CONSTRUCTING RATIONAL MAPS FROM SUBDIVISION RULES

J. W. CANNON, W. J. FLOYD, R. KENYON, AND W. R. PARRY

Abstract. Suppose $\mathcal{R}$ is an orientation-preserving finite subdivision rule with an edge pairing. Then the subdivision map $\sigma_{\mathcal{R}}$ is either a homeomorphism, a covering of a torus, or a critically finite branched covering of a 2-sphere. If $\mathcal{R}$ has mesh approaching 0 and $S_{\mathcal{R}}$ is a 2-sphere, it is proved in Theorem 3.1 that if $\mathcal{R}$ is conformal then $\sigma_{\mathcal{R}}$ is realizable by a rational map. Furthermore, a general construction is given which, starting with a one tile rotationally invariant finite subdivision rule, produces a finite subdivision rule $\mathcal{Q}$ with an edge pairing such that $\sigma_{\mathcal{Q}}$ is realizable by a rational map.

In this paper we illustrate a technique for constructing critically finite rational maps. The starting point for the construction is an orientation-preserving finite subdivision rule $\mathcal{R}$ with an edge pairing. For such a finite subdivision rule the CW complex $S_{\mathcal{R}}$ is a surface, and the map $\sigma_{\mathcal{R}}\colon S_{\mathcal{R}} \to S_{\mathcal{R}}$ is a branched covering. If $S_{\mathcal{R}}$ is orientable, then unless $\sigma_{\mathcal{R}}$ is a homeomorphism or a covering of the torus, $S_{\mathcal{R}}$ is a 2-sphere and $\sigma_{\mathcal{R}}$ is critically finite. In the latter case, $S_{\mathcal{R}}$ has an orbifold structure $\mathcal{O}_{\mathcal{R}}$ and $\sigma_{\mathcal{R}}$ induces a map $\tau_{\mathcal{R}}\colon \mathcal{T}(\mathcal{O}_{\mathcal{R}}) \to \mathcal{T}(\mathcal{O}_{\mathcal{R}})$ on the Teichmüller space of the orbifold. By work of Thurston, $\sigma_{\mathcal{R}}$ can be realized by a rational map exactly if $\tau_{\mathcal{R}}$ has a fixed point. Alternatively, we prove in Theorem 3.1 that $\sigma_{\mathcal{R}}$ can be realized by a rational map if $\mathcal{R}$ has mesh approaching 0 and is conformal.

We next give a general construction which, starting with a one tile rotationally invariant finite subdivision rule $\mathcal{R}$, produces an orientation-preserving finite subdivision rule $\mathcal{Q}$ with an edge pairing such that $\mathcal{Q}$ is conformal if and only if $\mathcal{R}$ is conformal; we then show in Theorem 3.2 that $\sigma_{\mathcal{Q}}$ is realizable by a rational map. We next give several examples of orientation-preserving finite subdivision rules with edge pairings. For each example $\mathcal{R}$ for which the associated map $\sigma_{\mathcal{R}}$ can be realized by a rational map, we explicitly construct a rational map realizing it. We conclude with some questions.

A motivation for this work is the Bowers-Stephenson paper [1]. In that paper they construct an expansion complex for the pentagonal subdivision rule (see Figure 4) and numerically approximate the expansion constant. In Example 4.4 we consider an associated finite subdivision rule $\mathcal{Q}$ with an edge pairing and construct a rational map $f_{\mathcal{Q}}(z) = \frac{2z(z+9/16)^5}{27(z-3/128)^3(z-1)^2}$ which realizes $\sigma_{\mathcal{Q}}$. The expansion constant for the pentagonal subdivision rule is $(f'_{\mathcal{Q}}(0))^{1/5} = (-324)^{1/5}$.

We thank Curt McMullen, Kevin Pilgrim, and the referee for helpful comments.

Date: March 28, 2003.

1991 *Mathematics Subject Classification.* Primary 37F10, 52C20; Secondary 57M12.

Key words and phrases. finite subdivision rule, rational map, conformality.

This research was supported in part by NSF grants DMS-9803868, DMS-9971783, and DMS-10104030.

1. Finite subdivision rules

The theory of finite subdivision rules arose from an ongoing attempt by three of the authors and their coworkers to resolve the following:

Conjecture 1.1. Suppose G is a Gromov-hyperbolic discrete group whose space at infinity is the 2-sphere. Then G acts properly discontinuously, cocompactly, and isometrically on hyperbolic 3-space $\mathbb{H}^3$.

Conjecture 1.1 is closely related to Thurston's Hyperbolization Conjecture, which states that if M is a closed 3-manifold whose fundamental group is infinite, is not a free product, and does not contain a subgroup isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$, then M admits a hyperbolic structure.

Our approach to the conjecture is through Cannon's definition (see [4]) of conformality for a sequence of shinglings (locally finite covers by compact, connected sets) of a topological surface X. Suppose X is a surface and $\mathcal{S}$ is a shingling of X. A *weight function* on $\mathcal{S}$ is a nonzero function $\rho\colon \mathcal{S}\to\mathbb{R}$ such that $\rho(s)\geq 0$ for all $s\in\mathcal{S}$. For $s\in\mathcal{S}$, $\rho(s)$ is called the *weight* of s. One can use a weight function to give combinatorial definitions of length and area, and can then define the modulus of an annulus by optimizing height^2/area or area/circumference^2. If R is an annulus in X and ρ is a weight function on $\mathcal{S}$, then the *area* $A(R,\rho)$ of R is the sum of the squares of the weights of the shingles that intersect R, the *length* $L(\alpha,\rho)$ of a curve α in R is the sum of the weights of the shingles that intersect α, the *height* $H(R,\rho)$ of R is the minimum length of a curve in R joining the ends of R, and the *circumference* $C(R,\rho)$ of R is the minimum length of a simple closed curve in R separating the ends of R. The *combinatorial moduli* are $M(R,\mathcal{S})=\sup_\rho\{\frac{H(R,\rho)^2}{A(R,\rho)}\}$ and $m(R,\mathcal{S})=\inf_\rho\{\frac{A(R,\rho)}{C(R,\rho)^2}\}$. One can similarly define combinatorial moduli for quadrilaterals in R.

Now suppose that $\{\mathcal{S}_i\}_{i=1}^\infty$ is a sequence of shinglings of a topological surface X with mesh locally approaching 0. That is, if X is endowed with a metric that induces the topology on X then in each compact set the largest diameter of a shingle converges to 0. The sequence $\{\mathcal{S}_i\}_{i=1}^\infty$ is *conformal* if there is a positive real number K satisfying the following conditions.

- **Axiom I:** For each annulus R in X, there is a positive real number r such that $m(R,\mathcal{S}_i), M(R,\mathcal{S}_i)\in[r,Kr]$ for sufficiently large i.
- **Axiom II:** Given a point $x\in X$, a neighborhood N of X, and an integer J, there is an annulus R in N which separates x and ∂N such that $m(R,\mathcal{S}_i)>J$ and $M(R,\mathcal{S}_i)>J$ for sufficiently large i.

Cannon's combinatorial Riemann mapping theorem [4] states that if X is a topological 2-sphere and $\{\mathcal{S}_i\}$ is a conformal sequence of shinglings on X, then there is a quasiconformal structure (a collection of charts for which the transition functions are uniformly quasiconformal) on X such that for each annulus R in X the analytic modulus of R lies within a multiplicative bound (independent of the annulus) of the asymptotic combinatorial moduli of R.

Now suppose that G is a negatively curved group whose space at infinity is a 2-sphere. Let Γ be a locally finite Cayley graph for G, and let $\mathcal{O}$ be a vertex of Γ. Then given a geodesic ray $F\colon [0,\infty)\to\Gamma$ with $F(0)=\mathcal{O}$ and a positive integer n, one can define a disk at infinity $D(F,n)$ corresponding to the half space of points closer to the tail of the ray than to the initial segment of the ray (for example, see

[5] for a definition). For each positive integer n, the collection

$$\mathcal{D}(n) = \{D(F,n) \colon F \text{ is a geodesic ray in } \Gamma \text{ with } F(0) = \mathcal{O}\}$$

is a finite cover of the sphere at infinity of G. In [5, Theorem 2.3.1], Cannon and Swenson prove that G acts properly discontinuously, cocompactly, and isometrically on hyperbolic 3-space if and only if the sequence $\{\mathcal{D}(n)\}_{n=1}^{\infty}$ of disk covers is conformal in the above sense. Cannon and Swenson also show in [5] that for every integer $n \geq 2$ the elements of $\mathcal{D}(n)$ can be obtained from the elements of $\mathcal{D}(n-1)$ by a finite recursion.

Finite subdivision rules were developed to give models for the above sequences of disk covers. While the resulting sequences of subdivisions are not as general as the sequences of disk covers, the reduction from considering sequences of disk covers to considering subdivisions coming from finite subdivision rules does not appear to be an essential simplification. Much of the basic theory of finite subdivision rules is developed in [8].

A *finite subdivision rule* $\mathcal{R}$ consists of a finite CW complex $S_{\mathcal{R}}$ which is the union of its closed 2-cells, a subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$, and a continuous map $\sigma_{\mathcal{R}} \colon \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$ whose restriction to every open cell is a homeomorphism onto an open cell. (In particular, $\sigma_{\mathcal{R}}$ is cellular.) Furthermore, $S_{\mathcal{R}}$ must have the property that for each closed 2-cell $\tilde{t}$ of $S_{\mathcal{R}}$, there is a cell structure t on a 2-disk such that t has at least three vertices, all of the vertices and edges of t are in ∂t, and the characteristic map $\psi_t \colon t \to S_{\mathcal{R}}$ takes each open cell homeomorphically onto an open cell. The cell complex t is called a *tile type* of $\mathcal{R}$. Similarly, if $\tilde{e}$ is a closed 1-cell of $S_{\mathcal{R}}$ then a 1-disk e equipped with a characteristic map $\psi_e \colon e \to S_{\mathcal{R}}$ is called an *edge type* of $\mathcal{R}$. See, for example, Spanier [22], for basic details about CW complexes. A CW complex Y is a *subdivision* of a CW complex X if they have the same underlying space and every closed cell of Y is contained in a closed cell of X. A finite subdivision rule $\mathcal{R}$ is *orientation preserving* if there is an orientation on the union of the open 2-cells of $S_{\mathcal{R}}$ such that the restriction of $\sigma_{\mathcal{R}}$ to each open 2-cell of $\mathcal{R}(S_{\mathcal{R}})$ is orientation preserving.

Example 1.2. We give a preliminary example to illustrate the definition. For this finite subdivision rule $\mathcal{R}$ the complex $S_{\mathcal{R}}$ is the 2-sphere $\widehat{\mathbb{C}}$, with a cell structure consisting of 4 vertices, three 1-cells, and one 2-cell. The complexes $S_{\mathcal{R}}$ and $\mathcal{R}(S_{\mathcal{R}})$ are shown in Figure 1. The vertices of $S_{\mathcal{R}}$ are labeled 0, 1, α, and ∞, and the 1-cells of $S_{\mathcal{R}}$ are labeled e_1, e_2, and e_3. The vertices 0, 1, α, and ∞ of $\mathcal{R}(S_{\mathcal{R}})$ all map to 0 under $\sigma_{\mathcal{R}}$, the vertices a_i all map to α, the vertex b maps to ∞, and the vertices c_j all map to 1. The 1-cells of $\mathcal{R}(S_{\mathcal{R}})$ are labeled by the labels of their images under $\sigma_{\mathcal{R}}$. The map $\sigma_{\mathcal{R}}$ is orientation preserving. The single tile type is a hexagon. It is shown in Figure 2, with its 1-cells labeled by their images in $S_{\mathcal{R}}$.

Suppose $\mathcal{R}$ is a finite subdivision rule. An $\mathcal{R}$-*complex* is a 2-dimensional CW complex X which is a union of its closed 2-cells together with a continuous map $f \colon X \to S_{\mathcal{R}}$ such that the restriction of f to each open cell is a homeomorphism onto an open cell. (In particular, f is cellular.) In this case there is a subdivision $\mathcal{R}(X)$ of X such that the induced map $f \colon \mathcal{R}(X) \to \mathcal{R}(S_{\mathcal{R}})$ restricts to a homeomorphism on each open cell. Furthermore, $\mathcal{R}(X)$ is also an $\mathcal{R}$-complex with associated map $\sigma_{\mathcal{R}} \circ f \colon \mathcal{R}(X) \to S_{\mathcal{R}}$. One can inductively define $\mathcal{R}^n(X)$ for $n > 1$ by $\mathcal{R}^n(X) = \mathcal{R}(\mathcal{R}^{n-1}(X))$, with associated map $\sigma_{\mathcal{R}}^n \circ f \colon \mathcal{R}^n(X) \to S_{\mathcal{R}}$. Note that $S_{\mathcal{R}}$ is an $\mathcal{R}$-complex with associated map the identity map. Also, each tile type of $\mathcal{R}$ is an

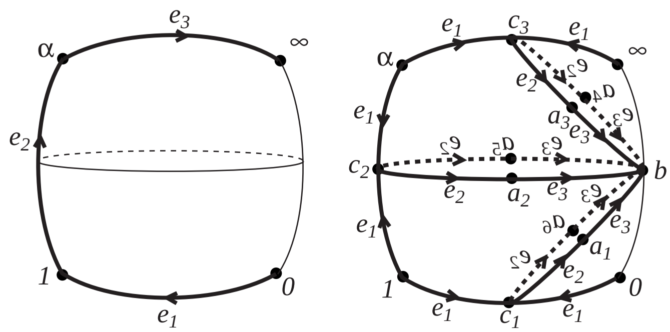


FIGURE 1. The complexes $S_{\mathcal{R}}$ and $\mathcal{R}(S_{\mathcal{R}})$ for Example 1.2

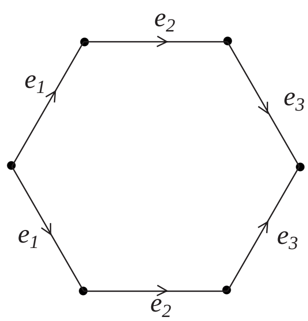


FIGURE 2. The tile type t for Example 1.2

$\mathcal{R}$-complex with associated map its characteristic map. The first three subdivisions $\mathcal{R}(t)$, $\mathcal{R}^2(t)$, and $\mathcal{R}^3(t)$ of the tile type t for Example 1.2 are shown in Figure 3. Figure 3 was drawn using Stephenson's program CirclePack [21].

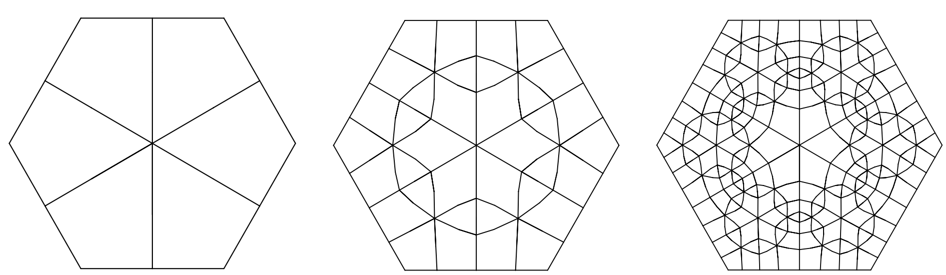


FIGURE 3. The subdivisions $\mathcal{R}(t)$, $\mathcal{R}^2(t)$, and $\mathcal{R}^3(t)$ for Example 1.2

A finite subdivision rule $\mathcal{R}$ has *bounded valence* if there is an upper bound to the set of valences of vertices of $\mathcal{R}^n(S_{\mathcal{R}})$, the n^{th} subdivision of $S_{\mathcal{R}}$, where n is any positive integer. Suppose $\mathcal{R}$ is a finite subdivision rule and X is an $\mathcal{R}$-complex that is also a surface. For each nonnegative integer n, let $\mathcal{S}^n(X)$ be the shingling of X whose elements are the closed tiles of $\mathcal{R}^n(X)$. Then $(X, \mathcal{R})$ is *conformal* if $\{\mathcal{S}^n(X)\}$ is conformal in $\mathrm{int}(X)$. If $(X, \mathcal{R})$ is conformal whenever X is a bounded valence $\mathcal{R}$-complex that is a surface, then the finite subdivision rule $\mathcal{R}$ is *conformal*.

Suppose that $\mathcal{R}$ and $\mathcal{Q}$ are finite subdivision rules with associated complexes $S_{\mathcal{R}}$ and $S_{\mathcal{Q}}$ and maps $\sigma_{\mathcal{R}}\colon \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$ and $\sigma_{\mathcal{Q}}\colon \mathcal{Q}(S_{\mathcal{Q}}) \to S_{\mathcal{Q}}$. Then $\mathcal{R}$ and $\mathcal{Q}$ are *isomorphic* if there is a cellular homeomorphism $h\colon S_{\mathcal{R}} \to S_{\mathcal{Q}}$ such that $\sigma_{\mathcal{Q}} \circ h = h \circ \sigma_{\mathcal{R}}$. If there are cellularly isotopic cellular homeomorphisms $g, h\colon S_{\mathcal{R}} \to S_{\mathcal{Q}}$ such that $\sigma_{\mathcal{Q}} \circ g = h \circ \sigma_{\mathcal{R}}$, then $\mathcal{R}$ and $\mathcal{Q}$ are *weakly isomorphic*.

A finite subdivision rule $\mathcal{R}$ has *mesh approaching* 0 if given an open cover $\mathcal{U}$ of $S_{\mathcal{R}}$ there is a positive integer n such that each tile of $\mathcal{R}^n(S_{\mathcal{R}})$ is contained in an element of $\mathcal{U}$. If a finite subdivision rule has mesh approaching 0, then for any point $x \in S_{\mathcal{R}}$, $\{x\} = \cap\{t\colon t$ is a closed 2-cell of $\mathcal{R}^n(S_{\mathcal{R}})$ for some nonnegative integer n and $x \in t\}$. This condition is very convenient, since it implies that each point of $S_{\mathcal{R}}$ is determined by its forward orbit under $\sigma_{\mathcal{R}}$. It can be difficult to verify, however, since it depends on more than just the combinatorics of $S_{\mathcal{R}}$, $\mathcal{R}(S_{\mathcal{R}})$, and $\sigma_{\mathcal{R}}$. From the point of view of subdivision rules, the following definition is easier to work with.

A finite subdivision rule $\mathcal{R}$ has *mesh approaching* 0 *combinatorially* if there is a positive integer n satisfying the following: i) each closed edge of $S_{\mathcal{R}}$ is properly subdivided in the subdivision $\mathcal{R}^n(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$; ii) if t is a tile type and e_1 and e_2 are disjoint edges of t, then no tile of $\mathcal{R}^n(t)$ contains an edge of the n^{th} subdivision of e_1 and an edge of the n^{th} subdivision of e_2. It is shown in [8] that if a finite subdivision rule $\mathcal{R}$ has mesh approaching 0 combinatorially, then $\mathcal{R}$ is weakly isomorphic to a finite subdivision rule $\mathcal{Q}$ with mesh approaching 0 and hence for each nonnegative integer n $\mathcal{R}^n(S_{\mathcal{R}})$ and $\mathcal{Q}^n(S_{\mathcal{Q}})$ are cellularly homeomorphic.

Let $\mathcal{R}$ be an orientation-preserving finite subdivision rule with bounded valence and mesh approaching 0, and let the tile types be given the orientations induced from the orientations on the open tiles of $S_{\mathcal{R}}$. We say that $\mathcal{R}$ is a *one tile rotationally invariant finite subdivision rule* if it satisfies the following two conditions:

1. If s and t are tile types of $\mathcal{R}$, then there exists an orientation-preserving cellular isomorphism from s to t which takes $\mathcal{R}(s)$ to $\mathcal{R}(t)$.
2. If t is a tile type of $\mathcal{R}$ with q edges, then there exists an orientation-preserving cellular automorphism of t of order q which is also a cellular automorphism of $\mathcal{R}(t)$.

If in addition for each tile type t of $\mathcal{R}$ there is an orientation-reversing cellular automorphism of t that is also a cellular automorphism of $\mathcal{R}(t)$, then $\mathcal{R}$ is a *one tile dihedrally invariant finite subdivision rule*. A straightforward argument (which is given in [10]) shows that if $\mathcal{R}$ is a one tile rotationally invariant finite subdivision rule, t_1 and t_2 are tile types of $\mathcal{R}$, e_1 is an edge of t_1, and e_2 is an edge of t_2, then there is an orientation preserving complete cellular isomorphism from t_1 to t_2 which takes e_1 to e_2. (A cellular isomorphism from t_1 to t_2 is a *complete cellular isomorphism* if for every positive integer n it is a cellular isomorphism from $\mathcal{R}^n(t_1)$ to $\mathcal{R}^n(t_2)$.)

Let $\mathcal{R}$ be an orientation-preserving finite subdivision rule. We say that $\mathcal{R}$ has an *edge pairing* if $S_{\mathcal{R}}$ is a closed surface. If this is true, then $\sigma_{\mathcal{R}}$ is a branched covering. By the Riemann-Hurwitz formula, if $S_{\mathcal{R}}$ is connected and orientable then either i) $\sigma_{\mathcal{R}}$ is a homeomorphism, ii) $S_{\mathcal{R}}$ is a torus and $\sigma_{\mathcal{R}}$ is a covering map, or iii) $S_{\mathcal{R}}$ is a 2-sphere. Case i) cannot occur if $\mathcal{R}$ properly subdivides any tile type. Case ii) occurs, for example, for the binary square subdivision rule $\mathcal{L}$ of Example 4.3. In this example, the rational functions that realize $\sigma_{\mathcal{L}}$ include a classical example due to Lattès [14] of a rational map whose Julia set is the 2-sphere. We are most interested in case iii), and wish to understand for that case when $\sigma_{\mathcal{R}}$ can be realized by a rational map. We first recall Thurston's characterization theorem for critically finite branched maps.

2. Thurston's characterization theorem

We give here a brief summary of Thurston's characterization theorem for critically finite branched maps. Our sources are [25], [11], and [16]. Let $f\colon S^2 \to S^2$ be an orientation-preserving branched map. Given $x \in S^2$, let $\deg_x(f)$ be the topological degree of f at x. The *critical set* of f is $\Omega_f = \{x\colon \deg_x(f) > 1\}$, and the *post-critical set* is $P_f = \cup_{n>0} f^{\circ n}(\Omega_f)$. The mapping f is called *critically finite* if P_f is finite. Two maps $f, g\colon S^2 \to S^2$ are called *equivalent* if there is a homeomorphism $h\colon S^2 \to S^2$ such that $h(P_f) = P_g$, $(h \circ f)\big|_{P_f} = (g \circ h)\big|_{P_f}$, and $h \circ f$ is isotopic, rel P_f, to $g \circ h$. In this case, if g is a rational map then we also say that f is *realized* by g.

Suppose that $f\colon S^2 \to S^2$ is an orientation-preserving, critically finite branched map. For each $x \in S^2$, let $D_f(x) = \{n \in \mathbb{Z}_+\colon$ there exists a positive integer m such that $f^{\circ m}$ has degree n at some $y \in S^2$ with $f^{\circ m}(y) = x\}$. We have $D_f(x) \neq \{1\}$ if and only if $x \in P_f$. Define $\nu_f\colon S^2 \to \mathbb{Z}_+ \cup \{\infty\}$ by

$$\nu_f(x) = \begin{cases} \operatorname{lcm}(D_f(x)) & \text{if } D_f(x) \text{ is finite,} \\ \infty & \text{if } D_f(x) \text{ is infinite.} \end{cases}$$

Let $\mathcal{O}_f$ be the orbifold (S^2, ν_f). A point $x \in \mathcal{O}_f$ with $\nu_f(x) > 1$ is called a *distinguished point*. These are the points in P_f. The Euler characteristic of $\mathcal{O}_f$ is

$$\chi(\mathcal{O}_f) = 2 - \sum_{x \in P_f} \left(1 - \frac{1}{\nu_f(x)}\right).$$

That is, the definition of the Euler characteristic of the orbifold is like the definition of the Euler characteristic of the underlying space except that the contribution of a vertex x to the Euler characteristic is $1/\nu_f(x)$ instead of 1. With this definition, the orbifold Euler characteristic is multiplicative for orbifold covering maps. An orbifold is *hyperbolic* if it has a hyperbolic structure in the complement of the set of distinguished points and in a neighborhood of a distinguished point x the metric is the metric of a hyperbolic cone with cone angle $2\pi/\nu_f(x)$. The orbifold $\mathcal{O}_f$ is hyperbolic if and only if $\chi(\mathcal{O}_f) < 0$.

We consider the Teichmüller space $\mathcal{T}(\mathcal{O}_f)$ of $\mathcal{O}_f$ as the space of complex structures on $\mathcal{O}_f$, up to the equivalence of isotopy fixing the distinguished points. A complex structure on $\mathcal{O}_f$ pulls back under f to a complex structure on $(S^2, f^{-1}(\nu_f))$, and this extends to a complex structure on $\mathcal{O}_f$. In this way we obtain a map $\tau_f\colon \mathcal{T}(\mathcal{O}_f) \to \mathcal{T}(\mathcal{O}_f)$. The map τ_f is analytic and does not increase the distances between points in $\mathcal{O}_f$. Douady and Hubbard show in [11] that if $\mathcal{O}_f$ is hyperbolic then $\tau_f^{\circ 2}$ decreases the distances between points in $\mathcal{O}_f$.

Theorem 2.1 (Thurston)**.** *An orientation-preserving critically finite branched map $f\colon S^2 \to S^2$ is equivalent to a rational map if and only if τ_f has a fixed point.*

When $\mathcal{O}_f$ is hyperbolic, Thurston gives the following topological characterization of when τ_f has a fixed point. To state this, we first need some terminology.

An *f-stable curve system* is a finite set Γ of simple, closed, disjoint, essential, non-peripheral, non-homotopic curves in $S^2 \setminus P_f$ such that for each $\gamma \in \Gamma$, each component of $f^{-1}(\gamma)$ is either null-homotopic, peripheral, or homotopic in $S^2 \setminus P_f$ to an element of Γ. Suppose Γ is an f-stable curve system. The matrix $A^\Gamma \colon \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is defined in coordinates by

$$A^\Gamma_{\gamma\delta} = \sum_\alpha \frac{1}{\deg(f \colon \alpha \to \delta)},$$

where the sum is taken over components α of $f^{-1}(\delta)$ which are isotopic to γ in $S^2 \setminus P_f$. (We think of the matrix A^Γ as estimating the change in moduli under f^{-1} of a family of annuli around the curves in Γ.) Since A^Γ has non-negative entries, by the Perron-Frobenius theorem its spectral radius $\lambda(\Gamma)$ is an eigenvalue with a non-negative eigenvector.

Theorem 2.2 (Thurston's characterization theorem)**.** *An orientation-preserving critically finite branched map $f \colon S^2 \to S^2$ with hyperbolic orbifold is equivalent to a rational function if and only if for any f-stable curve system Γ, $\lambda(\Gamma) < 1$. In that case, the rational function is unique up to conformal conjugation.*

If $\mathcal{O}_f$ is not hyperbolic, then $\mathcal{T}(\mathcal{O}_f)$ is a single point unless $\mathcal{O}_f$ is the orbifold $(2,2,2,2)$ (the rectangular pillowcase). In that case, $\mathcal{O}_f$ is double-covered by a torus T_f, f lifts to a covering map of this torus, and there is a 2×2 matrix A_f which represents the induced map $H_1(T_f, \mathbb{Z}) \to H_1(T_f, \mathbb{Z})$. In this case, Thurston's characterization is the following.

Theorem 2.3 (Thurston)**.** *An orientation-preserving critically finite branched map $f \colon S^2 \to S^2$ with $\mathcal{O}_f$ the orbifold $(2,2,2,2)$ is equivalent to a rational map if and only if A_f is either a multiple of the identity or the eigenvalues of A_f are not real.*

3. Constructing rational maps from finite subdivision rules

Let $\mathcal{R}$ be an orientation-preserving finite subdivision rule with an edge pairing, and assume furthermore that $S_\mathcal{R}$ is a 2-sphere. The branched covering $\sigma_\mathcal{R}$ preserves orientation and is cellular as a map from $\mathcal{R}(S_\mathcal{R})$ to $S_\mathcal{R}$, and so in particular it takes vertices of $S_\mathcal{R}$ to vertices of $S_\mathcal{R}$. Since the critical points of $\sigma_\mathcal{R}$ are all vertices of $\mathcal{R}(S_\mathcal{R})$, this implies that $\sigma_\mathcal{R}$ is critically finite. We are interested in understanding when $\sigma_\mathcal{R}$ can be realized by a rational map $f_\mathcal{R}$. For convenience, we will denote the orbifold $\mathcal{O}_{\sigma_\mathcal{R}}$ by $\mathcal{O}_\mathcal{R}$ and we will denote the map $\tau_{\sigma_\mathcal{R}} : \mathcal{T}(\mathcal{O}_\mathcal{R}) \to \mathcal{T}(\mathcal{O}_\mathcal{R})$ by $\tau_\mathcal{R}$.

By Theorem 2.1, $\sigma_\mathcal{R}$ can be realized by a rational map exactly if $\tau_\mathcal{R}$ has a fixed point. In particular, $\sigma_\mathcal{R}$ can be realized by a rational map if the orbifold $\mathcal{O}_\mathcal{R}$ has at most three distinguished points. When $\mathcal{T}(\mathcal{O}_\mathcal{R})$ has more than one point, Theorems 2.2 and 2.3 give topological characterizations of when $\tau_\mathcal{R}$ has a fixed point.

In some cases, one can also use techniques from finite subdivision rules to determine when $\sigma_\mathcal{R}$ can be realized by a rational map.

Theorem 3.1. *Let $\mathcal{R}$ be an orientation-preserving finite subdivision rule with an edge pairing such that $S_\mathcal{R}$ is a 2-sphere and the mesh of $\mathcal{R}$ approaches 0. If $\mathcal{R}$ is conformal, then $\sigma_\mathcal{R}$ is realizable by a rational map.*

Proof. Suppose that $\mathcal{R}$ is conformal. By the combinatorial Riemann mapping theorem [4], there is a quasiconformal structure on $S_\mathcal{R}$ such that the analytic moduli of annuli in $S_{\mathcal{R}}$ are uniformly approximated by their asymptotic combinatorial moduli. We adopt an argument from Cannon and Swenson [5] to show that the sequence $\{\sigma_{\mathcal{R}}^{\circ n}\}$ is uniformly quasiregular with respect to the quasiconformal structure on $S_{\mathcal{R}}$ that is given above. By a theorem of Kuusalo [13] the quasiconformal structure on $S_{\mathcal{R}}$ is quasiconformally equivalent to a conformal structure. Hence $S_{\mathcal{R}}$ has a conformal structure so that there is a positive constant K such that for any annulus A in $S_{\mathcal{R}}$, there is a positive real number r so that for n sufficiently large the analytic modulus $\mathrm{mod}(A)$ of A and the combinatorial moduli $M(A, \mathcal{S}^n(S_{\mathcal{R}}))$ and $m(A, \mathcal{S}^n(S_{\mathcal{R}}))$ lie in the interval $[r, Kr]$, where $\mathcal{S}^n(S_{\mathcal{R}})$ denotes the shingling of $S_{\mathcal{R}}$ by the tiles of $\mathcal{R}^n(S_{\mathcal{R}})$.

We show that for each positive integer k, $\sigma_{\mathcal{R}}^{\circ k}$ is K^2-quasiregular. Let k be a positive integer, let $x \in S_{\mathcal{R}}$ be a point which is not a branch point of $\sigma_{\mathcal{R}}^{\circ k}$, let U be a neighborhood of x such that $\sigma_{\mathcal{R}}^{\circ k}\big|_U$ is injective, and let A be an annulus in U. Then there are positive real numbers r_1 and r_2 such that for n sufficiently large i) $\mathrm{mod}(A)$ and $M(A, \mathcal{S}^n(S_{\mathcal{R}}))$ lie in the interval $[r_1, Kr_1]$ and ii) $\mathrm{mod}(\sigma_{\mathcal{R}}^{\circ k}(A))$ and $M(\sigma_{\mathcal{R}}^{\circ k}(A), \mathcal{S}^n(S_{\mathcal{R}}))$ lie in the interval $[r_2, Kr_2]$. For each positive integer n, the cell structure on $\mathcal{S}^{n+k}(S_{\mathcal{R}})$ is the inverse image under $\sigma_{\mathcal{R}}^{\circ k}$ of the cell structure on $\mathcal{S}^n(S_{\mathcal{R}})$. Since $\sigma_{\mathcal{R}}^{\circ k}$ is injective on A, for each positive integer n $M(A, \mathcal{S}^{n+k}(S_{\mathcal{R}})) = M(\sigma_{\mathcal{R}}^{\circ k}(A), \mathcal{S}^n(S_{\mathcal{R}}))$ and so $\frac{1}{K^2} \leq \frac{\mathrm{mod}(A)}{\mathrm{mod}(\sigma_{\mathcal{R}}^{\circ k}(A))} \leq K^2$. Hence the restriction of $\sigma_{\mathcal{R}}^{\circ k}$ to U is K^2-quasiconformal, $\sigma_{\mathcal{R}}^{\circ k}$ is K^2-quasiregular, and $\{\sigma_{\mathcal{R}}^{\circ k}\}$ is uniformly quasiregular. By a theorem of Sullivan [24, Theorem 9], $\sigma_{\mathcal{R}}$ is realizable by a rational map. $\square$

Since $\sigma_{\mathcal{R}}$ is critically finite, if $\sigma_{\mathcal{R}}$ can be realized by a rational map $f_{\mathcal{R}}$ then it follows from Sullivan's classification of stable Fatou regions (see, for example, [18, Corollary 16.5]) that the only possible periodic Fatou domains are superattracting domains. The Fatou set for $f_{\mathcal{R}}$ has a superattracting domain exactly if $\mathcal{R}$ does not have bounded valence. Since by Sullivan's nonwandering theorem [23] every Fatou domain is eventually periodic, if $\mathcal{R}$ has bounded valence then the Julia set of $f_{\mathcal{R}}$ is the 2-sphere.

Before giving specific examples in Section 4, we present here a general construction of finite subdivision rules with edge pairings that are realized by rational maps. Let $\mathcal{R}$ be a one tile rotationally invariant finite subdivision rule. Figures 4 and 5 show the subdivisions of the tile types for two such finite subdivision rules, the pentagonal subdivision rule and the twisted pentagonal subdivision rule.

Since $\mathcal{R}$ has mesh approaching 0, there is a tile type t such that there is a tile of type t in the interior of a subdivision $\mathcal{R}^n(u)$ of some tile type u. Let $n(t)$ be the number of edges of t. Then there is an orientation-preserving cellular automorphism $\theta \colon t \to t$ which has order $n(t)$ and is also a cellular automorphism of $\mathcal{R}(t)$. As remarked earlier, we can assume that θ is a complete cellular isomorphism. Since θ has finite order, it has a unique fixed point. We define the barycenter $b(t)$ of t to be this fixed point. Choose orientations for the open tiles of $S_{\mathcal{R}}$ with respect to which $\sigma_{\mathcal{R}}$ is orientation preserving, and choose an orientation for each tile type s so that its characteristic map ψ_s is orientation preserving. For each tile type s use the orientation on s to orient the edges of ∂s. By possibly replacing θ by a power, we can assume that θ maps each vertex of ∂t to the vertex that follows it in the orientation of ∂t. Let α be an edge of t. Let t' be a tile type such that there is a tile of type t' adjacent in $\mathcal{R}^n(u)$ to a tile of type t along an edge

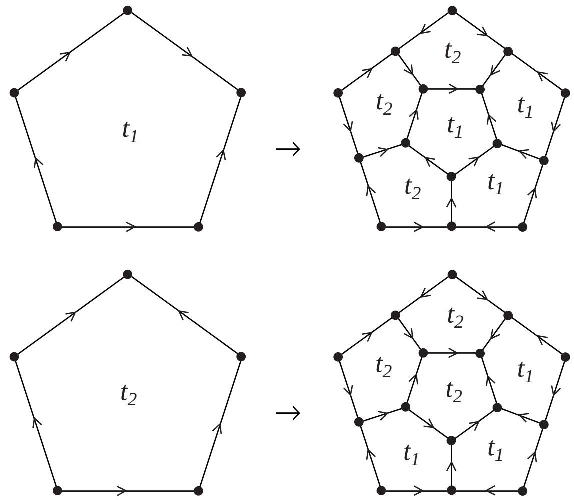


FIGURE 4. The subdivisions of the tile types for the pentagonal subdivision rule

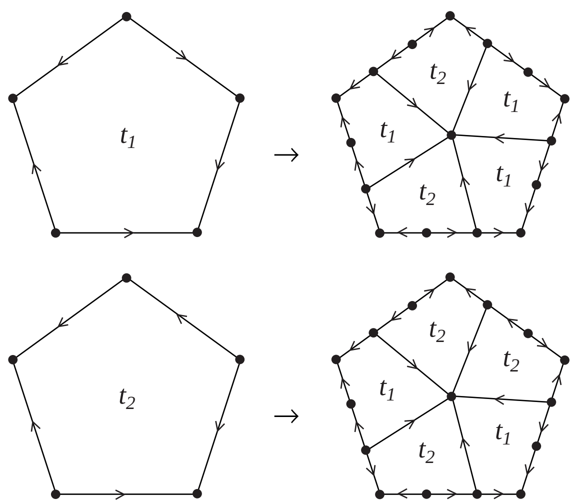


FIGURE 5. The subdivisions of the tile types for the twisted pentagonal subdivision rule

which corresponds to α in t and which corresponds to an edge α' in t'. Then there is a complete cellular isomorphism $h\colon t \to t'$ such that $h(\alpha) = \alpha'$. Then $\left(\psi_t\big|_{\mathrm{int}(\alpha)}\right)^{-1} \circ \psi_{t'}\big|_{\mathrm{int}(\alpha'))} \circ h\big|_{\mathrm{int}(\alpha)}$ is an orientation-reversing homeomorphism of $\mathrm{int}(\alpha)$ which extends to a homeomorphism $\iota_\alpha\colon \alpha \to \alpha$. We define the barycenter $b(\alpha)$ of α to be the fixed point of ι_α. For each tile type s we define the barycenter $b(s)$ to be the image of $b(t)$ under a complete cellular isomorphism that takes t to s. For each edge e of a tile type s we define the barycenter $b(e)$ to be the image of $b(\alpha)$ under a complete cellular isomorphism from t to s which takes α to e. Now subdivide the tile type t by adding the barycenter $b(t)$ to t, for each edge e of t

adding the barycenter $b(e)$ of e, for each vertex v of t adding an edge from v to $b(t)$, and for each edge e of t adding an edge from $b(e)$ to $b(t)$. This gives a subdivision $\triangle(t)$ of t into triangles, and one can do this so that θ is a cellular map from $\triangle(t)$ to $\triangle(t)$. For every other tile type s we define a subdivision $\triangle(s)$ by taking the image of $\triangle(t)$ under a complete cellular isomorphism from t to s. This induces a subdivision of every tile of $\mathcal{R}(s)$, and so we obtain a subdivision $\triangle(\mathcal{R}(s))$ of $\mathcal{R}(s)$ by triangles. Figure 6 shows the subdivisions $\triangle(t)$ and $\triangle(\mathcal{R}(t))$ for a tile type of the pentagonal subdivision rule, and Figure 7 shows the same thing for a tile type of the twisted pentagonal subdivision rule.

Fix a vertex v of t. It is easy to see that there is an edgepath ρ in $\triangle(\mathcal{R}(t))$ that joins v to $b(t)$ and has $\rho \cap \partial t = \{v\}$. Let ρ_v be an edgepath in $\triangle(\mathcal{R}(t))$ of minimal length such that $\rho_v \cap \partial t = \{v\}$ and ρ_v joins v and $b(t)$. Let $e_1, \dots, e_m$ be the edges of ρ_v, and for $1 \le i \le m$ let v_{i-1} be the initial vertex of e_i and let v_i be the terminal vertex of e_i. Then $v_0 = v$ and $v_m = b(t)$. By minimality of m, $v_i \ne v_j$ if $i \ne j$ and hence ρ_v is a simple edgepath. We prove by contradiction that if $1 \le k < n(t)$ then $\rho_v \cap \theta^{\circ k}(\rho_v) = \{b(t)\}$. Suppose not. Then there are integers i, j, k such that $1 \le k < n(t)$, $0 \le i, j < m$, and $v_i = \theta^{\circ k}(v_j)$. Since $\rho_v \cap \partial t = \{v_0\}$ and $\theta^{\circ k}(\rho_v) \cap \partial t = \{\theta^{\circ k}(v_0)\}$, $i, j > 0$. Since $v_m = b(t)$ is the only fixed point of $\theta^{\circ k}$, $i \ne j$. If $i < j$, then $e_1, \dots, e_i, \theta^{\circ k}(e_{j+1}), \dots, \theta^{\circ k}(e_m)$ is a shorter edgepath from v to $b(t)$, contradicting minimality of m. If $i > j$, then $e_1, \dots, e_j, \theta^{\circ -k}(e_{i+1}), \dots, \theta^{\circ -k}(e_m)$ is a shorter edgepath from v to $b(t)$, contradicting minimality of m. Hence if $1 \le k < n(t)$ then $\theta^{\circ k}(\rho_v) \cap \rho_v = \{b(t)\}$.

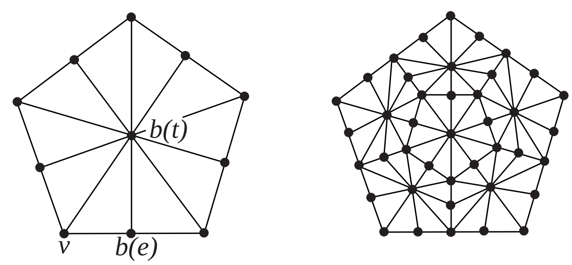


FIGURE 6. The pentagons subdivided into triangles for the pentagonal subdivision rule

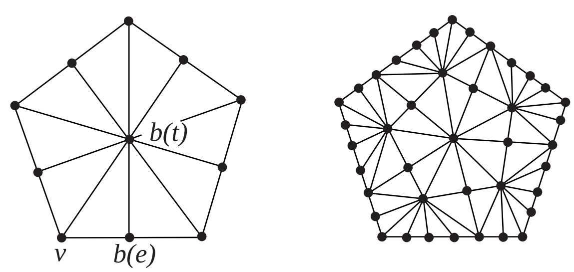


FIGURE 7. The pentagons subdivided into triangles for the twisted pentagonal subdivision rule

Let e be the edge of t with $\partial e = \{v, \theta(v)\}$. By the previous paragraph there exists an edgepath ρ_v in $\triangle(\mathcal{R}(t))$ such that ρ_v joins v and $b(t)$, $\rho_v \cap \partial t = \{v\}$, and

$\rho_v \cap \theta^{\circ k}(\rho(v)) = \{b(t)\}$ for $k \in \{1, \dots, n(t)-1\}$. We say that $\mathcal{R}$ is *amply triangulated* if for some such edgepath ρ_v there is an edgepath ρ_e in $\triangle(\mathcal{R}(t))$ from $b(e)$ to $b(t)$ such that $\rho_e \cap \partial t = b(e)$ and for each $k \in \{1, \dots, n(t)\}$, $\theta^{\circ k}(\rho_v) \cap \rho_e = \{b(t)\}$. The existence of such an edgepath ρ_e is independent of the choice of vertex v in t, but it does depend on the choice of the edgepath ρ_v. It is clear from Figure 8 that the pentagonal subdivision rule is amply triangulated but the twisted pentagonal subdivision rule is not amply triangulated.

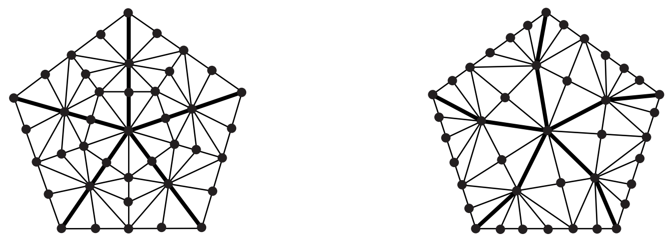


FIGURE 8. Choices of ρ_v and its cyclic images for a tile type of the pentagonal subdivision rule and for a tile type of the twisted pentagonal subdivision rule

We first suppose that $\mathcal{R}$ is amply triangulated, and let ρ_v and ρ_e be suitable edgepaths as above. Let e_1 be the edge in $\triangle(t)$ from $b(e)$ to v, let e_1' be the edge in $\triangle(t)$ from $b(e)$ to $\theta(v)$, let e_2 be the edge in $\triangle(t)$ from v to $b(t)$, let e_2' be $\theta(e_2)$, and let e_3 be the edge in $\triangle(t)$ from $b(t)$ to $b(e)$. Let t_1 be the triangle in $\triangle(t)$ whose boundary consists of e_1, e_2, and e_3, and let t_2 be the triangle in $\triangle(t)$ whose boundary consists of e_1', e_2', and e_3.

We will define a finite subdivision rule $\mathcal{Q}$. We define the complex $S_{\mathcal{Q}}$ to be the CW complex obtained from $t_1 \cup t_2$ by identifying e_2 with e_2' via θ and identifying the edge e_1 with e_1' by $\iota_e\big|_{e_1}$. The map $p\colon t_1 \cup t_2 \to S_{\mathcal{Q}}$ extends to a cellular map $p'\colon \triangle(t) \to S_{\mathcal{Q}}$ with $p' = p' \circ \theta$. Using the complete cellular isomorphisms between different tile types, for each tile type s we can define a map $p_s'\colon \triangle(s) \to S_{\mathcal{Q}}$. These maps descend to a map $p''\colon S_{\mathcal{R}} \to S_{\mathcal{Q}}$. Let $f\colon t \to t$ be a homeomorphism such that f is the identity on ∂t, $\theta \circ f = f \circ \theta$, $f(e_2) = \rho_v$, and $f(e_3) = \rho_e$. The cell structure on $\triangle(\mathcal{R}(t))$ pulls back under f to a subdivision of $\triangle(t)$, and this subdivision projects under p' to a subdivision $\mathcal{Q}(S_{\mathcal{Q}})$ of $S_{\mathcal{Q}}$. We next define $\sigma_{\mathcal{Q}}\colon \mathcal{Q}(S_{\mathcal{Q}}) \to S_{\mathcal{Q}}$. Let $x \in \mathcal{Q}(S_{\mathcal{Q}})$. Then there exists a point $y \in \triangle(t)$ with $p'(y) = x$. Define $\sigma_{\mathcal{Q}}(x) = p''(\sigma_{\mathcal{R}}(\psi_t(f(y))))$. One can check that $\sigma_{\mathcal{Q}}$ is well defined and that with this definition $\mathcal{Q}$ is a finite subdivision rule. Figure 9 shows the subdivision of the tile types for the finite subdivision rule $\mathcal{Q}$ associated with the pentagonal subdivision rule. Here we are labeling the edge types by e_1, e_2, and e_3 and are labeling each edge by its edge type. Since $\sigma_{\mathcal{Q}}$ is orientation preserving, one can tell the tile type of each 2-cell from the edge labels.

The subdivision rules $\mathcal{R}$ and $\mathcal{Q}$ are closely related. If X is an $\mathcal{R}$-complex, then it has a triangulation $\triangle(X)$ that is a $\mathcal{Q}$-complex. For any positive integer n, $\triangle(\mathcal{R}^n(X))$ is combinatorially isomorphic to $\mathcal{Q}^n(\triangle(X))$. Hence it follows from [6, Theorem 6.2.7] that $\mathcal{R}$ is conformal if and only if $\mathcal{Q}$ is conformal.

Now suppose that $\mathcal{R}$ is not amply triangulated. Since the mesh of $\mathcal{R}$ approaches 0, there is a positive integer m such that $\operatorname{star}(b(t), \mathcal{R}^m(t)) \cap \partial t = \emptyset$, $\cup\{\operatorname{star}\big(\theta^k(b(e)), \mathcal{R}^m(t)\big) : k = 1, \dots, n(t)\}$ does not separate v and $b(t)$, and there

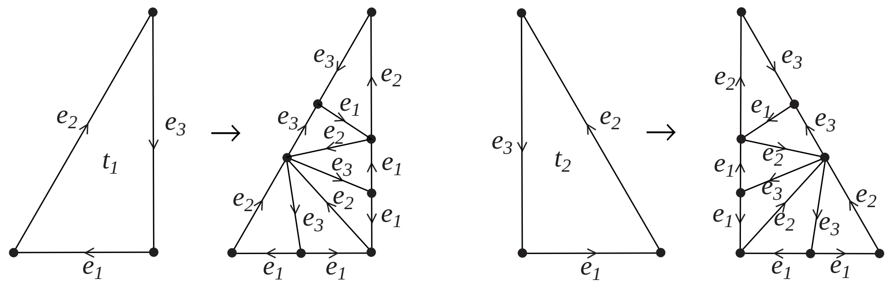


FIGURE 9. The subdivision of the tile types for the finite subdivision rule $\mathcal{Q}$ associated with the pentagonal subdivision rule

is an edgepath in $\mathcal{R}^m(t)$ from star $(b(e), \mathcal{R}^m(t))$ to star $(b(t), \mathcal{R}^m(t))$ which is disjoint from ∂t. Let ρ be a minimal edgepath from star $(b(e), \mathcal{R}^m(t))$ to star $(b(t), \mathcal{R}^m(t))$ which is disjoint from ∂t. By minimality, ρ is disjoint from its images under $\theta^{\circ k}$ for $1 \leq k < n(t)$. Note that if s is a tile of $\mathcal{R}^m(t)$ and w and w' are barycenters of distinct edges of s, then there is an edgepath in $\triangle(\mathcal{R}^m(t))$ which joins w and w' and whose interior is in the interior of s. Using this, one can show that there are edgepaths ρ_v and ρ_e in $\triangle(\mathcal{R}^m(t))$ such that ρ_v is an arc from v to $b(t)$, ρ_e is an arc from $b(e)$ to $b(t)$, $\rho_v \cap \theta^{\circ k}(\rho_v) = \{b(t)\}$ for $1 \leq k < n(t)$, $\rho_e \cap \theta^{\circ k}(\rho_e) = \{b(t)\}$ for $1 \leq k < n(t)$, and $\rho_e \cap \theta^{\circ k}(\rho_v) = \{b(t)\}$ for $0 \leq k < n(t)$.

Let $\mathcal{R}'$ be the finite subdivision rule defined by $S_{\mathcal{R}'} = S_{\mathcal{R}}$, $\mathcal{R}'(S_{\mathcal{R}'}) = \mathcal{R}^m(S_{\mathcal{R}})$, and $\sigma_{\mathcal{R}'} = \sigma_{\mathcal{R}}^{\circ m}$. Then $\mathcal{R}'$ is a one tile rotationally invariant finite subdivision rule. Furthermore, $\mathcal{R}'$ is amply triangulated. We let $\mathcal{Q}$ be the finite subdivision rule obtained from the above construction starting with $\mathcal{R}'$. It easily follows from [6, Theorem 6.2.7] that $\mathcal{R}$ is conformal if and only if $\mathcal{R}'$ is conformal, and hence that $\mathcal{R}$ is conformal if and only if $\mathcal{Q}$ is conformal.

Theorem 3.2. *Let $\mathcal{R}$ be a one tile rotationally invariant finite subdivision rule, and let $\mathcal{Q}$ be the finite subdivision rule obtained from $\mathcal{R}$ as described above. Then $\sigma_{\mathcal{Q}}$ can be realized by a rational map.*

Proof. By construction, $\sigma_{\mathcal{Q}}$ is orientation-preserving and has an edge pairing. Since $\mathcal{O}_{\mathcal{Q}}$ has at most three distinguished points, $\mathcal{T}(\mathcal{O}_{\mathcal{Q}})$ has a single point. Hence $\tau_{\mathcal{Q}}$ has a fixed point and by Theorem 2.1 $\sigma_{\mathcal{Q}}$ can be realized by a rational map. $\square$

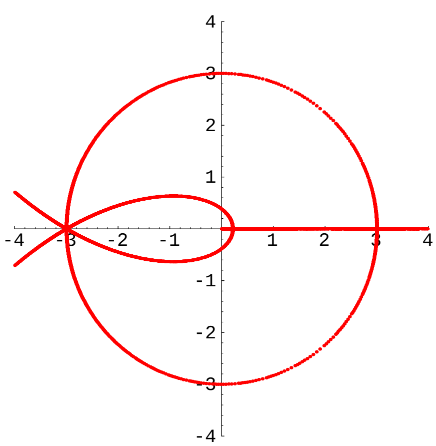
Theorem 3.2 also follows from [9, 10], where it is proved that $\mathcal{R}$ is conformal (and hence $\mathcal{Q}$ is conformal), and Theorem 3.1. In [17] D. Meyer shows how in certain cases a concrete construction of the corresponding rational map can be obtained.

4. Examples

In this section we give several examples of finite subdivision rules with edge pairings. For each such example $\mathcal{R}$, if $\sigma_{\mathcal{R}}$ can be realized by a rational map we explicitly construct a rational map realizing it. When we construct a rational map $f_{\mathcal{R}}$ realizing $\sigma_{\mathcal{R}}$, we also identify $S_{\mathcal{R}}$ with $\widehat{\mathbb{C}}$ so that for every positive integer n the cell complex $f_{\mathcal{R}}^{\circ -n}(S_{\mathcal{R}})$ subdivides $f_{\mathcal{R}}^{\circ -(n-1)}(S_{\mathcal{R}})$ and $f_{\mathcal{R}}^{\circ -n}(S_{\mathcal{R}})$ is combinatorially equivalent to $\mathcal{R}^n(S_{\mathcal{R}})$ in a way which respects $f_{\mathcal{R}}$ and $\sigma_{\mathcal{R}}$. If $\sigma_{\mathcal{R}}$ is topologically conjugate to $f_{\mathcal{R}}$, then it is possible to identify $S_{\mathcal{R}}$ with $\widehat{\mathbb{C}}$ so that we may assume the much stronger condition that $\sigma_{\mathcal{R}} = f_{\mathcal{R}}$.

Example 4.1. We first consider the finite subdivision rule $\mathcal{R}$ that was shown in Example 1.2. The subdivision complex $S_{\mathcal{R}}$ and its subdivision $\mathcal{R}(S_{\mathcal{R}})$ are shown in Figure 1, the single tile type is shown in Figure 2, and the first three subdivisions of the tile type are shown in Figure 3.

The branching data are as follows: $0 \mapsto 0$ with degree 1, $1 \mapsto 0$ with degree 2, $\alpha \mapsto 0$ with degree 2, $\infty \mapsto 0$ with degree 1, for each i $a_i \mapsto \alpha$ with degree 1, $b \mapsto \infty$ with degree 6, and for each j $c_j \mapsto 1$ with degree 2. We can easily see in two ways that $\sigma_{\mathcal{R}}$ can be realized by a rational map $f_{\mathcal{R}}$. It follows from the branching data that $\mathcal{O}_{\mathcal{R}}$ is the orbifold $(2,6,12)$. Hence $\mathcal{T}(\mathcal{O}_{\mathcal{R}})$ is a single point and by Theorem 2.1 $\sigma_{\mathcal{R}}$ can be realized by a rational map. Alternatively, since $\mathcal{R}$ has dihedral symmetry it follows from [8, Theorem 6.4] that $\mathcal{R}$ is conformal and hence by Theorem 3.1 that $\sigma_{\mathcal{R}}$ can be realized by a rational map. If follows from the branching data that $f_{\mathcal{R}} = \frac{kz(z-1)^2(z-\alpha)^2}{(z-b)^6}$ for some constants k, α, and b. Let $p(z)$ be the numerator of $\left(\frac{f'_{\mathcal{R}}(z)}{f_{\mathcal{R}}(z)}\right)^2$, and let $q(z)$ be the numerator of $f_{\mathcal{R}}(z)-1$. Then p and q are polynomials of degree 6 that have c_1, c_2, and c_3 as zeroes of degree 2. Using Mathematica[1], one can solve for k, α, and b to get the solution $f_{\mathcal{R}} = \frac{108z(z-1)^2(z-9)^2}{(z+3)^6}$. Furthermore, $c_1 = 3(7-4\sqrt{3})$, $c_2 = 3$, and $c_3 = 3(7+4\sqrt{3})$. One can choose $e_1 = [0,1]$, $e_2 = [1,9]$, and $e_3 = [9,\infty]$. (Here, for $x \in \mathbb{R}$, we are denoting $[x,\infty) \cup \{\infty\}$ by $[x,\infty]$.) It easily follows that $f_{\mathcal{R}}^{\circ -1}([0,\infty])$ is combinatorially equivalent to the 1-skeleton of $\mathcal{R}(S_{\mathcal{R}})$. It follows inductively that for each positive integer n, $f_{\mathcal{R}}^{\circ -n}([0,\infty])$ is combinatorially equivalent to the 1-skeleton of $\mathcal{R}^n(S_{\mathcal{R}})$, the n^{th} subdivision of $S_{\mathcal{R}}$. Figure 10 shows Mathematica approximations of the intersection of the rectangle $[-4,4] \times [-4,4]$ with $f_{\mathcal{R}}^{\circ -1}([0,\infty])$ and $f_{\mathcal{R}}^{\circ -2}([0,\infty])$. Since $\mathcal{R}$ has bounded valence, the Julia set of $f_{\mathcal{R}}$ is the 2-sphere.



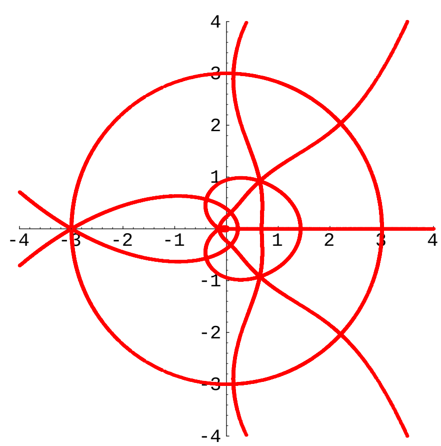


FIGURE 10. $f_{\mathcal{R}}^{\circ -1}([0,\infty])$ and $f_{\mathcal{R}}^{\circ -2}([0,\infty])$

Example 4.2. We next consider a variant $\mathcal{S}$ of the binary square subdivision rule with a single tile type t and two edge types. The subdivision of the tile type is shown

[1]A computer software system available from Wolfram Research, Inc., 100 Trade Center Drive, Champaign, IL 61820, USA.

in Figure 11. This is an orientation-preserving finite subdivision rule with mesh approaching 0, bounded valence, and an edge pairing. Each edge of the tile type t and of $\mathcal{S}(t)$ is labeled by its edge type and oriented so that $\sigma_{\mathcal{S}}$ preserves the induced orientations of the edges of $S_{\mathcal{S}}$ and $\mathcal{S}(S_{\mathcal{S}})$. Since $\mathcal{S}$ is orientation preserving, the edge labels and orientations determine $\sigma_{\mathcal{S}}$ up to a cellular isomorphism of $\mathcal{S}(S_{\mathcal{S}})$ fixing the vertices.

The surface $S_{\mathcal{S}}$ is topologically a 2-sphere. We can assume without loss of generality that $S_{\mathcal{S}}$ is $\widehat{\mathbb{C}}$ and that its vertices are 1, 0, and ∞. The branched covering $\sigma_{\mathcal{S}}\colon S_{\mathcal{S}} \to S_{\mathcal{S}}$ is shown in Figure 12. It is cellular as a map from $\mathcal{S}(S_{\mathcal{S}})$ to $S_{\mathcal{S}}$ and preserves the labels and orientations of the edges. The branching data are as follows: $0 \mapsto 0$ with degree 1, $1 \mapsto 0$ with degree 2, $\infty \mapsto 0$ with degree 1, $a \mapsto \infty$ with degree 4, $b \mapsto 1$ with degree 2, and $c \mapsto 1$ with degree 2. It follows that $\mathcal{O}_{\mathcal{S}}$ is the orbifold $(2,4,4)$. Since $\mathcal{T}(\mathcal{O}_{\mathcal{S}})$ is a single point, we know from Theorem 2.1 that $\sigma_{\mathcal{S}}$ can be realized by a rational map $f_{\mathcal{S}}$. (Since $\mathcal{S}$ has dihedral symmetry, this also follows from [8, Theorem 6.4] and Theorem 3.1.)

By the branching data, $f_{\mathcal{S}}(z) = \frac{kz(z-1)^2}{(z-a)^4}$ for some constants $a, k \in \mathbb{C}$ with $a \notin \{0,1\}$ and $k \neq 0$. Let $p(z)$ be the numerator of $\left(\frac{f'_{\mathcal{S}}(z)}{f_{\mathcal{S}}(z)}\right)^2$, and let $q(z)$ be the numerator of $f_{\mathcal{S}}(z)-1$. Then p and q are each polynomials of degree 4 which have b and c as zeroes of order 2. Since $p(0)=a^2$ and $q(0)=-a^4$, $g(z)=a^2p(z)+q(z)$ is the zero function. A straightforward computation shows that

$$\begin{aligned} g(z) = &\left(k+10a^3-6a^4\right)z + \left(-2k+3a^2-20a^3+9a^4\right)z^2 \\ &+ \left(k+4a-6a^2+6a^3\right)z^3 + \left(a^2-1\right)z^4. \end{aligned}$$

Since g is the zero function, $a^2-1=0$ and so $a=-1$. This implies that $k=16$ and

$$f_{\mathcal{S}}(z) = \frac{16z(z-1)^2}{(z+1)^4}.$$

Since $\mathcal{S}$ has bounded valence, the Julia set of $f_{\mathcal{S}}$ is the 2-sphere. One can easily see that $f_{\mathcal{S}}$ maps each of the intervals $[0, 3-2\sqrt{2}]$, $[3-2\sqrt{2}, 1]$, $[1, 3+2\sqrt{2}]$, and $[3+2\sqrt{2}, \infty]$ bijectively onto the interval $[0,1]$. This leads us to choose $e_1 = [1,\infty]$ and $e_2 = [0,1]$. As for the previous example, one can show that for each positive integer n, $f_{\mathcal{S}}^{\circ -n}([0,\infty])$ is combinatorially equivalent to the 1-skeleton of $\mathcal{S}^n(S_{\mathcal{S}})$, the n^{th} subdivision of $S_{\mathcal{S}}$. Figure 13 shows Mathematica approximations of the intersection of the rectangle $[-3,7]\times[-5,5]$ with $f_{\mathcal{S}}^{\circ -1}([0,\infty])$ and $f_{\mathcal{S}}^{\circ -2}([0,\infty])$.

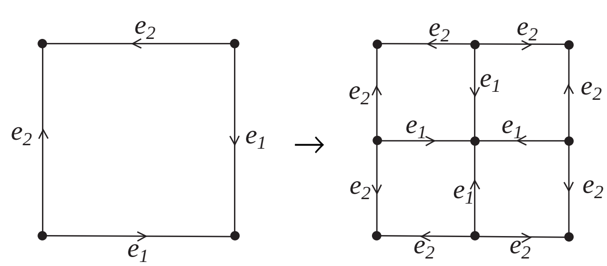


FIGURE 11. The subdivision of the tile type for the binary square subdivision rule $\mathcal{S}$ with one tile type

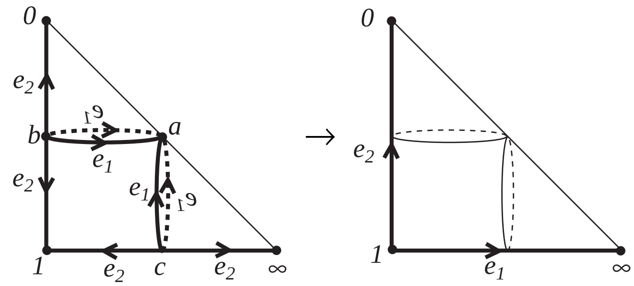


FIGURE 12. The branched covering $\sigma_{\mathcal{S}}$

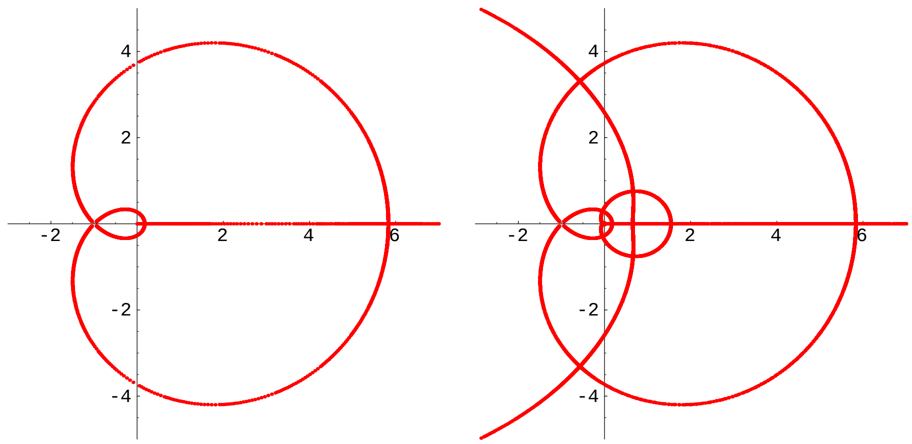


FIGURE 13. $f_{\mathcal{S}}^{\circ -1}([0,\infty])$ and $f_{\mathcal{S}}^{\circ -2}([0,\infty])$

Example 4.3. We next consider the binary square subdivision rule $\mathcal{L}$ with two tile types, as shown in Figure 14. There are two tile types (t_1 and t_2) and four edge types (e_1, e_2, e_3, and e_4). Since the edge and tile labels on $\mathcal{L}(S_{\mathcal{L}})$ determine the map $\sigma_{\mathcal{L}}$ up to a cellular isomorphism fixing the vertices, we have also labeled the edges and tiles of $\mathcal{L}(t_1)$ and $\mathcal{L}(t_2)$ to indicate the map $\sigma_{\mathcal{L}}$. This finite subdivision rule is orientation preserving and has an edge pairing.

$S_{\mathcal{L}}$ is a 2-sphere obtained by gluing t_1 and t_2 by a homeomorphism of their boundaries which preserves the labels and the orientations of the edges (i.e., $S_{\mathcal{L}}$ is a rectangular pillowcase). We can topologically identify $S_{\mathcal{L}}$ with $\widehat{\mathbb{C}}$; we do this in such a way that 0, 1, and ∞ are vertices of the cell structure on $S_{\mathcal{L}}$. Figure 15 indicates the map $\sigma_{\mathcal{L}}\colon S_{\mathcal{L}} \to S_{\mathcal{L}}$; it is cellular as a map from $\mathcal{L}(S_{\mathcal{L}})$ to $S_{\mathcal{L}}$ and preserves labels of edges and tiles. The branching data are as follows: $0 \mapsto \infty$ with degree 1, $1 \mapsto \infty$ with degree 1, $\infty \mapsto \infty$ with degree 1, $\alpha \mapsto \infty$ with degree 1, $a \mapsto 1$ with degree 2, $b \mapsto \alpha$ with degree 2, $c \mapsto 1$ with degree 2, $d \mapsto \alpha$ with degree 2, $g \mapsto 0$ with degree 2, and $h \mapsto 0$ with degree 2. $\mathcal{O}_{\mathcal{L}}$ is the orbifold $(2,2,2,2)$. Since $A_{\sigma_{\mathcal{L}}} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, by Theorem 2.3 $\sigma_{\mathcal{L}}$ can be realized by a rational map $f_{\mathcal{L}}$. (Once again, this also follows from [8, Theorem 6.4] and Theorem 3.1.) Again by the branching data, $f_{\mathcal{L}}(z) = \frac{k(z-g)^2(z-h)^2}{z(z-1)(z-\alpha)}$ for some constants k, g, h,

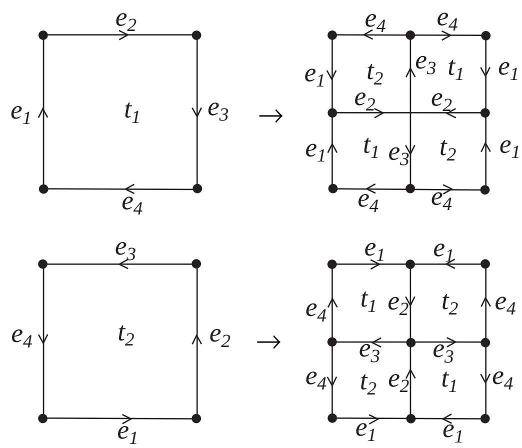


FIGURE 14. The subdivisions of the tile types for the binary square subdivision rule $\mathcal{L}$ with two tile types

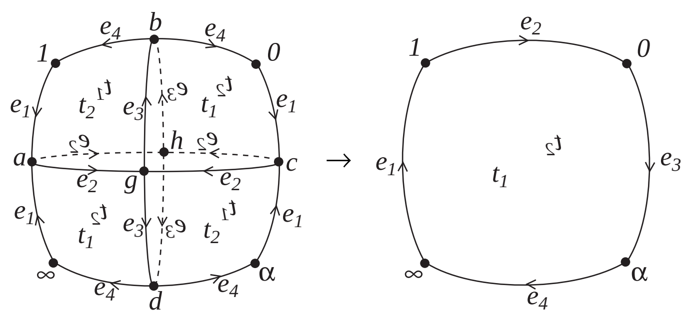


FIGURE 15. The branched covering $\sigma_{\mathcal{L}}$

and α. Let $p(z)$ be the numerator of $\left(\frac{f'_{\mathcal{L}}(z)}{f_{\mathcal{L}}(z)}\right)^2$, and let $q(z)$ be the numerator of $(f_{\mathcal{L}}(z)-1)(f_{\mathcal{L}}(z)-\alpha)$. Then p and q are each polynomials of degree 8 which have a, b, c, and d as zeroes of order two.

We will not try to characterize all of the rational functions that realize this branching data. One can get an infinite family of solutions as follows: given $\alpha \in \mathbb{C} \setminus \{0,1\}$, let

$$f_{\mathcal{L},\alpha}(z) = \frac{(z^2-\alpha)^2}{4z(z-1)(z-\alpha)}.$$

When $\alpha = -1$, this is Lattès's example [14] from 1918 of a rational map whose Julia set is the 2-sphere. Since $\mathcal{L}$ has bounded valence, for each choice of α the Julia set is the 2-sphere.

In the Lattès example with $\alpha = -1$, we may take the 1-skeleton of $S_{\mathcal{L}}$ to be $\mathbb{R} \cup \{\infty\}$. One can verify that for each positive integer n, $f_{\mathcal{L},-1}^{\circ -n}(\mathbb{R} \cup \{\infty\})$ is combinatorially equivalent to the 1-skeleton of $\mathcal{L}^n(S_{\mathcal{L}})$. Figure 16 shows Mathematica approximations of the intersection of the rectangle $[-3,3] \times [-3,3]$ with $f_{\mathcal{L},-1}^{\circ -1}(\mathbb{R} \cup \{\infty\})$ and $f_{\mathcal{L},-1}^{\circ -2}(\mathbb{R} \cup \{\infty\})$.

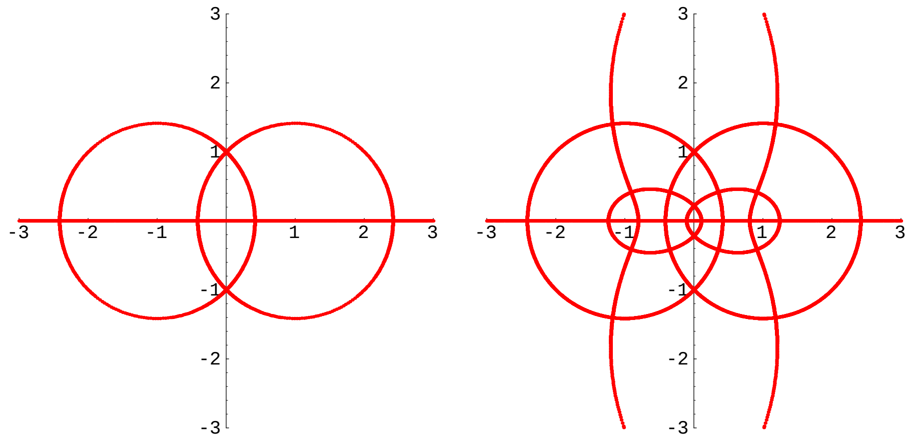


FIGURE 16. $f_{\mathcal{L},-1}^{\circ -1}(\mathbb{R} \cup \{\infty\})$ and $f_{\mathcal{L},-1}^{\circ -2}(\mathbb{R} \cup \{\infty\})$ for the Lattès example

Example 4.4. We next consider the pentagonal subdivision rule $\mathcal{P}$, which is a one tile dihedrally invariant finite subdivision rule with two tile types and one edge type. Each tile type is a pentagon that is subdivided into six pentagons, and the edge type is subdivided into two subedges. The subdivisions of the tile types are shown in Figure 4. The subdivision complex $S_{\mathcal{P}}$ is obtained from $t_1 \cup t_2$ by identifying all of the edges together preserving orientations. It is easy to see that $\mathcal{P}$ does not have an edge pairing.

One can identify t_1 with the central pentagon of $\mathcal{P}(t_1)$, and hence can identify $\mathcal{P}^i(t_1)$ with a subcomplex of $\mathcal{P}^j(t_1)$ if $i < j$. Figure 17 shows $\mathcal{P}(t_1)$, $\mathcal{P}^2(t_1)$, and $\mathcal{P}^3(t_1)$; all three figures were drawn using Stephenson's program CirclePack [21]. We denote by $E_{\mathcal{P}}$ the direct limit of the $\mathcal{P}^i(t_1)$'s and call $E_{\mathcal{P}}$ the *expansion complex* (see [9] for details about expansion complexes). There is an expansion map $\varphi\colon E_{\mathcal{P}} \to E_{\mathcal{P}}$ such that $\mathcal{P}^{i+1}(t_1)$ is the $\mathcal{P}$-subdivision of $\varphi(\mathcal{P}^i(t_1))$ for each $i \geq 0$. This complex was studied by Bowers and Stephenson in [1], where they constructed a conformal structure on $E_{\mathcal{P}}$ in which all of the pentagons are conformally regular. They then showed that $E_{\mathcal{P}}$ is conformally equivalent to $\mathbb{C}$, and that under this equivalence the expansion map φ corresponds to a dilation $z \mapsto \lambda z$ of $\mathbb{C}$. Using circle packing methods, they gave the estimate $|\lambda| \approx 3.2$.

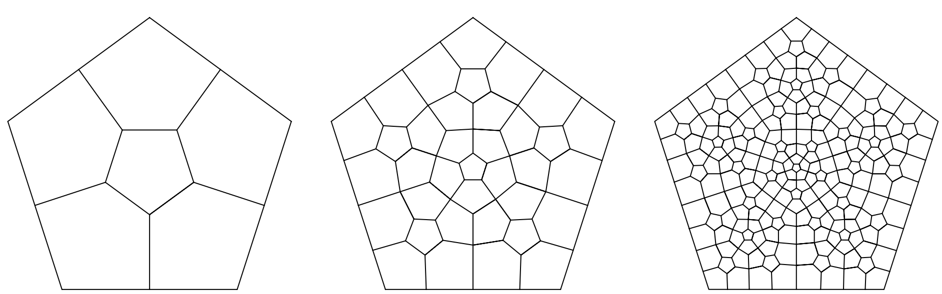


FIGURE 17. $\mathcal{P}(t_1)$, $\mathcal{P}^2(t_1)$, and $\mathcal{P}^3(t_1)$

Although the pentagonal subdivision rule does not have an edge pairing, as we saw in Section 3 we can construct from it an orientation-preserving finite subdivision rule $\mathcal{Q}$ with two tile types which does have an edge pairing. The subdivision of the tile types of $\mathcal{Q}$ is shown in Figure 9. We can assume without loss of generality that $S_{\mathcal{Q}}$ is $\widehat{\mathbb{C}}$ and its vertices are 0, 1, and ∞. By Theorem 3.2 or Theorem 3.1, $\sigma_{\mathcal{Q}}$ can be realized by a rational map.

We can subdivide the expansion complex $E_{\mathcal{P}}$ to get a $\mathcal{Q}$-complex $E_{\mathcal{Q}}$ as follows. Since each pentagon in $E_{\mathcal{P}}$ is conformally regular, it has a unique conformal center and is invariant under a conformal action of the dihedral group of order ten. Each pentagon in $E_{\mathcal{P}}$ can be uniquely subdivided, as in the left-hand part of Figure 6, into ten triangles, each of which is equivalent to its adjacent triangles under anticonformal reflections. Doing this for each pentagon gives a triangulation $E_{\mathcal{Q}}$ of $E_{\mathcal{P}}$. Furthermore, one can define a map $f\colon E_{\mathcal{Q}} \to S_{\mathcal{Q}}$ which turns $E_{\mathcal{Q}}$ into an expansion $\mathcal{Q}$-complex with expansion map $z \mapsto \lambda z$.

Since the tiling $E_{\mathcal{Q}}$ is invariant under the action of $\mathbb{Z}_5$ by rotations fixing the origin, its image under the map $z \mapsto z^5$ gives a tiling E of $\mathbb{C}$ by triangles such that each triangle can be mapped to any adjacent triangle by an anticonformal reflection. E is an expansion $\mathcal{Q}$-complex with expansion map $z \mapsto \lambda^5 z$.

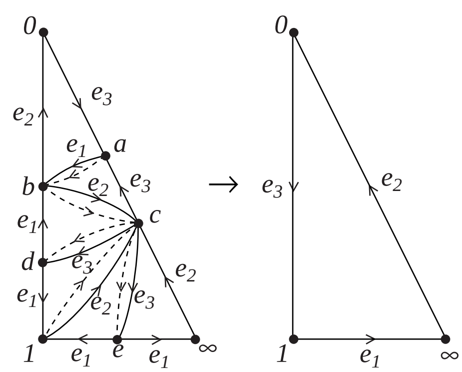


FIGURE 18. The branched cover $\sigma_{\mathcal{Q}}$

The branched cover $\sigma_{\mathcal{Q}}\colon S_{\mathcal{Q}} \to S_{\mathcal{Q}}$ is shown in Figure 18. The branching data are as follows: $0 \mapsto 0$ with degree 1, $b \mapsto \infty$ with degree 3, $c \mapsto 0$ with degree 5, $1 \mapsto \infty$ with degree 2, $\infty \mapsto \infty$ with degree 1, $a \mapsto 1$ with degree 2, $d \mapsto 1$ with degree 2, and $e \mapsto 1$ with degree 2. It easily follows that $\mathcal{O}_{\mathcal{Q}}$ is the orbifold $(5, 2, 12)$ and that the rational function is $f_{\mathcal{Q}}(z) = \dfrac{kz(z-c)^5}{(z-b)^3(z-1)^2}$ for some constants k, b, and c. Since there are three double roots of $f_{\mathcal{Q}}(z) - 1$, the numerator $p(z)$ of $\left(\frac{f'_{\mathcal{Q}}(z)}{f_{\mathcal{Q}}(z)}\right)^2$ and the numerator $q(z)$ of $f_{\mathcal{Q}}(z) - 1$ are multiples of each other. Using Mathematica, one can solve this to get $k = 2/27$, $b = 3/128$, and $c = -9/16$, which gives

$$f_{\mathcal{Q}}(z) = \frac{2z(z+9/16)^5}{27(z-3/128)^3(z-1)^2}.$$

Simply by viewing $f_{\mathcal{Q}}$ as a function of a real variable and considering its behavior at zeros and poles, it is easy to see that $f_{\mathcal{Q}}$ has a local maximum between c and

0, a local minimum between b and 1, and a local minimum greater than 1. These local extrema must be a, d, and e. We take a to be the local maximum between c and 0, we take d to be the local minimum between b and 1, and we take e to be the local minimum greater than 1. We take $\mathbb{R}\cup\{\infty\}$ to be the 1-skeleton of $S_{\mathcal{Q}}$. It is now a routine matter to verify that $f_{\mathcal{Q}}^{\circ -1}(S_{\mathcal{Q}})$ is combinatorially equivalent to $\mathcal{Q}(S_{\mathcal{Q}})$ in a way which respects $f_{\mathcal{Q}}$ and $\sigma_{\mathcal{Q}}$. It follows by induction that $f_{\mathcal{Q}}^{\circ -n}(S_{\mathcal{Q}})$ is combinatorially equivalent to $\mathcal{Q}^n(S_{\mathcal{Q}})$ in a way which respects $f_{\mathcal{Q}}$ and $\sigma_{\mathcal{Q}}$ for every positive integer n. Since the two triangles of $S_{\mathcal{Q}}$ are complex conjugates of each other, it follows that each triangle of $f_{\mathcal{Q}}^{\circ -n}(S_{\mathcal{Q}})$ can be mapped to any adjacent triangle by an anticonformal reflection for every positive integer n.

Let $\widetilde{\mathcal{O}}_{\mathcal{Q}}$ be the universal covering orbifold of $\mathcal{O}_{\mathcal{Q}}$, and suppose that $p \in \widetilde{\mathcal{O}}_{\mathcal{Q}}$ maps to $0 \in \mathcal{O}_{\mathcal{Q}}$. The multivalued function inverse to $f_{\mathcal{Q}}$ lifts to a function $\widetilde{f}_{\mathcal{Q}}^{\circ -1}\colon \widetilde{\mathcal{O}}_{\mathcal{Q}} \to \widetilde{\mathcal{O}}_{\mathcal{Q}}$ which fixes p. The orbifold $\widetilde{\mathcal{O}}_{\mathcal{Q}}$ is hyperbolic, and $\widetilde{f}_{\mathcal{Q}}^{\circ -1}$ strictly decreases hyperbolic distances. The $\mathcal{Q}$-complex structure on $\mathcal{O}_{\mathcal{Q}}$ lifts to a $\mathcal{Q}$-complex structure on $\widetilde{\mathcal{O}}_{\mathcal{Q}}$. For every positive integer n the complex $\widetilde{f}_{\mathcal{Q}}^{\circ -n}(\widetilde{\mathcal{O}}_{\mathcal{Q}})$ is combinatorially equivalent to $\mathcal{Q}^n(\widetilde{\mathcal{O}}_{\mathcal{Q}})$, and the diameters of the cells of $\widetilde{f}_{\mathcal{Q}}^{\circ -n}(\widetilde{\mathcal{O}}_{\mathcal{Q}})$ converge to 0 uniformly with respect to n. This yields the following conclusion. Given any finite subcomplex C of E and any neighborhood U of 0 in $S_{\mathcal{Q}}$ there is a positive integer n such that U contains a subcomplex of $f_{\mathcal{Q}}^{\circ -n}(S_{\mathcal{Q}})$ that is cellularly isomorphic to C. Furthermore, using the Koenigs linearization theorem (see, for example, [18, Theorem 8.2]) about the repelling fixed point 0 of $f_{\mathcal{Q}}$, one can define an expansion $\mathcal{Q}$-complex E' which is combinatorially equivalent to E and has expansion map $z \mapsto f'_{\mathcal{Q}}(0)z$. Since each triangle of E' can be mapped to any adjacent triangle by an anticonformal reflection and the same is true of E, it follows that E and E' are also conformally equivalent, and so $\lambda^5 = f'_{\mathcal{Q}}(0)$. Hence the expansion constant for the pentagonal subdivision rule is $\left(f'_{\mathcal{Q}}(0)\right)^{1/5} = (-324)^{1/5}$; its modulus is approximately 3.178.

Since one can take $\mathbb{R}\cup\{\infty\}$ to be the 1-skeleton of $S_{\mathcal{Q}}$, for each positive integer n $f_{\mathcal{Q}}^{\circ -n}(\mathbb{R}\cup\{\infty\})$ is combinatorially equivalent to the 1-skeleton of $\mathcal{Q}^n(S_{\mathcal{Q}})$. Since the edges of $\mathcal{Q}^n(S_{\mathcal{Q}})$ that correspond to the pentagonal subdivision rule $\mathcal{P}$ are those labeled by e_1, for each positive integer n $f_{\mathcal{Q}}^{\circ -n}([1,\infty])$ gives the edges of $f_{\mathcal{Q}}^{\circ -n}(S_{\mathcal{Q}})$ that correspond to pentagonal edges. One can get a glimpse of the above construction by looking at the inverse images of these pentagonal edges under the map $z \mapsto z^5$. Figure 19 shows Mathematica approximations of the intersection of the rectangle $[-0.5,0.5]\times[-0.5,0.5]$ with the inverse images under $z \mapsto z^5$ of $f_{\mathcal{Q}}^{\circ -1}([1,\infty])$, $f_{\mathcal{Q}}^{\circ -2}([1,\infty])$, and $f_{\mathcal{Q}}^{\circ -3}([1,\infty])$.

Example 4.5. In this example, the finite subdivision rule $\mathcal{H}$ has a single tile type t and three edge types, e_1, e_2, and e_3. The tile type t is a hexagon, and it is subdivided in $\mathcal{H}(t)$ into seven tiles. See Figure 20. Since the edge labels on $\mathcal{H}(S_{\mathcal{H}})$ determine the map $\sigma_{\mathcal{H}}$ up to a cellular isomorphism fixing the vertices, we have also labeled the edges of $\mathcal{H}(t)$ to indicate the map $\sigma_{\mathcal{H}}$. Figure 20 shows the edge labels for the tile type t and for the edges in $\mathcal{H}(t)$. The subdivision rule $\mathcal{H}$ is orientation preserving and has an edge pairing.

The surface $S_{\mathcal{H}}$ is a 2-sphere. The branched map $\sigma_{\mathcal{H}}$ is indicated in Figure 21; it is cellular as a map from $\mathcal{H}(S_{\mathcal{H}})$ to $S_{\mathcal{H}}$ and preserves edge labels and orientations. $\mathcal{O}_{\mathcal{H}}$ is the orbifold $(2,2,2,6)$. Since $\mathcal{T}(\mathcal{O}_{\mathcal{H}})$ is not just a point, one cannot conclude trivially from Theorem 2.1 that $\sigma_{\mathcal{H}}$ can be realized by a rational map. Figure 22

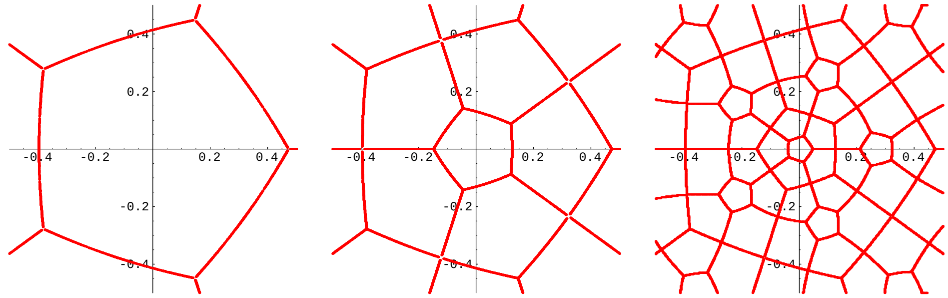


FIGURE 19. The inverse images under $z \mapsto z^5$ of $f_{\mathcal{Q}}^{\circ -1}([1,\infty])$, $f_{\mathcal{Q}}^{\circ -2}([1,\infty])$ and $f_{\mathcal{Q}}^{\circ -3}([1,\infty])$

shows a $\sigma_{\mathcal{H}}$-stable curve system $\Gamma = \{\gamma\}$ and its inverse image $\{\gamma_1, \gamma_2, \gamma_3\}$. To help the reader, edges of $\mathcal{H}(S_{\mathcal{H}})$ which are not edges of $S_{\mathcal{H}}$ are shown as dashed arcs and vertices of $\mathcal{H}(S_{\mathcal{H}})$ which are not vertices of $S_{\mathcal{H}}$ are shown as hollow dots. Note that $A^\Gamma = \left(\frac{1}{3} + \frac{1}{2} + \frac{1}{2}\right) = \left(\frac{4}{3}\right)$. Hence by Theorem 2.2 the branched map $\sigma_{\mathcal{H}}$ can not be realized by a rational map.

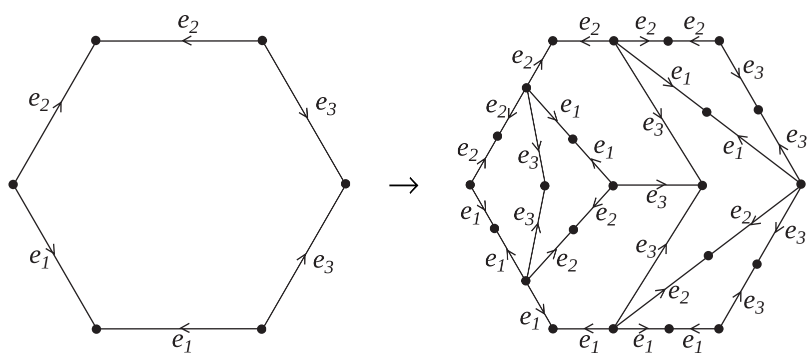


FIGURE 20. The subdivision of the tile type for the subdivision rule $\mathcal{H}$

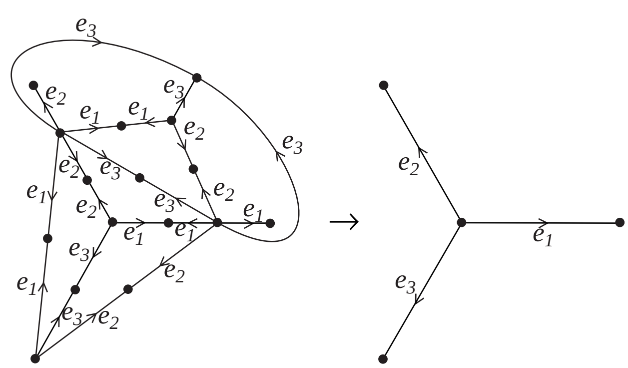


FIGURE 21. The branched covering $\sigma_{\mathcal{H}}$

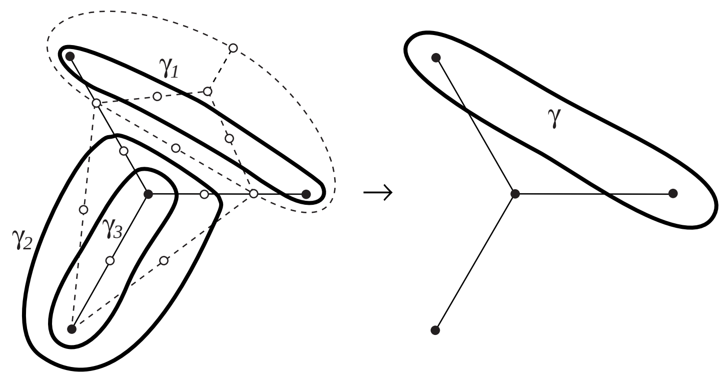


FIGURE 22. A $\sigma_{\mathcal{H}}$-stable curve system and its inverse image

Example 4.6. For a final example, consider the barycentric subdivision rule $\mathcal{B}$, as shown in Figure 23. This finite subdivision rule is orientation preserving and has an edge pairing.

$S_{\mathcal{B}}$ is a 2-sphere with exactly two tiles (it is a triangular pillowcase). We can assume without loss of generality that $S_{\mathcal{B}} = \widehat{\mathbb{C}}$ and that the vertices of $S_{\mathcal{B}}$ are 0, 1, and ∞. The map $\sigma_{\mathcal{B}} \colon S_{\mathcal{B}} \to S_{\mathcal{B}}$ is shown in Figure 24. The branching data are as follows: $0 \mapsto \infty$ with degree 2, $1 \mapsto \infty$ with degree 2, $\infty \mapsto \infty$ with degree 2, $a \mapsto 0$ with degree 3, $b \mapsto 0$ with degree 3, $c \mapsto 1$ with degree 2, $d \mapsto 1$ with degree 2, and $e \mapsto 1$ with degree 2. The associated orbifold $\mathcal{O}_{\mathcal{B}}$ is the orbifold $(2, 3, \infty)$. Since its Teichmüller space is a single point, it follows from Theorem 2.1 that $\sigma_{\mathcal{B}}$ can be realized by a rational map $f_{\mathcal{B}}$. From the branching data, $f_{\mathcal{B}}(z) = \frac{k(z-a)^3(z-b)^3}{z^2(z-1)^2}$ for some constants k, a, and b. Using the same overall strategy that was used before, one can solve for the constants and get that

$$f_{\mathcal{B}}(z) = \frac{4(z^2 - z + 1)^3}{27z^2(z-1)^2}.$$

Since there is a periodic critical point, the Julia set of $f_{\mathcal{B}}$ is not the entire 2-sphere. A Mathematica approximation of the Julia set, produced by plotting preimages of a repelling fixed point, is shown in Figure 25. We can take $\mathbb{R} \cup \{\infty\}$ to be the 1-skeleton of $S_{\mathcal{B}}$. One can verify that for each positive integer n $f_{\mathcal{B}}^{\circ -n}(\mathbb{R} \cup \{\infty\})$ is combinatorially equivalent to the 1-skeleton of $\mathcal{B}^n(S_{\mathcal{B}})$, the n^{th} barycentric subdivision of $S_{\mathcal{B}}$. Figure 26 shows Mathematica approximations of the intersections of the rectangle $[-1.5, 2.5] \times [-2, 2]$ with $f_{\mathcal{B}}^{\circ -1}(\mathbb{R} \cup \{\infty\})$ and with $f_{\mathcal{B}}^{\circ -2}(\mathbb{R} \cup \{\infty\})$. This example seems especially interesting because of the modular equation (see, for example, [12, Section 69] or [15, Section 1.1E])

$$J = \frac{4(\lambda^2 - \lambda + 1)^3}{27\lambda^2(\lambda - 1)^2}$$

relating the J-invariant and the modular function λ.

The appearance of J and λ can be explained as follows. We use the upper half complex plane H as our model for the hyperbolic plane. The function J is the unique conformal function which bijectively maps the hyperbolic triangle T with vertices $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, i, and ∞ to H with $J(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 0$, $J(i) = 1$, and $J(\infty) = \infty$. The triangle T is a fundamental domain for the action of $PGL(2, \mathbb{Z})$

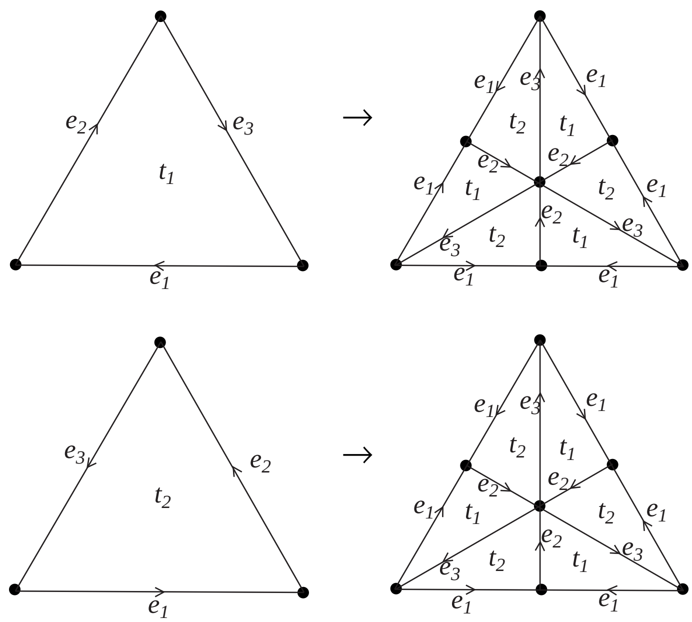


FIGURE 23. The subdivisions of the tile types for the barycentric subdivision rule $\mathcal{B}$

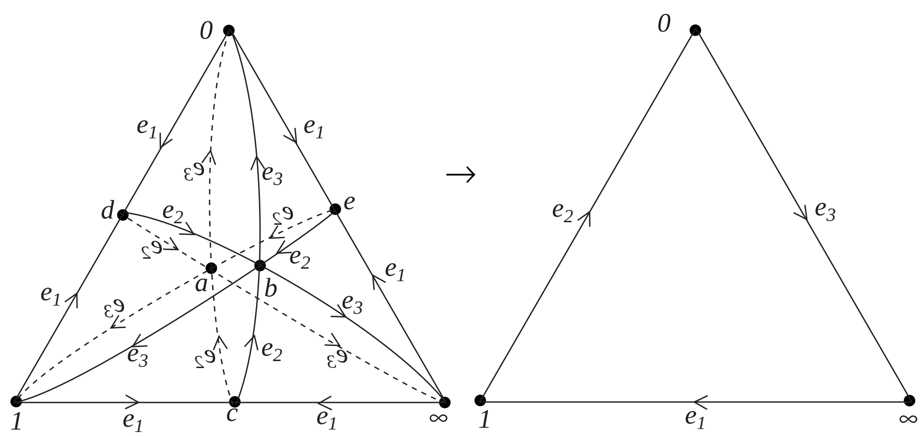


FIGURE 24. The branched covering $\sigma_{\mathcal{B}}$

(the (2,3,∞)-Coxeter group) on H. The figure on page 5 of [15] shows part of the corresponding tesselation of H. The domain of definition of J can be extended to H using this tesselation and the reflection principle. The function J is then a branched covering from H to $\mathbb{C}$ with $\deg_i(J) = 2$ and $\deg_{-\frac{1}{2}+\frac{\sqrt{3}}{2}i}(J) = 3$. In other words, we may identify $\mathcal{O}_{\mathcal{B}}$ with $\mathbb{C}$ and we may identify the universal cover of $\mathcal{O}_{\mathcal{B}}$ with H so that J is the universal covering. The group of covering transformations is $PSL(2,\mathbb{Z})$. The function λ is the unique conformal function which bijectively maps the hyperbolic triangle with vertices ∞, 0, and 1 to H with $\lambda(\infty) = 0$, $\lambda(0) = 1$,

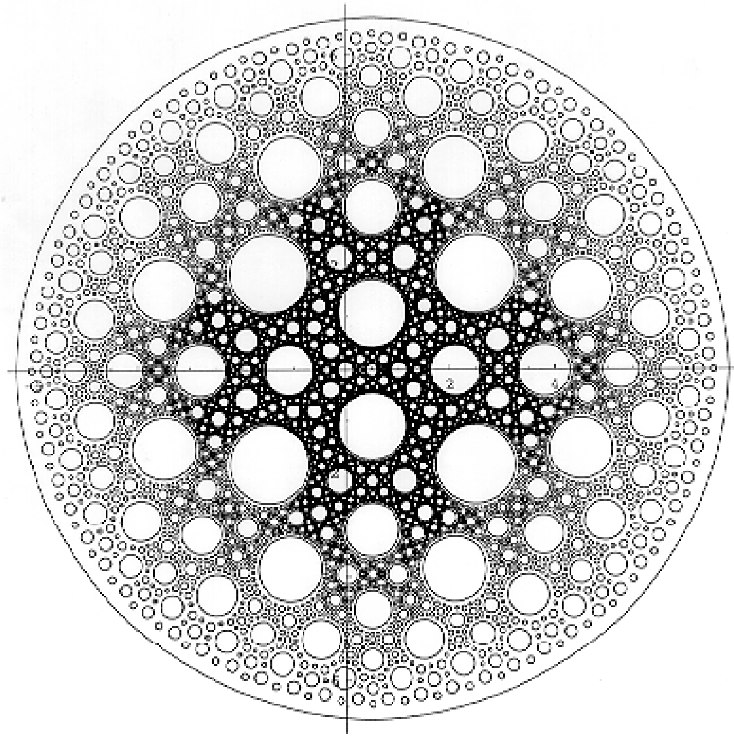


FIGURE 25. An approximation of the Julia set of $f_{\mathcal{B}}$

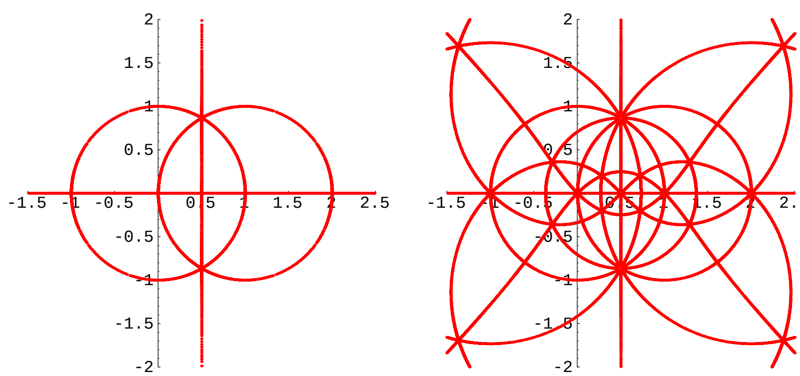


FIGURE 26. $f_{\mathcal{B}}^{\circ -1}(\mathbb{R} \cup \{\infty\})$ and $f_{\mathcal{B}}^{\circ -2}(\mathbb{R} \cup \{\infty\})$

and $\lambda(1) = \infty$. The domain of definition of λ can also be extended to H using the reflection principle, and λ is also a branched covering. Let $\Gamma(2)$ denote the subgroup of $SL(2, \mathbb{Z})$ consisting of all matrices in $SL(2, \mathbb{Z})$ which are congruent to the identity matrix modulo 2, and let $\overline{\Gamma}(2)$ denote the image of $\Gamma(2)$ in $PSL(2, \mathbb{Z})$. The branched covering λ is regular and $\overline{\Gamma}(2)$ is its group of covering transformations. The figure on page 358 of [15] shows a fundamental domain for $\overline{\Gamma}(2)$. This fundamental domain is the union of the hyperbolic triangle with vertices 0, 1, and ∞ ($\lambda^{\circ -1}(t_1)$) and the hyperbolic triangle with vertices 0, -1, and ∞ ($\lambda^{\circ -1}(t_2)$). The tesselation of H corresponding to $PGL(2,\mathbb{Z})$ subdivides these two triangles barycentrically. It follows that $J \circ \lambda^{\circ -1}$, which is $f_{\mathcal{B}}$, is an analytic function from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$ such that the pullback of $S_{\mathcal{B}}$ under this function subdivides $S_{\mathcal{B}}$ barycentrically.

We next show that the Julia set of $f_{\mathcal{B}}$ is a Sierpinski carpet, and hence that each component of the boundary of the Fatou set is a Jordan curve. By Corollary 5.18 of Pilgrim's thesis [19], to prove this it suffices to show that if $\alpha\colon (I,\partial I) \to (S^2, P_{f_{\mathcal{B}}})$ is an essential arc with $\alpha(\mathrm{int}(I)) \cap P_{f_{\mathcal{B}}} = \emptyset$ and if n is a positive integer, then no component of $f_{\mathcal{B}}^{\circ -n}(\alpha(\mathrm{int}(I)))$ has closure the image of an arc which is isotopic to α rel $P_{f_{\mathcal{B}}}$. Since this condition only depends on the isotopy class of α in $(S^2, P_{f_{\mathcal{B}}})$, one can assume that $\alpha(I)$ is either an edge of $S_{\mathcal{B}}$ or a loop based at a vertex of $S_{\mathcal{B}}$ which is a union of two medians. For such an arc α and any positive integer n, the closure of a component of $f_{\mathcal{B}}^{\circ -n}(\mathrm{int}(I))$ is either an edge in $\mathcal{B}^n(S_{\mathcal{B}})$ or the union of two medians in adjacent triangles in $\mathcal{B}^n(S_{\mathcal{B}})$. Since $\mathcal{B}^n(S_{\mathcal{B}})$ is the n^{th} barycentric subdivision of $S_{\mathcal{B}}$, it follows easily that it is impossible for one of these inverse images to have closure isotopic to α rel $P_{f_{\mathcal{B}}}$. Hence the Julia set of $f_{\mathcal{B}}$ is a Sierpinski carpet.

Let D_∞ be the component of the Fatou set which contains ∞. Since D_∞ is simply connected and ∂D_∞ is disjoint from $P_{f_{\mathcal{B}}}$, it follows from [3, Theorem 9.1] that ∂D_∞ cannot be differentiable at a single point unless it is either a line or a circle. Furthermore, it follows from the proof of [3, Theorem 9.1] that ∂D_∞ cannot be a line or a circle unless it is the entire Julia set. Thus ∂D_∞ cannot be differentiable at a single point. Since each boundary component of the Fatou set is mapped by an iterate of $f_{\mathcal{B}}$ onto ∂D_∞, none of the boundary components of the Fatou set can be differentiable at a single point.

Figure 25 was created by taking inverse images under $f_{\mathcal{B}}$ of the repelling fixed point p. While this is not the best method for approximating the Julia set in terms of exhibiting fractal properties of boundary curves, approximations using the escape-time algorithm have failed to show signs of logarithmic spiralling in ∂D_∞. The Julia set of $f_{\mathcal{B}}$ does contain repelling periodic points with logarithmic spiralling, but possibly no such point is in the boundary of a component of the Fatou set.

5. Questions

We conclude with several questions.

Question 5.1. Is there an efficient method for explicitly constructing a rational map which realizes a given critically finite branched map?

While we managed to explicitly construct rational maps for all of our examples that were realizable by rational maps, we do not know of an algorithm to explicitly construct a rational map realizing specific branching data. Note that a critically finite rational map with hyperbolic orbifold is conformally conjugate to one with algebraic coefficients. This follows, for example, from Theorems 3.6, 3.17, and 3.20 of [2]. An affirmative answer to the question is known for quadratic polynomials and for certain other classes of rational maps.

Question 5.2. Given a critically finite branched map $f\colon S^2 \to S^2$, how can you decide whether or not there is a finite subdivision rule $\mathcal{R}$ with an edge pairing such that f is equivalent to $\sigma_{\mathcal{R}}$?

If f is a critically finite rational map such that all of the coefficients of f are real, $|P_f| \geq 2$, and all of the post-critical points of f are real, then there is a finite subdivision rule $\mathcal{R}$ such that $S_\mathcal{R}$ is a CW complex on $\widehat{\mathbb{C}}$ with vertices the postcritical points and with open edges the components of $(\mathbb{R} \cup \infty) \setminus P_f$. The 1-skeleton of $\mathcal{R}(S_\mathcal{R})$ is $f^{\circ -1}(\mathbb{R} \cup \infty)$, and $\sigma_\mathcal{R} = f$. The finite subdivision rules $\mathcal{Q}$ in Example 4.4 and $\mathcal{B}$ in Example 4.6 could have been discovered from their corresponding rational maps by this construction.

If a critically finite rational map f has Julia set the 2-sphere, then it is expanding with respect to an orbifold metric on its Julia set. Hence f is a quotient of a subshift of finite type, and satisfies a subdivision or replacement rule. But it's not clear from this that it comes from a finite subdivision rule.

Question 5.3. Suppose $\mathcal{R}$ is an orientation-preserving finite subdivision rule with an edge pairing, and suppose that $S_\mathcal{R}$ is a 2-sphere. How do you tell from $\mathcal{R}$ whether $\sigma_\mathcal{R}$ can be realized by a rational map?

Pilgrim [20] shows that one can strengthen Theorem 2.2 by showing that it suffices to check the obstruction on a single, canonically defined f-stable curve system, but it isn't clear how to effectively find this curve system in terms of $\mathcal{R}$.

If $\mathcal{R}$ has mesh approaching 0, by Theorem 3.1 if $\mathcal{R}$ is conformal then $\sigma_\mathcal{R}$ can be realized by a rational map. We believe that the converse is true if $\mathcal{R}$ has bounded valence, but it is not true in general. Indeed, the barycentric subdivision rule from Example 4.6 is not conformal by [6, Theorem 6.3.1.1], but the subdivision map is realizable by a rational map. We also believe that one should be able to refine the definition of conformality for a finite subdivision rule with unbounded valence so that if $\mathcal{R}$ is a finite subdivision rule with unbounded valence, mesh approaching 0, and an edge pairing, and if $S_\mathcal{R}$ is a 2-sphere, then $\mathcal{R}$ is conformal exactly if $\sigma_\mathcal{R}$ can be realized by a rational map.

The expansion constant for the pentagonal subdivision rule, which is $(-324)^{1/5}$, is algebraic because the corresponding rational map $f_\mathcal{Q}$ has algebraic (in fact rational) coefficients. Hence if z is a periodic point of period n for $f_\mathcal{Q}$, then $(f_\mathcal{Q}^{\circ n})'(z)$ is algebraic. We conjecture that this algebraicity of derivatives is always the case for a finite subdivision rule with an invariant (partial) conformal structure. We first introduce some terminology.

Let $\mathcal{R}$ be an orientation-preserving finite subdivision rule. For each tile type t we define a triangulation $\boxtimes(t)$ of t by adding a barycenter $b(t)$ to t and joining $b(t)$ to each vertex of t by an arc. These triangulations push forward under the characteristic maps to give a triangulation $\boxtimes(S_\mathcal{R})$ of $S_\mathcal{R}$. A *butterfly* in $S_\mathcal{R}$ consists of an open edge e in $S_\mathcal{R}$ together with two open triangles t_1 and t_2 in $\boxtimes(S_\mathcal{R})$ such that e is contained in the closure of t_1 and of t_2, and t_1 and t_2 induce opposite orientations on e.

We define a "partial conformal structure" on $S_\mathcal{R}$ via charts. Let $\mathcal{F} = \{$open tiles of $S_\mathcal{R}\} \cup \{$butterflies of $S_\mathcal{R}\}$. A *chart* for an element $s \in \mathcal{F}$ is an open set $\widehat{s} \subseteq \mathbb{C}$ together with a orientation-preserving homeomorphism $\mu_s \colon s \to \widehat{s}$. A *partial conformal structure* on $S_\mathcal{R}$ is an atlas $\mathcal{A} = \{\mu_s \colon s \in \mathcal{F}\}$ of charts which satisfies the following compatibility condition: if s_1 is an open tile in $\mathcal{F}$ and s_2 is a butterfly in $\mathcal{F}$ such that $s_1 \cap s_2 \neq \emptyset$, then the map $\mu_{s_2} \circ \mu_{s_1}^{-1} \colon \mu_{s_1}(s_1 \cap s_2) \to \mu_{s_2}(s_1 \cap s_2)$ is conformal if it is orientation preserving and is anticonformal if it is orientation reversing.

A partial conformal structure $\mathcal{A}$ on $S_{\mathcal{R}}$ is called *$\mathcal{R}$-invariant* if it satisfies the following. Suppose $s \in \mathcal{F}$, $x \in s$, U is an neighborhood of $\mu_s(x)$ in $\widehat{s}$, and $t \in \mathcal{F}$ such that $\sigma_{\mathcal{R}}$ maps $\mu_s^{-1}(U)$ injectively into t. Then $\mu_t \circ \sigma_{\mathcal{R}} \circ \mu_s^{-1}\big|_U$ is conformal if it is orientation preserving and is aniconformal if it is orientation reversing.

Question 5.4. Let $\mathcal{R}$ be an orientation-preserving finite subdivision rule with bounded valence and mesh approaching zero. Suppose $\mathcal{A} = \{\mu_s \colon s \in \mathcal{F}\}$ is an $\mathcal{R}$-invariant partial conformal structure, and let z be a periodic point for $\sigma_{\mathcal{R}}$ with period n such that z is not a vertex of $S_{\mathcal{R}}$. Let $s \in \mathcal{F}$ with $z \in s$. Is $\left(\mu_s \circ \sigma_{\mathcal{R}}^{\circ n} \circ \mu_s^{-1}\right)' (\mu_s(z))$ an algebraic number?

References

[1] P. L. Bowers and K. Stephenson, *A "regular" pentagonal tiling of the plane*, Conform. Geom. Dyn. **1** (1997), 58–68 (electronic).

[2] E. Brezin, R. Byrne, J. Levy, K. Pilgrim, and K. Plummer, *A census of rational maps*, Conform. Geom. Dyn. **4** (2000), 35–74 (electronic).

[3] H. Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. **6** (1965), 103–144.

[4] J. W. Cannon, *The combinatorial Riemann mapping theorem*, Acta Math. **173** (1994), 155–234.

[5] J. W. Cannon and E. L. Swenson, *Recognizing constant curvature discrete groups in dimension* 3, Trans. Amer. Math. Soc. **350** (1998), 809–849.

[6] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Squaring rectangles: the finite Riemann mapping theorem*, The mathematical legacy of Wilhelm Magnus: groups, geometry and special functions (Brooklyn, NY, 1992), Amer. Math. Soc., Providence, RI, 1994, pp. 133–212.

[7] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Sufficiently rich families of planar rings*, Ann. Acad. Sci. Fenn. Math. **24** (1999), 265–304.

[8] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Finite subdivision rules*, Conform. Geom. Dyn. **5** (2001), 153–196 (electronic).

[9] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Expansion complexes for finite subdivision rules I*, preprint.

[10] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Expansion complexes for finite subdivision rules II*, in preparation.

[11] A. Douady and J. H. Hubbard, *A proof of Thurston's topological characterization of rational functions*, Acta Math. **171** (1993), 263–297.

[12] L. R. Ford, Automorphic Functions, Mc-Graw Hill, New York, 1929.

[13] T. Kuusalo, *Verallgemeinerter Riemannscher Abbildungssatz and quasikonforme Mannigfaltgkeiten*, Ann. Acad. Sci. Fenn. Ser. A I **409** (1967), 24 pages.

[14] S. Lattès, *Sur l'iteration des substitutions rationelles et les fonctions de Poincaré*, C. R. Acad. Sci. Paris **16** (1918), 26–28.

[15] J. Lehner, Discontinuous Groups and Automorphic Functions, Mathematical Surveys, Number VIII, American Mathematical Society, Providence, RI, 1964.

[16] C. T. McMullen, Complex Dynamics and Renormalization, Annals of Mathematics Studies 135, Princeton University Press, Princeton, NJ, 1994.

[17] D. Meyer, *Quasisymmetric embedding of self similar surfaces and origami with rational maps*, Ann. Acad. Sci. Fenn. Math. **27** (2002), 461–484.

[18] J. Milnor, Dynamics in One Complex Variable: Introductory Lectures, Vieweg, Braunschweig/Wiesbaden, 1999.

[19] K. M. Pilgrim, *Cylinders for iterated rational maps*, Ph.D. thesis, University of California, Berkeley, 1994.

[20] K. M. Pilgrim, *Canonical Thurston obstructions*, Adv. Math. **158** (2001), 154–168.

[21] K. Stephenson, *CirclePack*, software, available from http://www.math.utk.edu/~kens.

[22] E. H. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.

[23] D. Sullivan, *Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou-Julia problem on wandering domains*, Ann. Math. **122** (1985), 401–418.

[24] D. Sullivan, *Conformal dynamical systems*, Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math. **1007**, Springer, Berlin, 1983, pp. 725–752.

[25] W. P. Thurston, *Lecture notes*, CBMS Conference, University of Minnesota at Duluth, 1983.

Department of Mathematics, Brigham Young University, Provo, UT 84602, U.S.A.
E-mail address: `cannon@math.byu.edu`

Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, U.S.A.
E-mail address: `floyd@math.vt.edu`
URL: `http://www.math.vt.edu/people/floyd/`

Laboratoire de Topologie, Université Paris-Sud, Bat. 425, 91405 Orsay Cedex - FRANCE
URL: `http://topo.math.u-psud.fr/~kenyon/`

Department of Mathematics, Eastern Michigan University, Ypsilanti, MI 48197, U.S.A.
E-mail address: `walter.parry@emich.edu`