THE LENGTH-AREA METHOD AND DISCRETE RIEMANN MAPPINGS

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This article is based on a talk given by the first author at the Ahlfors Celebration at Stanford University in September, 1997. That talk, as well as many others from the Ahlfors Celebration, can be seen from the streaming video archive at MSRI (http://www.msri.org).

J.R.R. Tolkien, author of *The Hobbit* and *The Lord of the Rings*, despaired of interruptions that took him away from his work. It's not so much the interruptions alone, he said, as the fear of interruptions. We all have our tricks to find uninterrupted hours for mathematics. When I was in Wisconsin it got really cold during the winter and it stayed really cold. The way I found uninterrupted time was to buy some shoes with heavy soles and a coat with a tunnel hood and walk to and from school. I felt guilty about avoiding people, because surely our relationships with others are the most important part of our lives. But then I remembered that many of our most valued relationships are those that we have with people that we have never seen and people that we've never talked with. Because of writing, those of us who live far from the centers of mathematics have our chance to spend our time with those who are truly great. This was exactly my relationship with Lars Ahlfors. I spent many, many hours learning complex variables from his book *Complex Analysis* [1], and later I read very carefully his book *Conformal Invariants* [2]. This was how I came to know Ahlfors and appreciate him.

In 1912/1913, in two consecutive issues of the *Mathematische Annalen*, Carathéodory published three very interesting papers. The first of these papers [13] was about the dependence of the Riemann mapping on boundary data. The second paper [14] was on continuous extensions to the boundary if the conformal mapping was defined on a Jordan domain. In the third [15], he took care of the problem when the boundary was not a simple closed curve and he defined the notion of prime end.

These three papers set the stage for the work of Lars Ahlfors in the following way. These results were published in 1922 in the textbook *Funktionentheorie* by Hurwitz and Courant [19]. But the proof that Hurwitz and Courant gave of the continuous extension to the boundary was different from the proof given by Carathéodory. They used the method which has come to be known as the length-area method. They say this about the source. "The explanation of the questions treated in §7 is due to *Osgood* and above all to *Carathéodory*, who in particular recognized the significance of prime ends. The presentation given here follows work of the author."[19, p. 353] That's as far as we've been able to trace the length-area method

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and certainly that's where Ahlfors says that he learned it in the next few years. At any rate, this sets the stage.

1. The length-area method

Lars Ahlfors, age 21, at the Eidgenössische Technische Hochschule at Zurich, in 1928 proved his first result with the help of Nevanlinna and Polya. Ahlfors's first paper, and many of his subsequent ones, had as salient feature what is now called the length-area method. We proceed to describe how the length-area method is used to create a combinatorial version of the Riemann mapping theorem.

For a rectangle or a right circular cylinder, there is a natural height H, a natural width W, and an area $A = H \cdot W$. The quotient H/W is what in computer science is called the aspect ratio and what we call the modulus. Note that

$$\frac{H^2}{A} = \frac{H \cdot H}{H \cdot W} = \frac{H}{W} = \frac{H \cdot W}{W \cdot W} = \frac{A}{W^2}.$$

These expressions are scale invariant. If we multiply the metric by λ, the new height is $H_\lambda = \lambda H$, the new width is $W_\lambda = \lambda W$, and the new area is $A_\lambda = \lambda^2 A$. The modulus of the scaled rectangle or right circular cylinder is

$$\frac{H_\lambda}{W_\lambda} = \frac{\lambda H}{\lambda W} = \frac{H}{W}.$$

So the modulus does not change under scaling. Since conformal mappings are just local changes of scale, there is some hope that this expression will then tell us something about conformal mappings.

The Riemann mapping theorem, certainly the most beautiful and amazing theorem that I know of, deals with what happens when we change a complex variable conformally. A conformal change is simply a local scaling, so we multiply the old metric not by a constant but by a positive function $\rho(z)$. This multiplies the arc-length element by $\rho(z)$ and the area element by $\rho^2(z)$. That is, we replace $|dz|$ by $\rho(z)|dz|$ and dA by $\rho^2(z)dA$. The Riemann mapping theorem, as applied to a topological quadrilateral (a disk with four distinguished boundary points), simply says that it's possible to choose the multiplier $\rho(z)$ in such a way that the metric turns the quadrilateral into a geometric rectangle. As applied to a ring (annulus), the Riemann mapping theorem says that it's possible to choose the multiplier so as to turn the ring into a right circular cylinder. Certainly a wonderful theorem is that the modulus, this length-area comparison, of the geometric rectangle or right circular cylinder is a complete conformal invariant.

Now we come to a result of Ahlfors and Beurling, the direct or intrinsic characterization of the modulus $M(Q)$ of a quadrilateral or ring Q. If Q is a quadrilateral, then we call one of its four sides the top and the opposite side the bottom. If Q is a ring, then we call one of its boundary components the top and the other boundary component the bottom. Since the modulus is a complete conformal invariant, we would hope to be able to define it without carrying out the Riemann mapping. Ahlfors and Beurling say that this is indeed true. For this, look at all possible conformal changes of metric. Each change gives us a new height H_ρ (the infimum of the length $\int_\sigma \rho(z)|dz|$ of a curve σ joining the top and the bottom) and a new area A_ρ ($\int_Q \rho(z)^2 dA$). The theorem is that

$$M(Q) = \sup_\rho \frac{H_\rho^2}{A_\rho}.$$

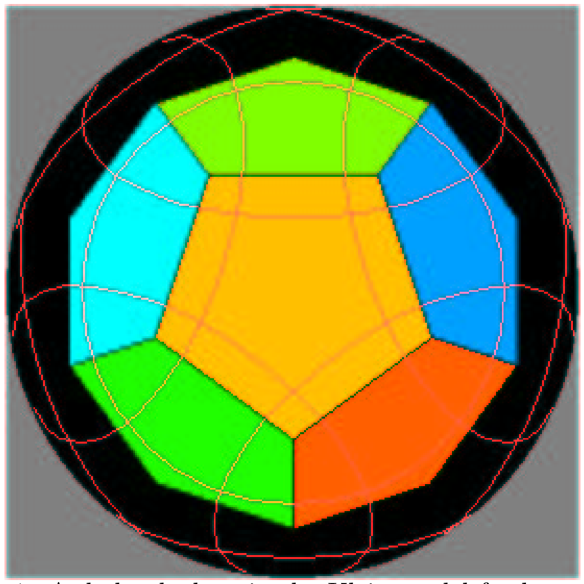

FIGURE 1. A dodecahedron in the Klein model for hyperbolic 3-space and its corresponding circles at infinity.

2. 3-MANIFOLD GROUPS

We wish to apply this length-area estimate to topology. Let M be a closed 3-manifold. From Thurston we have realized the importance of the following question: "When is the fundamental group of M Kleinian?" By Kleinian we mean only that the group is a discrete group of conformal homeomorphisms of the 2-sphere; we are not assuming that there is a nonempty domain of discontinuity. In this case the Kleinian group will have as its limit set the entire 2-sphere. There are two obvious (or almost obvious, or intuitively obvious) necessary conditions. First, the group itself must be the right kind of group. It has to be Gromov hyperbolic (see [17]), or negatively curved, as we would prefer to call it (see [6]). Second, the space at infinity of this Gromov hyperbolic group must be the 2-sphere. So these obvious necessary conditions become hypotheses and we find ourselves facing the task: find group invariant complex coordinates on the boundary 2-sphere.

In our attempt to solve this problem, we follow Jacobi's admonition, "You must always invert" [3, p. 323]. So we ask ourselves what the Kleinian structure would give us. When we analyze the geometry we find that there is at least a natural sequence of finite closed covers of the boundary, with a given cover subdividing the previous one. This sequence of covers, which comes naturally from the group, has a number of interesting properties. We next look at two examples.

We take a regular dodecahedron in hyperbolic 3-space with right angles. For our group we take the group generated by the reflections in the faces of our dodecahedron. This is a typical group of the kind we are talking about. We get a picture at infinity from this group by taking the faces of the dodecahedron and extending the planes containing them to infinity to get circles. These circles bound twelve closed disks which cover the 2-sphere. This is shown in Figure 1, which was created by the computer program SnapPea [25] from a modification written by Jeff Weeks. We obtain a tiling of hyperbolic 3-space by reflecting our dodecahedron in its faces

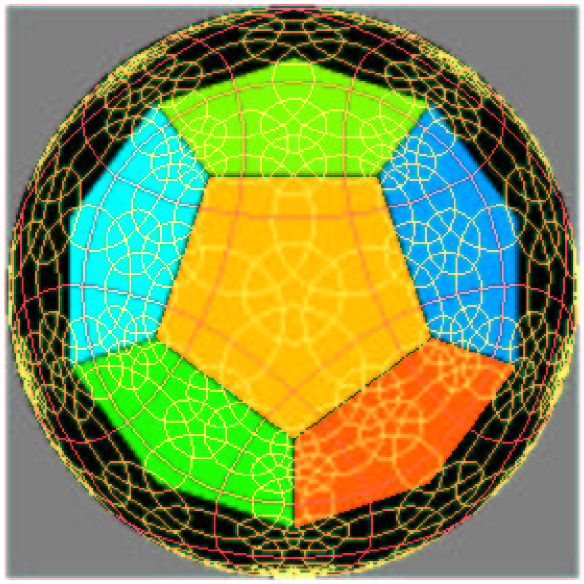

FIGURE 2. The pattern of circles at the second stage.

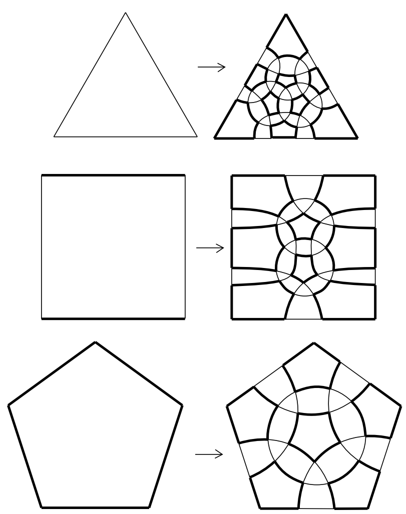

FIGURE 3. The subdivision rule for the dodecahedral reflection group.

and recursively reflecting the resulting dodecahedra in their faces. If we take all the dodecahedra that touch the initial one and repeat our construction of circles using their union, we get a more complex family of circles and a corresponding cover of the 2-sphere by closed disks. See Figure 2. If we examine how the circles fit together we find that there is a simple subdivision rule that tells us how to go from one cover to the next. See Figure 3. If one applies this rule one gets more and more complicated pictures at later stages, which we find very beautiful.

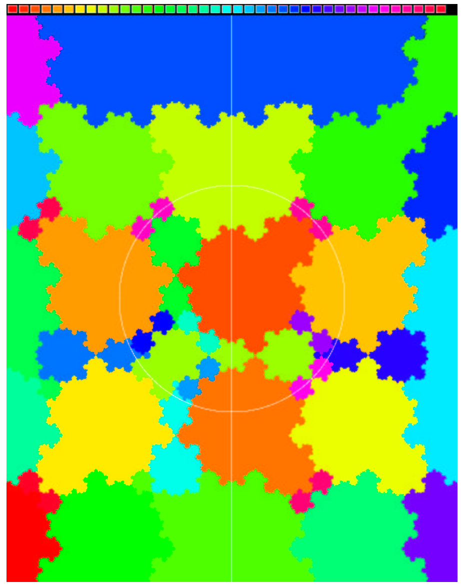

FIGURE 4. Part of a cover of the 2-sphere arising from a hyperbolic group.

So this is the kind of thing that we get from such a group. Lest the reader think that we always get circles, we show another example in Figure 4. This figure was created by André Rocha and Ian Redfern. In this case, the group is the group generated by the reflections in the faces of a hyperbolic tetrahedron with one ideal vertex. The figure shows part of one of the covers in the sequence of finite covers of the 2-sphere for this case. Somehow naturally in the group structure we always get beautiful structures at infinity. In this case we get closed disks which have fractal boundaries. But the disks look fairly substantial, and in fact this is always true. Furthermore, these disks are well spread out so they cover the 2-sphere fairly efficiently [12].

The above two examples of hyperbolic groups exhibit the kind of sequences of substantial finite covers that are suitable for a theory of differentiation. For example, one can read about Radon-Nikodym structures in Rudin's *Real and Complex Analysis* [22, Chapter 8]. So somehow out of the structure of the group alone we are able to get covers of the 2-sphere at infinity that are of the kind that can be used to build derivatives if we had a measure, which we should have if the group is truly Kleinian. Looking at our problem in this light, we find that we have an idea. We might seek a proof of the Riemann mapping theorem where the covers supply the derivatives and the image supplies the coordinates. We have no coordinates in the domain. We only have rough approximate combinatorial data, but of the kind suitable for constructing derivatives.

3. The Riemann mapping theorem

At this point I started looking around for proofs of the Riemann mapping theorem. I read Riemann's original paper [21], which is absolutely wonderful. I looked at Hilbert's *Über das Dirichletsche Prinzip* [18, pp. 10-14,15–37], and read *Conformal Invariants* [2], with its uniformization, by Ahlfors. Now once I was on a plane and was lucky enough to sit next to Fred Gehring. I can't remember how this happened;

he certainly didn't know me. I was a brand-new young topologist doing wild and tame topology, and he was a respected complex variables person. I told him that I had heard Bers's talks at Pennsylvania and they were wonderful and how would I learn about them. Gehring said that I should go read *Quasikonforme Abbildungen* by Lehto and Virtanen [20]. By reading Lehto and Virtanen, I learned the Ahlfors-Beurling method of direct characterization of modulus. Modulus is the supremum over ρ of H_ρ^2/A_ρ, and this supremum is realized when ρ is equal to the absolute value of the derivative of the Riemann mapping. This is really a remarkable thing. We are taking a supremum of something over all suitable nonnegative functions on a quadrilateral. That has nothing to do with coordinates and, not only is the supremum realized for some function ρ, but the function ρ is the absolute value of the derivative of the Riemann mapping. Somehow there is a Riemann mapping hidden in this supremum. So it seems that maybe we could use this supremum to characterize the derivative, then use the derivative to define the Riemann mapping, and then we would have the Riemann mapping theorem without coordinates. We might ask: How does one of these optimal ρ's, the function giving the supremum, determine the Riemann mapping?

So let's just look at how that happens for a quadrilateral. We take an optimal ρ, and we analyze the quadrilateral. We pick an arbitrary point x, and we measure its distance from the top, call it a, and its distance from the bottom, call it b. Then we claim that $a+b$ is the height of the quadrilateral with respect to this optimal ρ. The proof is very simple. The height H is the infimum of the lengths of paths joining the top and the bottom, so $a+b$ can't be less than the height. If $a+b = H+\epsilon$ for some positive ϵ, then we simply reduce the multiplier on the $\epsilon/4$ neighborhood of the point x. This reduces the area because we've made the multiplier smaller, but we haven't reduced the height. And since we've reduced the denominator without reducing the numerator, the modulus increases, which is a contradiction.

So now we know that in terms of distance to the ends we exactly fill up this quadrilateral by level lines. A wonderful theorem from plane topology shows that these level lines are arcs joining the sides of our quadrilateral. We cut the quadrilateral into ϵ-strips along level lines. Then we try to analyze the shapes of these strips. For example, take a cantaloupe and a pocket knife. Then peel the cantaloupe. In so doing you are cutting the boundary of the cantaloupe into strips of constant width. If these strips are fairly narrow they lie down flat in the plane. You can try it. It's a little messy, but you could do it instead with apples or oranges. At any rate, we cut our quadrilateral into ϵ-strips. The claim is that these strips have square ends. If some strip doesn't have square ends, then looking at the appropriate endpoint of its level line and a point a little distance away, we find that the strip really isn't of constant width. This contradicts our assumption. So the ends have to be square.

We next claim that the strips have to be straight. If they weren't straight, they would locally look like rainbows. So let's take one rainbow and see what happens to that rainbow. Cut the rainbow in the middle by height. This cuts it in half by height, but not in half by area because the area of the top half is bigger than the area of the bottom half. Now we redress this wrong by multiplying the metric in the bottom half by $1+\epsilon$ to make the area there bigger and in the top half by $1-\epsilon$ to make the area there smaller. This multiplies the area of the top half by $(1-\epsilon)^2$ and multiplies the area of the bottom half by $(1+\epsilon)^2$. Nothing happens to the height of the strip because we've multiplied by canceling factors, but the area of the strip changes. Calculating the area of the strip, we find that the dominant term is the linear term and it has a negative coefficient. In other words, the area goes down, the height hasn't changed, the modulus goes up, and this is a contradiction to the fact that we have a maximum. So that means that the strip is straight. If the strip curves both ways, then we have to modify this argument and work a little harder.

Now we look at the quadrilateral with this metric. What does the quadrilateral look like? It's made up of straight planar strips with square ends. So we stack them together, and what do we get? We get a rectangle. That's the Riemann mapping theorem. That's how the Riemann mapping theorem is hidden in this length-area expression.

It's beautiful! This argument has only a little snake oil in it, and that actually can be taken care of.

4. The combinatorial Riemann mapping theorem

Now we want to make this combinatorial. We have this expression for modulus that requires for its understanding heights and areas. But we only have combinatorial approximations to these things, so we want to be able to define heights and areas by means of covers instead of coordinates. So we do the most naive thing imaginable. We fix a finite cover, and we assign a weight (size) to each element in our cover. How are we going to figure out an intrinsic way to do it? Well, we are going to eventually change these weights conformally, because this is a conformal problem. And what does it mean to change weight conformally? That's to multiply by a nonnegative number. So, it doesn't matter what weights we give to these elements. We just give each element a weight, and then we'll change the weights later if we don't like them.

So we assign a weight to each element in our finite cover. Now we can decide what the length of a curve is. The length of a curve is the sum of the weights of the elements that it intersects. Similarly, the area of a set is the sum of the squares of the weights of the elements that it intersects. Now we can define modulus. Let Q be a quadrilateral or ring. Let ρ be a nonzero nonnegative real-valued function on our finite cover which gives weights. Then the modulus $M(Q, \rho)$ of Q with respect to ρ is the height of Q squared over the area of Q. The modulus $M(Q)$ of Q is the supremum of the moduli $M(Q, \rho)$ over all choices of ρ. So now we have a modulus.

Does this combinatorial modulus relate to a Riemann mapping? Well, at the time that I was doing this, I just had faith that it did. I didn't know. But after I proved the combinatorial Riemann mapping theorem in [7], this was solved independently by John Robertson, by Oded Schramm [23], and by us [8]. The modulus $M(Q)$ of a quadrilateral Q, this supremum, is actually achieved for an optimal weight function that is unique up to scaling, and such an optimal weight function determines a Riemann mapping. We illustrate this with the following example. Figure 5 shows a topological quadrilateral Q, namely the continental United States. We have to choose corners for Q, so we obviously choose Washington, California, Florida, and Maine. The tiling by states gives us our cover by closed subsets. An optimal weight function tells us, in a way that is unique up to scaling, the exact size to make each state. We make each state a square with side length its weight. After all, we defined height (of every open tile) as weight and area as square of weight. These squares fit together precisely to form a rectangle, and the combinatorial modulus $M(Q)$ is the height of this rectangle divided by its width, just as in the continuous situation. These squares have the same adjacencies as in the original picture, except that

FIGURE 5. A topological quadrilateral and its squared rectangle.

that's impossible. We must allow certain things to happen. After all, we can't have six distinct squares meeting at a corner, which would be necessary for a tiling having a vertex of valence 6. We have to allow vertices to explode into vertical intervals. This happens at the four corners area of Utah, Colorado, New Mexico, and Arizona. Another thing that we must allow is that certain tiles are allowed to collapse to size 0. You will notice that some people's favorite states, in an area that we will always refer to as the Northeast, are absolutely inessential in this picture. They have collapsed to the upper right-hand corner. All of New England is cut off by New York; New York makes New England irrelevant. So these optimal weight functions definitely give Riemann mappings. This is a finite version. There is also a beautiful version for finite electrical circuits. In that case Kirchhoff's laws imply that the solution also gives a squaring of a rectangle [8].

So that's our first step. At this point we can handle one finite cover. Our next task is to take a limit over a sequence of finite covers to try to get complex coordinates. What would have to happen to be able to obtain complex coordinates by means of such a limit? We would need a uniform constant $K > 0$ such that the following condition is satisfied. Let Q be a quadrilateral or ring. Then for almost every cover in our sequence of finite covers the combinatorial modulus of Q with respect to our finite cover and the modulus of Q determined by the complex coordinates lie within a multiplicative factor of K of one another. This condition says that we're measuring conformal shapes in a compatible way. Now this condition leaves us with some difficulties. Namely, compatibility with complex coordinates is impossible to achieve if the combinatorial moduli either approach 0 (because a true quadrilateral or ring is never given modulus 0 in complex coordinates) or if they approach infinity. Moreover if the combinatorial moduli are self-incompatible then they can't be compatible with those given by complex coordinates. So our hypotheses must require that our combinatorial moduli are self-compatible and don't degenerate to 0 or infinity. Now these natural requirements seem like an innocent thing, but that's exactly the defect of [7]. Namely, it is really hard to tell whether our combinatorial data lead to degeneracies or to incompatibility.

For example, take barycentric subdivision. Start with a triangulation of the plane and then recursively subdivide barycentrically. For any starting triangulation, this leads to degenerate moduli [8]. In fact, every ring has modulus approaching infinity. One might think that this is because, as we subdivide barycentrically, valences of vertices become arbitrarily large, but there are examples with bounded valence that degenerate as well. Furthermore, there is a subdivision rule — it's not a finite subdivision rule in the sense of [9], but it is a local replacement rule — that leads to a quadrilateral that has infinitely oscillating moduli. In other words, there is a finite local rule producing a sequence of finite tilings of a quadrilateral which conformally gets long and skinny, and then it gets fatter and fatter, and then exponentially longer and skinnier, and then exponentially fatter, and so forth.

So bad things can happen with a subdivision rule, but we do have the following combinatorial Riemann mapping theorem [7]. Roughly speaking, if the combinatorial moduli arising from our sequence of covers are nondegenerate and self-compatible, then in fact compatible complex coordinates exist. Furthermore, by a theorem of Cannon-Swenson [12], if our covers come from a negatively curved group whose space at infinity is the 2-sphere in the way that we indicated by example, then these combinatorial moduli are nondegenerate and self-compatible if and only if the group is Kleinian.

Proving the combinatorial Riemann mapping theorem of [7] is of course a problem of solving a differential equation without coordinates to give derivatives. Now, according to Clifford Taubes [24], there are two ways to solve a differential equation. They involve an Arzela-Ascoli argument or the contraction mapping principle. Our proof is no exception; it uses an Arzela-Ascoli argument. We construct a sequence of finite approximations to a metric and get the sequence to converge. Once we get it to converge, then we have to show that the limit metric is somehow quasiconformal. This involves a combinatorial version of the theorem that says that a limit of K-quasiconformal mappings is K-quasiconformal. To prove this we imitate the standard proof in Lehto and Virtanen [20]. But there's a problem. That proof uses a logarithm, and logarithms are defined in terms of coordinates. So we have to build a combinatorial logarithm. There is no intrinsic difficulty in building a combinatorial logarithm, but I had considerable difficulty in showing that this combinatorial logarithm is well behaved. This difficulty kept me stuck for at least a month (probably longer, maybe three months). Many of the proofs in [7] involve calculus of variations on finite sets, so they are very standard types of proofs. But the calculus of variations can be applied additively or multiplicatively, and realizing that I should work multiplicatively instead of additively was the key to controlling the combinatorial logarithm.

This brings us to the end game. There are miracles in the end game, and I can't get over these miracles. We are working with a cylinder, and we want this cylinder in the limit metric to be a perfect right-circular cylinder with complex coordinates. After analyzing our cylinder, we realize that, yes, it has constant height by essentially the argument that we gave in the continuous case. Then we analyze the level curves. The level curves are nice topological simple closed curves that separate top from bottom, and fit together continuously. Beautiful, it looks like a right circular cylinder. Then we analyze the lengths of those curves. They have finite length; great! But we don't want just finite length, we want constant length. So we look at the lengths, and we find that the lengths vary discontinuously in some examples. What a downer! But then a miracle happens! Circumference should be the derivative of area with respect to height. So we change the metric on the cylinder in the horizontal direction to get what we want by taking the derivative. What derivative? We prove that the derivative exists, and it turns this thing into an exact right-circular cylinder. It's just like someone came along and fixed all of our errors, just ironed them right out, and it just goes "whusst!", right-circular cylinder, complex coordinates, quasiconformal change of coordinates. Everything works, and that proves the theorem.

5. How should we view examples?

So that's the combinatorial Riemann mapping theorem. It's a lot of fun, but it doesn't prove what we wanted. What we wanted is to prove that if we start with a negatively curved group with space at infinity the 2-sphere, then we do have compatible nondegenerate moduli and we do have coordinates, and so the group is Kleinian. But we have to prove that those moduli are nondegenerate and compatible. Actually, in [11] we prove in the case of a negatively curved group with space at infinity the 2-sphere that it suffices to bound the moduli from 0. Despite this considerable simplification, there is still much to prove. So now we start trying to understand how to measure those moduli and how to approximate them. This is a very difficult problem. One of the difficulties is that it is really hard to write down examples of groups and examine them exactly, because as we have seen, we are immediately led to very complex subdivisions. So we have been looking at simple subdivision rules unrelated to groups. It is easy to get some quite simple ones; it's just hard to get ones that come from groups. The theory of subdivision rules is really a fascinating theory; it has all of complex analysis built into it in finite terms. There are lots of beautiful problems that come up in trying to understand combinatorial things conformally.

Let's finish by looking at various ways of viewing subdivision rules. Figure 6 shows a simple subdivision rule, the pentagonal subdivision rule. We take edges and bisect them, and we take pentagons and divide them into six pentagons as shown. Notice that we can't do it so that the pentagons are similar. How should we view this to construct future subdivisions? Well, there's the affine approach. We just subdivide every edge by inserting its midpoint and extend affinely. Scott

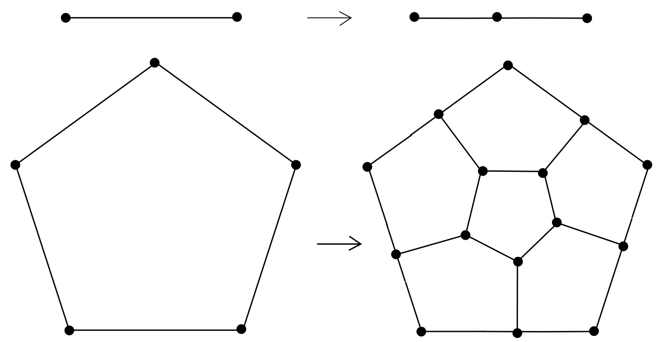

FIGURE 6. The pentagonal subdivision rule.

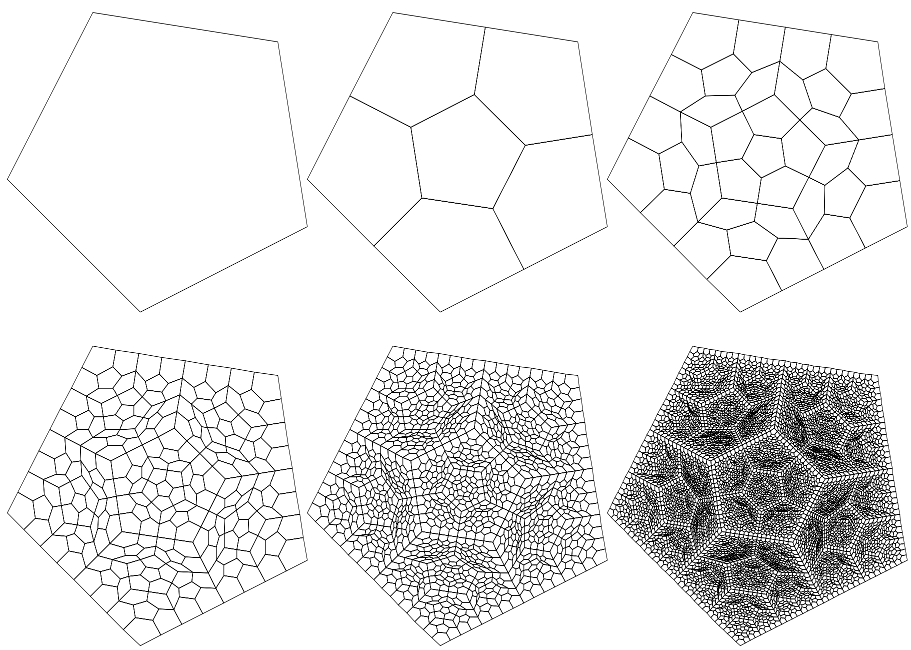

FIGURE 7. Subdividing the pentagons affinely.

Sheffield, for a student project one summer, programmed a computer to draw these affine subdivisions. Figure 7 shows output from his program for the pentagonal subdivision rule. Notice that stresses and strains start to develop. In trying to understand these conformally, we are asking the following questions. Are these stresses and strains intrinsic? Can they be avoided? By changing somehow our way of viewing the picture, can we make them disappear? Can we see that they are really irrelevant? (Answers: No. Yes. Yes. Yes.)

We could try to understand the shapes by squaring rectangles. The problem with squaring is that we always change things into rectangles, and the rectangles look worse than what we started with. For example, take a pentagon, and view it as a quadrilateral Q by choosing one edge to be the bottom and the pair of edges disjoint from it to be the top. Figure 8 is the squared rectangle (created using the cyclic algorithm from [8]) for the sixth subdivision of Q. Isn't that enlightening! This is what the length-area method gives. But if we keep track of what happens to

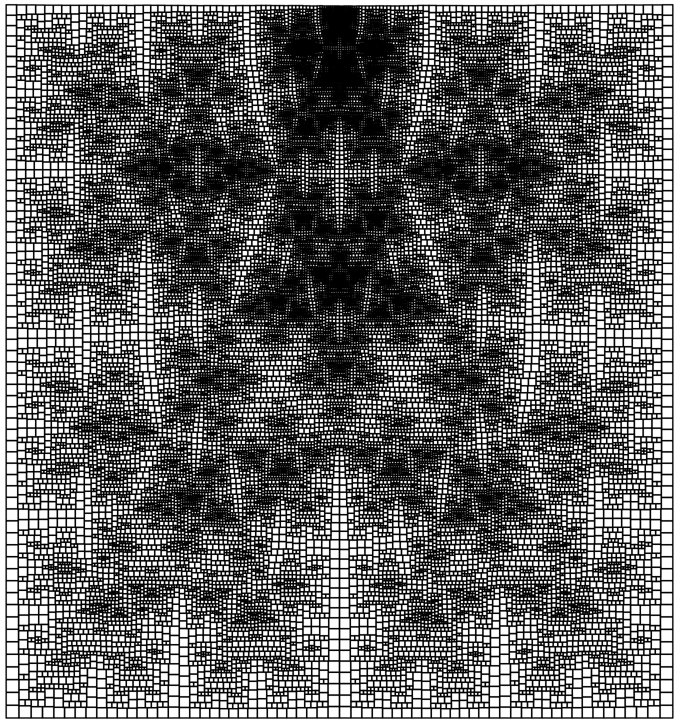

Figure 8. A squared rectangle for the sixth subdivision of the pentagonal subdivision rule. There are 46,656 squares.

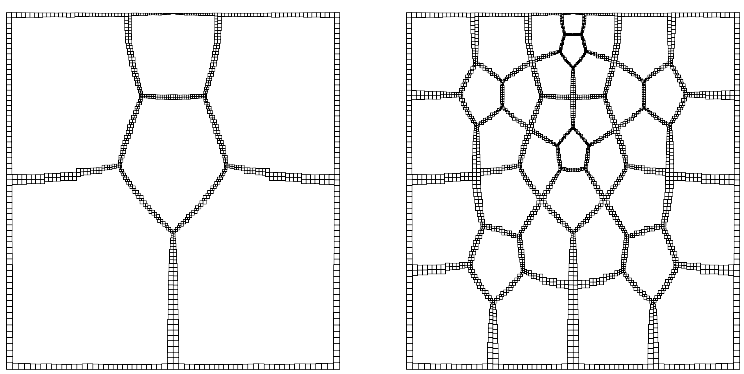

Figure 9. Squares corresponding to tiles that meet the boundary of the first subdivision and of the second subdivision.

our boundaries in earlier stages we find that there really is conformal information hidden in this. Figure 9 shows this for the traces of the edges of the first subdivision and for the traces of the edges of the second subdivision. This is starting to get really close to what we think is the conformal shape of the tiles under subdivision.

Several years ago, Ken Stephenson suggested using circle packings to view the shapes. (See, for example, [4] or [16] for details about circle packings.) Given a tiling of a disk, triangulate it by inserting a barycenter in each face and joining the

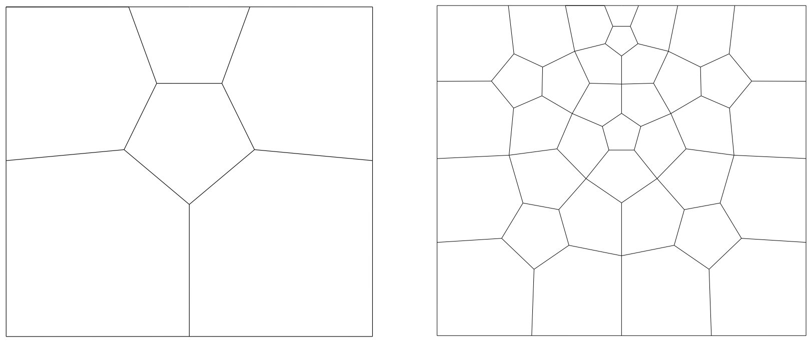

FIGURE 10. Circle packing pictures for the first two subdivisions of the pentagonal subdivision rule.

barycenter with every vertex of the face by an edge. Then realize the triangulation by a circle packing and draw the part of the carrier of the packing corresponding to the tiling (that is, don't draw the edges that were added to get a triangulation). Figure 10 shows the output of this for the first and second subdivision of a pentagon.

The circle packing method and the length-area method are both giving something that seems to be right in the quasiconformal sense. So they seem about equivalent in the end. But certainly we get more quickly at things that look good by circle packing. Figure 11 shows a squared rectangle and a circle packing picture for the third subdivision of a quadrilateral with respect to the dodecahedral subdivision rule (Figure 3). The squared rectangle isn't nearly as good as the circle packing picture.

Ken Stephenson's circle packing program, CirclePack, is available from his web page, www.math.utk.edu/∼kens. Our programs, subdivide.c and tilepack.c, which implement the subdivision of a quadrilateral by a finite subdivision rule and write the input to CirclePack, are available from www.math.vt.edu/people/floyd. The program tilepack.c writes a script file for packing a quadrilateral (it was used, for example, with CirclePack to produce Figure 10), but the output can be amended to pack a disk or a different polygon. For example, Figure 12 is a circle packing picture, from a script file written by tilepack.c, of the third subdivision of a pentagon with respect to the pentagonal subdivision rule.

Bowers and Stephenson [5] looked at successive subdivisions of a pentagon with respect to the pentagonal subdivision rule. They thought in terms of an expansion process instead of a subdivision process. That is, they thought of each subdivision as being contained naturally in the next one and then took a direct limit. They discovered that the direct limit complex has a natural conformal structure, yielding the complex plane, in which all of the pentagons are conformally equivalent and the expansion map is given by multiplication by a complex number. Even though the seed (starting pentagon) is unique, it is still true that the entire conformal complex can be generated from any one pentagon by successive reflections in the edges. Their ideas look promising for our work and raise the possibility of solving our problem (showing that a negatively curved group with space at infinity a 2-sphere is Kleinian) by finding a fixed point of a map of some Teichmüller space.

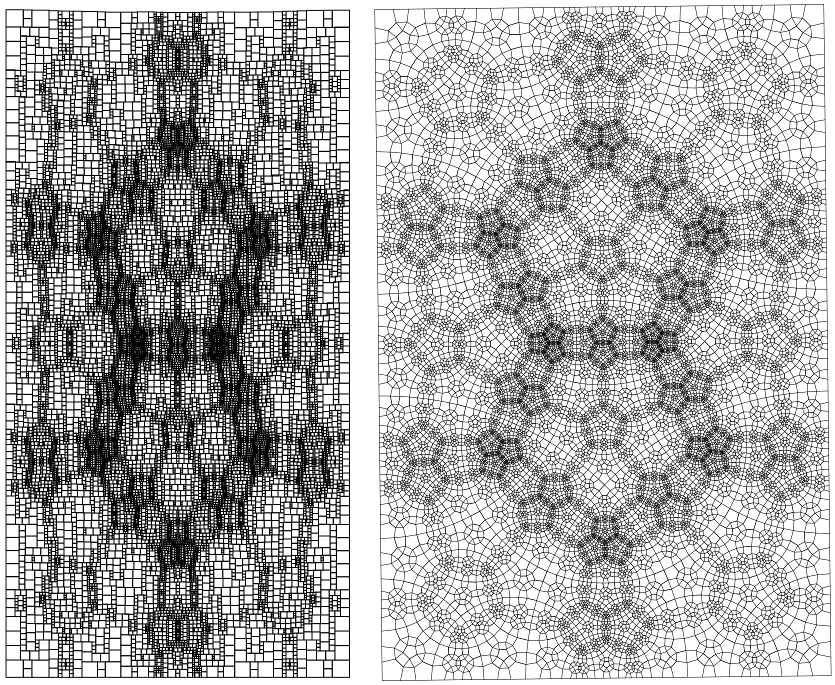

FIGURE 11. A squared rectangle and a circle packing picture for the dodecahedral subdivision rule. There are 27,839 tiles.

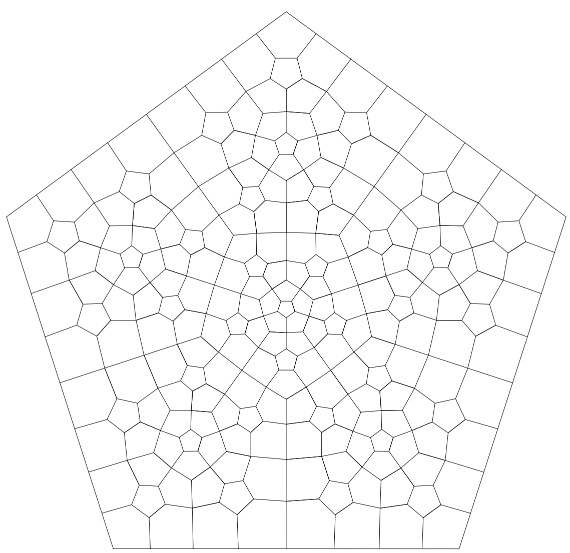

FIGURE 12. A circle packing picture of the third subdivision of a pentagon with respect to the pentagonal subdivision rule.

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