# COMBINATORIALLY REGULAR POLYOMINO TILINGS

J. W. CANNON, W. J. FLOYD, AND W. R. PARRY

Abstract. Let $\mathcal{T}$ be a regular tiling of $\mathbf{R}^2$ which has the origin 0 as a vertex, and suppose that $\varphi\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a homeomorphism such that i) $\varphi(0) = 0$, ii) the image under $\varphi$ of each tile of $\mathcal{T}$ is a union of tiles of $\mathcal{T}$, and iii) the images under $\varphi$ of any two tiles of $\mathcal{T}$ are equivalent by an orientation-preserving isometry which takes vertices to vertices. It is proved here that there is a subset $\Lambda$ of the vertices of $\mathcal{T}$ such that $\Lambda$ is a lattice and $\varphi|_\Lambda$ is a group homomorphism.

The tiling $\varphi(\mathcal{T})$ is a tiling of $\mathbf{R}^2$ by polyiamonds, polyominos, or polyhexes. These tilings occur often as expansion complexes of finite subdivision rules. The above theorem is instrumental in determining when the tiling $\varphi(\mathcal{T})$ is conjugate to a self-similar tiling.

## 1. Introduction

There are three regular tilings of the plane: the tiling by equilateral triangles in which six meet at each vertex; the tiling by squares in which four meet at each vertex; and the tiling by regular hexagons in which three meet at each vertex. We are interested here in tilings of the plane whose tiles are congruent in an orientation-preserving way and each tile is an amalgamation of tiles from one of the regular tilings. Moreover, our tilings satisfy one of the following: each tile is a polyiamond with three vertices (but possibly many more corners), and six tiles meet at each vertex; each tile is a polyomino with four vertices, and four tiles meet at each vertex; each tile is a polyhex, and three tiles meet at each vertex. Because our tilings are isomorphic to regular tilings, we say that they are combinatorially regular. For example, Figure 1 shows parts of two of our combinatorially regular tilings

---

*Date*: September 11, 2003.

1991 *Mathematics Subject Classification.* Primary 05B50, 52C20.

*Key words and phrases.* finite subdivision rule, polyomino.

This research was supported in part by NSF grants DMS-9803868, DMS-9971783, DMS-10104030, and DMS-0203902.

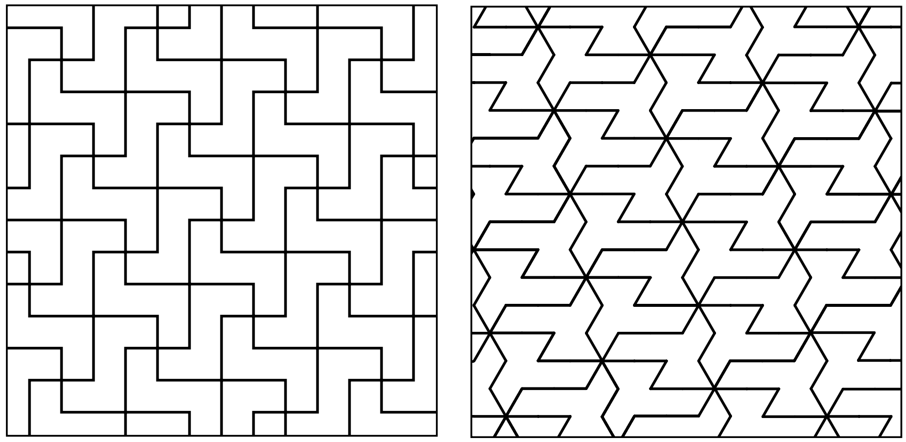

FIGURE 1. A combinatorially regular polyomino tiling and a combinatorially regular polyiamond tiling

We are interested in these tilings from the point of view of renormalization. Suppose $\mathcal{S} = \mathcal{S}_1$ is a combinatorially regular tiling as described above which is obtained from amalgamating tiles of a regular tiling $\mathcal{T}$ with the origin as a vertex. In many (if not all) cases, one can encode the way a tile of $\mathcal{S}$ is decomposed into tiles of $\mathcal{T}$ by means of a finite subdivision rule. One can rescale $\mathcal{S}_1$ to get a tiling $\mathcal{S}_1'$ so that $(0,0)$ and $(1,0)$ are adjacent vertices of a tile of $\mathcal{S}_1'$. Using the data of the finite subdivision rule, one can obtain a new tiling $\mathcal{S}_2$ by amalgamating tiles of $\mathcal{S}_1'$, and can then rescale $\mathcal{S}_2$ to a tiling $\mathcal{S}_2'$ so that $(0,0)$ and $(1,0)$ are vertices of a tile of $\mathcal{S}_2'$. One can continue this renormalization process indefinitely to obtain a sequence $\{\mathcal{S}_i'\}$ of combinatorially regular tilings by polyiamonds, polyominos, or polyhexes. Special cases of this were considered in [2] and (without the terminology of finite subdivision rules) in [4, 5].

Our interest in this centers on the problem of determining when such a sequence $\{\mathcal{S}_i'\}$ of tilings limits to a (self-similar) tiling of the plane. A potential problem is that tiles get flatter and flatter as $i$ increases, and do not converge to tiles in the limit. Since the initial tiling $\mathcal{S}$ is combinatorially regular, there is a homeomorphism $\varphi\colon \mathbf{R}^2 \to \mathbf{R}^2$ which fixes the origin and takes each tile of $\mathcal{T}$ to a tile of $\mathcal{S}$. In this paper we show that $\varphi$ restricts to a group homomorphism on a subset of the vertices of $\mathcal{T}$ that is a lattice. Hence we can associate to $\varphi$ a $2 \times 2$ matrix. In [3] we show that the sequence $\{\mathcal{S}_i'\}$ limits to a tiling exactly if this matrix is either a scalar matrix or its eigenvalues are not real.

Here is a precise statement of the main theorem.

**Main Theorem.** *Let $\mathcal{T}$ be a regular tiling of $\mathbf{R}^2$; the tiles of $\mathcal{T}$ are either equilateral triangles with six meeting at every vertex, squares with four meeting at every vertex, or regular hexagons with three meeting at every vertex. Suppose that the origin 0 is a vertex of a tile of $\mathcal{T}$. Let $\varphi\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a homeomorphism such that*

1. *$\varphi(0)=0$;*
2. *if $t$ is a tile of $\mathcal{T}$, then $\varphi(t)$ is a union of tiles of $\mathcal{T}$;*
3. *if $s$ and $t$ are tiles of $\mathcal{T}$, then there exists an orientation-preserving isometry $\tau\colon \varphi(s) \to \varphi(t)$ such that $\varphi^{-1} \circ \tau \circ \varphi$ maps the vertices of $s$ to the vertices of $t$.*

*Then there exists a subset $\Lambda$ of the set of vertices of tiles of $\mathcal{T}$ such that $\Lambda$ is a lattice in $\mathbf{R}^2$ and $\varphi\big|_\Lambda$ is a group homomorphism.*

The reader might look at the beginning of Section 4, where we discuss the fact that our proof of the main theorem actually proves something a bit stronger. See also the next-to-last paragraph of this introduction.

We assume that a tiling of $\mathbf{R}^2$ is a set of closed topological disks called tiles which cover $\mathbf{R}^2$ and that the interiors of distinct tiles are disjoint.

Maintaining the assumptions of the main theorem, let $\mathcal{S} = \{\varphi(t) : t \in \mathcal{T}\}$, a combinatorially regular tiling of $\mathbf{R}^2$. In the case of squares, condition 2 implies that every tile of $\mathcal{S}$ is a polyomino. Hence in the case of squares, we are dealing with combinatorially regular polyomino tilings of $\mathbf{R}^2$. This explains the title of this paper. The tiles of $\mathcal{S}$ are polyiamonds in the case of equilateral triangles, and they are polyhexes in the case of regular hexagons.

The tiles of $\mathcal{T}$ have vertices and edges. Abusing terminology, we refer to these vertices and edges as vertices and edges of $\mathcal{T}$. If $v$ is a vertex of a tile $t$ of $\mathcal{T}$, then we call $\varphi(v)$ a vertex of $\varphi(t)$. If $e$ is an edge of a tile $t$ of $\mathcal{T}$, then we call $\varphi(e)$ an edge of $\varphi(t)$. Abusing terminology again, we refer to these vertices and edges as vertices and edges of $\mathcal{S}$. Condition 3 states that the tiles of $\mathcal{S}$ are mutually congruent by means of orientation-preserving isometries which map vertices to vertices and edges to edges.

In Section 2 we define an automorphism of $\mathcal{S}$ to be an orientation-preserving isometry $\sigma\colon \mathbf{R}^2 \to \mathbf{R}^2$ which maps tiles of $\mathcal{S}$ to tiles of $\mathcal{S}$. The set of all automorphisms of $\mathcal{S}$ is a group $\mathrm{Aut}(\mathcal{S})$. In Section 2 we also define a vertex automorphism of $\mathcal{S}$ to be an orientation-preserving isometry $\sigma\colon \mathbf{R}^2 \to \mathbf{R}^2$ for which there exists a function $F_\sigma\colon \mathcal{S} \to \mathcal{S}$ such that if $S \in \mathcal{S}$, then $\sigma$ maps the vertices of $S$ to the vertices of $F_\sigma(S)$. We show that the set of all vertex automorphisms of $\mathcal{S}$ is a group $\mathrm{Aut}_V(\mathcal{S})$. While $\mathrm{Aut}(\mathcal{S})$ might be trivial, there is a natural action of $\mathrm{Aut}_V(\mathcal{S})$ on $\mathcal{S}$ and our proof of the main theorem shows that this action of $\mathrm{Aut}_V(\mathcal{S})$ on $\mathcal{S}$ is transitive.

We maintain the assumptions of the main theorem throughout this paper. We also let $\mathcal{S} = \{\varphi(t) : t \in \mathcal{T}\}$, and we let $T$ be a fixed tile of $\mathcal{S}$.

## 2. Isometries

This section consists mainly of definitions together with some elementary results concerning the action of isometries on $\mathcal{S}$. Let $\mathrm{Isom}^+(\mathbf{R}^2)$ be the group of all orientation-preserving isometries of $\mathbf{R}^2$. We orient the edges of $T$ in the counterclockwise direction.

We say that edges $E$ and $F$ of $T$ are **congruent** if there exists $\sigma \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\sigma(E) = F$. We say that edges $E$ and $F$ of $T$ are **properly congruent** if there exists $\sigma \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\sigma(E) = F$ and $\sigma\big|_E$ preserves orientation. We say that edges $E$ and $F$ of $T$ are **improperly congruent** if there exists $\sigma \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\sigma(E) = F$ and $\sigma\big|_E$ reverses orientation. We say that edges $E$ and $F$ of $T$ are **parallel** if there exists a translation $\sigma \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\sigma(E) = F$ and $\sigma\big|_E$ reverses orientation. We say that edges $E$ and $F$ of $T$ **match** if there exist distinct tiles $T_1$ and $T_2$ of $\mathcal{S}$ and orientation-preserving isometries $\sigma_1\colon T \to T_1$ and $\sigma_2\colon T \to T_2$ such that $\sigma_1(E) = \sigma_2(F)$.

The notion of matching puts a symmetric relation on the set of edges of $T$: two edges of $T$ are related if and only if they match. This relation generates an equivalence relation. We refer to the equivalence classes of this equivalence relation as **matching classes**.

We say that $T$ has an **edge pairing** if every edge of $T$ matches exactly one edge of $T$ (possibly itself).

We say that two distinct edges of $T$ are **opposite** if the corresponding edges of $\varphi^{-1}(T)$ are parallel.

We say that an isometry $\sigma \in \mathrm{Isom}^+(\mathbf{R}^2)$ is an **automorphism** of $\mathcal{S}$ if $\sigma$ maps tiles of $\mathcal{S}$ to tiles of $\mathcal{S}$. The set of all automorphisms of $\mathcal{S}$ is a group, denoted by $\mathrm{Aut}(\mathcal{S})$. We likewise have a group of automorphisms $\mathrm{Aut}(\mathcal{T})$.

We say that an element $\sigma$ of $\mathrm{Isom}^+(\mathbf{R}^2)$ is a **vertex automorphism** of $\mathcal{S}$ if there exists a function $F_\sigma\colon \mathcal{S} \to \mathcal{S}$ such that if $S \in \mathcal{S}$, then $\sigma$ maps the vertices of $S$ to the vertices of $F_\sigma(S)$. Let $\mathrm{Aut}_V(\mathcal{S})$ denote the set of vertex automorphisms of $\mathcal{S}$.

Let $\sigma \in \mathrm{Aut}_V(\mathcal{S})$. The map $F_\sigma$ is clearly injective. If $S$ and $S'$ are tiles of $\mathcal{S}$ with an edge in common, then the definition of vertex automorphism implies that $F_\sigma(S)$ and $F_\sigma(S')$ have two vertices and hence

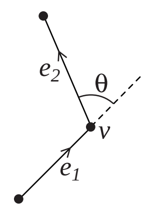


FIGURE 2. Defining turning angles

an edge in common. In other words, $F_\sigma$ preserves edge adjacency. A straightforward argument using this shows that $F_\sigma$ is surjective. Thus $F_\sigma$ is bijective. This implies that $\sigma^{-1} \in \mathrm{Aut}_V(\mathcal{S})$.

Now we see that $\mathrm{Aut}_V(\mathcal{S})$ is a group. It is clear that $\mathrm{Aut}(\mathcal{S})$ is a subgroup of $\mathrm{Aut}_V(\mathcal{S})$.

Let $\sigma \in \mathrm{Aut}_V(\mathcal{S})$, and let $S \in \mathcal{S}$. We obtain an action of $\mathrm{Aut}_V(\mathcal{S})$ on $\mathcal{S}$ by setting $\sigma S = F_\sigma(S)$.

Let $\sigma \in \mathrm{Aut}_V(\mathcal{S})$, and let $t \in \mathcal{T}$. Then $\varphi^{-1} \circ \sigma \circ \varphi$ maps the vertices of $t$ to the vertices of $\varphi^{-1}(F_\sigma(\varphi(t))) \in \mathcal{T}$. Since $\varphi$ maps tiles with an edge in common to tiles with an edge in common and $F_\sigma$ preserves edge adjacency, it follows that there exists $\tau \in \mathrm{Aut}(\mathcal{T})$ such that $\varphi^{-1} \circ \sigma \circ \varphi(v) = \tau(v)$ for every vertex $v$ of $\mathcal{T}$. The map $\sigma \mapsto \tau$ is a group homomorphism: there exists an injective group homomorphism $\omega\colon \mathrm{Aut}_V(\mathcal{S}) \to \mathrm{Aut}(\mathcal{T})$ such that if $\sigma \in \mathrm{Aut}_V(\mathcal{S})$, then $\varphi^{-1} \circ \sigma \circ \varphi(v) = \omega(\sigma)(v)$ for every vertex $v$ of $\mathcal{T}$.

## 3. Curvature

This section deals with curvature of oriented piecewise linear arcs and simple closed curves in $\mathbf{R}^2$.

Let $\gamma$ be an oriented piecewise linear arc or simple closed curve in $\mathbf{R}^2$. We view $\gamma$ as a 1-complex with vertices and edges. Let $v$ be an interior vertex of $\gamma$. In other words, $\gamma$ contains edges $e_1$ and $e_2$ so that $e_2$ immediately follows $e_1$ relative to the orientation of $\gamma$ and $v = e_1 \cap e_2$. We define the **turning angle** of $\gamma$ at $v$ to be the oriented angle $\theta$ from an extension of $e_1$ to $e_2$ such that $-\pi < \theta < \pi$. We orient angles so that counterclockwise is the positive direction and clockwise is the negative direction. See Figure 2, which shows a positive turning angle $\theta$.

With $\gamma$ as in the previous paragraph, we define the **total curvature** $K(\gamma)$ of $\gamma$ to be the sum of the turning angles of the interior vertices of $\gamma$. As is well known, the Euler formula for a closed topological disk with the structure of a simplicial complex implies that if $\gamma$ is a simple closed curve, then $K(\gamma) = 2\pi$.

The tile $T$ is a union of tiles of $\mathcal{T}$, and so every edge of $T$ is a union of edges of $\mathcal{T}$. Hence if $E$ is an edge of $T$, then we may speak of the edges of ($\mathcal{T}$ in) $E$. The counterclockwise orientation of $\partial T$ induces an orientation on every edge $E$ of $T$, and so we may speak of the initial and terminal edges of ($\mathcal{T}$ in) $E$.

**Lemma 3.1.**

1. *Let $t$ be a tile of $\mathcal{T}$. Then the turning angle of $\partial T$ at every vertex of $T$ is equal to the turning angle of $\partial t$ at every vertex of $t$.*
2. *If $E$ is an edge of $T$ such that $E$ is improperly congruent to itself, then $K(E) = -K(E)$, and so $K(E) = 0$.*

*Proof.* This is clear. □

**Lemma 3.2.** *Let $v$ be a vertex of $T$. Let $E_1$ be the edge of $T$ immediately preceding $v$, and let $E_2$ be the edge of $T$ immediately following $v$. Let $\gamma_1$ be the line segment joining the vertices of $E_1$, and let $\gamma_2$ be the line segment joining the vertices of $E_2$. Let $\gamma$ be the oriented arc consisting of $\gamma_1$ followed by $\gamma_2$. Let $\phi$ be the turning angle of $\gamma$ at $v$, and let $\theta$ be the turning angle of $\partial T$ at $v$.*

1. *If $E_1$ and $E_2$ are improperly congruent, then $\phi = \theta$.*
2. *If $E_1$ and $E_2$ are properly congruent, then $\phi \equiv \theta + K(E_1)$ modulo $2\pi$.*

*Proof.* If $E_1$ and $E_2$ are improperly congruent, then there exists a rotation $\sigma \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\sigma(v) = v$ and $\sigma(E_1) = E_2$. Hence $\sigma$ rotates $\gamma_1$ to $\gamma_2$, and $\sigma$ rotates the terminal edge of $E_1$ to the initial edge of $E_2$. Thus $\phi = \theta$. This proves statement 1.

To prove statement 2, suppose that $E_1$ and $E_2$ are properly congruent. See Figure 3. Let $\alpha$ be the angle from the initial edge of $E_1$ to $\gamma_1$ with $-\pi < \alpha \le \pi$. Let $\beta$ be the angle from an extension of $\gamma_1$ to an extension of the terminal edge of $E_1$ with $-\pi < \beta \le \pi$. Then

$$\alpha + \beta \equiv K(E_1) \mod 2\pi.$$

Because $E_1$ and $E_2$ are properly congruent, $\alpha$ is the angle from the initial edge of $E_2$ to $\gamma_2$. Since $\phi$ is the angle from an extension of $\gamma_1$ to an extension of the terminal edge of $E_1$ to the initial edge of $E_2$ to $\gamma_2$,

$$\phi = \beta + \theta + \alpha \equiv \theta + K(E_1) \mod 2\pi.$$

This proves Lemma 3.2. □

**Lemma 3.3.** *Let $E_1$ and $E_2$ be improperly congruent disjoint edges of $T$. Let $\gamma$ be the oriented subarc of $\partial T$ whose initial edge is the terminal edge of $E_1$ and whose terminal edge is the initial edge of $E_2$. Suppose that $K(\gamma) = \pi$. Then $E_1$ and $E_2$ are parallel.*

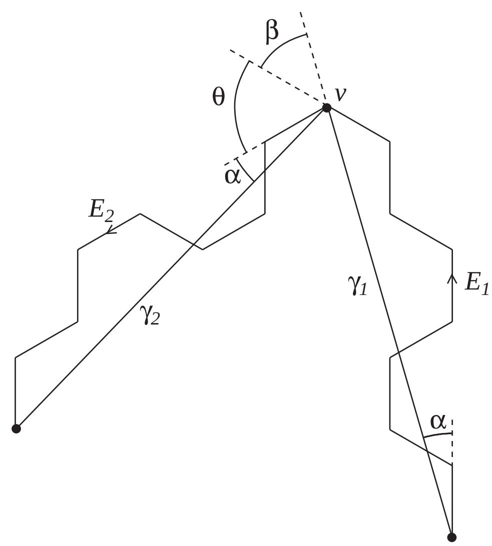


Figure 3. Proving Lemma 3.2

*Proof.* Because $E_1$ and $E_2$ are improperly congruent, there exists $\sigma \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\sigma$ takes the terminal edge of $E_1$ to the initial edge of $E_2$. Because $K(\gamma) = \pi$, it follows that $\sigma$ is a translation. This proves Lemma 3.3. □

## 4. The three possibilities

Our proof of the main theorem actually proves something stronger. We prove that one of the following three statements holds.

1. The tile $T$ has an edge pairing.
2. The vertices of $T$ are the vertices of a regular polygon $P$ in order.
3. The tile $T$ has at least four edges; if two distinct edges of $T$ match, then they are opposite; and at most two edges of $T$ are not parallel to the opposite edges of $T$.

In statement 2 the expression "in order" means that adjacent vertices of $T$ are adjacent vertices of $P$ and that counterclockwise orientation is preserved. These are the three possibilities mentioned in the title of this section. In this section we show that each of these three statements implies the conclusion of the main theorem. Lemmas 4.1, 4.2, and 4.3 show that each of the above three statements implies that $\mathrm{Aut}_V(\mathcal{S})$ acts transitively on $\mathcal{S}$. In other words, we prove that the hypotheses of the main theorem imply that $\mathrm{Aut}_V(\mathcal{S})$ acts transitively on $\mathcal{S}$. Lemma 4.4 shows that if $\mathrm{Aut}_V(\mathcal{S})$ acts transitively on $\mathcal{S}$, then the conclusion of the main theorem is true. We begin with Lemma 4.1.

**Lemma 4.1.** *Suppose that $T$ has an edge pairing. Then $Aut(\mathcal{S})$ acts transitively on $\mathcal{S}$, and so $Aut_V(\mathcal{S})$ acts transitively on $\mathcal{S}$.*

*Proof.* The hypotheses of the main theorem imply that if $S_1, S_2 \in \mathcal{S}$, then there exists $\psi \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\psi(S_1) = S_2$ and $\psi$ maps the vertices of $S_1$ to the vertices of $S_2$. To prove Lemma 4.1, it suffices to prove that $\psi \in \mathrm{Aut}(\mathcal{S})$, which is what we do. A straightforward argument shows that to prove that $\psi \in \mathrm{Aut}(\mathcal{S})$, it suffices to prove the following. Let $\psi \in \mathrm{Isom}^+(\mathbf{R}^2)$, and suppose that $R$ and $S$ are tiles of $\mathcal{S}$ such that $R \cap S$ is an edge of $\mathcal{S}$ and that $\psi$ maps $R$ to a tile of $\mathcal{S}$ taking vertices of $R$ to vertices of $\psi(R)$. Then $\psi$ maps $S$ to a tile of $\mathcal{S}$ taking vertices of $S$ to vertices of $\psi(S)$.

So suppose that $\psi \in \mathrm{Isom}^+(\mathbf{R}^2)$ and that $R$ and $S$ are tiles of $\mathcal{S}$ such that $R \cap S$ is an edge $E$ of $\mathcal{S}$ and that $\psi$ maps $R$ to a tile $R'$ of $\mathcal{S}$ taking vertices of $R$ to vertices of $R'$. Let $E' = \psi(E)$, and let $S'$ be the tile of $\mathcal{S}$ such that $R' \cap S' = E'$. The assumptions of the main theorem imply that there exist $\rho, \sigma, \tau \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\rho$ maps $T$ to $R$ taking vertices to vertices, $\sigma$ maps $T$ to $S$ taking vertices to vertices, and $\tau$ maps $S$ to $S'$ taking vertices to vertices.

Then $\rho^{-1}(E)$ matches $\sigma^{-1}(E)$ and $\rho^{-1} \circ \psi^{-1}(E')$ matches $\sigma^{-1} \circ \tau^{-1}(E')$. Since $\rho^{-1}(E) = \rho^{-1} \circ \psi^{-1}(E')$ and $T$ has an edge pairing, $\sigma^{-1}(E) = \sigma^{-1} \circ \tau^{-1}(E')$. Hence $\tau(E) = E'$. Thus the isometries $\psi$ and $\tau$ agree on $E$, and so they are equal.

This proves Lemma 4.1. $\square$

**Lemma 4.2.** *Suppose that the vertices of $T$ are the vertices of a regular polygon in order. Then $Aut_V(\mathcal{S})$ acts transitively on $\mathcal{S}$.*

*Proof.* We proceed as in the proof of Lemma 4.1. The hypotheses of the main theorem imply that if $S_1, S_2 \in \mathcal{S}$, then there exists $\psi \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\psi(S_1) = S_2$ and $\psi$ maps the vertices of $S_1$ to the vertices of $S_2$. To prove Lemma 4.2, it suffices to prove that $\psi \in \mathrm{Aut}_V(\mathcal{S})$, which is what we do. To prove that $\psi \in \mathrm{Aut}_V(\mathcal{S})$, it suffices to prove the following. Let $\psi \in \mathrm{Isom}^+(\mathbf{R}^2)$, and suppose that $R$ and $S$ are tiles of $\mathcal{S}$ such that $R \cap S$ is an edge of $\mathcal{S}$ and that $\psi$ maps the vertices of $R$ to the vertices of a tile of $\mathcal{S}$ in order. Then $\psi$ maps the vertices of $S$ to the vertices of a tile of $\mathcal{S}$ in order. But this is clear.

This proves Lemma 4.2. $\square$

**Lemma 4.3.** *Suppose that $T$ has at least four edges, that if two distinct edges of $T$ match, then they are opposite, and that at most two edges of $T$ are not parallel to the opposite edges of $T$. Then $Aut_V(\mathcal{S})$ acts transitively on $\mathcal{S}$.*

*Proof.* One verifies that the parallel edge condition implies that there exists a rotation in $\mathrm{Isom}^+(\mathbf{R}^2)$ of order 2 which maps vertices of $T$ to vertices of $T$ in order. If there exists a rotation in $\mathrm{Isom}^+(\mathbf{R}^2)$ of order greater than 2 which maps vertices of $T$ to vertices of $T$ in order, then the vertices of $T$ are the vertices of a regular polygon in order. Hence Lemma 4.2 implies that $\mathrm{Aut}_V(\mathcal{S})$ acts transitively on $\mathcal{S}$. Thus we may assume that there does not exist a rotation in $\mathrm{Isom}^+(\mathbf{R}^2)$ of order greater than 2 which maps vertices of $T$ to vertices of $T$ in order.

In this paragraph we partition the edges of $\mathcal{S}$ into $q/2$ types, where $q$ is the number of edges of $T$. We say that two edges of $T$ have the same type if and only if they are either equal or opposite. Let $E$ be an edge of $\mathcal{S}$. Let $S$ be a tile of $\mathcal{S}$ containing $E$, and let $\sigma\colon S \to T$ be an orientation-preserving isometry which maps vertices to vertices. We define the type of $E$ to be the type of $\sigma(E)$. We must show that this definition is independent of the choices of $\sigma$ and $S$. If $\tau\colon S \to T$ is an orientation-preserving isometry which maps vertices to vertices, then $\sigma \circ \tau^{-1}(T) = T$. By the previous paragraph, the order of $\sigma \circ \tau^{-1}$ is either 1 or 2. Hence $\sigma(E)$ and $\tau(E)$ are either equal or opposite, and so our definition is independent of the choice of $\sigma$. Now let $S'$ be the tile of $\mathcal{S}$ other than $S$ such that $E \subseteq S'$, and let $\sigma'\colon S' \to T$ be an orientation-preserving isometry which maps vertices to vertices. Then $\sigma'(E)$ matches $\sigma(E)$. Hence $\sigma'(E)$ and $\sigma(E)$ are either equal or opposite. Thus we have partitioned the edges of $\mathcal{S}$ into $q/2$ types. This partition has the following property. If $S_1$ and $S_2$ are tiles of $\mathcal{S}$, if $E$ is an edge of $S_1$, and if $\rho\colon S_1 \to S_2$ is an orientation-preserving isometry which maps vertices to vertices, then $\rho(E)$ has the same type as $E$.

Now we proceed as in Lemmas 4.1 and 4.2. The hypotheses of the main theorem imply that if $S_1, S_2 \in \mathcal{S}$, then there exists $\psi \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\psi(S_1) = S_2$ and $\psi$ maps the vertices of $S_1$ to the vertices of $S_2$. To prove Lemma 4.3, it suffices to prove that $\psi \in \mathrm{Aut}_V(\mathcal{S})$, which is what we do. To prove that $\psi \in \mathrm{Aut}_V(\mathcal{S})$, it suffices to prove the following. Let $\psi \in \mathrm{Isom}^+(\mathbf{R}^2)$, and suppose that $R$, $S$ and $R'$ are tiles of $\mathcal{S}$ such that $R \cap S$ is an edge of $\mathcal{S}$ and that for every edge $E$ of $R$ there exists an edge $E'$ of $R'$ such that $E'$ has the same type as $E$ and $\psi$ maps the vertices of $E$ to the vertices of $E'$ in order. Then there exists a tile $S'$ of $\mathcal{S}$ so that for every edge $E$ of $S$ there exists an edge $E'$ of $S'$ such that $E'$ has the same type as $E$ and $\psi$ maps the vertices of $E$ to the vertices of $E'$ in order.

So suppose that $\psi \in \mathrm{Isom}^+(\mathbf{R}^2)$ and that $R$, $S$, and $R'$ are tiles of $\mathcal{S}$ such that $R \cap S$ is an edge of $\mathcal{S}$ and that for every edge $E$ of $R$ there exists an edge $E'$ of $R'$ such that $E'$ has the same type as $E$ and $\psi$ maps the vertices of $E$ to the vertices of $E'$ in order. Let $E = R \cap S$, and let

$E'$ be the edge of $R'$ such that $\psi$ maps the vertices of $E$ to the vertices of $E'$. Let $S'$ be the tile of $\mathcal{S}$ such that $R' \cap S' = E'$. The assumptions of the main theorem imply that there exists $\tau \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\tau$ maps $S$ to $S'$ taking vertices to vertices. Then $E'$ and $\tau(E)$ both have the same type as $E$. So $E'$ and $\tau(E)$ are edges of $S'$ with the same type. It follows that either $\tau(E) = E'$ or there exists a rotation $\rho \in \mathrm{Isom}^+(\mathbf{R}^2)$ of order 2 which maps the vertices of $S'$ to the vertices of $S'$ in order such that $\rho \circ \tau(E) = E'$. If $\tau(E) = E'$, then $\psi$ and $\tau$ agree on the vertices of $E$. This implies that $\psi = \tau$, which proves Lemma 4.3 in this case. In the other case $\psi$ and $\rho \circ \tau$ agree on the vertices of $E$. This implies that $\psi = \rho \circ \tau$, which proves Lemma 4.3 in this case.

This proves Lemma 4.3. $\square$

**Lemma 4.4.** *Suppose that $\mathrm{Aut}_V(\mathcal{S})$ acts transitively on $\mathcal{S}$. Then the conclusion of the main theorem is true.*

*Proof.* Let $\omega\colon \mathrm{Aut}_V(\mathcal{S}) \to \mathrm{Aut}(\mathcal{T})$ be the group homomorphism from the end of Section 2. As in [1, 1.7.5.2], because $\mathrm{Aut}_V(\mathcal{S})$ acts transitively on $\mathcal{S}$, $\mathrm{Aut}_V(\mathcal{S})$ contains a subgroup $G$ generated by two translations which translate by vectors which are linearly independent over $\mathbf{R}$. As in [1, 9.3.4], every element of $\mathrm{Isom}^+(\mathbf{R}^2)$ is either a translation or a rotation. Every rotation in $\mathrm{Aut}_V(\mathcal{S})$ or $\mathrm{Aut}(\mathcal{T})$ has finite order and every nontrivial translation has infinite order. Hence a nontrivial element of $\mathrm{Aut}_V(\mathcal{S})$ or $\mathrm{Aut}(\mathcal{T})$ is a translation if and only if it has infinite order. Thus every element of $\omega(G)$ is a translation. It follows that if $\sigma$ and $\tau$ are both elements of $G$ or both elements of $\omega(G)$, then $\sigma \circ \tau(0) = \sigma(0) + \tau(0)$.

Let $\Lambda'$ be the orbit of 0 under $G$. It follows that $\Lambda'$ is a lattice. Let $\lambda_1, \lambda_2 \in \Lambda'$. Then there exist $\gamma_1, \gamma_2 \in G$ such that $\lambda_1 = \gamma_1(0)$ and $\lambda_2 = \gamma_2(0)$. We have for every $\sigma \in \mathrm{Aut}_V(\mathcal{S})$ that $\omega(\sigma)(0) = \varphi^{-1}(\sigma(\varphi(0))) = \varphi^{-1}(\sigma(0))$. Moreover

$$\begin{aligned}
\varphi^{-1}(\lambda_1 + \lambda_2) &= \varphi^{-1}(\gamma_1(0) + \gamma_2(0)) = \varphi^{-1}(\gamma_1 \circ \gamma_2(0)) = \omega(\gamma_1 \circ \gamma_2)(0) \\
&= \omega(\gamma_1) \circ \omega(\gamma_2)(0) = \omega(\gamma_1)(0) + \omega(\gamma_2)(0) \\
&= \varphi^{-1}(\gamma_1(0)) + \varphi^{-1}(\gamma_2(0)) = \varphi^{-1}(\lambda_1) + \varphi^{-1}(\lambda_2).
\end{aligned}$$

This shows that $\varphi^{-1}\big|_{\Lambda'}$ is a group homomorphism. Let $\Lambda = \varphi^{-1}(\Lambda')$. Then $\Lambda$ consists of vertices of $\mathcal{T}$, and $\Lambda$ is a lattice because it is a discrete subgroup of $\mathbf{R}^2$ isomorphic to $\mathbf{Z}^2$. Moreover, $\varphi\big|_{\Lambda}$ is a group homomorphism.

This proves Lemma 4.4. $\square$

**Corollary 4.5.** *Suppose that one of the three displayed statements at the beginning of this section holds. Then the conclusion of the main theorem is true.*

*Proof.* This follows from Lemmas 4.1, 4.2, 4.3, and 4.4. □

## 5. Proof of the main theorem

Our proof of the main theorem proceeds by way of a case analysis. In every case we show either that the assumptions of that case lead to a contradiction or that one of the three displayed statements at the beginning of Section 4 holds. The conclusion of the main theorem then follows from Corollary 4.5.

We denote the edges of $T$ by $a, b, c, \dots$ in counterclockwise order. In the cases which we consider we make assumptions on the edges of $T$. Suppose that the assumptions in one case are given by a logical proposition $P(a, b, c, \dots)$. If the conclusion of the main theorem is true assuming $P(a, b, c, \dots)$, then it is also true for $P$ with its variables permuted cyclically in any way. By reflecting both $\mathcal{T}$ and $\mathcal{S}$, we see that the conclusion of the main theorem is also true for $P$ with the order of its variables reversed. After we prove that the conclusion of the main theorem is true for $P(a, b, c, \dots)$, we say that by symmetry it is true for these other orderings of the variables of $P$.

Let $S$ be a tile of $\mathcal{S}$. Then there exists an orientation-preserving isometry $\sigma \colon T \to S$ (possibly not unique) which maps vertices of $T$ to vertices of $S$. This induces a labeling of the edges of $S$ using the letters $a, b, c, \dots$ . Conversely, $\sigma$ is determined by this labeling. We draw diagrams with edge labels as in Figure 4 to indicate one way in which the tiles of $\mathcal{S}$ can be identified with $T$.

First suppose that the tiles of $\mathcal{T}$ are equilateral triangles. If every edge of $T$ matches only itself, then $T$ has an edge pairing, and we are done. Otherwise there exist two distinct edges of $T$ which match each other. These two edges of $T$ have a vertex in common. Statement 1 of Lemma 3.2 implies that the vertices of $T$ are the vertices of an equilateral triangle in order. This proves the main theorem if the tiles of $\mathcal{T}$ are equilateral triangles.

Now suppose that the tiles of $\mathcal{T}$ are squares.

*Case 1. Edges a and b match only themselves.* This implies that $\mathbf{R}^2$ is a union of infinite strips labeled as in Figure 4. It follows that $T$ has an edge pairing, and we are done.

*Case 2. Edge a matches edge b.* Statement 1 of Lemma 3.2 implies that the vertices of $a \cup b$ are three vertices of a square in order. If $c$ matches $b$, then for the same reason, the vertices of $T$ are the vertices of

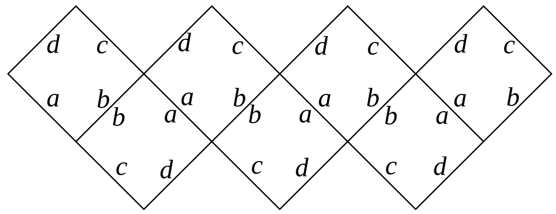

FIGURE 4. Part of an infinite strip for Case 1 (squares)

a square in order, and we are done. If $c$ matches $d$, then the vertices of $c \cup d$ are three vertices of a square in order. Hence the vertices of $T$ are the vertices of a square in order, and we are done. So we may assume that $c$ matches only either $a$ or $c$ and by symmetry that $d$ matches only either $b$ or $d$. By Case 1 and symmetry we may assume that $c$ matches $a$. If $d$ matches itself, then $K(d) = 0$ by Lemma 3.1, and then $a$ and $c$ are parallel by Lemmas 3.1 and 3.3. Since the vertices of $a \cup b$ are three vertices of a square in order, it follows that the vertices of $T$ are the vertices of a square in order, and we are done. If $d$ matches $b$, then because $b$ matches $a$ and $a$ matches $c$, it follows that $c$ and $d$ are improperly congruent. Lemma 3.2 implies that the vertices of $c \cup d$ are three vertices of a square in order. Thus the vertices of $T$ are the vertices of a square in order, and we are done.

*Case 3. If two distinct edges of $T$ match, then they are opposite.* If $T$ has an edge pairing, then we are done. So by symmetry we may assume that $a$ matches itself and $c$. If $b$ matches itself, then $K(b) = 0$ by Lemma 3.1, and so $a$ and $c$ are parallel by Lemmas 3.1 and 3.3. Hence statement 3 at the beginning of Section 4 is true, and so we are done. If $b$ matches $d$, then we use the fact that $K(a) = 0$ and argue in the same way.

By symmetry, Case 2 and Case 3 prove the main theorem if the tiles of $\mathcal{T}$ are squares.

Now suppose that the tiles of $\mathcal{T}$ are regular hexagons.

*Case 1. Edges $a$ and $b$ match only themselves.* This implies that $\mathbf{R}^2$ is a union of infinite strips labeled as in Figure 5. It follows that $T$ has an edge pairing, and we are done.

*Case 2. Edges $a$ and $c$ match only themselves.* This implies that $\mathbf{R}^2$ is a union of infinite strips labeled as in Figure 6. Again it follows that $T$ has an edge pairing, and we are done.

*Case 3. Edges $a$ and $d$ match only each other.* This implies that $\mathbf{R}^2$ is a union of infinite strips labeled as in Figure 7. If $T$ has an edge pairing, then we are done, and so we may assume that the matching classes are $\{a, d\}$, $\{b, e\}$, $\{c, f\}$ and that $b$, $c$, $e$, and $f$ are improperly

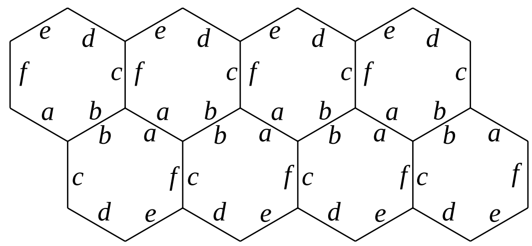


FIGURE 5. Part of an infinite strip for Case 1 (hexagons)

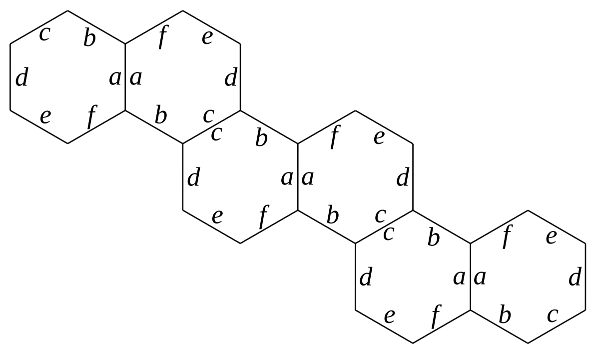


FIGURE 6. Part of an infinite strip for Case 2

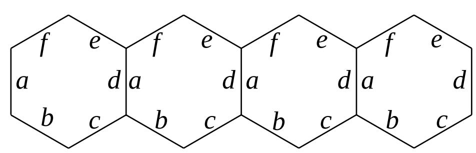


FIGURE 7. Part of an infinite strip for Case 3

congruent to themselves. Lemma 3.1 implies that $K(b) = K(c) = 0$. Now Lemmas 3.1 and 3.3 combine to imply that $a$ and $d$ are parallel. The assumptions imply that $b$ and $e$ are properly congruent to each other, and so there exists $\sigma \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\sigma(b) = e$ and $\sigma\big|_b$ preserves orientation. There likewise exists $\tau \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\tau(c) = f$ and $\tau\big|_c$ preserves orientation. We see that $\sigma(b \cap c) = \tau(b \cap c)$, and statement 1 of Lemma 3.1 implies moreover that $\sigma = \tau$. So $\sigma(a \cap b) = d \cap e$ and $\sigma(c \cap d) = a \cap f$. Because $a$ and $d$ are parallel, this implies that $\sigma$ is a rotation of order 2. Hence since $\sigma(b) = e$, there exists a translation which takes the vertices of $b$ to the vertices of $e$. Because $b$ and $e$ match, this translation in fact takes $b$ to $e$. Thus $b$ and $e$ are parallel. We therefore are in the situation of statement 3 at the beginning of Section 4, and so we are done.

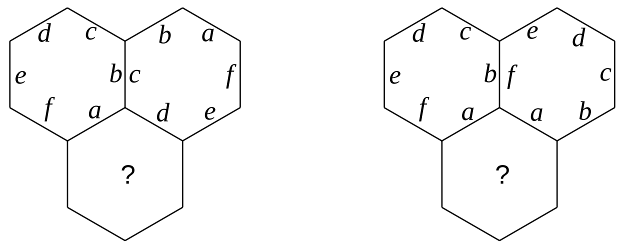


FIGURE 8. Ruling out possibilities in Case 4

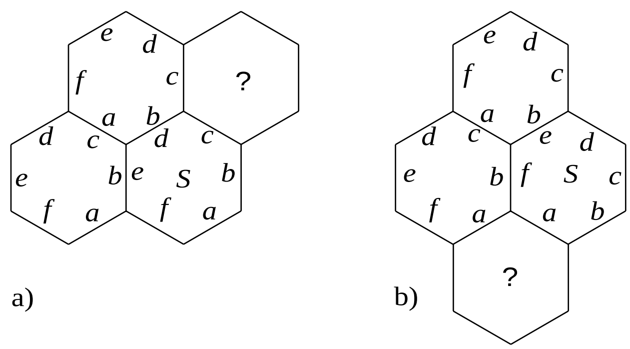


FIGURE 9. Showing that Case 5 is impossible

*Case 4. The set* $\{a, d\}$ *is a union of matching classes.* Figure 8 shows that it is impossible for $b$ to match either $c$ or $f$. This and symmetry imply that $\{b, e\}$ and $\{c, f\}$ are unions of matching classes. By Case 3 and symmetry we may assume that every edge of $T$ is improperly congruent to itself. By Case 1 and symmetry we may assume that $a$ matches $d$ and $b$ matches $e$. Lemma 3.1 implies that every edge of $T$ has total curvature 0. Lemma 3.3 implies that $a$ is parallel to $d$ and $b$ is parallel to $e$. Hence we are in the situation of statement 3 at the beginning of Section 4, and so we are done.

Thus far we have proved the main theorem for regular hexagons if $T$ has at least two edges which match only themselves.

*Case 5. The set* $\{a, c\}$ *is a matching class.* We choose two tiles as in either part a) or part b) of Figure 9 with a common edge labeled with $a$ and $c$ and reduce the labeling of the tile $S$ to one of the two labelings shown. This case is impossible.

*Case 6. One matching class contains only one edge, and one matching class contains at least four edges.* By symmetry we may assume that $a$ matches only itself. Since we have proved the main theorem if $T$ has two edges which match only themselves, we may assume that

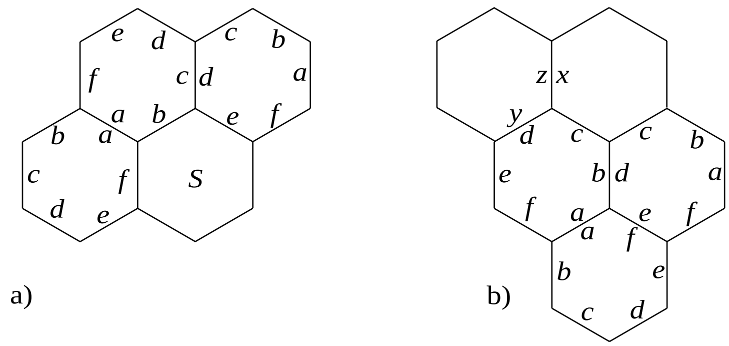


FIGURE 10. Verifying Case 7

$\{b, c, d, e, f\}$ is a matching class. We have that $K(\partial T) = 2\pi$. We obtain $K(\partial T)$ by summing the total curvatures of the edges of $T$ plus the turning angles of $\partial T$ at the vertices of $T$. Lemma 3.1 shows that these six turning angles are all $\pi/3$ and $K(a) = 0$. Hence the sum of the total curvatures of $b$, $c$, $d$, $e$ and $f$ is 0. These total curvatures have the same absolute value. It follows that they are 0. Now we apply Lemma 3.2 to every vertex of $T$ not in $a$. We conclude that the vertices of $T$ are the vertices of a regular hexagon in order, and so we are done.

*Case 7. Some matching class contains only one edge.* We may assume that $a$ matches only itself. We may assume that $d$ does not match only itself by Case 4. If $d$ matches $c$, then part a) of Figure 10 shows that no matter how the edges of tile $S$ are labeled, there is a matching class with at least four edges, and so we are done by Case 6. So we may assume that $d$ does not match $c$ and, by symmetry, that $d$ does not match $e$. By symmetry we may assume that $d$ matches $b$. Part b) of Figure 10 shows that $e$ matches $f$. Because neither $a$ nor $d$ matches $c$, label $x$ in part b) of Figure 10 is either $a$ or $d$. If $x = d$, then it is impossible to find $y$ and $z$ because $d$ does not match $a$, $c$ or $e$. If $x = a$, then $y = f$. This implies that $b$, $d$, $e$, $f$ are in a matching class, and we are done by Case 6.

*Case 8. All matching classes have two edges.* By Cases 4 and 5 and symmetry, we may assume that the matching classes are $\{a, b\}$, $\{c, d\}$ and $\{e, f\}$. If some edge of $T$ matches itself, then we may assume that $a$ matches itself. Figure 11 shows that this is impossible. Hence $T$ has an edge pairing, and so we are done.

*Case 9. One matching class has two edges, and one matching class has four edges.* By Cases 4 and 5 and symmetry, we may assume that the matching classes are $\{a, b\}$ and $\{c, d, e, f\}$. If one of $c$, $d$, $e$, or $f$ matches itself, then $c$, $d$, $e$, and $f$ are improperly congruent to each other. In this situation Lemma 3.2 implies that the vertices of

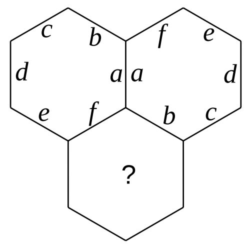


FIGURE 11. Ruling out a possibility in Case 8

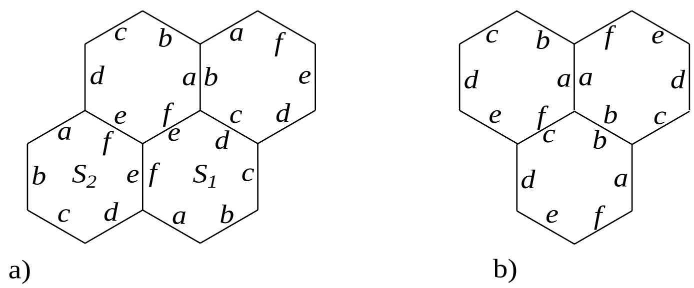


FIGURE 12. Verifying Case 9

$c \cup d \cup e \cup f$ are five vertices of a regular hexagon in order. The vertices of $a \cup b$ are likewise three vertices of a regular hexagon in order. Hence the vertices of $T$ are the vertices of a regular hexagon in order, and so we are done. This shows that whenever we have two tiles as in part a) of Figure 12 for which $a$ and $b$ match, we may assume that the edges of $S_1$ are labeled as indicated. It follows that if $a$ matches only $b$, then $f$ matches only $e$, and so the edges of $S_2$ are labeled as indicated. This implies that $e$ matches only $f$, and so $\{e, f\}$ is a matching class, which is not true. Hence $a$ matches itself, and so $\mathcal{S}$ has a configuration of tiles with edges labeled as in part b) of Figure 12. Hence $c$ is improperly congruent to $d$, which is improperly congruent to $e$ (by means of three matches), which is improperly congruent to $f$. Lemma 3.2 again shows that the vertices of $c \cup d \cup e \cup f$ are five vertices of a regular hexagon in order. Again it follows that the vertices of $T$ are the vertices of a regular hexagon in order, and so we are done.

We have reduced the proof of the main theorem for regular hexagons to the case in which every matching class has at least three edges.

*Case 10. There is a matching class with three edges which are not improperly congruent to themselves.* Then two of them match only the third. We may assume that the third is $a$. Part a) of Figure 13 shows that if $b$ matches only $a$, then $a$ matches only $b$, which is impossible. Part b) of Figure 13 shows that it is impossible for $c$ to match only $a$.

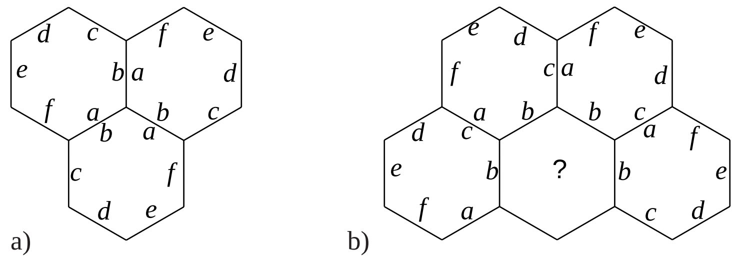


FIGURE 13. Showing that Case 10 is impossible

So it is impossible for either $b$ or $c$ to match only $a$. By symmetry the same is true for $e$ and $f$. Thus Case 10 is impossible.

*Case 11. The matching classes are* $\{a, b, c\}$ *and* $\{d, e, f\}$. By Case 10 the edge $a$ is improperly congruent to $b$, which is improperly congruent to $c$. Hence Lemma 3.2 implies that the vertices of $a \cup b \cup c$ are four vertices of a regular hexagon in order. The same is true for $d \cup e \cup f$. Hence the vertices of $T$ are the vertices of a regular hexagon in order, and so we are done.

*Case 12. The matching classes are* $\{a, b, d\}$ *and* $\{c, e, f\}$. By Case 10, the edges $a$, $b$ and $d$ are properly and improperly congruent to each other. The same is true of $c$, $e$ and $f$. Statement 2 of Lemma 3.1 implies that $K(b) = K(c) = 0$. Now Lemmas 3.1 and 3.3 imply that $a$ and $d$ are parallel. Since $a$ and $d$ are properly congruent, there exists $\sigma \in \mathrm{Isom}^+(\mathbf{R}^2)$ such that $\sigma(a) = d$ and $\sigma\big|_a$ preserves orientation. Because $a$ and $d$ are parallel, $\sigma$ is a rotation of order 2. Because the turning angle of $\partial T$ at $c \cap d$ equals the turning angle of $\partial T$ at $a \cap f$ and the edges $c$ and $f$ are properly congruent, $\sigma(c) = f$. So $\sigma$ permutes the vertices of $T$ in order. Statement 1 of Lemma 3.2 implies that the vertices of $a \cup b$ are three vertices of a regular hexagon in order. Applying $\sigma$, we see that the same is true of the vertices of $d \cup e$. As for $a \cup b$, the vertices of $e \cup f$ are three vertices of a regular hexagon in order. From this we see that the vertices of $d \cup e \cup f$ are four vertices of a regular hexagon in order. Applying $\sigma$, we see that the vertices of $a \cup b \cup c$ are four vertices of a regular hexagon in order. Thus the vertices of $T$ are the vertices of a regular hexagon in order, and we are done.

*Case 13. There are two matching classes containing three edges.* Suppose that the matching classes are $\{a, c, e\}$ and $\{b, d, f\}$. By symmetry we may assume that $a$ matches $c$. Figure 14 shows that this is impossible. Hence there exists a matching class with two adjacent edges. We may assume that $a$ and $b$ are in a matching class. Cases 11 and 12 and symmetry handle all the possibilities.

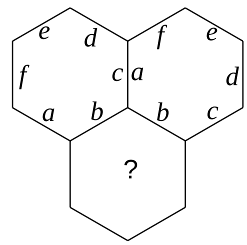


FIGURE 14. Ruling out one possibility in Case 13

We have proved the main theorem if there are at least two matching classes.

*Case 14. Edges $a$, $c$ and $e$ are properly congruent to each other and improperly congruent to $b$, $d$ and $f$.* Lemma 3.2 implies that every three consecutive vertices of $T$ are three consecutive vertices of a regular hexagon in order. Hence the vertices of $T$ are the vertices of a regular hexagon in order, and we are done.

*Case 15. Edges $a$, $b$ and $d$ are properly congruent to each other and improperly congruent to $c$, $e$ and $f$.* As in Case 14, Lemma 3.2 implies that the vertices of $a \cup f$ are three vertices of a regular hexagon in order. Similarly, the vertices of $b \cup c \cup d \cup e$ are the vertices of a regular hexagon in order. It follows that the vertices of $T$ are the vertices of a regular hexagon in order, and we are done.

*Case 16. Edges $a$, $b$ and $c$ are properly congruent to each other and improperly congruent to $d$, $e$ and $f$.* Since $K(a) = -K(d)$, by symmetry we may assume that $0 \leq K(a) \leq \pi$ modulo $2\pi$. If $K(a) \equiv \pi$ modulo $2\pi$, then Lemma 3.2 implies that the turning angle at $a \cap b$ determined by the two line segments joining $a \cap f$, $a \cap b$ and $b \cap c$ is congruent to $4\pi/3$ modulo $2\pi$. The same is true at $b \cap c$. But then the vertices of $a \cup b \cup c$ are the vertices of an equilateral triangle, which is impossible. If $K(a) \equiv 2\pi/3$ modulo $2\pi$, then Lemma 3.2 implies that this turning angle at $a \cap b$ is congruent to $\pi$ modulo $2\pi$. This means that $a \cap f = b \cap c$, which is impossible. If $K(a) \equiv \pi/3$ modulo $2\pi$, then the vertices of $a \cup b$ are again the vertices of an equilateral triangle, which is impossible. Thus we may assume that $K(a) \equiv 0$ modulo $2\pi$, and so each edge has total curvature 0 modulo $2\pi$. Now Lemma 3.2 implies that the vertices of $T$ are the vertices of a regular hexagon in order, and we are done.

*Case 17. There is only one matching class.* The total curvatures of the edges of $T$ have the same absolute value. As we have seen in Case 6, the sum of these total curvatures is 0. Cases 14, 15 and 16 and symmetry handle the cases in which three of these total curvatures have some value $K \neq 0$ and the other three total curvatures equal $-K$. Hence we may assume that the total curvature of every edge of $T$ is 0. Now Lemma 3.2 implies that the vertices of $T$ are the vertices of a regular hexagon in order, and we are done.

This proves the main theorem.

## References


[1] M. Berger, *Geometry I*, Springer-Verlag, Berlin Heidelberg New York, 1987.

[2] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Finite subdivision rules*, Conform. Geom. Dyn. **5** (2001), 153–196 (electronic).

[3] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Expansion maps for finite subdivision rules II*, preprint, available from http://www.math.vt.edu/people/floyd.

[4] Giles, J., Jr. Construction of replicating superfigures, J. Combinat. Theory A **26** (1979), 328–334.

[5] Giles, J., Jr., Superfigures replicating with polar symmetry, J. Combinat. Theory A **26** (1979), 335–337.



Department of Mathematics, Brigham Young University, Provo, UT 84602, U.S.A.

*E-mail address*: cannon@math.byu.edu

Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, U.S.A.

*E-mail address*: floyd@math.vt.edu

*URL*: http://www.math.vt.edu/people/floyd

Department of Mathematics, Eastern Michigan University, Ypsilanti, MI 48197, U.S.A.

*E-mail address*: walter.parry@emich.edu