminaries [prelim]

In this section we collect some well-known facts about Dehn surgery which will be used later.

We first discuss twist moves. They appear on page 162 of [5] as Rolfsen twists, they appear in Sections 16.4, 16.5 and 19.4 of [9] as Fenn-Rourke moves, and they appear in Section 9.H of [10]. For this let L be a link in S^3 framed by the elements of $\mathbf{Q}\cup\{\infty\}$. Let J be an unknotted component of L. Then $L\setminus J$ is contained in a closed solid torus T, which is the complement in S^3 of a regular neighborhood of J. Let τ be a right hand Dehn twist of T. Let $n\in\mathbf{Z}$. Let L' be the link gotten from L by applying τ^n to $L\setminus J$. We frame L' as follows. If the L-framing of J is r, then the L'-framing of J is $\frac{1}{n+\frac{1}{r}}$. If K is a component of L other than J with framing r, then the image of K in L' has framing $r+n\mathrm{lk}^2(J,K)$, where $\mathrm{lk}(J,K)$ is the linking number of J and K. When $n=1$, we say that L' is obtained from L by performing a twist move about J. In general we obtain L' by performing n twist moves about J. We are interested in twist moves because the manifold obtained by Dehn surgery on L' is homeomorphic to the manifold obtained by Dehn surgery on L.

We next discuss slam-dunks. These appear on page 163 of [5]. Let L be a framed link in S^3. Suppose that one component K of L is a meridian of another component J and that K is contained in a topological ball in S^3 which meets no component of L other than J and K. Suppose that the framing of J is $n\in\mathbf{Z}$ and that the framing of K is $r\in\mathbf{Q}\cup\{\infty\}$. Let L' be the framed link obtained from L by deleting K and changing the framing of J to $n-\frac{1}{r}$. We say that L' is obtained from L by performing the slam-dunk which removes K. The manifold obtained by Dehn surgery on L' is homeomorphic to the manifold obtained by Dehn surgery on L.

2. Connected sums of corridor complex links

This section is devoted to establishing the fact that the links obtained from the corridor construction in [4] are closed under the operation of connected sum in a certain restricted sense.

We begin with two faceted 3-balls P_1 and P_2. For $i\in\{1,2\}$ let ϵ_i be an orientation-reversing face-pairing on P_i with multiplier function mul_i, and let $M_i=M(\epsilon_i,\mathrm{mul}_i)$. Theorem 6.2.2 of [4] deals with framed links obtained from corridor constructions. Let L_i be such an unframed link so that, after L_i is appropriately framed, Dehn surgery on L_i yields M_i for $i\in\{1,2\}$. Recall that every component of L_i is either a face component or an edge component, that is, every component of L_i corresponds to either a face-pair of P_i or an edge cycle of P_i for $i\in\{1,2\}$. Let C_i be an edge component of L_i for $i\in\{1,2\}$. Let e_i be an edge of P_i which lies in the ϵ_i-edge cycle corresponding to C_i for $i\in\{1,2\}$. We assume that either e_1 has distinct vertices or e_2 has distinct vertices. Let P_i' be the faceted 3-ball obtained from P_i by replacing e_i with a digon D_i for $i\in\{1,2\}$. See Figure 1. Because either e_1 has distinct vertices or e_2 has distinct vertices, we obtain a faceted 3-ball P from P_1' and P_2' by cellularly identifying D_1 and D_2. We refer to P as a connected sum of P_1 and P_2 along e_1 and e_2. The face-pairings ϵ_1 and ϵ_2 induce a face-pairing ϵ

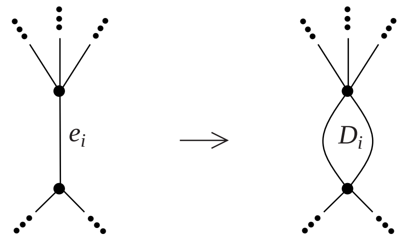

FIGURE 1. Replacing e_i with a digon D_i.

on P. Except for choices to be made involving corridors along either e_1 or e_2, the corridor constructions for (P_1, ϵ_1) and (P_2, ϵ_2) which give rise to L_1 and L_2 induce a corridor construction for (P, ϵ), which gives rise to an unframed link L. The isotopy type of L is uniquely determined by L_1, L_2 and the identification of D_1 and D_2. It is easy to see that L is a connected sum of L_1 and L_2 which joins C_1 and C_2. We summarize this paragraph in the following theorem.

Theorem 2.1. *Let P_1 and P_2 be faceted 3-balls with orientation-reversing face-pairings ϵ_1 and ϵ_2. Let L_1 and L_2 be corresponding unframed corridor complex links. Let C_1 be an edge component of L_1, and let C_2 be an edge component of L_2. Let e_1 be an edge of P_1 which lies in the ϵ_1-edge cycle corresponding to C_1, and let e_2 be an edge of P_2 which lies in the ϵ_2-edge cycle corresponding to C_2. Suppose that either e_1 has distinct vertices or e_2 has distinct vertices. Let P be a connected sum of P_1 and P_2 along e_1 and e_2, and let L be a connected sum of L_1 and L_2 which joins C_1 and C_2. Then L is an unframed corridor complex link associated to the orientation-reversing face-pairing on P induced by ϵ_1 and ϵ_2.*

Proof. This is clear from the previous paragraph. □

3. Reflection face-pairings

In this section we consider face-pairings of a very special sort. We assume that our model faceted 3-ball P can be identified with the closed unit ball in $\mathbf{R}^3$ so that the following holds. The intersection of the unit sphere with the xy-plane is a union of edges of P and the model face-pairing ϵ on P is given by reflection in the xy-plane. In other words, we have cell structures on both the northern and southern hemispheres of the unit sphere in $\mathbf{R}^3$, and the face-pairing maps of the model face-pairing ϵ are given by the map $(x, y, z) \mapsto (x, y, -z)$, which is therefore a cellular automorphism of P. In this case we call P a **reflection faceted 3-ball**, and we call ϵ a **reflection face-pairing**. Using the identification of P with the closed unit ball in $\mathbf{R}^3$, we speak of the **equator** of P and the **northern** and **southern hemispheres** of P.

Let P be a reflection faceted 3-ball with reflection face-pairing ϵ and multiplier function mul. As in Figure 2, we can describe P, ϵ, and mul using a diagram which consists of a cellular decomposition of a closed disk together with a positive integer for every edge. We view this closed disk as the northern hemisphere of P. Hence we have the cellular decomposition of the northern hemisphere of P, which therefore determines the cellular decomposition of the southern hemisphere of P, and the positive integer attached to the edge e is the multiplier of the ϵ-edge cycle of e. We sometimes allow ourselves the liberty of attaching 0 to an edge as well as positive

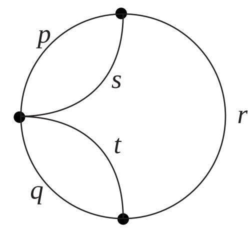

FIGURE 2. The diagram corresponding to P, ϵ and mul.

integers. Attaching 0 to an edge means that every edge in the corresponding ϵ-edge cycle collapses to a vertex.

Let P be a reflection faceted 3-ball with reflection face-pairing ϵ. Suppose given a multiplier function mul for ϵ, and let M be the associated twisted face-pairing manifold. Theorem 6.2.2 in [4] produces a framed link in the 3-sphere S^3 such that Dehn surgery on this framed link gives M. In this paragraph we describe another framed link L in S^3 such that Dehn surgery on L also gives M. We construct L as follows. We identify P with the closed unit ball in $\mathbf{R}^3$ as in the definition of reflection faceted 3-ball. For every edge e of the northern hemisphere of P we choose an open topological ball $B_e \subseteq \mathbf{R}^3$ such that $B_e \cap \partial P$ is a topological disk which meets e and is disjoint from every edge of P other than e. We assume that such topological balls corresponding to distinct edges are disjoint. For every face f of the northern hemisphere of P we construct an unknot C_f in the interior of f such that if e is an edge of f, then C_f meets B_e. These unknots are all components of L with framings 0. We call these components of L **face components**. Let $\sigma \in \{\pm 1\}$. Every edge e of P in the northern hemisphere also gives a component C_e of L, called an **edge component**, as follows. Let e be an edge in the equator of P contained in the face f of the northern hemisphere. The ϵ-edge cycle of e is just $\{e\}$. We define C_e to be a meridian of C_f contained in B_e with framing $\sigma/\text{mul}(\{e\})$. Now let e be an edge of the northern hemisphere of P not contained in the equator. Let f and g be the faces of P which contain e. Let x be a point of $f \cap B_e$ separated by C_f from ∂f, and let $y \neq x$ be a point of $g \cap B_e$ separated by C_g from ∂g. The ϵ-edge cycle of e is $\{e, \epsilon_f(e)\}$. We define C_e to be an unknot in B_e with framing $\sigma/\text{mul}(\{e, \epsilon_f(e)\})$ such that $P \cap C_e$ is a properly embedded arc in $P \cap B_e$ joining x and y. This defines L.

Example 3.1. *Let P be the reflection faceted 3-ball with reflection face-pairing and multiplier function given by the diagram in Figure 2. Figure 3 shows the framed link L constructed above from these data using nonnegative framings.*

Theorem 3.2. *Let P be a reflection faceted 3-ball with reflection face-pairing ϵ. Suppose given a multiplier function for ϵ, and let M be the associated twisted face-pairing manifold. Let L be the framed link in S^3 constructed above. Then Dehn surgery on L gives M.*

Proof. Theorem 6.2.2 in [4] produces a framed link L' in S^3 such that Dehn surgery on L' gives M. We review the construction of L' and indicate how to adapt the proof of Theorem 6.2.2 in [4] to the present situation.

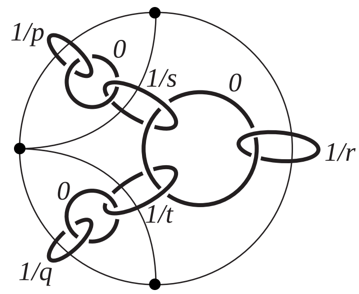

Figure 3. The framed link L.

The link L' is constructed essentially as follows. We identify P with the closed unit ball in $\mathbf{R}^3$ as in the definition of a reflection faceted 3-ball. We construct a handlebody H' by attaching handles to P, one handle for every pair of faces of P. One end of every handle lies in a face f of P, and the other end lies in the face of P paired with f by ϵ. This is done so that the closure of $S^3 \setminus H'$ is also a handlebody. Certain components of L', called edge components, are curves in the interior of H' constructed roughly as follows. Choose a face f of P and an edge e of f. We construct a curve in $\partial H'$ which begins at e, goes to the handle of H' attached to f, goes across this handle to the face $\widehat{f}$ of P paired with f and then to the edge e' of $\widehat{f}$ corresponding to e. This curve crosses e' and proceeds in this way until we obtain a simple closed curve in $\partial H'$. We isotop this curve slightly into the interior of H'. The result is a component of L'. In this way every edge cycle of ϵ gives rise to a component of L'. The framing of this component of L' is the framing determined by $\partial H'$ plus or minus $1/\mathrm{mul}(E)$, where E is the ϵ-edge cycle which contains e. The remaining components of L' are called face components. They are constructed as follows. Choose a handle of H', and choose a meridian curve in $\partial H'$ for this handle. This meridian curve is a component of L'. It has framing 0.

To prove Theorem 3.2, we replace H' by another handlebody H. We construct H as follows. We still identify P with the closed unit ball in $\mathbf{R}^3$. Let B be the topological ball which is the closure in S^3 of $S^3 \setminus P$. We construct H by attaching handles to B as follows. Let f be a face of P, and let $\widehat{f}$ be the face of P paired with f by ϵ. Then f and $\widehat{f}$ are joined by a vertical circular cylinder. We attach such a cylinder to B. Doing this for every pair of faces of P yields our handlebody H. It is clear that the closure in S^3 of $S^3 \setminus H$ is also a handlebody.

We identify the components of L with curves in ∂H in a straightforward way. It is easy to see that the face components of L correspond to the face components of L' and that the edge components of L correspond to the edge components of L'. Furthermore the framing determined by ∂H of every edge component of L is 0. As in the proof of Theorem 6.2.2 in [4], it follows that Dehn surgery on L gives M. □

4. Lens spaces

In this section we show that every lens space is a twisted face-pairing manifold. We begin by defining the notion of a scallop. A **scallop** is a reflection faceted 3-ball P (defined in Section 3) whose northern hemisphere has a cell structure essentially as indicated in Figure 4. More precisely, every vertex of a scallop P lies on the

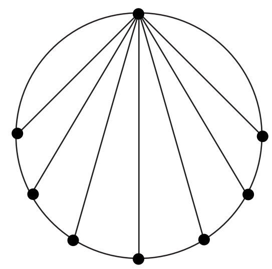

FIGURE 4. Top view of a scallop.

equator of P, P contains a vertex v such that every edge of P not contained in the equator of P joins v with another vertex and every vertex of P other than v is joined with v by at least one edge. We call v the **apex** of the scallop. So the northern hemisphere of a scallop might consist of just a monogon. Otherwise it is subdivided into digons and triangles, in which case it has at least two digons, but it may have arbitrarily many digons.

Theorem 4.1. *Let P be a scallop with k faces in its northern hemisphere. Let ϵ be a reflection face-pairing on P, let mul be a multiplier function for ϵ, and let $M = M(\epsilon, mul)$. Suppose that P, ϵ, and mul are given by the diagram in Figure 5, where $m_1 > 0$, $m_k > 0$, and $m_i \geq 0$ for $i \in \{2, \ldots, k-1\}$. (If a multiplier is 0, then the corresponding edge in Figure 5 collapses to a vertex of P.) Define integers $a_1, \ldots, a_k$ so that $a_1 = m_1$ if $k = 1$ and if $k > 1$, then $a_1 = m_1 + 1$, $a_k = m_k + 1$, and $a_i = m_i + 2$ for $i \in \{2, \ldots, k-1\}$. Then there exist relatively prime positive integers $p \geq q$ such that M is homeomorphic to the lens space $L(p, q)$, where*

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_{k-1} - \cfrac{1}{a_k}}}}.$$

(It is possible that $p = q = 1$, in which case we obtain the 3-sphere.) Furthermore, given relatively prime positive integers p and q with $p \geq q$, then there exists a unique sequence of integers $m_1, \ldots, m_k$ as above such that the above continued fraction equals p/q.

Proof. Theorem 3.2 implies that M is given by Dehn surgery on the framed link in Figure 6. We repeat that if $m_i = 0$ for some $i \in \{2, \ldots, k-1\}$, then the corresponding edge in Figure 5 collapses to a vertex of P. In this case the corresponding component of the link in Figure 6 is to be removed. This is consistent with the fact that any component with framing ∞ may be removed from a framed link without changing the resulting manifold. We next use Kirby calculus to simplify the framed link in Figure 6. For every $i \in \{1, \ldots, k\}$ we perform the slam-dunk which removes the component with framing $-1/m_i$. In doing this, the component linked with the given component acquires the framing m_i. We next perform a twist move about every component shown in Figure 6 with framing -1. Every such component is

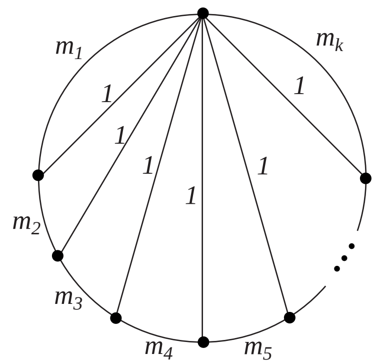

FIGURE 5. The diagram for P, ϵ and mul.

then removed, and 1 is added to the framing of the components linked with it. The resulting framed link is in Figure 7. It follows from page 272 of [10] or page 108 of [9] or just by iterating slam-dunks that M is the lens space as stated in Theorem 4.1.

The uniqueness statement is well known. For this, first note that if $k = 1$, then a_1 is an arbitrary positive integer. If $k > 1$, then $a_1, \ldots, a_k$ are arbitrary integers with $a_i \geq 2$ for $i \in \{1, \ldots, k\}$. Given p and q, we calculate $a_1, \ldots, a_k$ by modifying the division algorithm usually used to calculate continued fractions. Instead of taking the greatest integer less than or equal to our given number, we take the least integer greater than or equal to our given number. The details are left to the reader.

This proves Theorem 4.1. □

Corollary 4.2. *Every lens space is a twisted face-pairing manifold.*

5. Torus bundles over a circle

Theorem 5.1. *Every orientable torus bundle over a circle with Solv geometry is a twisted face-pairing manifold.*

Proof. We begin with a scallop. If the scallop is simply the union of two monogons, then we insert two new vertices in the scallop's edge. Otherwise we insert one new vertex in each of the two equatorial edges which contain the scallop's apex. Two edges of the resulting faceted 3-ball therefore lie in the equator and contain the scallop's apex. Let P be the faceted 3-ball obtained by replacing each of these two edges with a digon. Figure 8 shows ∂P with one equatorial point at infinity. Let ϵ_1 and ϵ_2 be face-pairings on P which extend the reflection face-pairing on the original scallop so that ϵ_1 pairs the new digons fixing the apex and ϵ_2 pairs the new digons without fixing the apex. Figure 9 shows corridor complexes and corridor complex framed links L_1 and L_2 for the pairs (P, ϵ_1) and (P, ϵ_2).

We use Kirby calculus to simplify L_1 and L_2 just as we simplified the link in the proof of Theorem 4.1. The resulting framed links are shown in Figure 10, where $a_1, \ldots, a_k$ are positive integers. If $k = 1$, then it is not difficult to see that $a_1 \geq 1$ for (P, ϵ_1) and $a_1 \geq 5$ for (P, ϵ_2). If $k > 1$, then $a_i \geq 2$ for $i \in \{1, \ldots, k\}$, and $a_i \geq 3$ for some i because our original scallop has at least one equatorial edge. According to

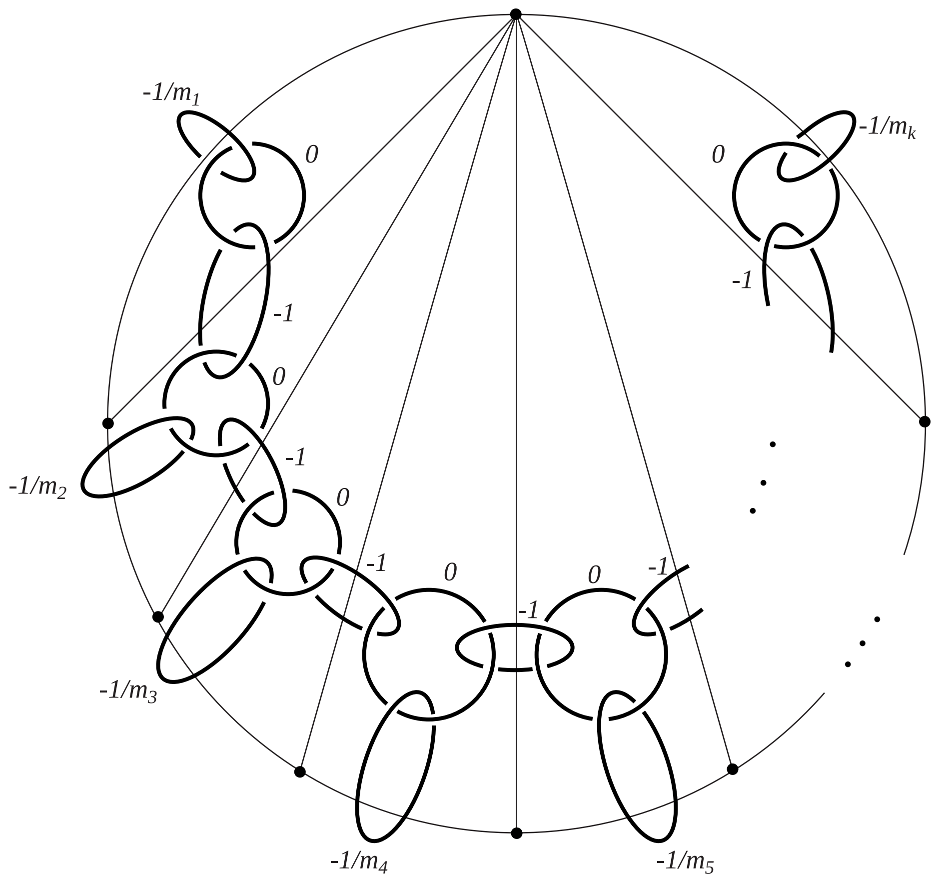

FIGURE 6. The framed link corresponding to Figure 5.

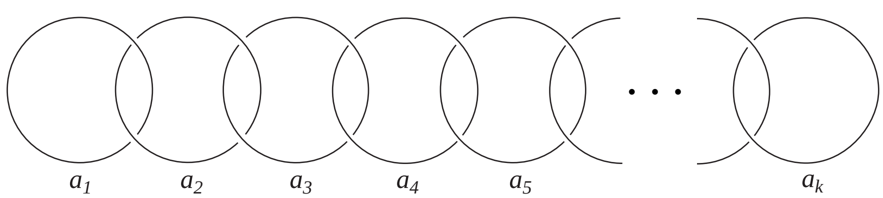

FIGURE 7. Dehn surgery on this framed link gives M.

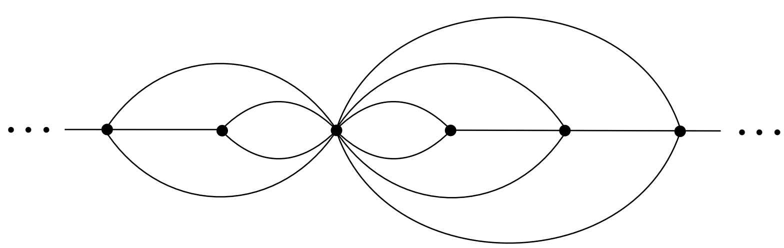

FIGURE 8. The faceted 3-ball P.

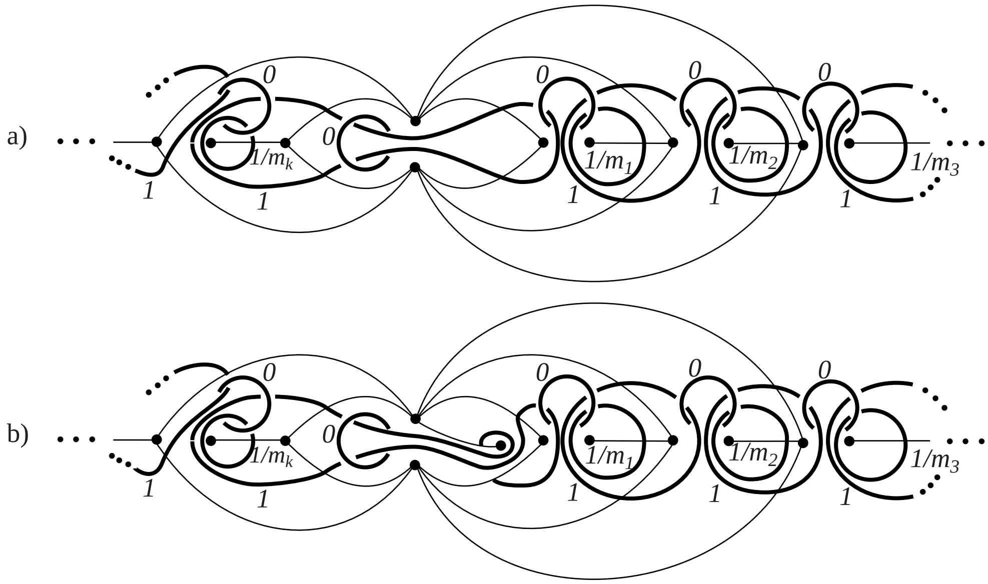

FIGURE 9. Corridor complexes and the framed links L_1 and L_2.

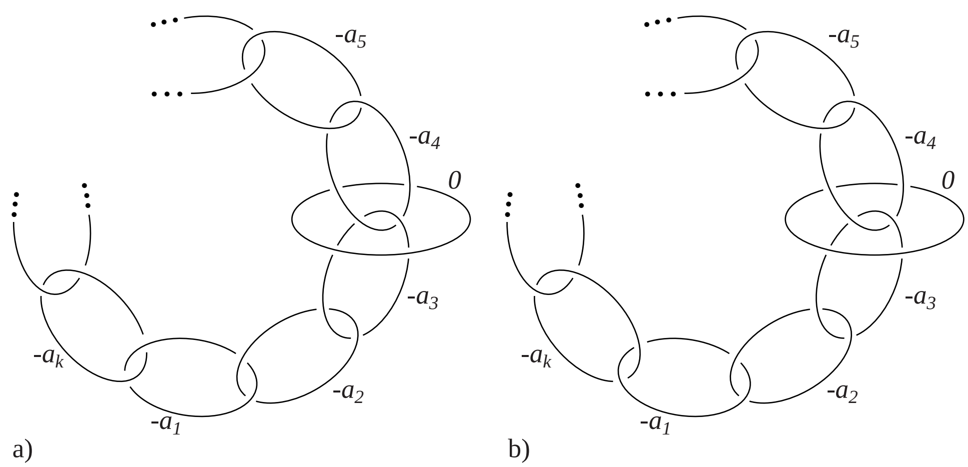

FIGURE 10. Simplifications of L_1 and L_2.

the discussion on pages 308 and 309 of [8] or Sections 4.6 and 6.1 of [5], these links correspond to the plumbing graphs in Figure 11, where $b_i \geq 2$ for $i \in \{1, \ldots, k\}$ and $b_i \geq 3$ for some i. Now statement IV of Theorem 6.1 of [8] implies that the twisted face-pairing manifolds which arise from (P, ϵ_1) are all torus bundles over a circle and that these torus bundles are those whose structure matrices have traces at least 3. Statement V of Theorem 6.1 of [8] gives the corresponding result for (P, ϵ_2) and traces at most -3.

This easily proves Theorem 5.1. □

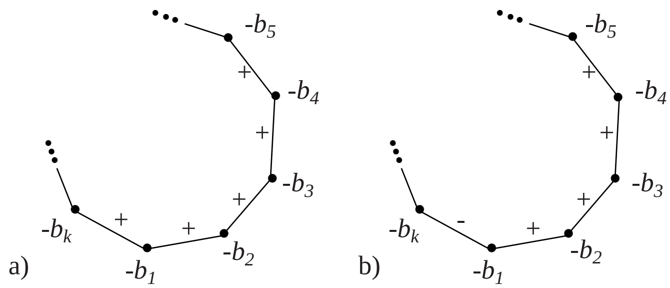

FIGURE 11. The corresponding plumbing graphs.

6. Generalizing framings of corridor complex links

We construct corridor complex links in Section 6.2 of [4] by means of link projections. Some components, the face components, of a corridor complex link correspond to the face-pairs of our model face-pairing, and the remaining components, edge components, correspond to the edge cycles of our model face-pairing. We frame corridor complex links as follows. Let C be a component of a corridor complex link L. If C is a face component, then we define the framing of C to be 0. If C is an edge component, then we define the framing of C to be the blackboard framing of C plus or minus the reciprocal of the multiplier of the edge cycle corresponding to C. The sign is chosen to be either plus for every C or minus for every C. Theorem 6.2.2 of [4] states that performing Dehn surgery on L with this framing obtains our twisted face-pairing manifold. The following theorem states that if we redefine the framing of L by replacing every edge cycle multiplier with an arbitrary positive rational number, then Dehn surgery on L still obtains a twisted face-pairing manifold (usually constructed from a different faceted 3-ball).

Theorem 6.1. *Let L be an unframed corridor complex link. We frame L as follows. Let C be a component of L. If C is a face component, then we define the framing of C to be 0. If C is an edge component, then we define the framing of C to be the blackboard framing of C plus a nonzero rational number. These rational numbers are chosen to have the same sign. Then Dehn surgery on L with this framing obtains a twisted face-pairing manifold.*

Proof. Theorem 6.2.2 of [4] states that if the framing of every edge component of L is its blackboard framing minus the reciprocal of a positive integer, then Dehn surgery on L obtains a twisted face-pairing manifold. Let C be an edge component of L with blackboard framing b. In the next paragraph we consider the case in which the framing of C is changed from b minus the reciprocal of a positive integer to b minus a rational number greater than 1.

We in effect change the framing of C by "attaching a scallop" to our model faceted 3-ball, proceeding as follows. Twisted face-pairing manifolds of scallops with reflection face-pairings are obtained by Dehn surgery on framed links such as the one in Figure 6. We subdivide one edge of our scallop, obtaining the framed link K in Figure 12. It is easy to see that K is a corridor complex link. Let J be the connected sum of L and K which joins C and the component of K with framing -1 which is parallel to the component with framing $-1/m_1$. Theorem 2.1 implies that J is a corridor complex link. The blackboard framing of the component of J

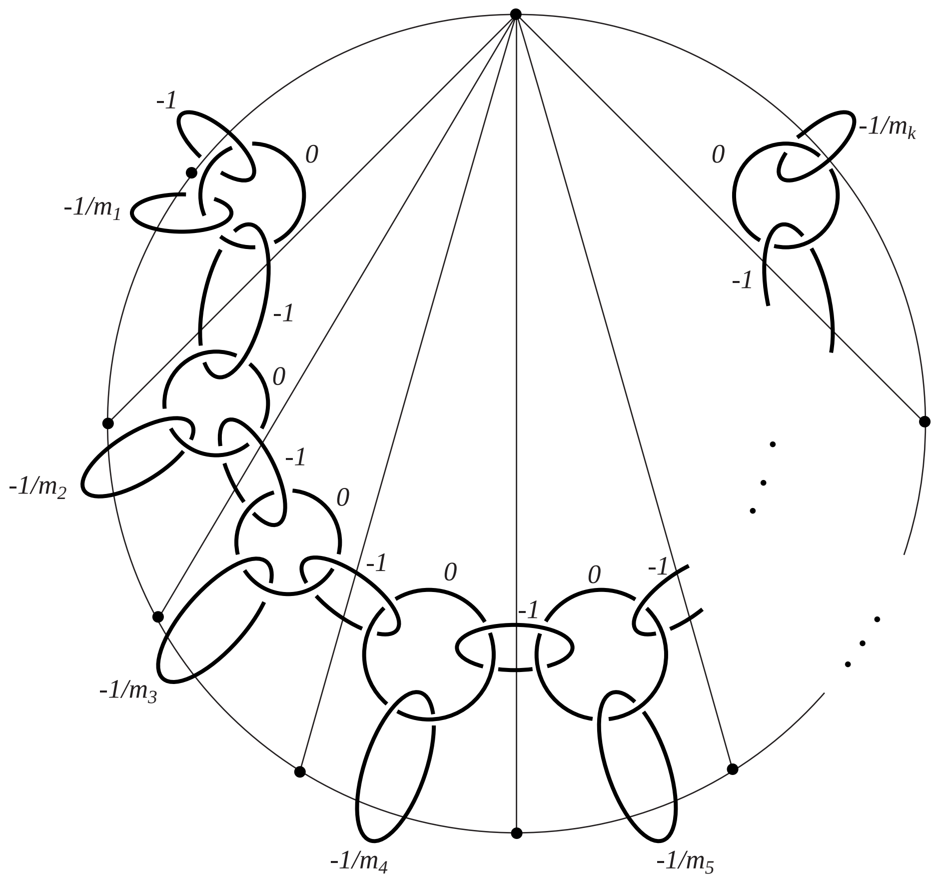

FIGURE 12. The framed link K.

corresponding to C is b. We take the framing of this component of J to be $b-1$. We next simplify the part of J corresponding to K as in the proof of Theorem 4.1. We obtain the framed link in Figure 13, where $a_1,\dots,a_k$ are integers, $a_1\geq 1$, and $a_i\geq 2$ for $i\in\{2,\dots,k\}$. As in the proof of Theorem 4.1, by performing $k-1$ slam-dunks, we may reduce J to the framed link in Figure 14, where r is an arbitrary positive rational number. One more slam-dunk reduces J to L, where the framing of C is b minus an arbitrary rational number greater than 1.

In this paragraph we show how to change the framing of C to b minus an arbitrary positive rational number which is not the reciprocal of an integer by "splitting edges". The link L in the statement of Theorem 6.1 is obtained from a model face-pairing ϵ on a faceted 3-ball. The component C corresponds to an ϵ-edge cycle. We construct a new faceted 3-ball by subdividing every edge in this ϵ-edge cycle into two edges. The face-pairing ϵ induces a face-pairing on this new faceted 3-ball. In terms of links, the result is to adjoin to L a new component D which is parallel to C in our corridor complex. Let C have framing $b-r$ for some rational number $r>1$, and let D have framing $b-1/m$ for some positive integer m. Part a) of Figure 15 shows parts of C and D. One slam-dunk leads to part b) of Figure 15.

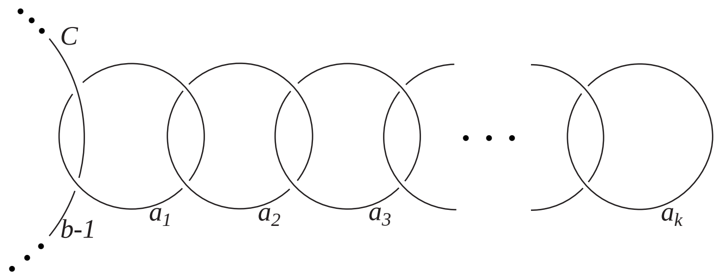

FIGURE 13. Simplifying the framed link J.

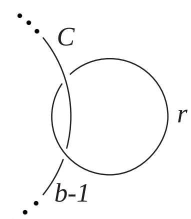

FIGURE 14. Simplifying the framed link J.

We give C and D opposite orientations as in part c) of Figure 15 to prepare for a type 2 Kirby move. The result of our type 2 Kirby move is in part d) of Figure 15; the framing of the new component is the framing of C plus the framing of D plus $2\text{lk}(C, D) = -2b$. An isotopy gives part e) of Figure 15. Performing m twist moves about the component with framing $-1/m$ in part e) gives part f) of Figure 15. Finally, a slam dunk gives part g) of Figure 15. It follows that if the framing of C is changed to any rational number less than b, then Dehn surgery on L with this new framing obtains a twisted face-pairing manifold.

This easily proves Theorem 6.1. □

7. Lune complexes

Let P be a faceted 3-ball with orientation-reversing face-pairing ϵ such that P is a regular CW-complex whose faces are all digons. Hence P has exactly two vertices and every edge of P joins the two vertices of P. We call P a **lune complex**. Given a multiplier function for ϵ, we obtain a twisted face-pairing manifold. This twisted face-pairing manifold does not depend on the specific homeomorphisms by which ϵ pairs the faces of P; it only depends on what faces are paired and the action of ϵ on vertices of face-pairs. It follows that we may, and do, assume that there exists a homeomorphism from P to the closed unit ball in $\mathbf{R}^3$ such that the vertices of P correspond to $(0, 0, \pm 1)$, the edges of P correspond to arcs of great circles, and every face-pairing map corresponds to an isometry. We refer to the inverse image in P of the xy-plane under this homeomorphism as an **equatorial disk** of P and ϵ.

Let D be an equatorial disk of a lune complex P with orientation-reversing face-pairing ϵ. Then D has a cell structure whose vertices are the intersections of D with the edges of P and whose edges are the intersections of D with the faces of

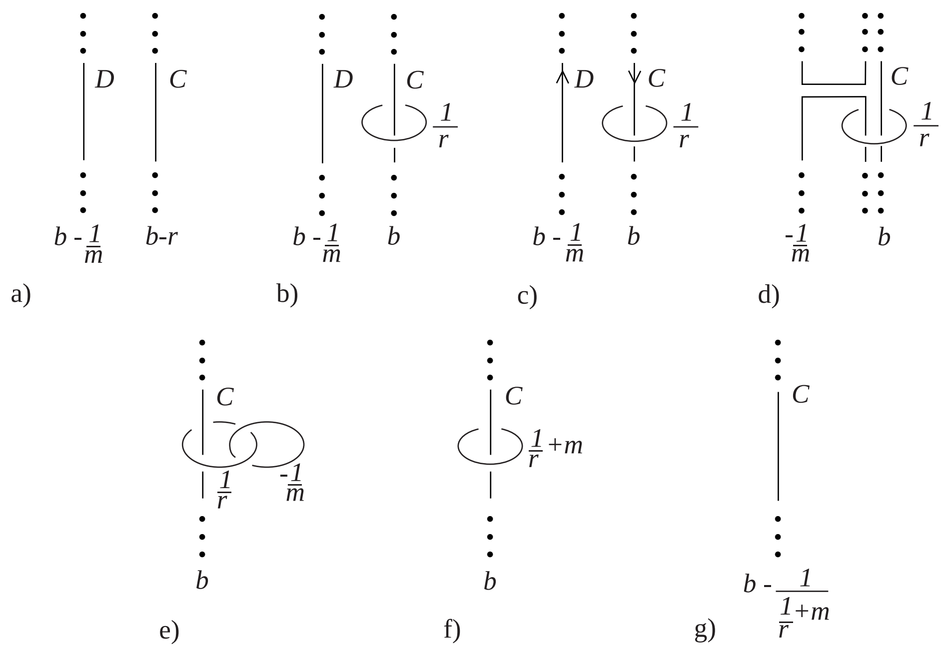

FIGURE 15. Simplifying the link gotten by adjoining the component D.

P. Furthermore, ϵ restricts to an edge pairing on D. The vertex cycles of D under its edge pairing are in canonical bijective correspondence with the edge cycles of ϵ. Conversely, suppose that the closed unit disk D in the xy-plane of $\mathbf{R}^3$ is given the cell structure of a polygon. An edge pairing on D in which every edge is paired with a different edge by an isometry determines a lune complex structure P on the closed unit ball in $\mathbf{R}^3$ with vertices $(0, 0, \pm 1)$ and edges arcs of great circles. It also determines an orientation-reversing face-pairing on P by isometries which restricts on the equator of P to the given edge pairing on D.

Let D be a polygon in $\mathbf{R}^3$ with an edge pairing, so that every edge of D is paired with a different edge. We may use the edge pairing on D to attach handles to D in a straightforward way, attaching one handle for every pair of edges of D. The resulting surface T in $\mathbf{R}^3$ is uniquely determined up to homeomorphism as an abstract surface, but its embedding in $\mathbf{R}^3$ is not. We call T a **handle surface** for D. The boundary components of T are in canonical bijective correspondence with the vertex cycles of D under its edge pairing. Hence if D is an equatorial disk of a lune complex with face-pairing ϵ, then the boundary components of T are in canonical bijective correspondence with the ϵ-edge cycles. Conversely, a surface T in $\mathbf{R}^3$ which consists of a polygon D with handles attached as above gives an edge pairing on D.

Theorem 7.1. *Let P be a lune complex with orientation-reversing face-pairing ϵ and multiplier function mul such that $P \subseteq \mathbf{R}^3$. Let D be an equatorial disk of P and ϵ. Let T be a handle surface for D. Suppose that T has a regular neighborhood in $\mathbf{R}^3$ which is a handlebody H such that the closure of $S^3 \setminus H$ is also a handlebody.*

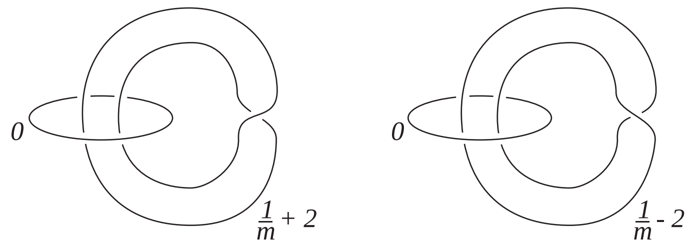

FIGURE 16. Two framed links for Example 7.2.

We choose handles for H to correspond to the handles of T in the straightforward way, and we choose a meridian curve for every handle of H. Let L be the link in S^3 whose components consist of the boundary components of T together with the meridian curves of H. We frame L so that the boundary component of T which corresponds to the ϵ-edge cycle E has framing $\pm 1/mul(E)$ plus the framing determined by T and every meridian curve of H has framing 0. The sign is either always plus or always minus. Then the manifold obtained by Dehn surgery on L is homeomorphic to $M = M(\epsilon, mul)$.

Proof. This follows easily from Theorem 6.1.2 of [4]. □

Example 7.2. *Let P be a lune complex with exactly two faces. Let ϵ be an orientation-reversing face-pairing on P which interchanges the two vertices of P. Then ϵ has only one edge cycle. Let this edge cycle have multiplier m. An equatorial disk D of P and ϵ has just two edges. Every such surface T is a Mobius band. Every such handlebody H is a solid torus. Two such framed links L are shown in Figure 16.*

We have seen that giving a lune complex with orientation-reversing face-pairing is equivalent to giving a polygon with edge pairing and that this is equivalent to giving a surface constructed by attaching handles to a polygon. Similarly, it is also equivalent to giving a closed surface with the structure of a CW-complex with exactly one 2-cell; the 2-cell corresponds to a polygon with an edge pairing. Given a lune complex P and an orientation-reversing face-pairing ϵ with equatorial disk D, we define an **equatorial surface** of P and ϵ to be a closed surface obtained by identifying the edges of D using the edge pairing on D induced by ϵ. When this equatorial surface is orientable, Theorem 7.1 can be interpreted as follows.

Corollary 7.3. *Let P be a lune complex with orientation-reversing face-pairing ϵ and multiplier function mul. Let S be an equatorial surface of P and ϵ, and suppose that S is orientable. We embed S in S^3 so that it bounds two handlebodies in S^3. We identify $S \times [-1, 1]$ with a regular neighborhood of S in S^3. We construct a framed link L in S^3 as follows. For every vertex v of S we choose closed disks D_v and E_v in S such that $v \in int(D_v)$, $D_v \subseteq int(E_v)$, and $E_u \cap E_v = \emptyset$ if $u \neq v$. The boundary of E_v is one component of L for every vertex v of S. Every vertex v of S corresponds to an ϵ-edge cycle E, and we take the framing of ∂E_v to be $1/mul(E)$. Every edge e of S gives a component of L as follows. Let u and v be the vertices of e. Let s be a closed segment of e whose endpoints lie in $\partial D_u \cup \partial D_v$ such that $e \setminus s \subseteq D_u \cup D_v$. Then $s \times [-1, 1] \subseteq S \times [-1, 1]$ is a closed disk in S^3, and we take the boundary of this disk to be a component of L with framing 0. This defines L. Then the manifold obtained by Dehn surgery on L is homeomorphic to $M = M(\epsilon, mul)$.*

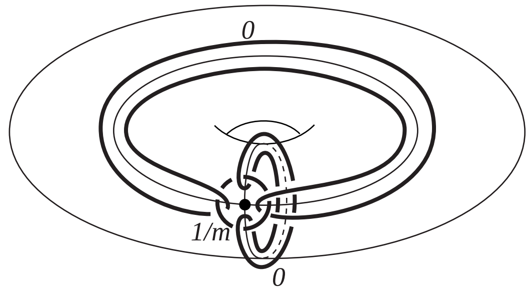

FIGURE 17. The equatorial surface S and the framed link L of Example 7.4.

Proof. We apply Theorem 7.1. We may identify the surface T of Theorem 7.1 with the surface gotten from S by deleting the interiors of the disks E_v. Let U be the surface gotten from S by deleting the interiors of the disks D_v. Because S bounds two handlebodies in S^3, it is not difficult to see that $H = U \times [-1, 1]$ is a handlebody in S^3 such that the closure of $S^3 \setminus H$ is also a handlebody. Corollary 7.3 now follows easily from Theorem 7.1. □

Example 7.4. *We return to Example 7.3 of* [4]*, which is also considered in Example 7.2 of* [2]*. Let P be a lune complex with four faces. Let ϵ be the orientation-reversing face-pairing on P which pairs opposite faces of P and fixes the vertices of P. Then there is exactly one ϵ-edge cycle. Let m be the multiplier of the ϵ-edge cycle. The induced edge pairing on an equatorial disk D for P and ϵ identifies opposite edges of D to give an equatorial surface S which is a torus. Figure 17 shows S with its vertex and two edges. Figure 17 also shows the framed link of Corollary 7.3. We see that L is the link of Borromean rings. As in Example 7.3 of* [4]*, it follows that our twisted face-pairing manifold is the Seifert fibered manifold $(Oo1|0; (m, 1))$. When $m = 1$, this is the Heisenberg manifold, the prototype of Nil geometry.*

Our next result shows that every twisted face-pairing manifold obtained from a lune complex is a Seifert fibered manifold, and so we now fix notation for Seifert fibered manifolds. We use the notation in Chapter 4 of [7]. Thus we denote a closed, connected, orientable, Seifert fibered manifold M by either

$$(Oog|b; (\alpha_1, \beta_1), \ldots, (\alpha_s, \beta_s)) \qquad \text{or} \qquad (Onk|b; (\alpha_1, \beta_1), \ldots, (\alpha_s, \beta_s)).$$

The O means that M is orientable. The o and g mean that the base surface of M is the orientable closed surface with g handles. The n and k mean that the base surface of M is the nonorientable closed surface with k crosscaps. Furthermore, b is an arbitrary integer, s is a nonnegative integer, and α_i and β_i are relatively prime positive integers with $\alpha_i > \beta_i$ for every $i \in \{1, \ldots, s\}$. The manifold M can be constructed as follows. Let S be the base surface of M. Let N be the orientable closed 3-manifold which is a circle bundle over S with a section. Then M is obtained by performing Dehn surgeries on $s + 1$ fibers of N with framings $1/b, \alpha_1/\beta_1, \ldots, \alpha_s/\beta_s$.

Theorem 7.5. *Let P be a lune complex with orientation-reversing face-pairing ϵ and multiplier function mul. Let S be an equatorial surface of P and ϵ. Suppose that the ϵ-edge cycle multipliers are $m_1, \ldots, m_{s+b}$ for some nonnegative integers s*

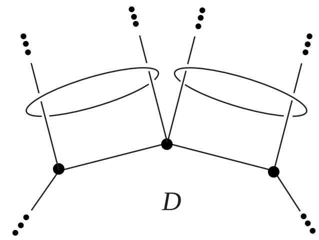

FIGURE 18. Two unpaired adjacent edges of D.

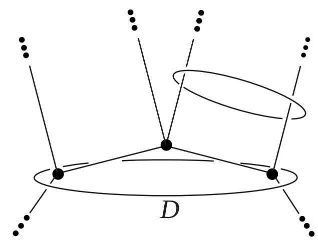

FIGURE 19. The result of performing a Kirby move.

and b *such that* $m_i > 1$ *for* $i \in \{1, \dots, s\}$ *and* $m_i = 1$ *for* $i \in \{s+1, \dots, s+b\}$*. Then* $M = M(\epsilon, mul)$ *is the Seifert fibered manifold with base surface homeomorphic to* S *given by either*

$$(Oog|b; (m_1, 1), \dots, (m_s, 1)) \qquad or \qquad (Onk|b; (m_1, 1), \dots, (m_s, 1)).$$

Proof. We prepare to apply Theorem 7.1. Without loss of generality we assume that $P \subseteq \mathbf{R}^3$. Let D be an equatorial disk of P. Let T be a handle surface for D so that T has a regular neighborhood in $\mathbf{R}^3$ which is a handlebody H such that the closure of $S^3 \setminus H$ is also a handlebody. Theorem 7.1 states that T and H give rise to a framed link L such that M is obtained by Dehn surgery on L.

In this paragraph we present a way to modify D, P, T, and L. Suppose given two adjacent edges of D which are not paired by the edge pairing on D. Figure 18 shows the two edges of D, the ends of the two handles of T attached to these edges, and the two corresponding meridian curves of H. These two meridian curves are components of L each with framing 0. Now we perform a type 2 Kirby move with these two components of L in a straightforward way to replace one of these components, say, the left one, as in Figure 19. The new component also has framing 0. Just as L is obtained from D, this new framed link is obtained from the disk with edge pairing gotten from D by cutting and pasting as in Figure 20. Thus we may modify D is this way, thereby modifying P, T, and L, without modifying M.

We have just modified D by cutting off a triangle from D and pasting it elsewhere. By combining several such operations we may cut D along any arc c which joins two of its vertices as long as some edge of D on one side of c is paired with an edge on the other side of c.

As is well known these last operations can be used to put D into one of the two normal forms shown in Figure 21. This is essentially done in Section 1.3 of [11]. More precisely, Section 1.3 of [11] shows how to obtain the normal forms in Figure 21

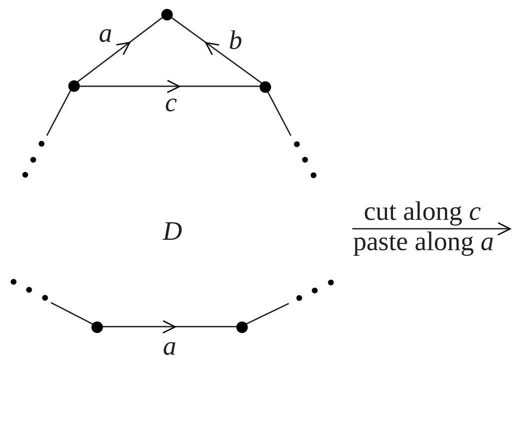

FIGURE 20. Modifying D.

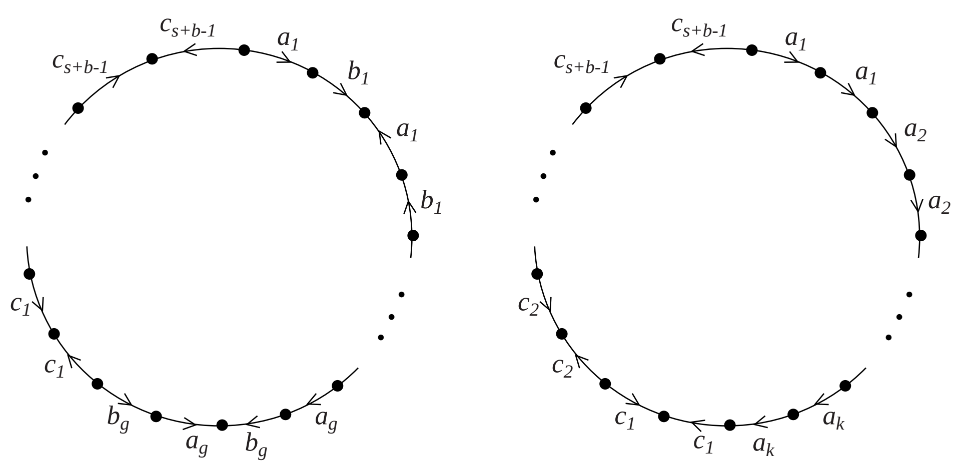

FIGURE 21. The normal forms for D.

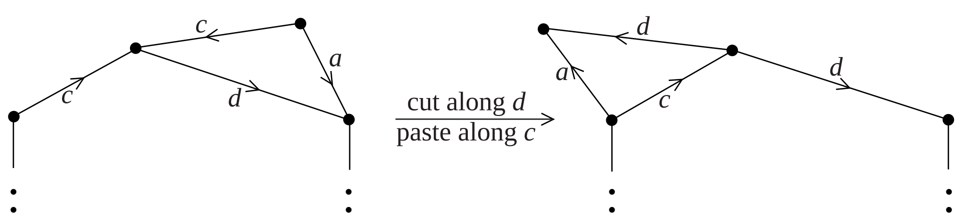

FIGURE 22. Gathering together $c_1, \ldots, c_{s+b-1}$.

except for gathering together the edges $c_1, \ldots, c_{s+b-1}$. The edges $c_1, \ldots, c_{s+b-1}$ can be gathered together using the operation shown in Figure 22.

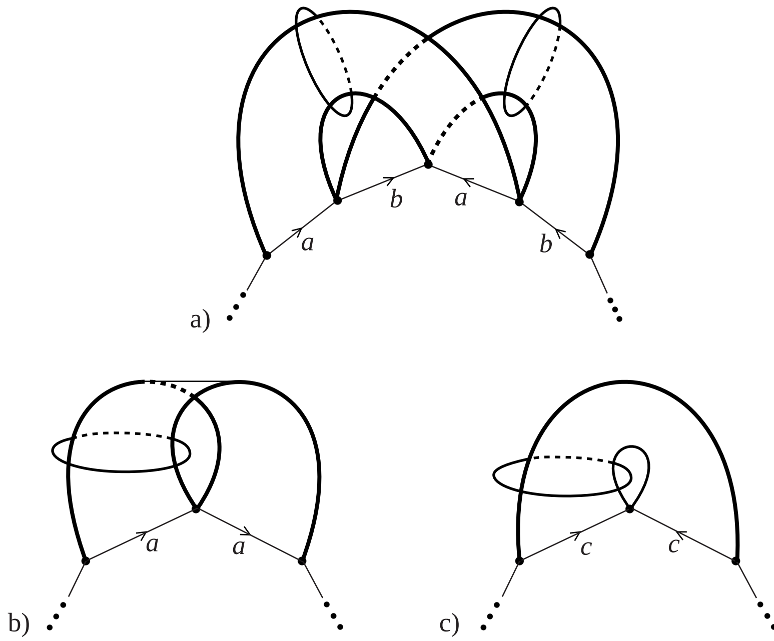

FIGURE 23. Attaching handles to D.

So we assume that D has one of the normal forms in Figure 21. We also assume that T is gotten from D by attaching handles as in Figure 23. In addition to showing how the handles of T attach to D, Figure 23 also shows the corresponding meridian curves of H. Part of one boundary component of T is drawn with thick arcs and dashes in Figure 23. We denote this boundary component of T by α. In addition to α, every handle of T as in part c) of Figure 23 gives rise to one boundary component of T.

So the link L consists of the meridian curves of H, the component α, and one component for every handle of T as in part c) of Figure 23. We next discuss the framing of L given by Theorem 7.1. Every meridian curve of H has framing 0. Every other component β of L corresponds to some ϵ-edge cycle, with multiplier, say, m. We take the framing of β to be $-1/m$ plus the framing determined by T. If $\beta \neq \alpha$, then the framing of β determined by T is 0. Handles of T as in parts a) and c) of Figure 23 contribute nothing to the framing of α, and every handle of T as in part b) of Figure 23 contributes -2 to the framing of α.

Now we apply an isotopy to L as follows. Every portion of L as shown in a part of Figure 23 is transformed to what is shown in the corresponding part of Figure 24 . In parts a) and c) of Figure 24 the component α is shown with framing f. In part b) of Figure 24 the component α is shown with framing $f-2$. The two link portions in Figure 25 can be seen to be equivalent by twisting -2 times about the component with framing $1/2$. This subtracts 2 from the framing of α and from the framing of the component with framing 2. After performing all such moves, the framing of α is $-1/m$, where m is the multiplier of the ϵ-edge cycle corresponding to α. Slam-dunks give the equivalences shown in Figures 26 and 27.

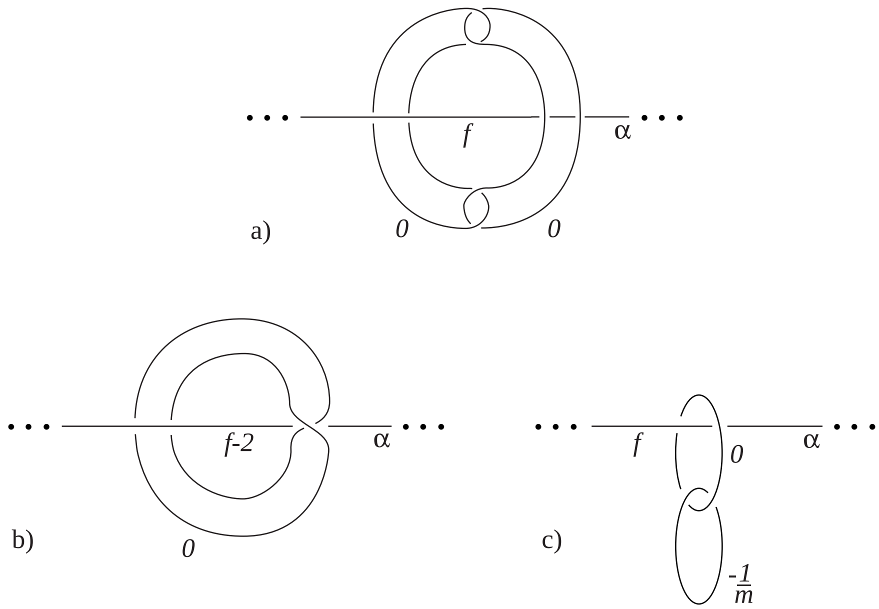

FIGURE 24. Isotoping portions of L.

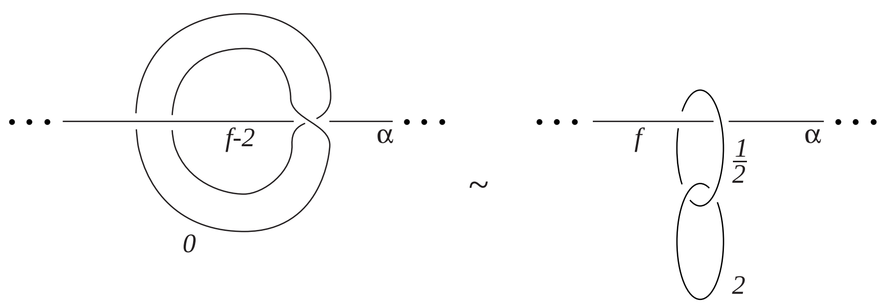

FIGURE 25. Using Kirby calculus to modify L.

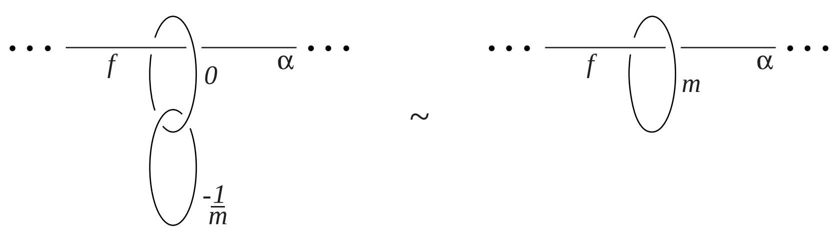

FIGURE 26. Using Kirby calculus to modify L.

The link L now has one of the two forms shown in Figure 28. It is easy to see in each case that the components with framings $m_{s+1}, \dots, m_{s+b}$ may be replaced with one component with framing $1/b$. The resulting links appear in Figure 12 on page 146 of [7]. According to [7], the manifold M is as stated in Theorem 7.5.

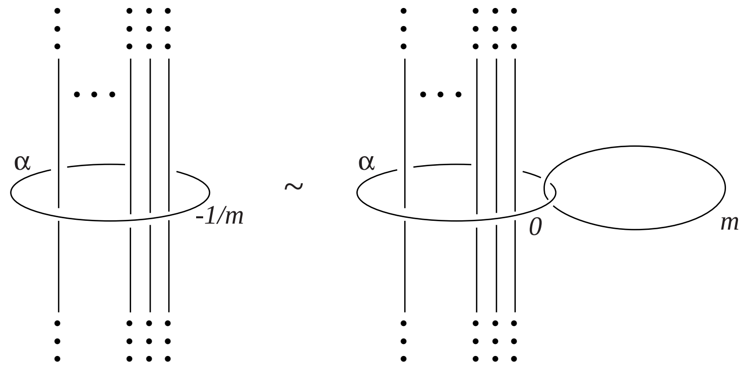

FIGURE 27. Modifying L by linking α with an unknot.

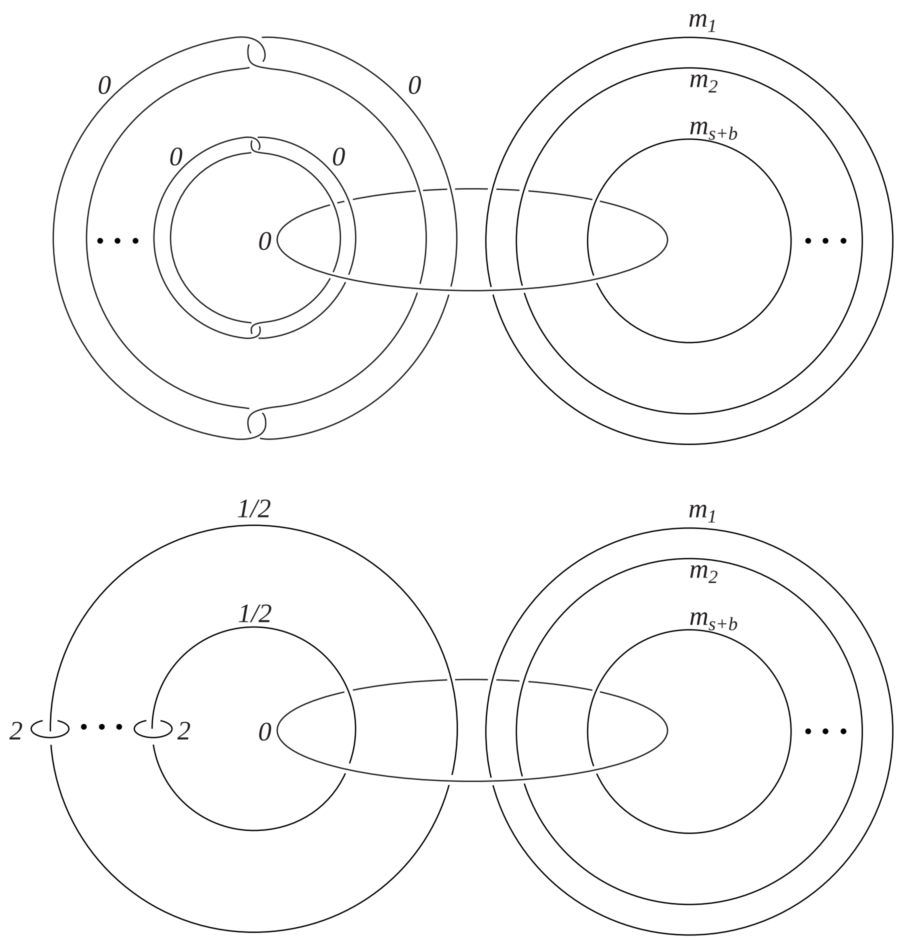

FIGURE 28. Almost the final form of L.

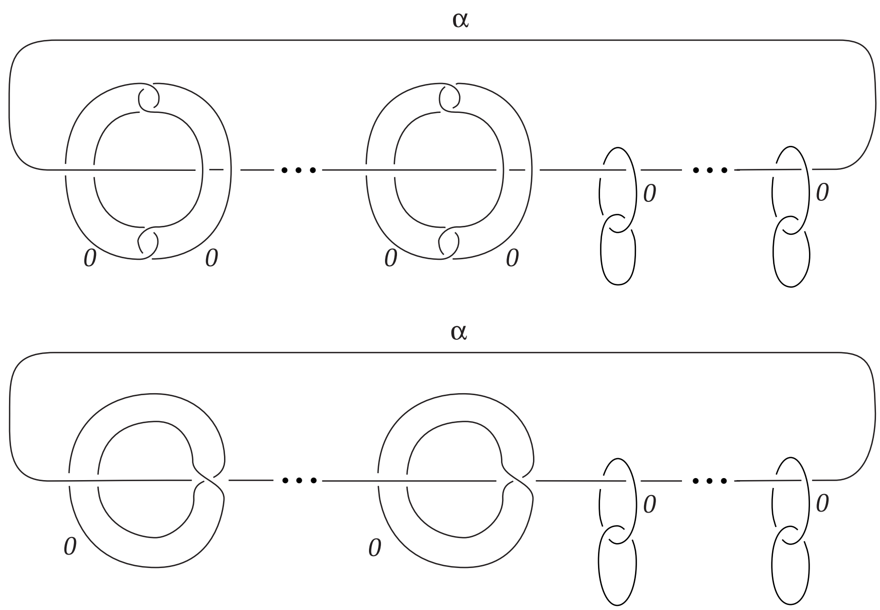

FIGURE 29. Some corridor complex links.

This proves Theorem 7.5. □

The following lemma, which is used later, follows from the proof of Theorem 7.5.

Lemma 7.6. *Every link shown in Figure 29 is a corridor complex link. The only component which must occur is the one labeled α, and the link cannot consist of just α. The components framed with 0 are face components. All other components, including α, are edge components. The framing of α determined by the corridor complex is -2 times the number of components in the link as in part b) of Figure 24. The framing determined by the corridor complex of every other component of the link is 0.*

Proof. We return to the proof of Theorem 7.5. We begin with the normal forms for D in Figure 21. We attach handles to D as in Figure 23. We then perform isotopies as in Figure 24. This easily proves Lemma 7.6. □

8. SEIFERT FIBERED MANIFOLD

In this section we prove that most closed, connected, orientable Seifert fibered manifolds are twisted face-pairing manifolds.

Theorem 8.1. *Let M be a Seifert fibered manifold given by either*

$$(Oog|b;(\alpha_1,\beta_1),\dots,(\alpha_s,\beta_s)) \qquad \text{or} \qquad (Onk|b;(\alpha_1,\beta_1),\dots,(\alpha_s,\beta_s)).$$

Suppose that $b \geq 0$ and that either $b > 0$ or $s > 0$. Then M is a twisted face-pairing manifold.

Proof. Define an integer t so that $t = s-1$ if $b = 0$ and $t = s$ if $b > 0$. The assumptions imply that $t \geq 0$. If $g = 0$ and $t = 0$, then either $M = (Oo0|0;(\alpha_1,\beta_1))$

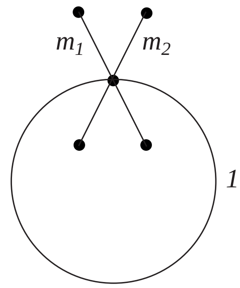

FIGURE 30. The faceted 3-ball P and edge cycle multipliers.

or $M = (Oo0|b)$. These manifolds are all lens spaces, and hence they are twisted face-pairing manifolds by Corollary 4.2. Thus we assume that if $g = 0$, then $t > 0$. Now we construct a link L which has the form of those in Figure 29. We take L to consist of the component α, t of the elements in part c) of Figure 24, and either g of the elements in part a) of Figure 24 or k of the elements in part b) of Figure 24. Lemma 7.6 implies that L is a corridor complex link. We next frame L. We define the framing of every face component of L to be 0. We define the framing of α to be an arbitrary negative rational number minus 2 times the number of components of L as in part b) of Figure 24. We frame every remaining component of L with an arbitrary negative rational number. Lemma 7.6 and Theorem 6.1 imply that Dehn surgery on L with this framing obtains a twisted face-pairing manifold. We transform L as indicated in Figures 25, 26, and 27 to obtain a link as in Figure 28 with the index $s + b$ in Figure 28 replaced by the current $t + 1$. The framings $m_1, \ldots, m_{t+1}$ are arbitrary positive rational numbers. According to Figure 12 on page 146 of [7], it follows that we can choose the framing of L to obtain M. This proves Theorem 8.1. □

9. Connected sums of twisted face-pairing manifolds

Theorem 9.1. *The connected sum of two twisted face-pairing manifolds is a twisted face-pairing manifold.*

Proof. Let P be the faceted 3-ball with just two faces which are degenerate pentagons as in Figure 30. Let ϵ be the face-pairing on P which fixes the edge common to the two faces, and let mul be the multiplier function for ϵ indicated in Figure 30. Figure 31 shows a corridor complex and a corridor complex framed link L for ϵ and mul.

Now let P_1 and P_2 be faceted 3-balls with face-pairings and multiplier functions which give rise to twisted face-pairing manifolds M_1 and M_2. We choose one of the two edges of P in the ϵ-edge cycle with multiplier m_1, and we form a connected sum P_1' of P and P_1 along this edge and any edge of P_1. Next we choose one of the two edges of P in the ϵ-edge cycle with multiplier m_2. This edge corresponds to an edge of P_1'. We form a connected sum P_2' of P_1' and P_2 along this edge and any edge of P_2. Theorem 2.1 easily implies that we obtain a twisted face-pairing manifold M which is the connected sum of M_1, M_2, and a manifold which is obtained by Dehn surgery on a framed link which consists of two simply linked unknots with

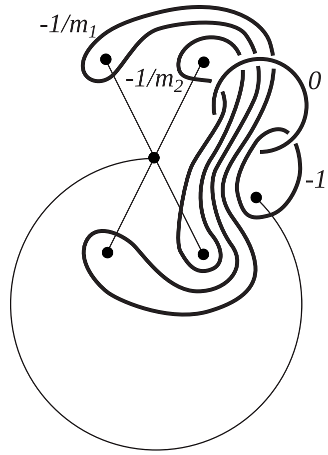

FIGURE 31. The framed corridor complex link L.

framings 0 and -1. This third connected summand is the 3-sphere. Thus M is the connected sum of M_1 and M_2.

This proves Theorem 9.1. □

10. SnapPea computations of hyperbolic manifolds with small volume

In this section we give a brief report on our SnapPea [12] computations. We seek twisted face-pairing manifolds which are hyperbolic and have small volume. For a working definition of "small volume" we find it convenient to focus on the first 75 manifolds in SnapPea's census of orientable closed hyperbolic 3-manifolds. In the introduction of [1] we reported that we had obtained approximately one fourth of these manifolds as twisted face-pairing manifolds. Now we can realize 36 of these 75 manifolds as twisted face-pairing manifolds.

We have two methods of computation. One method uses a program, pairsnap.c, which we wrote. It is freely available from http://www.math.vt.edu/people/floyd. The program pairsnap.c takes as input the data given by a faceted 3-ball with an orientation-reversing face-pairing and multiplier function and transforms it into a file suitable for input to SnapPea. So, to compute we first run pairsnap.c and then we run SnapPea. Our other method of computation uses our Dehn surgery description of twisted face-pairing manifolds. A faceted 3-ball with an orientation-reversing face-pairing and multiplier function gives rise to a corridor complex link, and Dehn surgery on this framed link in S^3 obtains our twisted face-pairing manifold. So to compute we draw our link (more likely, a simplification of it) in SnapPea, have SnapPea compute the link complement, and then have SnapPea perform the appropriate Dehn fillings corresponding to the components of our link. Theorem 6.1 provides this second method of computation with an advantage over the first method of computation because where the first method is in effect restricted to inverses of positive integers, the second method may use arbitrary positive rational numbers.

We illustrate our second method of computation in this paragraph. We begin with a model faceted 3-ball P which is a tetrahedron. Let e_1 and e_2 be two disjoint edges of P. We choose our orientation-reversing model face-pairing ϵ to fix e_i and interchange the two faces which contain e_i for $i \in \{1,2\}$. As in Example 7.2 of

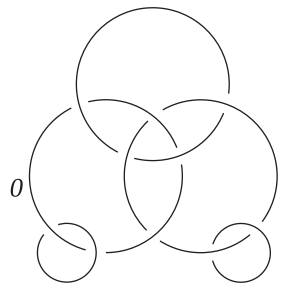

FIGURE 32. A corridor complex link L.

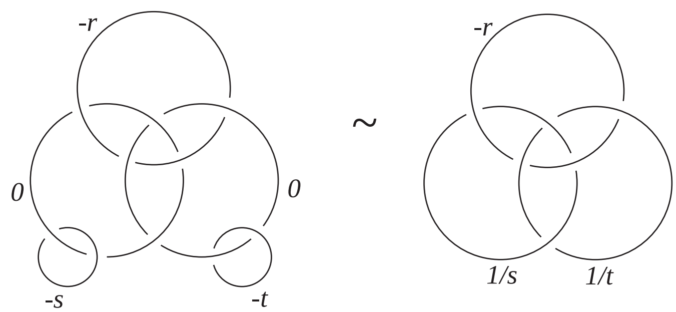

FIGURE 33. Framing and simplifying L.

[4], we see that the link L in Figure 32 is a corridor complex link for the pair (P, ϵ). The two components of L with framing 0 are face components of L, and the other three components of L are edge components with blackboard framings 0. Theorem 6.1 implies that Dehn surgery on the left framed link in Figure 33 obtains a twisted face-pairing manifold, where r, s and t are positive rational numbers. Two slam-dunks transform the left framed link in Figure 33 to the right one. Hence Dehn surgery on the Borromean rings yields a twisted face-pairing manifold if the framings are nonzero rational numbers not of the same sign. Applying SnapPea to the Borromean rings with such framings, we found 25 twisted face-pairing manifolds among the first 75 manifolds in SnapPea's census of orientable closed hyperbolic 3-manifolds.

As stated in the first paragraph of this section, we have found 36 twisted face-pairing manifolds among the first 75 manifolds in SnapPea's census of orientable closed 3-manifolds. The 11 not accounted for in the previous paragraph can be obtained using the method of the previous paragraph applied to face-pairings on model faceted 3-balls P such that P is either a tetrahedron, P is gotten from two tetrahedra by identifying a face of one with a face of the other (P is a hexahedron), P is an octahedron, or P is a tetrahedron with two disjoint edges bisected (which makes every face a quadrilateral instead of a triangle).

11. Questions

We conclude with some questions about twisted face-pairing 3-manifolds.

Question 11.1. Is every closed, connected, orientable 3-manifold a twisted face-pairing manifold?

It seems unlikely that the answer to this question is yes. In particular, are any of the following twisted face-pairing manifolds: the 3-torus, the Poincaré homology 3-sphere, the closed hyperbolic 3-manifold of smallest volume in the SnapPea [12] census, any closed 3-manifold with the geometry of $\mathbf{R}^3$ or of $\mathbf{H}^2 \times \mathbf{R}$? Is there an effective characterization of twisted face-pairing manifolds?

Question 11.2. Is every twisted face-pairing 3-manifold that comes from a regular faceted 3-ball irreducible?

This question seems approachable because of the explicit description of the Heegaard diagrams in [4]. A related question is whether every twisted face-pairing 3-manifold that comes from a regular faceted 3-ball has a nontrivial fundamental group. Question 11.2 is motivated in part by the following fact. Every twisted face-pairing manifold is a connected sum of finitely many (possibly zero) copies of $S^2 \times S^1$ (which is a twisted face-pairing manifold by Example 6.2.1 of [4]) and finitely many (possibly zero) twisted face-pairing manifolds which arise from regular faceted 3-balls. The last sentence can be proved using the results of [4], but our argument is too long to be presented conveniently here.

Question 11.3. Is every twisted face-pairing 3-manifold that comes from an ample faceted 3-ball hyperbolic?

The twisted face-pairing technique was discovered in a search for an easy combinatorial construction for test manifolds for Thurston's Hyperbolization Conjecture. By [3], any twisted face-pairing 3-manifold that comes from an ample faceted 3-ball has a Gromov-hyperbolic fundamental group, and hence is a test manifold for the Hyperbilization Conjecture.

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