ACTIONS OF CLASSICAL SMALL CATEGORIES

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Preface

It is our intent here to write an interpretive account of the study of the actions of the classical small categories of topology, reasonably complete in its topological presentation up through about 1975. We are far from expert in this area, but approach it from the related field of equivariant topology. Our feeling is that the colimits, homotopy colimits, and realizations of these categories have made large contributions to the core of topology, and that this language provides a good framework for thinking about these contributions.

The original plan for this project was an idea of Ed Floyd's from 1980. Because of his commitments as Dean of the Faculty and then Provost at the University of Virginia, he did not get to work seriously on the project for several years after that. We started working on the project together in 1987, and had a completed manuscript in 1989.

We started working on a major revision of the manuscript in 1990, at least partly to take into account work of Baues and Meiwes pertaining to the cubical category. It has become clear to me in the years since Ed's death at the end of 1990 that I would not get the revision finished. This book is essentially the manuscript that we finished in 1989.

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Bill Floyd

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Introduction

The Equivariant General Topology Setting

We first settle on a preferred category TOP of spaces and maps, the compactly generated spaces of McCord [1.3]. If G is a small category, then a *G-space* Y is a covariant functor $Y : G \to \text{TOP}$ and a *G^o-space* X is a contravariant functor $X : G \to \text{TOP}$. Thus Y assigns to each object p of G a compactly generated space $Y(p)$, and to each morphism $g : p \to q$ of G a map

$$g_* : Y(p) \to Y(q), \quad y \mapsto gy$$

such that the functorial conditions hold. Given G-spaces Y and Y', then a G-map $\phi : Y \to Y'$ is a natural transformation of functors, thus is a collection of maps $\phi_p : Y(p) \to Y'(p)$ commuting with the action maps g_*. We arrive at the category TOP^G of G-spaces, and the category TOP^{G^o} of G^o-spaces. These are classical examples of category theory.

Given a G^o-space X and a G-space Y, there is the space

$$X \times_G Y = \coprod X(p) \times Y(p)/\sim,$$

where $\sim$ is the least equivalence relation such that given

$$x \in X(p), \quad p \xleftarrow{g} q, \quad y \in Y(q)$$

then $(xg, y) \sim (x, gy)$. The space $X \times_G Y$ is often not compactly generated, so that this bifunctor is of the form

$$\times_G : \text{TOP}^{G^o} \times \text{TOP}^G \to \text{Top},$$

where Top is the category of k-spaces.

There is a terminal object in the category TOP^G, the G-space *Ob* G which associates with each object p of G the singleton $\{p\}$. Setting $Y = Ob\ G$ in $X \times_G Y$ yields the *colimit* of the G^o-space X, the space

$$X/G = colim\ X = \coprod X(p)/\sim.$$

A major purpose of this work is to consider the generalized colimits assigning to a G^o-space X the space $X \times_G Y$, where Y is some fixed G-space, usually with each $Y(p)$ contractible. The basic language outlined here is set up in Chapter 1.

The Category TOP^{Δ^o} of Simplicial Spaces

There is in a sense a universal setting for this equivariant general topology, the category TOP^{Δ^o} of *simplicial spaces.* Here Δ is the category whose objects are the non-negative integers, and whose morphisms $\delta : m \to n$ are the order preserving functions

$$\delta : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}.$$

The word *simplicial* comes from the Δ-space ∇ which assigns to n the standard n-simplex $\nabla(n)$, with ordered vertices $v_{n,0}, v_{n,1}, \cdots, v_{n,n}$, and to $\delta : m \to n$ the unique affine map $\delta_* : \nabla(m) \to \nabla(n)$ for which $\delta_*(v_{m,i}) = v_{n,\delta(i)}$. We thus have Milnor's realization functor [2.9]

$$|\diamond| : \mathrm{TOP}^{\Delta^o} \to \mathrm{TOP}, \quad X \mapsto |X| = X \times_\Delta \nabla,$$

perhaps the most important single generalized colimit.

A *topological category* G is a small category G whose sets *Ob* G of objects and Mor G of morphisms are compactly generated spaces, with the structure functions continuous. Denote by TOPCAT the category whose objects are the topological categories and whose morphisms are the continuous functors. The universality of TOP^{Δ^o} comes from Segal's nerve functor [2.11]

$$N : \mathrm{TOPCAT} \to \mathrm{TOP}^{\Delta^o}.$$

Here one interprets Δ as a category whose objects are categories

$$\underline{n} : \quad 0 \leftarrow 1 \leftarrow \cdots \leftarrow n,$$

and whose morphisms are the functors $\underline{m} \to \underline{n}$. Then given a topological category G, one gets the Δ^o-space $NG = \{G^{\underline{n}}\}$, where $G^{\underline{n}}$ denotes the space of functors $\underline{n} \to G$, equivalently the space of diagrams

$$p_0 \xleftarrow{g_1} p_1 \xleftarrow{g_2} \cdots \xleftarrow{g_n} p_n$$

in G.

For any small category G, there are functors

$$M_0 : \mathrm{TOP}^{G^o} \to \mathrm{TOPCAT}, \quad M_1 : \mathrm{TOP}^G \to \mathrm{TOPCAT},$$
$$M : \mathrm{TOP}^{G^o} \times \mathrm{TOP}^G \to \mathrm{TOPCAT}$$

due to Segal [2.11] and May [2.6]. Given a G-space Y, $M_1 Y$ has objects $y \in \coprod Y(p)$ and morphisms $gy \xleftarrow{(g,y)} y$ for

$$p \xleftarrow{g} q, \quad y \in Y(q),$$

while $M_0 X$ has objects $x \in \coprod X(p)$ and morphisms $x \xleftarrow{(x,g)} xg$ for

$$x \in X(p), \quad p \xleftarrow{g} q,$$

and $M(X,Y)$ has objects $(x,y) \in \coprod X(p) \times Y(p)$ and morphisms $(x,gy) \xleftarrow{(x,g,y)} (xg,y)$ for

$$x \in X(p), \quad p \xleftarrow{g} q, \quad y \in Y(q).$$

For a G-space Y and for a fixed object p of G, define a G^o-space X by letting $X(q)$ be the discrete space of morphisms $g : q \to p$. Applying the composition

$$\mathrm{TOP}^{G^o} \times \mathrm{TOP}^G \xrightarrow{M} \mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o} \xrightarrow{|\diamond|} \mathrm{TOP}$$

to the pair (X,Y), we get a compactly generated space which we denote by $(E_G Y)(p)$. Letting p vary, we get a G-space $E_G Y$. The colimit $B_G Y$ of $E_G Y$ is obtained by applying the composition

$$\mathrm{TOP}^G \xrightarrow{M_1} \mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o} \xrightarrow{|\diamond|} \mathrm{TOP}$$

to the G-space Y.

These classical applications of simplicial topology to equivariant topology are set up in Chapter 2.

The Homotopy Colimits of a G-Space

In the years 1965-1975, Boardman-Vogt [4.1], May [2.8], and Segal [2.11,4.4] codified in somewhat different ways a homotopy theory of actions of small categories. The fundamentals of this theory are reviewed in Chapter 4, the needed general homotopy theory having been reviewed in Chapter 3.

Both TOP^G and TOP^{G^o} have associated homotopy categories. Given a G-space Y, one can form a G-space $I \times Y$ by letting $(I \times Y)(p) = I \times Y(p)$, with action $g(t,y) = (t,gy)$. There results a homotopy relation on the G-maps $\phi : Y \to Y'$. The homotopy category corresponding to TOP^G has objects the G-spaces, and morphisms $Y \to Y'$ the homotopy classes of G-maps $[\phi] : Y \to Y'$. A *homotopy equivalence* $\phi : Y \to Y'$ *in* TOP^G is a G-map such that $[\phi]$ is an isomorphism in the homotopy category. If HE denotes the subcategory of TOP^G whose morphisms are the homotopy equivalences in TOP^G, then we denote the homotopy category by $\mathrm{TOP}^G\,[\mathrm{HE}^{-1}]$, as the result of inverting the homotopy equivalences in TOP^G.

A *weak homotopy equivalence* $\phi : Y \to Y'$ *in* TOP^G is a G^o-map for which each $\phi_p : Y(p) \to Y'(p)$ is a homotopy equivalence in TOP. Denote by WHE the subcategory of TOP^G whose morphisms are the weak homotopy equivalences in TOP^G.

Since the colimit functor $\mathrm{TOP}^G \to \mathrm{Top}$ is a delicate invariant, for example is often not compactly generated, we seek G-spaces whose colimits are well behaved. For each collection $A = \{A(p) | p \in Ob\ G\}$ of compactly generated spaces, form the G-space $G \times_{Ob\ G} A$ which has

$$(G \times_{Ob\ G} A)(p) = \{(g,a) | p \xleftarrow{g} q,\ a \in A(q)\},$$

with action $g'(g,a) = (g'g,a)$.

Define a G-space Y to be a *principal G-space* if there exists a closed filtration $Y = \bigcup Y_n$ in TOP^G with

(i) a homeomorphism in TOP^G of some $G \times_{Ob\ G} A_0$ onto Y_0, and

(ii) for $n > 0$ a relative homeomorphism

$$(G \times_{Ob\ G} A_n, G \times_{Ob\ G} B_n) \to (Y_n, Y_{n-1})$$

in TOP^G, for some collection $(A_n, B_n) = \{(A_n(p), B_n(p))\}$ of closed cofibered pairs in TOP.

Denote by PRINC TOP^G the full subcategory of TOP^G whose objects are the principal G-spaces. The following propositions largely adapted from Boardman-Vogt codify the fact that

$$colim : \mathrm{PRINC\ TOP}^G \to \mathrm{TOP}$$

makes an interesting substitute for $colim : \mathrm{TOP}^G \to \mathrm{Top}$.

1. If $\phi : Y \to Y'$ is a G-map joining principal G-spaces, then ϕ is a homotopy equivalence in TOP^G iff ϕ is a weak homotopy equivalence in TOP^G.
2. Given a diagram $Y' \xrightarrow{\phi} Y \xleftarrow{\theta} Y''$ in TOP^G, where Y' is a principal G-space and θ is a weak homotopy equivalence in TOP^G, then there exists a unique homotopy class $[\mu] : Y' \to Y''$ of G-maps with $[\phi] = [\theta\mu]$.
3. Call a *principalization* of a G-space Y a principal G-space EY together with a weak homotopy equivalence $\pi : EY \to Y$ in TOP^G. Every G-space Y has a principalization; in fact, the natural G-map $E_G Y \to Y$ is a principalization. Any two principalizations are joined by a natural homotopy class of homotopy equivalences in TOP^G; hence their colimits are joined by a natural homotopy class of homotopy equivalences in TOP.

Given a G-space Y, define its *standard homotopy colimit* to be the colimit $B_G Y$ of the principalization $E_G Y$. A *homotopy colimit* of Y is a compactly generated space C together with a homotopy class of homotopy equivalences $B_G Y \to C$ in TOP.

If Y is the terminal G-space $Ob\ G$, any principalization E of Y is called a *universal G-space* and its colimit E/G is called a *classifying space* B for G. There is the standard universal G- space E_G and the standard classifying space B_G.

Homotopy colimits of G^o-spaces have an analogous treatment, with the standard homotopy colimit of a G^o-space X denoted by $B_{G^o} X$. There is an identification

$$B_{G^o} X = X \times_G E_G,$$

and if E is any universal G-space then $X \times_G E$ is a homotopy colimit of X.

One can consider this theory as a part of the study of the category

$$\mathrm{TOP}^G\ [\mathrm{WHE}^{-1}]$$

in which the weak homotopy equivalences in TOP^G have been inverted. A precise model is the category whose objects are the G-spaces and whose morphisms $Y \to Y'$ are the homotopy classes $[\phi] : E_G Y \to E_G Y'$ of G-maps $\phi : E_G Y \to E_G Y'$. Equivalently the morphisms can be taken as the homotopy classes of G-maps $E_G Y \to Y'$.

It is useful to be able to exhibit a wide class of principal G-spaces. In Chapter 5, we adapt from May [2.6] and Quillen [5.5] that for each functor $\theta : H \to G$ of small categories one gets a principal G-space $\theta_\# E_H$. For each object p of G, let

$$(\theta_\# E_H)(p) = \{(g, r, x) | p \xleftarrow{g} \theta(r), r \in Ob\, H, x \in E_H(r)\}/ \sim,$$

where $\sim$ is the least equivalence relation such that if $h : s \to r$ is a morphism of H then $(g, r, hx) \sim (g\theta(h), s, x)$. If each $(\theta_\# E_H)(p)$ is contractible, then $\theta_\# E_H$ is a universal G-space and homotopy colimits of G^o-spaces are produced by

$$X \times_G \theta_\# E_H \simeq \theta^\# X \times_H E_H = B_{H^o} \theta^\# X,$$

where $\theta^\# X$ is the H-space given by $(\theta^\# X)(r) = X(\theta(r))$.

There is a dual development of homotopy limits. Homotopy limits are more difficult and therefore have been developed more recently, and in this review of older work we tend to concentrate on homotopy colimits.

Homotopy Colimits of Simplicial Spaces

These are considered in Chapter 6. In order to consider homotopy colimits of all simplicial spaces, we need a certain functor $\theta : H \to \Delta$ for which $\theta_\# E_H$ is a universal Δ-space. Denote by $Mono\, \Delta$ the subcategory of Δ whose morphisms are the order preserving monos

$$\delta : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}.$$

Then $E_{Mono\, \Delta}(n)$ is the standard simplex $\nabla(n)$, with the action maps $\delta_* : \nabla(m) \to \nabla(n)$ given by the face operators. If $i : Mono\ \Delta \to \Delta$ is the inclusion functor, we show that $i_\# E_{Mono\, \Delta}$ is a universal Δ-space. By mild abuse of language, we then exhibit the homotopy colimit

$$BX = X \times_{Mono\, \Delta} \nabla$$

of Δ^o-spaces X; this we have learned from Segal [4.4].

We then consider the natural map

$$X \times_{Mono\, \Delta} \nabla \to |X| = X \times_\Delta \nabla,$$

and place conditions on X which ensure that the map is a homotopy equivalence, i.e. that $|X|$ is a homotopy colimit of X. For each order preserving epi $\delta : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}$, there is the closed pair $(X(m), \delta^* X(n))$. The Δ^o-space X *satisfies the cofibration condition for simplicial spaces* if each of these is a cofibered pair. For such simplicial spaces X, $|X|$ is a homotopy colimit of X.

We now generalize TOP^G and TOP^{G^o} to the case where G is a topological category, and do so in two slightly different ways. In the first case, say that a topological category G *satisfies the cofibration condition for topological categories* if the pair $(Mor\ G, Id\ G)$ is a cofibered pair of spaces over $Ob\ G \times Ob\ G$. In this case, one can simply mimic the treatment of TOP^G for G an untopologized small category, using the composition

$$\mathrm{TOP}^G \xrightarrow{M_1} \mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o} \xrightarrow{|\diamond|} \mathrm{TOP}$$

to produce $B_G Y$,etc.

For all topological categories, one can follow a second option by replacing $|\diamond| : \mathrm{TOP}^{\Delta^o} \to \mathrm{TOP}$ by $B : \mathrm{TOP}^{\Delta^o} \to \mathrm{TOP}$, otherwise following the previous pattern. We denote the new standard models in this case by $\mathcal{E}_G Y$, $\mathcal{B}_G Y$, $\mathcal{E}_G$, and $\mathcal{B}_G$.

We consider the case in which G is a *topological monoid*, i.e. a topological category with a single object. Then all the structure is subsumed under $Mor\ G$, and we denote $Mor\ G$ simply by G. The nerve NG of G is then $NG = \{G^n\}$, where G^0 is the singleton $\{1\}$. We prove that if G is a topological monoid with homotopy inverses, then the natural inclusion

$$G \to \Omega\mathcal{B}_G$$

is a homotopy equivalence in TOP.

We then obtain the theorem of James [6.4]. Denote by TOP_* the category of compactly generated spaces A with base point a_0. There is the functor of James

$$J : \mathrm{TOP}_* \to \mathrm{TOP}\ MON, \quad X \mapsto JX$$

where $JX = \coprod X^n/\sim$, with $\sim$ the least equivalence relation allowing the deletion of any coordinate which is the base point. If X has cofibered base point, then the reduced suspension SX is a classifying space for JX. If also JX is path connected and of the homotopy type of a CW-complex, then JX has homotopy inverses. Hence if all these conditions hold, the natural map $JX \to \Omega SX$ is a homotopy equivalence.

Converting Simplicial Spaces into Topological Categories

We now use homotopy colimits to produce a functor

$$W : \mathrm{TOP}^{\Delta^o} \to \mathrm{TOPCAT}.$$

If we use colimits instead of homotopy colimits, and ignore topology, we obtain the Gabriel-Zisman functor [2.4]

$$\mathrm{SET}^{\Delta^o} \to \mathrm{CAT}$$

adjoint to the nerve functor. If we consider the composition

$$\mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o} \xrightarrow{W} \mathrm{TOPCAT},$$

we obtain an explosion functor

$$\mathrm{TOPCAT} \to \mathrm{TOPCAT}$$

analogous to a Boardman-Vogt construction [4.1]. If we consider the composition

$$\mathrm{TOP}^{\Delta^o} \xrightarrow{W} \mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o},$$

we obtain a functor

$$\mathrm{TOP}^{\Delta^o} \to \mathrm{TOP}^{\Delta^o},$$

which is analogous to a construction of Segal and which converts a simplicial space into the nerve of a topological category. Thus in Chapter 7 we are extending work of Gabriel-Zisman, Boardman-Vogt, and Segal.

If homotopy colimits of G^o-spaces X are to have more structure than that of a space, then additional structure must be placed on G, or X, or both.

A small category G is *strictly monoidal* if we are given a bifunctor

$$\oplus : G \times G \to G$$

which is associative, and an object 0 of G which is an identity element for $Ob\, G$ and is such that 1_0 is an identity element for $Mor\, G$.

Let G be a strictly monoidal small category, and let X be a G^o-space. From $\oplus : G \times G \to G$ we get

$$\oplus^{\#} : \mathrm{TOP}^{G^o} \to \mathrm{TOP}^{G^o \times G^o}$$

and the $G^o \times G^o$-space $\oplus^{\#} X$ with

$$(\oplus^{\#} X)(p, q) = X(p \oplus q).$$

There is also the $G^o \times G^o$-space $X \times X$. Then X is *comultiplicative* if we are given a $G^o \times G^o$-map $\phi : \oplus^{\#} X \to X \times X$, equivalently equivariant maps $\phi_{p,q} : X(p \oplus q) \to X(p) \times X(q)$, which is associative and has both compositions

$$X(p) = X(p \oplus 0) \xrightarrow{\phi_{p,0}} X(p) \times X(0) \xrightarrow{proj} X(p),$$

$$X(p) = X(0 \oplus p) \xrightarrow{\phi_{0,p}} X(0) \times X(p) \xrightarrow{proj} X(p)$$

the identity. Using the projections onto $X(0)$ instead of the projections onto $X(p)$, each $X(p)$ receives the structure of a space over $X(0) \times X(0)$. The maps $\phi_{p,q}$ then become maps

$$\phi_{p,q} : X(p \oplus q) \to X(p) \times_{X(0)} X(q)$$

of spaces over $X(0) \times X(0)$. The G^o-space X is *strictly comultiplicative* if each of these maps is a homeomorphism. Denote by COMULT TOP^{G^o} the category whose objects are the comultiplicative G^o-spaces and whose morphisms are the structure preserving G^o-maps. Denote by STR COMULT TOP^{G^o} the full subcategory whose objects are the strictly comultiplicative G^o-spaces.

If G is a strictly monoidal small category, then we get a functor

$$\text{STR COMULT } \mathrm{TOP}^{G^o} \to \mathrm{CAT}$$

which associates with X a small category whose set of objects is $X(0)$ and whose set of morphisms is the colimit of X and we get a functor

$$\text{STR COMULT } \mathrm{TOP}^{G^o} \to \mathrm{TOPCAT}$$

which associates with X a topological category whose space of objects is $X(0)$ and whose space of morphisms is the standard homotopy colimit $B_{G^o} X$.

Consider now the subcategory Λ of Δ whose morphisms are the order preserving functions

$$\lambda : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}$$

for which $\lambda(0) = 0$ and $\lambda(m) = n$. Then Λ is strictly monoidal, by letting $m \oplus n = m + n$ and by letting $\lambda \oplus \mu$ for $\lambda : m \to m'$ and $\mu : n \to n'$ be given by

$$(\lambda \oplus \mu)(i) = \begin{cases} \lambda(i), & \text{for } 0 \leq i \leq m \\ m' + \mu(i - m), & \text{for } m \leq i \leq m + n. \end{cases}$$

We then have an equivalence of categories

$$i^{\#} : \mathrm{TOP}^{\Delta^o} \to \mathrm{COMULT\ TOP}^{\Lambda^o}$$

which assigns to the simplicial space X the Λ^o-space $i^{\#}X$, where $i : \Lambda \to \Delta$ is inclusion, with comultiplication

$$(i^{\#}X)(m + n) = X(m + n) \to X(m) \times X(n)$$

the map sending $x \in X(m + n)$ into (x', x'') where x' is the front m-face of x and x'' is the back n-face.

Now define $\wr\Lambda$ to be the category whose objects A are all the subsets

$$\{0, m\} \subset A \subset \{0, 1, \cdots, m\}.$$

Given $\{0, n\} \subset B \subset \{0, 1, \cdots, n\}$, define the morphisms $A \to B$ to be all the morphisms $\lambda : m \to n$ in Λ for which $\lambda(A) \supset B$. The category $\wr\Lambda$ is strictly monoidal. Given objects A and B as above, one can use $\oplus$ for Λ to construct $A \oplus B$ with

$$\{0, m + n\} \subset A \oplus B \subset \{0, 1, \cdots, m + n\},$$

and one can go on to define $\oplus$ on morphisms. There is a natural inclusion $\Lambda \to \wr\Lambda$ obtained by identifying each m with $A = \{0, m\}$. Then one can think of $Ob\ \wr\Lambda$ as the free monoid generated by the non-zero objects of Λ, thus one can enumerate the objects of $\wr\Lambda$ as 0 together with all $(m_1, \cdots, m_k)$ where each m_i is a positive integer.

There is an equivalence of categories

$$\wr : \mathrm{TOP}^{\Delta^o} \to \mathrm{STR\ COMULT\ TOP}^{(\wr\Lambda)^o}$$

which associates with a simplicial space X a strictly comultiplicative $(\wr\Lambda)^o$-space $\wr X$ given by

$$(\wr X)(m_1, \cdots, m_k) = X(m_1) \times_{X(0)} \cdots \times_{X(0)} X(m_k),$$

$$(\wr X)(0) = X(0).$$

Applying previous remarks, we get

$$\mathrm{TOP}^{\Delta^o} \xrightarrow{\wr} \mathrm{STR\ COMULT\ TOP}^{(\wr\Lambda)^o} \to \mathrm{TOPCAT}$$

and the composition

$$W : \mathrm{TOP}^{\Delta^o} \to \mathrm{TOPCAT}$$

assigning to X a topological category WX. The space of objects of WX is $X(0)$ and the space of morphisms is $B_{(\wr\Lambda)^o}(\wr X)$.

Simplicial Spaces Determined up to Homotopy by $X(0)$ and $X(1)$

The simplest simplicial spaces X are those with each $X(n)$ determined up to homotopy by $X(0)$, i.e. those for which the unique $n \to 0$ induces a homotopy equivalence $X(0) \to X(n)$ for all n. Then BX is naturally homotopy equivalent to $X(0)$. As examples, consider the functor

$$\diamond^{\nabla} : \mathrm{TOP} \to \mathrm{TOP}^{\Delta^o}, \quad A \mapsto A^{\nabla} = \{A^{\nabla(n)}\},$$

where clearly each $X = A^{\nabla}$ has $X(n)$ determined up to homotopy by $X(0) = A$, and thus the natural map $BX \to A$ is a homotopy equivalence.

Denote by PAIR TOP the category of closed pairs (A, A_0) in TOP. For each n, let $\nabla_0(n) \subset \nabla(n)$ denote the set of vertices of $\nabla(n)$. Then there is the functor

$$\diamond^{(\nabla,\nabla_0)} : \mathrm{PAIR\ TOP} \to \mathrm{TOP}^{\Delta^o}, \quad (A, A_0) \mapsto X = \{(A, A_0)^{(\nabla(n),\nabla_0(n))}\},$$

assigning to (A, A_0) the singular simplices in A all of whose vertices are in A_0. It is readily checked that each $X(n)$ is determined up to homotopy by $X(0)$ and $X(1)$. That is, consider each $X(n)$ as a space over $X(0) \times X(0)$ by assigning to a singular simplex its first vertex and last vertex. Then each of the maps

$$X(m + n) \to X(m) \times_{X(0)} X(n)$$

is a homotopy equivalence of spaces over $X(0) \times X(0)$. This is what we mean by $X(n)$ being determined up to homotopy by $X(0)$ and $X(1)$, since it follows that for such X we have

$$X(n) \sim X(1) \times_{X(0)} \cdots \times_{X(0)} X(1)$$

as spaces over $X(0) \times X(0)$.

Given a simplicial space X, let X_0 denote the 0-skeleton of X, that is, the simplicial space which has $X_0(n) = \delta^* X(0)$ where δ is the unique morphism $n \to 0$. Then we get

$$\mathrm{TOP}^{\Delta^o} \to \mathrm{PAIR\ TOP}, \quad X \mapsto (BX, BX_0).$$

In Chapter 8, we show that if we apply the composition

$$\mathrm{PAIR\ TOP} \to \mathrm{TOP}^{\Delta^o} \to \mathrm{PAIR\ TOP}$$

to a pair (A, A_0) with A a CW-complex and with A_0 a subcomplex intersecting each path component of A, then the result is homotopy equivalent to (A, A_0) in PAIR TOP. Given such a pair, we get from

$$\mathrm{PAIR\ TOP} \to \mathrm{TOP}^{\Delta^o} \xrightarrow{W} \mathrm{TOPCAT}$$

a topological category G whose space of objects is A_0 and whose space of morphisms is homotopy equivalent to the space of paths in A whose ends are in A_0. Moreover, the classifying space $\mathcal{B}_G$ is homotopy equivalent to A. In particular, if A is a path connected CW-complex with base point a vertex, we receive a topological monoid G which is homotopy equivalent to the loop space ΩA and whose classifying space $\mathcal{B}_G$ is homotopy equivalent to A. Such theorems are an integral part of our topic, and in fact go back to Milnor's theorem [8.1,8.2] that one can choose G to be a CW-group.

Such results can be regarded as a part of the general program of Stasheff [4.5,7.6] to replace homotopy associativity systems by strictly associativity systems.

Following Segal, in Chapter 7 we consider more generally the simplicial spaces X which have $X(n)$ determined up to homotopy by $X(0)$ and $X(1)$. For many purposes, these are the simplicial spaces which can be replaced by the nerve of a topological category, specifically by $NW(X)$. In particular, if also $X(0)$ is a singleton then X can be replaced for many purposes by the nerve of a topological monoid. If this topological monoid has homotopy inverses we get Segal's theorem that the natural inclusion

$$X(1) \to \Omega BX$$

is a homotopy equivalence.

Colimits and Homotopy Colimits of the $Mono\ \Sigma$-Spaces Corresponding to $\mathbf{TOP}_*$

Consider the category $Mono\ \Sigma$ whose objects are the non-negative integers, and whose morphisms $\sigma : m \to n$ are the one-to-one functions

$$\sigma : \{1, \cdots, m\} \to \{1, \cdots, n\}.$$

One sees that disjoint union gives a strictly monoidal structure to $Mono\ \Sigma$, and in fact this structure comes from a coproduct. The subcategory $Iso\ \Sigma$ of isomorphisms in $Mono\ \Sigma$ is just $\coprod \Sigma(n)$ where $\Sigma(n)$ is the symmetric group on n letters. Every morphism of $Mono\ \Sigma$ is uniquely expressed as an isomorphism followed by a mono

$$\delta : \{1, \cdots, m\} \to \{1, \cdots, n\}$$

which preserves order. We interpret this category as a suitable entry into the study of relationships between the symmetric groups and topology.

There is a natural functor

$$\mathrm{TOP}_* \to \mathrm{TOP}^{Mono\ \Sigma}, \quad A \mapsto A^\infty = \{A^n\},$$

where if $\sigma : m \to n$ is a morphism of $Mono\ \Sigma$ then

$$\sigma_*(a_1, \cdots, a_m) = (b_1, \cdots, b_n)$$

with $b_j = a_{\sigma^{-1}}(j)$ when $\sigma^{-1}(j) \neq \emptyset$ and b_j the base point otherwise. The identifications $A^{m+n} \simeq A^m \times A^n$ make A^∞ strictly comultiplicative. The fact that $A^0 = pt$ makes the colimit of A^∞ a monoid. The presence of the coproduct makes the colimit an abelian monoid. One can prove by filtration methods that the colimit is compactly generated. We have with this abstract language cited one of the classics of our subject, the functor

$$\mathrm{TOP}_* \to \mathrm{AB\ TOP}\ MON, \quad A \mapsto SP^\infty(A)$$

of Dold-Thom [9.2] which assigns to a compactly generated space A with base point the infinite symmetric product $SP^\infty A$, an abelian topological monoid, specifically the colimit of the $Mono\ \Sigma$-space A^∞.

Consider the case in which A has cofibered base point; then $\{1\} \subset SP^\infty(A)$ is cofibered. The space $B_{SP^\infty(A)}$ can be computed, and turns out to be $SP^\infty(SA)$.

If A is path connected and of the homotopy type of a CW-complex, the topological monoid $SP^\infty(A)$ has homotopy inverses. Thus if all of these are true, the natural inclusion

$$SP^\infty(A) \to \Omega SP^\infty(SA)$$

is a homotopy equivalence. For the spheres S^n, the most immediate $SP^\infty(S^n)$ is given by $SP^\infty(S^2) \simeq CP(\infty)$. Since $CP(\infty)$ is a classifying space for S^1 one has that $SP^\infty(S^2)$ is a $K(Z,2)$ and hence, for all $n > 0$, $SP^\infty(S^n)$ is a $K(Z,n)$.

These classic facts of Dold-Thom are early in Chapter 9. In Chapter 10, we return to this example and discuss a little the classic extension by May [2.8] of this example to the composition

$$\mathrm{TOP}_* \to \mathrm{TOP}^{Mono\,\Sigma} \xrightarrow{hocolim} \mathrm{TOP}.$$

Namely, if A has cofibered base point, is path connected, and is of the homotopy type of a CW-complex, then $\Omega^\infty S^\infty A$ is a homotopy colimit of the $Mono\,\Sigma$-space A^∞.

We first analyze in the fashion of May conditions on a $(Mono\ \Sigma)^o$- space X which assure that

$$X \times_{Mono\,\Sigma} A^\infty$$

is a homotopy colimit of A^∞ for all A with cofibered base point. It turns out to be sufficient to require that each $X(n)$ is homotopy equivalent in $\mathrm{TOP}^{(\Sigma(n))^o}$ to a universal $(\Sigma(n))^o$- space. In order to use this fact, May considers the $(Mono\ \Sigma)^o$- spaces X_k, due to Boardman-Vogt, where $X_k(n)$ consists of the n-tuples $(C_1, \cdots, C_n)$ of subcubes of $[-1,1]^k$ which have disjoint interiors. Here the action is given by

$$(C_1, \cdots, C_n)\sigma = (C_{\sigma(1)}, \cdots, C_{\sigma(m)}).$$

There is a natural map

$$X_k \times_{Mono\,\Sigma} A^\infty \to \Omega^k S^k A$$

and May proves that under the cited conditions this is a homotopy equivalence. One next defines a $(Mono\ \Sigma)^o$-map $X_k \to X_{k+1}$ given by

$$(C_1, \cdots, C_n) \mapsto (C_1 \times [-1,1], \cdots, C_n \times [-1,1]),$$

and defines $X = colim\ X_k$. Then one has the map

$$X \times_{Mono\,\Sigma} A^\infty \to \Omega^\infty S^\infty A,$$

which is a homotopy equivalence under the cited conditions. Moreover, each $X(n)$ is homotopy equivalent to a universal $(\Sigma(n))^o$- space, thus the left hand side is a homotopy colimit of A^∞.

Generalizations by McCord and Segal of the Infinite Symmetric Product

The major point of Chapter 9 is to consider Segal's category Γ which has objects the non-negative integers, equivalently the finite sets $\{0, 1, \cdots, n\}$ with base point 0, and morphisms $\gamma : m \to n$ the base point preserving functions. The identification

$$\{0, 1, \cdots, m\} \vee \{0, 1, \cdots, n\} \simeq \{0, 1, \cdots, m + n\}$$

gives by means of this coproduct a strictly monoidal structure to Γ.

If A is a space with base point, there is the Γ^o-space $\coprod A^n$, where A^n is interpreted as the base point preserving functions $\{0, 1, \cdots, n\} \to A$, which thus receives a Γ^o-structure. There is also a functor

$$\text{AB TOP } MON \to \text{TOP}^\Gamma$$

assigning to an abelian topological monoid G the Γ-space $\coprod G^n$ where $\gamma : m \to n$ gives $\gamma_* : G^m \to G^n$ by

$$\gamma(g_1, \cdots, g_m) = (g'_1, \cdots, g'_n)$$

with

$$g'_j = \begin{cases} \prod_{\gamma(i)=j} g_i, & \text{for } \gamma^{-1}(j) \neq \emptyset \\ 1, & \text{for } \gamma^{-1}(j) = \emptyset. \end{cases}$$

The reduced product $\coprod A^n \times_\Gamma \coprod G^n$ is compactly generated, and the identifications $A^{m+n} \simeq A^m \times A^n$ and $G^{m+n} \simeq G^m \times G^n$ together with the coproduct in Γ make it an abelian topological monoid. Thus we have McCord's extension [1.3] of the infinite symmetric product to a functor

$$\begin{gathered} SP^\infty : \text{TOP}_* \times \text{AB TOP } MON \to \text{AB TOP } MON, \\ (A, G) \mapsto SP^\infty(A; G) = \coprod A^n \times_\Gamma \coprod G^n. \end{gathered}$$

Letting $A = S^1$, we get

$$B_G = SP^\infty(S^1; G).$$

If G is an abelian toplogical monoid, then B_G is also an abelian topological monoid, thus one can define the abelian topological monoid B_G^{n+1} as the classifying space of B_G^n. From McCord's formula

$$SP^\infty(A; SP^\infty(B; G)) \simeq SP^\infty(A \wedge B; G)$$

we get $B_G^n \simeq SP^\infty(S^n; G)$. If G is a discrete abelian group, there is the homotopy equivalence

$$SP^\infty(S^n; G) \to \Omega SP^\infty(S^{n+1}; G)$$

and one has the model $SP^\infty(S^n; G)$ for the Ω-spectrum $K(G)$.

For any Γ-space Y, the unique morphism $n \to 0$ makes $Y(n)$ a space over $Y(0)$. The natural maps

$$m \longleftarrow m + n \longrightarrow n$$

also provide natural maps $Y(m + n) \to Y(m) \times_{Y(0)} Y(n)$, where the right hand side now denotes the product as spaces over $Y(0)$. It follows from the work of Segal that one should consider the Γ-spaces Y for which $Y(n)$ is determined up to homotopy by $Y(0)$ and $Y(1)$, interpreted to mean that the maps $Y(m+n) \to Y(m) \times_{Y(0)} Y(n)$ are homotopy equivalences of spaces over $Y(0)$.

Thus we follow Segal [4.4] through his consideration of Γ-spaces Y for which $Y(n)$ is determined up to homotopy by $Y(0)$ and $Y(1)$, with the condition that $Y(0) = pt$. There is then a spectrum

$$\{SP^\infty(S^n; Y)\}$$

where any $SP^\infty(A; Y)$ is given by

$$SP^\infty(A; Y) = \coprod A^n \times_\Gamma Y.$$

Moreover, if each $Y(n)$ is of the homotopy type of a CW-complex then the resulting spectrum is an Ω-spectrum, disregarding the first map $Y(1) \to SP^\infty(S^1; G)$. If in addition the monoid $\pi_0(Y(1)$ is a group then the first map is also a homotopy equivalence. The maps of the spectrum are best understood by formulating an extension

$$\mathrm{TOP}_* \times \mathrm{TOP}^\Gamma \to \mathrm{TOP}^\Gamma$$

of the McCord functor

$$\mathrm{TOP}_* \times \mathrm{AB\ TOP}\ MON \to \mathrm{AB\ TOP}\ MON.$$

Segal extends this result to the case in which $Y(0)$ is contractible, and the maps $Y(m+n) \to Y(m) \times Y(n)$ are homotopy equivalences.

We include as an example a version of Segal's Γ-space Y which yields stable homotopy theory. Consider the diagram

$$\Gamma \to \mathrm{TOP}_* \to \mathrm{TOP}^{Mono\ \Sigma} \xrightarrow{hocolim} \mathrm{TOP},$$

whose composition yields a Γ-space Y. It is seen that $Y(0)$ is the classifying space $B_{Mono\ \Sigma}$ and one can thus check that $Y(0)$ is contractible. Less evident is the fact that the maps $Y(m+n) \to Y(m) \times Y(n)$ are homotopy equivalences, but it can be checked. It remains to exhibit $Y(1)$, which is the standard homotopy colimit of the $Mono\ \Sigma$-space $(S^0)^\infty$. Any standard homotopy colimit can be computed by converting the G-space by means of M_1 into a topological category C and computing B_C. Here the objects of C are all subsets $S \subset \{1, \cdots, m\}$ for all m, and for each $\sigma : m \to n$ in $Mono\ \Sigma$ there is a morphism $\sigma : S \to \sigma(S)$ in C. Then $B_C \sim B_D$, where D is the subcategory whose objects are $S = \{1, \cdots, m\}$ and whose morphisms are the permutations. Thus $Y(1) \sim \coprod B_{\Sigma(n)}$. It is also the case, for example from the work of May, that if A is path connected, of the homotopy type of a CW-complex, and has cofibered base point, then

$$SP^\infty(A; Y) \sim \Omega^\infty S^\infty A.$$

We conclude Chapter 10, and the tract, with a review of the models for $B_{\Sigma(n)}$ which come from the work of Nakamura [10.2].

CHAPTER I

The General Topology Background

We assume at least the equivalent of a one-semester course in general topology, but fill in here some elementary topics which represent special needs. An overall goal of the work is for a framework in which to present a wide variety of specific models for topological spaces and continuous functions joining such spaces; the emphasis here is on diagrams of spaces and maps as a technique for producing models. Moreover our attention is on topological models rather than algebraic models. In this chapter there is reviewed some of the general topology especially relevant to diagrams of spaces and maps.

It is our goal to expose the structure of the material, but to provide only a working outline of proofs. To some extent, we seek to provide only a workbook.

It is desirable to use the basic language of category theory, thus it is assumed here. A definitive and frequently used reference is the book of MacLane [1.2]. In this language, we will start with the category top whose objects are the topological spaces X, and whose morphisms $f : X \to Y$ are the continuous functions.

Products and Coproducts of Spaces

We first recall that top has arbitrary products and coproducts in the sense of category theory.

Given a family $\{X_p | p \in P\}$ of topological spaces indexed by a set P, there is the product space $\prod_{p\in P} X_p$ whose elements are the functions $x = \{x_p | p \in P\}$ which assign to $p \in P$ an element $x_p \in X_p$. There are also the projection maps

$$\pi_q : \prod X_p \to X_q$$

for each $q \in P$, defined by $\pi_q(x) = x_q$. If Y is any space and if for each $q \in P$ we are given a map $\nu_q : Y \to X_q$, then there is a unique map $\phi : Y \to \prod X_p$ with $\pi_q \phi = \nu_q$ for each $q \in P$, and ϕ is given by $\phi(y) = \{\nu_p(y)\}$. That is, the category of spaces and maps has products.

It also has coproducts. Given $\{X_p | p \in P\}$, there is the disjoint union

$$\coprod_{p \in P} X_p = \{(p, x) | p \in P,\ x \in X_p\}.$$

For each $q \in P$ there is the function $\rho_q : X_q \to \coprod X_p$ given by $\rho_q(x) = (q, x)$ for $x \in X_q$. Then $\coprod X_p$ is topologized so that the sets $\rho_q(U)$ form a basis for the topology as q ranges over all elements of P and U ranges over all open sets of X_q.

The Topology on Mapping Spaces

Given objects X and Y in a category, we must be clear on any special structure enjoyed by the set of morphisms from X to Y. For top, that means putting the best topology one can on the set of continuous functions from one given space to another.

Given spaces X and Y, denote by Y^X the set of all continuous functions $f : X \to Y$. For each compact subset C of X and each open subset U of Y, let

$$W(C, U) = \{f \in Y^X | f(C) \subset U\}.$$

Then Y^X is topologized by requiring the $W(C, U)$ to constitute a sub-basis. This is the *compact-open* topology.

(1.1) *Given a map (i.e. a continuous function) $\phi : X \to X'$ and a space Y, the function $\phi^{\#} : Y^{X'} \to Y^X$ sending f into the composition $f\phi$ in the diagram*

$$Y \xleftarrow{f} X' \xleftarrow{\phi} X$$

is continuous.

Given f and a sub-basis element $W(C, U)$ of Y^X to which $f\phi$ belongs, we have $f(\phi(C)) \subset U$ hence f belongs to the sub-basis element $W(\phi(C), U)$ of $Y^{X'}$. It is seen that $\phi^{\#}$ maps $W(\phi(C), U)$ into $W(C, U)$ and hence that $\phi^{\#}$ is continuous. $\square$

(1.2) *If Y is a space and X is a compact Hausdorff space, then the evaluation function*

$$e : Y^X \times X \to Y, \qquad e(f, x) = f(x)$$

is continuous.

If U is open in Y and if $f(x) \in U$ for some map $f : X \to Y$, the fact that X is regular implies that there exists an open set V containing x with $f(\overline{V}) \subset U$. Since $\overline{V}$ is automatically compact, it follows that

$$e^{-1}(U) = \bigcup W(\overline{V}, U) \times V,$$

where the union is over all open sets V of X. $\square$

(1.3) *If X and Y are spaces, then for every map $h : C \to Y^X \times X$ where C is compact Hausdorff, the composition*

$$C \xrightarrow{h} Y^X \times X \xrightarrow{e} Y$$

is continuous.

Suppose the map $h : C \to Y^X \times X$ is given by $u \mapsto (f(u), g(u))$ where $f : C \to Y^X$ and $g : C \to X$. Then the composition

$$C \xrightarrow{\ d\ } C \times C \xrightarrow{f \times 1} Y^X \times C \xrightarrow{g^\# \times 1} Y^C \times C \xrightarrow{\ e\ } Y$$

is continuous, where d is the diagonal map $d(c) = (c, c)$. But this composition is eh. $\square$

The reader should check that given a map $f : X \times Y \to Z$ then for each fixed $x_0 \in X$ the function $y \mapsto f(x_0, y)$ is a map $Y \to Z$.

(1.4) *Given spaces X, Y and Z, there is the map*

$$L : Z^{X \times Y} \to (Z^Y)^X$$

defined by $[Lf(x)](y) = f(x, y)$.

The sentence preceding (1.4) gives a function $Lf : X \to Z^Y$ for each $f : X \times Y \to Z$. We first note that each Lf is continuous, i.e. that each $(Lf)^{-1}(W(D, U))$ is open in X. Now $(Lf)^{-1}(W(D, U))$ is the set of $x \in X$ with $f(x \times D) \in U$. Since D is compact, this set is open. Thus L is a well-defined function.

We have to prove that $L^{-1}V$ is open for each open subset V of $(Z^Y)^X$. It suffices to prove that $L^{-1}V$ is open for each V in a sub-basis for the topology of $(Z^Y)^X$. Hence it suffices to prove that each $L^{-1}(W(C, V'))$ is open. Here V' varies over all open subsets of Z^Y, but it suffices to check it on a sub-basis. Hence let $V' = W(D, U)$ where D is a compact subset of Y and U is an open subset of Z. Then it is seen that

$$L^{-1}(W(C, W(D, U)) = W(C \times D, U),$$

and the theorem follows. $\square$

(1.5) *Suppose that Y is a compact Hausdorff space and that X and Z are spaces. Then the map $L : Z^{X \times Y} \to (Z^Y)^X$ is one-to-one and onto.*

Fix $g : X \to Z^Y$. By (1.2), there is the composition of maps

$$X \times Y \xrightarrow{g \times 1} Z^Y \times Y \xrightarrow{e} Z$$

which yields an element in $L^{-1}g$. It is seen that there is at most one element, so we can denote the composition simply by $L^{-1}g$. $\square$

We summarize up to this point. Two very basic construction devices can be looked upon as bifunctors. These are given at the object level by

$$(X, Y) \mapsto X \times Y, \qquad (X, Y) \mapsto X^Y;$$

call these the product bifunctor and the map bifunctor respectively. The product bifunctor is covariant in both variables, the map bifunctor is covariant in the first variable and contravariant in the second, thus in arrow notation they are written as (covariant) functors

$$\times : \text{top} \times \text{top} \to \text{top}, \qquad exp : \text{top} \times (\text{top})^o \to \text{top},$$

where $(\text{top})^o$ denotes the opposite category of top. There is a defect in top. In categorical language, it is not *cartesian closed* [1.2,p.95]. That is, for all spaces X, Y and Z we have a natural map $Z^{X\times Y} \to (Z^Y)^X$ but it is not always an epi.

Quotient Maps and Inclusion Maps

The monomorphisms in top are the one-to-one continuous functions, and the epimorphisms are the continuous functions $f : X \to Y$ for which $f(X) = Y$. Restricted classes of monos and epis are more useful here than are the monos and epis.

Suppose that X is a given space and that we are also given a function f of X onto a set Y. Then the *topology on Y induced by f* has as open sets those subsets U for which $f^{-1}(U)$ is open in X. If X and Y are spaces and if f is a continuous function of X onto Y, then f is a *quotient map* if and only if the topology on Y coincides with that induced by f.

Each equivalence relation $\sim$ on X leads to a disjoint partitioning of X into the various equivalence classes $[x]$. Denote by $X/\sim$ the set of equivalence classes, and by $\pi : X \to X/\sim$ the function given by $\pi(x) = [x]$. Then we can put on $X/\sim$ the topology induced by π, and π is then a quotient map. Moreover, given any quotient map $f : X \to Y$ there is an equivalence relation $\sim$ with Y homeomorphic to $X/\sim$. Simply define $\sim$ by $x \sim x'$ if and only if $f(x) = f(x')$.

A useful fact about quotient maps is that if one has a commutative diagram of spaces and functions

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\ Z \arrow[r, equal] & Z \end{tikzcd}$$

where f is a quotient map and h is a map, then g is a map. Its proof is trivial.

(1.6) *Consider the commutative diagram*

$$\begin{tikzcd} Y \arrow[d, "f_1"'] & X \arrow[l, "f_2"'] \arrow[d, "g_2"] \\ Z & Y' \arrow[l, "g_1"] \end{tikzcd}$$

of spaces and maps, where f_2 and g_2 are quotient maps and where f_1 and g_1 are one-to-one continuous maps. Then there is a homeomorphism h of Y onto Y' such that $hf_2 = g_2$ and $g_1h = f_1$.

Check that the equivalence relations $\sim_{f_2}$ and $\sim_{g_2}$ on X induced by f_2 and g_2 coincide. It then follows that the functions

$$g_2 f_2^{-1} : Y \to Y', \quad f_2 g_2^{-1} : Y' \to Y$$

are well-defined functions. They are both continuous since f_2 and g_2 are quotient maps. $\square$

Every map $f : X \to Z$ can be written as $f = f_1 f_2$ where f_2 is a quotient map and where f_1 is one-to-one and continuous. For define $\sim$ to be the equivalence relation on X given by f and check that there is the factorization

$$Z \leftarrow X/\sim \leftarrow X$$

with the desired properties. Then (1.6) gives the uniqueness of the factorization up to natural homeomorphism of the middle space.

(1.7) *If $f : X \to X'$ is a quotient map and Y is compact Hausdorff, then $f \times 1 : X \times Y \to X' \times Y$ is a quotient map.*

Define an equivalence relation $\sim$ on $X \times Y$ by $(x, y) \sim (z, w)$ if $y = w$ and $f(x) = f(z)$. Let $Z = X \times Y/\sim$ and let $g : X \times Y \to Z$ be the quotient map. The diagram

$$Z \xleftarrow{g} X \times Y \xrightarrow{f \times 1} X' \times Y$$

yields the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle Lg}\big\downarrow & & \\ Z^Y & = & Z^Y \end{array}$$

and it is seen that there is a function $h : X' \to Z^Y$ making the diagram commutative. It follows from an earlier observation that h is continuous. Then by (1.5) we get the commutative diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{f \times 1} & X' \times Y \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle L^{-1}h} \\ Z & = & Z. \end{array}$$

By construction of Z, $f \times 1$ induces a continuous map $Z \to X' \times Y$ which is an inverse of $L^{-1}h$. Hence $f \times 1$ is a quotient map. $\square$

Dually, for each one-to-one function $f : X \to Y$ of a set X into a space Y, define a topology on X by defining the open sets to be the sets $\{f^{-1}(U)\}$ for all U open in Y. Call this the relative topology on the set X. Then a map $f : X \to Y$ is an *inclusion map* if it is one-to-one and if the topology on X coincides with the relative topology. In particular, if X is a space and if A is a subset of X, then A can be given the relative topology via $A \hookrightarrow X$, and the space A is then called a subspace of X. The inclusion maps $f : X \to Y$ can then be characterized as the compositions of homeomorphisms of X onto subspaces A of Y followed by $A \hookrightarrow Y$. The equivalent of (1.6) can then be proven for factorizations of a given map into an onto mapping followed by an inclusion map.

McCord [1.3] has given the following useful device for recognizing quotient maps and inclusion maps onto closed subsets. We leave its proof as an exercise.

(1.8) *Consider the diagram of spaces and maps*

$$\begin{array}{ccc} A & & X \\ {\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow \\ B & \xrightarrow{i} & Y \end{array}$$

where f is a quotient map and i is a one-to-one map. Suppose also that $[Z_j | j \in J]$ is a locally finite collection of closed subsets of X with $f^{-1}i(B) = \bigcup Z_j$, and that for each j we have a map $\sigma_j : Z_j \to A$ with $ih\sigma_j = f$ on Z_j. Then h is a quotient map and i is an inclusion map onto a closed subspace of Y.

In summary, we have pointed out in the category top the "strong epis", i.e. the quotient maps, and the "strong monos", i.e. the inclusion maps.

Limits and Colimits of Diagrams

We now put in a first crude form the two basic construction devices considered in this tract, which amount to two ways of systematically making spaces out of diagrams of spaces and maps. The notation "limit" and "colimit" for these primary constructions is taken from category theory, although we will not put these constructions in a fully categorical setting until later in the chapter.

A *diagram scheme* consists of a set P together with a set $D_{p,q}$ for each ordered pair (p, q) of elements of P. A diagram of spaces corresponding to a given diagram scheme is a pair consisting of a function associating with each element $p \in P$ a space $X(p)$ and a function associating with each $g \in D(p, q)$ a map $X(q) \to X(p)$ with functional values denoted by $x \mapsto gx$. Then the limit *lim* X is given by the subspace

$$\mathit{lim}\ X \subset \prod_{p\in P} X(p)$$

consisting of all $\{x_p\}$ such that for any $g \in D_{p,q}$ we have $gx_q = x_p$. The colimit *colim* X is defined to be the quotient space

$$\mathit{colim}\ X = \coprod_{p\in P} X(p)/\sim$$

where $\sim$ is the least equivalence relation on $\coprod X(p)$ such that for any $(q, x) \in \coprod X(p)$ and $g \in D_{p,q}$ we have $(q, x) \sim (p, gx)$. There are then the natural maps

$$\mathit{lim}\ X \to \prod X(p), \qquad \coprod X(p) \to \mathit{colim}\ X$$

and these are respectively inclusion maps, quotient maps.

The terms *graph* and *precategory* are also used for what we have called a diagram scheme; see [1.2,p.48].

Below are some frequently occurring limits and colimits, with their special names:

(i) *pushouts*, the colimits Y of diagrams $X \xleftarrow{f} A \xrightarrow{g} B$ in top; the space Y is then the quotient space $(X \sqcup A \sqcup B)/\sim$, where $\sim$ is the least equivalence relation on the disjoint union with $a \sim f(a)$ for $a \in A$ and $a \sim g(a)$ for $a \in A$; an important subcase are the pushouts of diagrams

$$X \xleftarrow{i} A \xrightarrow{g} B,$$

where A is closed in X and i is the inclusion map; the pushout is in this case denoted by $X \cup_g B$ and called an attaching space; the topology on the attaching space is better described by using (1.8) on

$$\begin{tikzcd} X \sqcup B \arrow[d, "k"'] & X \sqcup A \sqcup B \arrow[d, "h"] \\ X \cup_g B \arrow[r, equal] & X \cup_g B \end{tikzcd}$$

together with the closed covering $[X, A, B]$ of $X \sqcup A \sqcup B$ together with the maps $X \xrightarrow{1_X} X$, $A \xrightarrow{g} B$, and $B \xrightarrow{1_B} B$ which shows that k is a quotient map and that one may use k to define the topology of the attaching space. A *relative homeomorphism* is a map $h : (X, A) \to (Y, B)$, where (X, A) and (Y, B) are closed pairs, $h \sqcup i : X \sqcup B \to Y$ is a quotient map, and h maps $X - A$ one-to-one onto $Y - B$; in this case, Y is naturally homeomorphic to $X \cup_{h|A} B$;

(ii) *pullbacks*, the limits of diagrams of the form $X \xrightarrow{f} A \xleftarrow{g} B$ in top; the pullback is the subspace

$$\{(x, a, b) \in X \times A \times B | f(x) = a = g(b)\},$$

of $X \times A \times B$ but this space is naturally homeomorphic to

$$\{(x, b) \in X \times B | f(x) = g(b)\};$$

one prefers the more economical presentation;

(iii) *coequalizers*, the colimits of diagrams $X \rightrightarrows Y$; also *equalizers*, the limits of such diagrams; the reader should present these explicitly;

(iv) *filtered spaces* $X = \bigcup X_n$ in top, where X is the colimit of a diagram

$$X_0 \to \ldots \to X_n \to \ldots$$

with each X_{n-1} a closed inclusion in X_n; here A is closed in X if and only if $A \cap X_n$ is closed in X for all n.

The General Topology of k-spaces

The category top of all spaces and maps has lead to an incomplete framework. For example, (1.3), (1.5) and (1.7) have compactness hypotheses which are inconvenient for a general framework. One seeks a large full subcategory of top which is big enough to be useable for a general framework and in which the product bifunctor and the map bifunctor are better behaved. The clue is the presence of compactness hypotheses in the above.

Let X be a topological space, and let T be its topology; i.e. its collection of open sets. Define T' to be the topology containing T whose open sets U are the subsets of X such that if $\sigma : C \to X$ is any map of a compact Hausdorff space C into X then $\sigma^{-1}(U)$ is open in C. Denote by kX the set X taken with the topology T'. A space X is a *k-space* if and only if $kX = X$.

Consideration of such spaces is due to Kelley [1.1], and other references are in Steenrod's paper [1.4]. The presentation of Kelley used the compact subspaces of the space X even if X is non-Hausdorff. The variant presented above uses compactness only when accompanied by the Hausdorff property, so that one has to look outside the space, at maps $\sigma : C \to X$ where C is compact Hausdorff. See for example McCord [1.3] or Boardman-Vogt [4.1] for the variant we have used.

Since the topology T' above contains the topology T, it follows immediately that the function $kX \to X$ given by $x \mapsto x$ is continuous. In particular, if C is a compact Hausdorff space and $\sigma : C \to kX$ is continuous, then $\sigma : C \to X$ is also continuous. The converse of this can be checked, thus there are precisely the same maps of compact Hausdorff spaces into the spaces X and kX.

(1.9) *A space X is a k-space if and only if there is a quotient map $f : D \to X$ with D a locally compact Hausdorff space.*

Let X be a k-space. Enumerate all the nonopen subsets of X as $\{M_j | j \in J\}$. For each $j \in J$ select a compact Hausdorff space C_j and a map $\sigma_j : C_j \to X$ for which $\sigma_j^{-1} M_j$ fails to be open in C_j. Augment the collection of maps $\sigma_j : C_j \to X$, with C_j compact Hausdorff, so that every $x \in X$ is in the image of some σ_j. One can do the augmentation by including enough maps of singleton spaces into X. There is then the coproduct $\coprod_{j \in J} C_j$ and the map $f : \coprod C_j \to X$ defined by $f(j, x) = \sigma_j(x)$ for $x \in C_j$. Then $\coprod C_j$ is locally compact Hausdorff and f is a quotient map.

Suppose now that there exists a quotient map $f : D \to X$ where D is locally compact Hausdorff. If M is a subset of X which is not open in X, then $f^{-1}(M)$ is not open in D, since f is a quotient map. There is then a compact subspace C of D such that $f^{-1}(M) \cap C$ is not open in C. There is then the map $f|C : C \to X$ such that $(f|C)^{-1}(M)$ is not open in C. Then the topology T' of kX is equal to the topology T of X, and X is a k-space. $\square$

This latter part of the above proposition can be generalized slightly. If X is a k-space and $f : X \to Y$ is a quotient map, then Y is a k-space. It is also easy to see that every kX is a k-space. One way to do so is to use the first paragraph of the above proof to construct a quotient map $\coprod C_j \to kX$ for any space X, from which it follows from (1.9) that kX is a k-space.

(1.10) *Let X be a k-space and let C be a compact Hausdorff space. Then $C \times X$ is a k-space.*

There exists from (1.9) a locally compact Hausdorff space D and a quotient map $f : D \to X$. By (1.7), $1 \times f : C \times D \to C \times X$ is a quotient map. Since $C \times D$ is locally compact Hausdorff, it follows from (1.9) that $C \times X$ is a k-space. $\square$

(1.11) *Let X and Y be spaces and let $f : X \to Y$ be a function. Then $f : kX \to kY$ is continuous if and only if for every map $\sigma : C \to X$, where C is compact Hausdorff, the composition of*

$$C \xrightarrow{\sigma} X \xrightarrow{f} Y$$

is continuous.

We prove only the half that we use. Suppose every composition $f\sigma$ as above is continuous. As in the proof of (1.9), there is a quotient map $\sigma : \coprod C_j \to kX$ with every C_j compact Hausdorff. In the commutative diagram of functions

$$\begin{array}{ccc} \coprod C_j & \xrightarrow{\sigma} & kX \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ kY & = & kY \end{array}$$

one can see that h is continuous by the hypothesis and the fact that there are the same continuous maps from a compact Hausdorff space to Y as to kY. Continuity of f then follows from an earlier observation about quotient maps. $\square$

(1.12) *Let X and Y be spaces. Then if e is the evaluation function $Y^X \times X \to Y$, it follows that*

$$e : k(Y^X \times X) \to kY$$

is continuous.

This is a combination of (1.3) and the above. $\square$

(1.13) *Given k-spaces X,Y,Z and a map $f : k(X \times Y) \to Z$, there is a unique map $Lf : X \to k(Z^Y)$ with $[(Lf)x]y = f(x, y)$.*

Let $C \to X$ be a map of a compact Hausdorff space into X. Then we have the composed map, denoted by g, of

$$C \times Y = k(C \times Y) \to k(X \times Y) \to Z.$$

By (1.4), there is the map $Lg : C \to Z^Y$. Use of (1.11) shows that $X \to k(Z^Y)$ is continuous. $\square$

The Category Top of k-spaces

We have now gathered together enough to proceed to a full subcategory of top in which the basic framework is improved. Denote by Top the full subcategory of top whose objects are the k-spaces.

First of all, one retopologizes function spaces. For a brief moment denote by $(Y^X)_{\text{top}}$ the space of maps from any space X to any space Y, where the topology is the compact-open topology. If X and Y are k-spaces, then again for a brief moment denote by $(Y^X)_{\text{Top}}$ the space $k[(Y^X)_{\text{top}}]$. Similarly let $[X_p : p \in P]$ be a collection of k-spaces. There is the product $(\prod X_p)_{\text{top}}$ in top. Define the product in Top by

$$(\prod X_p)_{\text{Top}} = k((\prod X_p)_{\text{top}}).$$

Suppose now that we denote $(\coprod X_p)_{\mathrm{Top}}$ simply by $\coprod X_p$, and the two-fold product in Top by $X \times Y$. These are products in Top in the sense of category theory. Similarly for X and Y in Top let Y^X denote $(Y^X)_{\mathrm{Top}}$. Then one has in Top the following remarkable improvements over top.

Theorem (1.14) *For k-spaces X and Y, suppose Y^X and $X \times Y$ are topologized as k-spaces as above. Then the following hold.*

1. *The evaluation function $Y^X \times X \to Y$ is always continuous in Top.*
2. *Composition $Z^Y \times Y^X \to Z^X$ is always continuous in Top.*
3. *If $f : X \to Y$ and $g : X' \to Y'$ are quotient maps in Top, then the product*
$$f \times g : X \times X' \to Y \times Y'$$
is also a quotient map in Top.
4. *These are all related to an adjointness between products and morphism spaces in Top. If we have a map $f : X \times Y \to Z$ in Top, then we get a map $X \to Z^Y$ given by $x \mapsto f_x$ where $f_x(y) = f(x,y)$. From each map $f : X \times Y \to Z$ we thus get $Lf : X \to Z^Y$ and the resulting function*
$$L : Z^{X \times Y} \to (Z^Y)^X$$
is a natural homeomorphism.

The proofs are an exercise. Number 1 follows easily from previous facts. One also has from previous facts that given $f : X \times Y \to Z$ in Top, one gets the map $Lf : X \to Z^Y$. With these starting facts, and with the fact that a composition of two quotient maps is a quotient map, one proves them all.

Coproducts and colimits work optimally in Top; the coproducts and colimits in top serve without change in Top. The disjoint union $\coprod X_p$ of a collection of k-spaces is again a k-space. If $f : X \to Y$ is a quotient map and if X is a k-space, then Y is a k-space.

Inclusion maps exist in Top, but one may have to change topology. Given a space X, and a subset A of X, denote for the moment by A_{top} the set A together with the relative topology of top. Then the corresponding subspace topology for Top is given by $A_{\mathrm{Top}} = k(A_{\mathrm{top}})$. If one accepts this as the k-space topology of a subset A of X, then one has inclusion maps and limits in Top. For closed subsets of X, one gets the same topology in Top as in top.

(1.15) *Let $f : X \to Y$ be a quotient map in Top, and let Z be a k-space. Then the map $f^{\#} : Z^Y \to Z^X$ given by $\phi \mapsto \phi f$ is an inclusion map in Top.*

It is clear that $f^{\#}$ is one-to-one and continuous. We must prove that given a map $g : A \to Z^X$ with the image of g contained in the image of $f^{\#}$, then we can express g as a composition

$$A \to Z^Y \xrightarrow{f^{\#}} Z^X.$$

Consider the associated map $L^{-1}g : A \times X \to Z$. One checks that in the diagram

$$A \times Y \xleftarrow{1 \times f} A \times X \xrightarrow{L^{-1}g} Z$$

each $(L^{-1}g)((1\times f)^{-1}(a,y)$ is a singleton, thus since $1\times f$ is a quotient map that $L^{-1}g$ can be factored as a composition

$$A\times X \xrightarrow{1\times f} A\times Y \xrightarrow{h} Z,$$

from which it follows that g is the composition

$$A \xrightarrow{Lh} Z^Y \xrightarrow{f^\#} Z^X. \quad \square$$

Compactly Generated Spaces

As we go on, we will move to the general topology of homotopy theory, where it is convenient to have an appropriate separation property on the spaces considered. We make a choice, which amounts to choosing an appropriate full subcategory of Top, whose spaces we will call the compactly generated spaces. Be warned that Steenrod's paper [1.4] or G.W. Whitehead's book [1.6] use a slightly different choice but the same name. The version we use is due to Moore; see for example McCord's paper [1.3]. The original version defined the Hausdorff k-spaces to be the compactly generated spaces. Moore's variant used here will be a slightly weaker condition.

Theorem (1.16) *For a space Y in Top, the following conditions are all equivalent.*

- *(a) The diagonal $\mathcal{D}=\{(y,y)\}\subset Y\times Y$ is a closed subset of the product $Y\times Y$ in Top.*
- *(b) for each diagram $X\rightrightarrows Y$ in Top, the equalizer is a closed subspace of X; that is, if $f,g: X\to Y$, then $\{x\in X|f(x)=g(x)\}$ is a closed subset of X.*
- *(c) whenever $\sigma: C\to Y$ is a map of a compact Hausdorff space C into Y then $\sigma(C)$ is closed in Y.*

Proof. Suppose $\mathcal{D}$ is closed in $Y\times Y$, and that we have $f,g: X\to Y$ in Top. Then $(f\times g)^{-1}(\mathcal{D})$ is closed in $X\times X$. If $d: X\to X\times X$ is the diagonal map, then $d^{-1}((f\times g)^{-1}(\mathcal{D}))$ is closed in X. But this is the equalizer.

Suppose the equalizer condition is satisfied, and that $\sigma: C\to Y$ is a fixed map with C compact Hausdorff. In order to show that $\sigma(C)$ is closed in Y, we have to show that if $\tau: D\to Y$ where D is compact Hausdorff, then $\tau^{-1}(\sigma(C))$ is closed in D. There are the maps $f,g: C\times D\to Y$ given by $f(c,d)=\sigma(c)$ and $g(c,d)=\tau(d)$. The equalizer is closed, hence $\{(c,d)\in C\times D|\sigma(c)=\tau(d)\}$ is closed in the product $C\times D$ in top. Since the spaces are compact Hausdorff, the projection $C\times D\to D$ carries closed sets into closed sets and $\tau^{-1}(\sigma(C))$ is closed in D.

Suppose now that whenever we have a map $\sigma: C\to Y$ with C compact Hausdorff, then $\sigma(C)$ is closed in Y. It is then immediate that every such σ is a closed map. It follows readily from this that $\sigma(C)$ is Hausdorff in the relative topology of top, and we leave this to the reader. In order to show that $\mathcal{D}$ is closed in $Y\times Y$, it suffices to show that if we are given $f,g: C\to Y$, then the equalizer is closed in C. Since $f(C)\cup g(C)$ is a closed compact Hausdorff subspace of Y, this is reduced to the case in which Y is a Hausdorff space, where the conclusion is easy. $\square$

Definition. We call a space Y a *compactly generated* space if it is a k-space which satisfies any one of the above equivalent conditions.

(1.17) *A space Y in top is compactly generated if and only if every compact Hausdorff subspace C in top is closed in Y, and every set M in Y which intersects every compact Hausdorff subspace C in a closed set is itself closed in Y.*

Suppose that Y satisfies the above condition on compact Hausdorff subspaces. Then a compact subspace D of Y in top intersects every compact Hausdorff subspace C in a compact subspace. For given an open covering $\{U_j\}$ of $C \cap D$, one can augment it by the open set $Y - C$ of Y, get a finite open subcovering of D and then pass to a finite subcovering of $\{U_j\}$ for $C \cap D$. Hence every compact subspace of Y in top is closed. Then use (1.16).

If Y is compactly generated, then the proof of (1.16) shows that every continuous image in Y of a compact Hausdorff space is also Hausdorff as well as closed in Y. The condition on compact Hausdorff subspaces follows readily. $\square$

(1.18) *Let $X = \bigcup X_n$ be a filtered space in Top with each X_n compactly generated. Then every subspace C of X which is compact in the relative topology of top is contained in some X_n, and X is also compactly generated.*

Suppose C is a compact subspace of X and for each $n \geq 0$ that there exists $c_n \in C$ with $c_n \notin X_n$. Then $\{c_n\}$ intersects each X_n in a finite set, thus in a closed subset of X_n. Hence $\{c_n\}$ is a closed subset of X, has the discrete topology, and is compact. It is therefore finite, which is a contradiction. Hence each compact subspace of X is in some X_n, and the result follows from (1.16). $\square$

(1.19) *Let X be a k-space, and let $f : X \to Y$ be a quotient map. Then Y is compactly generated if and only if the subset $\{(x, y) \in X \times X | f(x) = f(y)\}$ is a closed subset of the product $X \times X$ in Top.*

It follows from (1.14) that $f \times f : X \times X \to Y \times Y$ is a quotient map. Also Y is compactly generated if and only if $\mathcal{D}$ is closed in $Y \times Y$. The remark follows readily. $\square$

(1.20) *Consider the diagram $X \xleftarrow{i} A \xrightarrow{f} B$ in Top, where A is a closed subset of X and i is the inclusion map. If X and B are compactly generated, so also is the pushout Y.*

There is the quotient map $h : X \sqcup B \to Y$, and one applies (1.19). $\square$

The Category TOP of Compactly Generated Spaces

Denote by TOP the full subcategory of Top whose objects are the compactly generated spaces. This is the general topology setting in which we eventually work.

Morphism sets, products and coproducts transfer without change of topology from Top to TOP. If X and Y are in Top, the mapping space $(Y^X)_{\text{Top}}$ is denoted simply by Y^X, it being implicit that the topology is that of Top. If X and Y are in Top with Y also compactly generated, then for each $x_0 \in X$ $\{(f, g) \in Y^X \times Y^X | f(x_0) = g(x_0)\}$ is the equalizer of two maps $Y^X \times Y^X \to Y$ and hence is closed in $Y^X \times Y^X$. Hence the diagonal in $Y^X \times Y^X$ is the intersection of closed sets and Y^X is compactly generated. If $\{X_p\}$ is a collection of compactly generated spaces, then the product $\prod X_p$ in Top is compactly generated, thus serves as product in TOP. Similarly for coproducts. Consider now the limit in Top of a diagram D in TOP. Fix one of the maps $g : X_q \to X_p$. Then the set of points $\{x_r\}$ in $\prod X_r$ for which $g(x_q) = x_p$ is the equalizer of maps

$$\prod X_r \rightrightarrows X_p$$

and is thus closed in the product. Hence the intersection over all g is closed in the product, and is therefore compactly generated. But this intersection is precisely the limit in Top. Hence it serves as the limit in TOP.

Colimits generally work badly in TOP; it is the nature of colimits to destroy separation properties, as also do quotient maps which do not satisfy (1.19). While the colimits in Top will not generally serve for TOP, there are useful exceptions of (1.18) and (1.20):

(i) if A is closed in X and $X \leftarrow A$ is inclusion, the attaching space $X \cup_f B$ of the diagram $X \leftarrow A \xrightarrow{f} B$ in TOP is also in TOP; equivalently, if $F : (X, A) \to (Y, B)$ is a relative homeomorphism of closed pairs in Top, then X, B compactly generated implies Y compactly generated;

(ii) if $X = \bigcup X_n$ is a filtered space in Top and each X_n is compactly generated, then X is compactly generated.

Actions of Groups and Monoids

We have now to put diagrams of spaces and maps into a more understandable setting. In order to do so, we need a little background in the equivariant general topology associated with Top.

Fix a group G whose operation is multiplication and whose identity element is 1. One could equally well fix a topological group, but at this time if we need G topologized we will take the discrete topology. A *left action* of G on a k-space X is a map

$$G \times X \to X, \qquad (g, x) \mapsto gx$$

such that $g(g'x) = (gg')x$ whenever $(g, g', x) \in G \times G \times X$, and $1x = x$ for all $x \in X$. A *right action* of G on a k-space X is a map

$$X \times G \to X, \qquad (x, g) \to xg$$

such that $(xg)g' = x(gg')$ and $x1 = x$. Technically one can regard a right action of G as a left action of the "opposite" group G^o whose multiplication is $(g, g') \mapsto g'g$.

A G-space is a pair consisting of a k-space X and a left action of G on X. If X and X' are G-spaces, an *equivariant map* or a *G-map* $\phi : X \to X'$ is a map with $\phi(gx) = g\phi(x)$ for all $(g, x) \in G \times X$. We denote by Top^G the category whose objects are the G-spaces and whose morphisms are the G-maps.

There are in this category two basic invariants of a a G-space X:

(i) the space of stationary points, the subspace of X in Top consisting of $x \in X$ such that $gx = x$ for all $g \in G$;
(ii) the orbit space, the k-space $X/G = X/\sim$ where $\sim$ is the equivalence relation on X given by $x \sim x'$ if and only if $x' = gx$ for some $g \in G$.

One thus has at the start the two functors

$$\mathrm{Top}^G \to \mathrm{Top},$$

assigning to X the stationary subspace of X and the orbit space of X.

Note the point of contact with diagram schemes and diagrams. Namely, every G-space can be interpreted as a diagram of spaces and maps. The diagram scheme has P a singleton $\{p\}$, thus there is just one $D_{p,p}$ which is defined to be G. The diagram assigns X to p, and to each $g \in G = D_{p,p}$ assigns the map $x \mapsto gx$ of X. The limit of the diagram will be precisely the stationary point set, and the colimit will be precisely the orbit space. In fact, we use the terms "limit" and "colimit" hereafter for "stationary subspace" and "orbit space".

A G^o-space similarly is a pair consisting of a k-space X and a right action of G on X, the G^o-maps are the maps $\phi : X \to X'$ with $\phi(xg) = (\phi(x))g$, and there is the category Top^{G^o}.

One can provide in this new setting a mapping bifunctor and a product bifunctor. The mapping bifunctor is obtained by assigning to each pair X and Y of G-spaces the subspace

$$(X^Y)_{\mathrm{Top}^G} \subset X^Y$$

in Top consisting of all equivariant maps $\phi : Y \to X$.

If one then fixes Y to be a singleton G-space, then $(X^Y)_{\mathrm{Top}^G}$ is precisely the stationary point set, i.e. the limit. As one proceeds through this work, one will see that "generalized limits" are obtained by replacing the singleton G-space Y by some G-space Y for which Y is a contractible as a space.

The new product bifunctor requires that one be given at the start a right G-space X and a left G-space Y. One can then form a "reduced"-product

$$X \times_G Y = (X \times Y)/\sim,$$

where $\sim$ is the least equivalence relation on $X \times Y$ such that if $(x, g, y) \in X \times G \times Y$ then $(xg, y) \sim (x, gy)$. If we now take again for Y a singleton G-space, the result is checked to be the orbit space of X. Our subject displays many instances in which one forms a "generalized colimit" by fixing a G-space Y for which Y is a contractible space, and then assigning the generalized colimit $X \times_G Y$ to X.

We will require operator domains which are not groups. Having fixed the letter G for the operator domain and the letter g for the individual operators, we do not change the symbols. As a transitional state, consider as operator domain a monoid G, i.e. require of the multiplication on G only that it be associative and have an identity element, and do not require inverse elements. The definition of orbit space has to be changed slightly in the above discussion. Let $\sim$ be the least equivalence relation on X such that $x \sim gx$ whenever $(g, x) \in G \times X$, since the relation used in the group case is not necessarily symmetric in the monoid case. All else in the discussion of groups can be unchanged.

Our subject takes a bigger jump than this; it allows for X being a family of k-spaces, and it allows the total action to be made up of individual actions taking one member of the family into another. The diagram schemes used earlier in this chapter do not suffice, for they do not allow for multiplications and identity elements. That is, we take in this tract as the appropriate basic model for equivariant general topology that in which G is a small category, where "small" is the technical term indicating that everything involved in the structure of G is a set. The small categories G with precisely one object are exactly the monoids, thus the monoid case and hence the group case will be a special case of considerable interest.

Actions of a Small Category

Fix a small category G. Denote its set of objects by $Ob\ G$, and denote individual objects by such letters as p,q, etc. Denote its set of morphisms by $Mor\ G$, and denote individual morphisms by letters such as g,h, etc. A morphism is also written in arrow notation as $p \xleftarrow{g} q$. Let

$$Mor\ G \times_{Ob\ G} Mor\ G \subset Mor\ G \times Mor\ G$$

denote all ordered pairs (g, g') for which the composition exists, i.e. all (g, g') of the form

$$p \xleftarrow{g} q \xleftarrow{g'} r.$$

The structure functions of the category G are then the functions

$$Ob\ G \to Mor\ G, \qquad p \mapsto 1_p,$$

$$Mor\ G \to Ob\ G \times Ob\ G, \qquad p \xleftarrow{g} q \mapsto (p, q),$$

$$Mor\ G \times_{Ob\ G} Mor\ G \to Mor\ G, \qquad (g, g') \mapsto gg'$$

subject to the standard requirements.

If G and H are small categories, there is the usual product category $G \times H$ with

$$Ob\ (G \times H) = Ob\ G \times Ob\ H, \quad Mor\ (G \times H) = Mor\ G \times Mor\ H.$$

There is also the *opposite* category H^o of H, in which target and source are interchanged, and the order of composition is inverted.

A G-space Y is defined to be a covariant functor $Y : G \to \text{Top}$. Thus a G^o-space is a contravariant functor $X : G \to \text{Top}$. One gets then a family of k-spaces $\{X(p)|p \in Ob\ G\}$, or $\{Y(p)|p \in Ob\ G\}$ by restricting the functor to objects. The action of G on the family is the result of restricting the functor to morphisms. For $q \xleftarrow{g} p$ in $Mor\ G$, there results for X a map $X(q) \to X(p)$ which we denote by g^* and whose values we denote by $x \mapsto xg$. For $q \xleftarrow{g} p$ in $Mor\ G$, there results for Y a map $Y(q) \xleftarrow{g_*} Y(p)$ whose values are denoted by $y \mapsto gy$. The usual rules on compositions and identity elements are satisfied.

Given G^o-spaces X and X', an *equivariant map* $\phi : X \to X'$ is defined to be a natural transformation of functors. That is, ϕ is a collection $\{\phi_p|p \in Ob\ G\}$

of maps $\phi_p : X(p) \to X'(p)$ such that given any morphism $q \overset{g}{\leftarrow} p$, the following diagram is commutative:

$$\begin{array}{ccc} X(q) & \xrightarrow{g^*} & X(p) \\ {\scriptstyle \phi_q}\Big\downarrow & & {\scriptstyle \phi_p}\Big\downarrow \\ X'(q) & \xrightarrow{g^*} & X'(p). \end{array}$$

These maps are also called G^o-maps. There is the similar definition of equivariant maps, or G-maps, joining G-spaces.

The category whose objects are G-spaces and whose morphisms are G-maps is denoted by Top^G. The category whose objects are G^o-spaces and whose morphisms are G^o-maps is denoted by Top^{G^o}.

Given a G-space Y, one gets a diagram of spaces and maps and thus the limit and colimit of the diagram. These are called the limit and colimit of the G-space Y. Hereafter, we will interpret limit and colimit to be the resulting two functors

$$\mathrm{Top}^G \to \mathrm{Top}.$$

The Reduced Product Bifunctor $\times_G : \mathbf{Top}^{G^o} \times \mathbf{Top}^G \to \mathbf{Top}$

Fix a small category G, and let X be a G^o-space in Top and Y a G-space in Top. Form the disjoint union

$$X \times_{Ob\ G} Y = \coprod_{p \in Ob\ G} X(p) \times Y(p)$$

and to ease notation suppose either that all the $X(p)$ are disjoint or all the $Y(p)$ are disjoint so that the disjoint union coincides as a set with the ordinary union. Let

$$X \times_{Ob\ G} G \times_{Ob\ G} Y = \coprod_{p,q \in Ob\ G} X(p) \times G(p,q) \times Y(q),$$

where $G(p,q)$ denotes the set of all morphisms $p \overset{g}{\leftarrow} q$ in G. Thus $X \times_{Ob\ G} G \times_{Ob\ G} Y$ is all triples (x, g, y) where

$$x \in X(p), \qquad p \overset{g}{\leftarrow} q, \qquad x \in Y(q).$$

There are then the maps

$$X \times_{Ob\ G} G \times_{Ob\ G} Y \rightrightarrows X \times_{Ob\ G} Y$$

given by $(x, g, y) \mapsto (xg, y)$ and $(x, g, y) \mapsto (x, gy)$ respectively. Then the *reduced product* $X \times_G Y$ is defined to be the coequalizer of

$$X \times_{Ob\ G} G \times_{Ob\ G} Y \rightrightarrows X \times_{Ob\ G} Y.$$

Specifically, we have

$$X \times_G Y = [\coprod X(p) \times Y(p)]/\sim,$$

where $\sim$ is the least equivalence relation on $\coprod X(p) \times Y(p)$ such that $(xg, y) \sim (x, gy)$ whenever

$$x \in X(p), \qquad p \xleftarrow{g} q, \qquad y \in Y(q).$$

There is the natural quotient map

$$\pi : \coprod X(p) \times Y(p) \to X \times_G Y$$

where $\pi(x, y) = x \times_G y$ is the equivalence class containing (x, y). We thus have the functor

$$\times_G : \mathrm{Top}^{G^o} \times \mathrm{Top}^G \to \mathrm{Top}.$$

We have stated the definition of $X \times_G Y$ in this simplest form for clarity, but also need it in a generalized form.

We need first an interpretation of $G \times H^o$-spaces X, whose verification we leave to the reader. As a collection of spaces, X is a family $\{X(p, q) | p \in Ob\ G, q \in Ob\ H\}$. For each fixed q, we have the G-space $X(\diamond, q) = \{X(p, q) | p \in Ob\ G\}$, where given $p' \xleftarrow{g} p$ we have the action map

$$X(p, q) \to X(p', q), \qquad (g, x) \mapsto gx.$$

The diamond here is used as a blank in which any object of G can be inserted. For each fixed $p \in Ob\ G$, we have the H^o-space $X(p, \diamond) = \{X(p, q) | q \in Ob\ H\}$, where given $q \xleftarrow{h} q'$ we have the action map

$$X(p, q) \to X(p, q'), \qquad (x, h) \mapsto xh.$$

Moreover, for x, g and h as above, we have $(gx)h = g(xh)$. These characterize $G \times H^o$-spaces.

The above is an informal description of isomorphisms

$$\mathrm{Top}^{G \times K} \simeq (\mathrm{Top}^K)^G \simeq (\mathrm{Top}^G)^K.$$

We leave the details to the reader, or see [1.2,p.37].

As an example, we can interpret G as providing a $G \times G^o$-space which we also denote simply by G. Interpreted in this way, G is the $G \times G^o$-space $\{G(p, q)\}$ with left composition giving the left action of G on its morphisms, and right composition giving the right action. Here $G(p, q)$ continues to denote the set of all morphisms $p \xleftarrow{g} q$.

We can now generalize the reduced product to a functor

$$\times_H : \mathrm{Top}^{G \times H^o} \times \mathrm{Top}^{H \times K^o} \to \mathrm{Top}^{G \times K^o}.$$

Given a $G \times H^o$-space X and an $H \times K^o$-space Y, define $X \times_H Y$ to be the family of spaces

$$(X \times_H Y)(p, r) = X(p, \diamond) \times_H Y(\diamond, r)$$

with actions

$$g(x \times_H y) = (gx) \times_H y, \qquad (x \times_H y)h = x \times_H (yh).$$

In the above, the diamond is used as a blank where the same variable object of H is to be filled in.

One can specialize this bifunctor to bifunctors

$$\mathrm{Top}^{G\times H^o} \times \mathrm{Top}^H \to \mathrm{Top}^G, \quad \mathrm{Top}^{G^o} \times \mathrm{Top}^{G\times H^o} \to \mathrm{Top}^{H^o}.$$

Theorem (1.21) *Let X be a G^o-space, let Y be a $G\times H^o$-space, and let Z be an H-space. Then $X\times_G Y$ is an H^o-space, thus we obtain the space $(X\times_G Y)\times_H Z$. Moreover, $Y\times_H Z$ is a G-space, thus we obtain the space $X\times_G (Y\times_H Z)$. We have a homeomorphism*

$$(X\times_G Y)\times_H Z \simeq X\times_G (Y\times_H Z), \qquad (x\times_G y)\times_H z \mapsto x\times_G (y\times_H z).$$

Proof. Let $W = X\times_{Ob\ G} Y\times_{Ob\ H} Z$ denote the disjoint union

$$W = \coprod_{p\in Ob\ G, q\in Ob\ H} X(p)\times Y(p,q)\times Z(q).$$

There are the diagrams of quotient maps

$$W \to \coprod_{q\in Ob\ H} (X\times_G Y(\diamond,q))\times Z(q) \to (X\times_G Y)\times_H Z,$$

$$W \to \coprod_{p\in Ob\ G} X(p)\times (Y(p,\diamond)\times_H Z) \to X\times_G (Y\times_H Z).$$

One can then work across the diagram

$$\begin{array}{ccccc} W & \longrightarrow & \coprod (X\times_G Y(\diamond,q))\times Z(q) & \longrightarrow & (X\times_G Y)\times_H Z \\ \| & & & & \\ W & \longrightarrow & \coprod X(p)\times (Y(p,\diamond)\times_H Z) & \longrightarrow & X\times_G (Y\times_H Z) \end{array}$$

to obtain vertical maps on the right, and can show that they are inverse to each other. □

As an exercise, the reader should check that if we fix the $G\times G^o$-space in

$$\times_G : \mathrm{Top}^{G\times G^o} \times \mathrm{Top}^G \to \mathrm{Top}^G$$

to be G then we get a natural G-homeomorphism

$$G\times_G Y \to Y$$

given by $g\times_G y \mapsto gy$.

The Extension Functor and the Restriction Functor

Let G be a small category, and let H be a subcategory of G. Denote by $i : H \to G$ the inclusion functor. There is then a natural functor

$$i^{\#} : \mathrm{Top}^G \to \mathrm{Top}^H$$

called *restriction*. Namely, given a G-space $X = \{X(p) | p \in Ob\ G\}$, simply let $i^{\#}X = \{X(p) | p \in Ob\ H\}$ and restrict the action to the morphisms of H. We can also consider restriction as applicable to right actions.

There is also a natural functor $i_{\#} : \mathrm{Top}^H \to \mathrm{Top}^G$ called *extension*. Out of G and H, one can make a $G \times H^o$-space which we denote by G_H. Namely,

$$G_H = \{G(p,q) | p \in Ob\ G, q \in Ob\ H\}$$

and let G act by left composition and H act by right composition. Then given an H-space X, define the extension $i_{\#}X$ by

$$i_{\#}X = G_H \times_H X.$$

We can also use G_H on the right to interpret the restriction $i^{\#} : \mathrm{Top}^{G^o} \to \mathrm{Top}^{H^o}$. In fact, if X is a G^o-space then it can be checked that

$$i^{\#}X = X \times_G G_H.$$

(1.22) *Let G be a small category, and let H be a subcategory. If X is a G^o-space and if Y is an H-space, then there is a natural homeomorphism*

$$i^{\#}X \times_H Y \simeq X \times_G i_{\#}Y.$$

This is an easy consequence of (1.21) used on

$$(X \times_G G_H) \times_H Y \simeq X \times_G (G_H \times_H Y). \quad \square$$

The Mapping Bifunctor $\mathbf{Top}^G \times (\mathbf{Top}^G)^o \to \mathbf{Top}$

Fix a small category G, and let X and Y be G-spaces in Top. There is then the set $(X^Y)_{\mathrm{Top}^G}$ of all G-maps $\phi : Y \to X$, and it is not hard to give this set a natural k-space topology. First, there is the space $\prod_{p \in Ob\ G} X(p)^{Y(p)}$ in Top, and we have the natural inclusion as sets

$$(X^Y)_{\mathrm{Top}^G} \subset \prod X(p)^{Y(p)}.$$

Simply give $(X^Y)_{\mathrm{Top}^G}$ the subspace topology in Top. This gives us the mapping bifunctor

$$\mathrm{Top}^G \times (\mathrm{Top}^G)^o \to \mathrm{Top}.$$

As with the reduced product bifunctor, we can extend the mapping bifunctor in various ways, for example to a bifunctor

$$\mathrm{Top}^G \times (\mathrm{Top}^{G \times H^o})^o \to \mathrm{Top}^H, \qquad (X,Y) \mapsto (X^Y)_{\mathrm{Top}^G}.$$

Here the H-space is $\{(X^{Y(\diamond,q)})_{\mathrm{Top}^G}\}$ for all $q \in Ob\ H$, where $\diamond$ denotes the variable. The reader should study the action of H, under which given an equivariant map $X \leftarrow Y(\diamond, q)$ and a morphism $q' \xleftarrow{h} q$ of H there is assigned the composition

$$X \leftarrow Y(\diamond, q) \xleftarrow{h^*} Y(\diamond, q').$$

One should also verify that if X is any G-space in Top, then

$$(X^G)_{\mathrm{Top}^G} \simeq X$$

in the case $\mathrm{Top}^G \times (\mathrm{Top}^{G\times G^o})^o \to \mathrm{Top}^G$.

The mapping bifunctor is well adapted to TOP. Let X be a G-space in TOP and let Y be a G-space in Top. Then the spaces $X(p)^{Y(p)}$ and $\prod X(p)^{Y(p)}$ are compactly generated. Moreover $(X^Y)_{\mathrm{Top}^G}$ is a closed subset of $\prod X(p)^{Y(p)}$. For each fixed $r \xleftarrow{g} q$ in $Mor\ G$, there are the maps

$$\prod X(p)^{Y(p)} \rightrightarrows X(r)^{Y(q)}$$

sending $\phi = \{\phi_p\}$ into $\phi_r g_*$ and $g_* \phi_q$ respectively. The equalizer is closed. The intersection of these closed sets is then closed, and is precisely

$$(X^Y)_{\mathrm{Top}^G} \subset \prod X(p)^{Y(p)}.$$

Hence if X is in TOP^G then the space $(X^Y)_{\mathrm{Top}^G}$ is in TOP, and we have the mapping bifunctor

$$\mathrm{TOP}^G \times (\mathrm{Top}^G)^o \to \mathrm{TOP}$$

as well as the more general form

$$\mathrm{TOP}^G \times (\mathrm{Top}^{G\times H^o})^o \to \mathrm{TOP}^H.$$

Thus we have introduced the primary object of this work, the study of TOP^G for G a small category. The primary invariants limit and colimit are very delicate for arbitrary G-spaces X. In the next chapter we plunge into semisimplicial topology in order to replace systematically a G-space X by an exploded G-space $E_G X$ whose colimit $B_G X$ is generally better behaved than that of X. Similarly, we will replace X by another exploded version $E^G X$ whose limit is better behaved than that of X. The next chapter is devoted to setting these up in a standard way, which requires study of simplicial spaces. It will take another couple of chapters before we have pinned down precisely how the limits and colimits of the exploded versions are better behaved.

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CHAPTER II

The General Topology of Simplicial Spaces

We now start our review of actions of small categories with a central and classic example, the study of the simplicial spaces. Here we study them in a preliminary way, including few homotopy properties.

Many have contributed to the development. We give special mention to several of these contributions. The seminal work, which set up the discrete version which has come to be called simplicial sets, is due to Eilenberg-Zilber [2.3,1950]. Milnor [2.9,1957] introduced his realizations which made an explicit bridge to topology for either simplicial sets or spaces. Segal [2.11,1968] put the work into equivariant topological terms such as we use in this tract, so that one could use the more general simplicial spaces in place of simplicial sets. There is an excellent summary of simplicial spaces in May's 1972 book [2.8,pp. 100-112].

There are noteworthy books treating simplicial sets: those of Lamotke [2.5, 1968], May [2.7,1967], Gabriel-Zisman [2.4,1967] and Bousfield-Kan [2.1,1972]. There is also Quillen's book [2.10,1967] on abstract homotopy theory which is closely interwoven with the subject. One of our purposes in adding to this excellent literature is to give a full presentation which from the first treats simplicial spaces rather than simplicial sets.

Our goal is to expose in the first five chapters, as quickly as makes sense to us, the various tools that we need so that in a more leisurely way we can go through, in chapters 6-10, important examples that relate to algebraic topology. Perhaps any sense of historical order lacking in the first part will be clearer in the second part. At this time we simply plunge into the middle of the subject.

The Simplicial Category Δ

The *simplicial category* Δ is the category whose objects are the nonnegative integers and whose morphisms $\delta : m \to n$ are the order-preserving functions

$$\delta : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\},$$

where *order preserving* means that $i \geq j$ implies $\delta(i) \geq \delta(j)$. The monomorphisms of Δ are those functions which are also one-to-one, and the epimorphisms are those which are also onto. Every morphism in Δ has a unique factorization into an epi followed by a mono. We sometimes denote the partially ordered set $\{0, 1, \cdots, n\}$ simply by $\underline{n}$.

(2.1) *The subcategory Epi Δ of all epis has pushouts. The subcategory Mono Δ of all monos has a restricted pullback condition: if the diagram $n_1 \xrightarrow{\alpha_1} m \xleftarrow{\alpha_2} n_2$ of monos is such that there exists at least one commutative diagram*

$$\begin{tikzcd} r \arrow[r] \arrow[d] & n_1 \arrow[d, "\alpha_1"'] \\ n_2 \arrow[r, "\alpha_2"] & m \end{tikzcd}$$

in Mono Δ, then there is a pullback in Mono Δ.

We start with the second statement. For a fixed m, the monos $n \to m$ are in natural one-to-one correspondence with the nonempty subsets of $\{0, 1, \cdots, m\}$. If two subsets have a nonempty intersection, then the intersection serves as the pullback in the category of nonempty subsets. This translates into the second assertion.

In order to show the first statement, consider first any epi $\gamma : m \to n$. Let $\sim$ denote the equivalence relation on $\{0, \cdots, m\}$ given by $i \sim j$ if and only if $\gamma(i) = \gamma(j)$. The equivalence classes are a disjoint partitioning of $\{0, \cdots, m\}$ into subintervals, and these occur in the same order on $\{0, \cdots, m\}$ as do their images in $\{0, \cdots, n\}$. Thus the equivalence relations whose equivalence classes are subintervals are entirely equivalent to the epis $\gamma : m \to n$. Thus convert the pushout assertion on epis into a pushout assertion on such equivalence relations, and prove it. $\square$

(2.2) *Given an epi $\gamma : m \to n$ in Δ, there exists a mono $\alpha : n \to m$ with $\gamma\alpha = 1_n$. Call such an α a section of γ. If two epis $\gamma_1, \gamma_2 : m \to n$ have precisely the same sections, then $\gamma_1 = \gamma_2$.*

The proof is an exercise.

Cosimplicial Spaces

Define the *standard m-simplex* $\nabla(m)$, for each $m \geq 0$, to be the subspace of Euclidean (m+1)-space consisting of all points $(t_0, t_1, \cdots, t_m)$ for which each $t_i \geq 0$ and for which $\sum_{0 \leq i \leq m} t_i = 1$. The standard simplices are then indexed by the objects of Δ; in addition, for each morphism $\delta : m \to n$ we get a corresponding map $\nabla(m) \to \nabla(n)$ taking $(t_0, \cdots, t_m) \in \nabla(m)$ into

$$\delta(t_0, \cdots, t_m) = (u_0, \cdots, u_n) \in \nabla(n),$$

where

$$u_j = \begin{cases} \sum_{\delta(i)=j} t_i, & \text{when } \delta^{-1}(j) \neq 0 \\ 0, & \text{when } \delta^{-1}(j) = \emptyset. \end{cases}$$

The map is also denoted by $\delta_* : \nabla(m) \to \nabla(n)$.

Denote the vertices of $\nabla(m)$ by

$$v_{0,m} = (1, 0, \cdots, 0),\ v_{1,m} = (0, 1, \cdots, 0),\ v_{m,m} = (0, \cdots, 0, 1).$$

Alternatively, the map $\delta_* : \nabla(m) \to \nabla(n)$ is the unique affine map which takes $v_{i,m}$ into $v_{\delta(i),n}$ for all $0 \le i \le m$. A composition of $r \xleftarrow{\delta'} n \xleftarrow{\delta} m$ in Δ has $\delta'(\delta y) = (\delta'\delta)y$ for each $y \in \nabla(m)$. Moreover, $1_m y = y$ for $y \in \nabla(m)$, where 1_m is the identity morphism.

In short, ∇ is a covariant functor $\Delta \to$ TOP; that is, ∇ is a Δ-space. The Δ-spaces are also called *cosimplicial spaces*. For the moment, the only Δ-space which interests us is ∇.

If Y is a cosimplicial space and if $y \in Y(m)$, then y is *degenerate* if there exist $n < m$, a mono $\alpha : n \to m$ and $v \in Y(n)$ with $y = \alpha v$. Otherwise y is nondegenerate. For ∇, $y \in \nabla(m)$ is degenerate if $y \in \partial\nabla(m)$.

(2.3) *The cosimplicial space ∇ has the following properties.*

(i) If $\alpha : n \to m$ is a mono, then α_* maps $\nabla(n)$ homeomorphically onto a closed subset of $\nabla(m)$.

(ii) Given $v \in \nabla(m)$, there exists a unique triple consisting of $n \le m$, a mono $\alpha : n \to m$, and a nondegenerate $u \in \nabla(n)$ with $v = \alpha u$.

(iii) If the diagram $n_1 \xrightarrow{\alpha_1} m \xleftarrow{\alpha_2} n_2$ of monos has no pullback, then

$$\alpha_{1*}\nabla(n_1) \cap \alpha_{2*}\nabla(n_2) = \emptyset.$$

(iv) If the above diagram of monos has a pullback diagram of monos

$$\begin{array}{ccc} r & \xrightarrow{\rho_1} & n_1 \\ {\scriptstyle\rho_2}\downarrow & & \downarrow{\scriptstyle\alpha_1} \\ n_2 & \xrightarrow{\alpha_2} & m \end{array}$$

and if $\alpha_1\rho_1 = \alpha_2\rho_2 = \beta$, then

$$\alpha_{1*}\nabla(n_1) \cap \alpha_{2*}\nabla(n_2) = \beta_*\nabla(r).$$

(v) If $\gamma : m \to n$ is an epi and if $v \in \nabla(m)$ is nondegenerate, then $\gamma v \in \nabla(n)$ is nondegenerate.

The proofs are left as an exercise.

The following is one way of stating a remarkable computation that goes back to Milnor's paper [2.9].

Theorem (2.4) *Consider Δ embedded diagonally as a subcategory of $\Delta \times \Delta$ by identifying an object k of Δ with (k,k) and a morphism δ of Δ with (δ,δ), thus obtaining an inclusion functor $i : \Delta \to \Delta \times \Delta$. We have then the extension*

$$i_\# : TOP^\Delta \to TOP^{\Delta\times\Delta},$$

and we have in particular a homeomorphism of $\Delta \times \Delta$-spaces

$$i_\#\nabla \simeq \nabla \times \nabla.$$

PROOF. We compute each side. We have on the left for each pair m, n of nonnegative integers,

$$(i_{\#}\nabla)(m,n) = (\Delta(m,\diamond) \times \Delta(n,\diamond)) \times_{\Delta} \nabla,$$

where $\diamond$ denotes a variable nonnegative integer. Here $\Delta(m,\diamond) \times \Delta(n,\diamond)$ can be interpreted also as all order preserving functions $\underline{m} \times \underline{n} \leftarrow \diamond$, where $\underline{m} \times \underline{n}$ is the partially ordered set which is the product of the partially ordered sets $\underline{m}$ and $\underline{n}$, and where now $\diamond$ denotes a variable among the various posets $\underline{k}$. Every order preserving function $\underline{m} \times \underline{n} \leftarrow \underline{k}$ can be factored uniquely as an order preserving composition

$$\underline{m} \times \underline{n} \xleftarrow{\alpha} \underline{p} \xleftarrow{\gamma} \underline{k},$$

where α is a monomorphism and γ is an epimorphism. Let $\{\underline{m} \times \underline{n} \leftarrow \diamond\}$ denote all the order preserving functions and let $\{\underline{m} \times \underline{n} \hookleftarrow \diamond\}$ denote all the order preserving monos. Then it is seen that

$$\{\underline{m} \times \underline{n} \leftarrow \diamond\} \simeq \{\underline{m} \times \underline{n} \hookleftarrow \diamond\} \times_{Mono\ \Delta} \Delta.$$

It is seen to follow from associativity of the reduced product that

$$i_{\#}\nabla(m,n) \simeq \{\underline{m} \times \underline{n} \hookleftarrow \diamond\} \times_{Mono\ \Delta} \nabla.$$

Thus we have computed the left hand side. Points of $i_{\#}\nabla(m,n)$ are uniquely expressed as $\alpha \times_{Mono\ \Delta} v$ where for some k, α is an order preserving mono $\underline{m} \times \underline{n} \hookleftarrow \underline{k}$ and where $v \in \nabla(k) - \partial\nabla(k)$.

We now interpret the right hand side $\nabla(m) \times \nabla(n)$ in a standard form as a finite simplicial complex; for full details of this simplicial decomposition, see Eilenberg and Steenrod [2.2]. There are the vertices $(v_{i,m}, v_{j,n})$, i.e. its set of vertices is in natural one-to-one correspondence with $\underline{m} \times \underline{n}$. Any simplex of dimension k is spanned by the set of vertices which correspond to an order preserving mono $\underline{k} \hookrightarrow \underline{m} \times \underline{n}$, and the correspondence between simplices of dimension k and such monos is one-to-one.

Thus the two sides $i_{\#}\nabla(m,n)$ and $\nabla(m) \times \nabla(n)$ are naturally homeomorphic. We leave it to the reader to ponder further this unusual theorem. $\square$

The Category $\mathbf{TOP}^{\Delta^o}$ of Simplicial Spaces

A Δ^o-space X in TOP is a contravariant functor $X : \Delta \to$ TOP. The Δ^o-spaces are also called *simplicial spaces*. These are the objects of TOP^{Δ^o}.

There is also the category SET^{Δ^o} whose objects are the contravariant functors $X : \Delta \to$ SET, where SET is the category of sets and functions, and whose morphisms are the natural transformations of functors. We regard this category as the full subcategory of TOP^{Δ^o} whose objects are the Δ^o-spaces X with each $X(m)$ a discrete space. Such X are also called *simplicial sets*.

We get a Δ^o-space A^{∇} for each space A in TOP; to be specific, $A^{\nabla} = \{A^{\nabla(n)} | n \geq 0\}$. Thus we have a functor

$$\diamond^{\nabla} : \mathrm{TOP} \to \mathrm{TOP}^{\Delta^o},$$

where given $A \xleftarrow{f} \nabla(n)$ and $n \xleftarrow{\delta} m$ in Δ, then $f\delta$ is the composition

$$A \xleftarrow{f} \nabla(n) \xleftarrow{\delta_*} \nabla(m).$$

The elements of $A^{\nabla(m)}$ are the *singular m-simplices* of A. The monomorphisms $\alpha : m \to n$ of Δ assign to $f \in A^{\nabla(n)}$ the various *faces* $\alpha^* f$ of f. The epimorphisms $\gamma : m \to n$ with $m > n$ assign to $f \in A^{\nabla(n)}$ the various *degenerate* m-simplices based on f.

One can equally well consider each $A^{\nabla(n)}$ as having the discrete topology. In this case, one may as well take A any topological space, thus if one ignores the topology on the mapping spaces one gets a functor

$$\mathrm{top} \to \mathrm{SET}^{\Delta^o}$$

sending A into A^{∇} with the discrete topology. These are the Δ^o-spaces used to approximate a space by a CW-complex.

Terminology natural to the above example is also applied to an arbitrary simplicial space X. For example, a point of $X(0)$ is called a *vertex* of X.

Topological Categories and the Functor TOPCAT $\to$ TOP$^{\Delta^o}$

There is another large class of simplicial spaces. Define a *topological category* G to be a small category G, together with compactly generated topologies on $Ob\ G$ and $Mor\ G$ in which the structure functions are all continuous. If G and H are topological categories, a *continuous* functor $H \to G$ is a functor for which both $Ob\ H \to Ob\ G$ and $Mor\ H \to Mor\ G$ are continuous.

TOPCAT denotes the category whose objects are the topological categories and whose morphisms are the continuous functors. This category is due to Segal [2.11,1968], who gave the functor

$$\mathrm{TOPCAT} \to \mathrm{TOP}^{\Delta^o}$$

that we consider now.

In order to present this functor, one needs to look at the category Δ in terms of categories and functors. For each $n \geq 0$, consider the category $\underline{n}$ whose objects are the elements of $\{0, 1, \cdots, n\}$, and which has precisely one morphism $\gamma_{j,i} : i \to j$ whenever $i \geq j$ and no morphism $i \to j$ if $i < j$. That is, $\underline{n}$ is now the category associated with the poset $\{0, \cdots, n\}$ in its natural linear order. We write

$$\underline{n} = \{0, 1, \cdots, n\}.$$

The functors $\underline{m} \to \underline{n}$ are in natural one-to-one correspondence with the order preserving functions

$$\delta : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}.$$

Given δ, there is the functor $\underline{m} \to \underline{n}$ which takes an object i into the object $\delta(i)$ and the morphism $\gamma_{j,i}$ for $j \leq i$ into the morphism $\gamma_{\delta(j),\delta(i)}$. Thus we could equally well have presented Δ as the category with objects $\underline{n}$ for all $n \geq 0$, and with morphisms the functors $\underline{m} \to \underline{n}$.

Given a topological category G and given a nonnegative integer n, consider all functors $\underline{n} \to G$. These are in natural correspondence with the diagrams

$$p_0 \xleftarrow{g_1} p_1 \xleftarrow{g_2} \cdots \xleftarrow{g_n} p_n$$

in G, for given such a diagram there is the functor F which sends an object i into $F(i) = p_i$ and the morphism $\gamma_{j,i}$ for $j \leq i$ into

$$F(\gamma_{j,i}) = \begin{cases} g_{j+1} \cdots g_i, & \text{if } j < i \\ 1_{p_i}, & \text{if } j = i. \end{cases}$$

For $n > 0$, the functors $\underline{n} \to G$ are in natural correspondence with the n-tuples $(g_1, g_2, \cdots, g_n)$ of morphisms of G for which the composition $g_1 g_2 \cdots g_n$ exists. It can be seen that the set of all such n-tuples is closed in $(Mor\ G)^n$, and therefore is a compactly generated space. If $n = 0$, then the functors $\underline{0} \to G$ can be identified with the space *Ob* G or with the subspace of $Mor\ G$ consisting of all the identity morphisms; these are homeomorphic spaces, so it doesn't matter which is used.

Given a functor $F : \underline{n} \to G$ represented by $(g_1, \cdots, g_n)$, and given a functor $\delta : \underline{m} \to \underline{n}$, then the composed functor $F\delta$ is seen to be represented by $(g'_1, \cdots, g'_m)$, where

$$g'_i = \begin{cases} g_{\delta(i-1)+1} \cdots g_{\delta(i)}, & \text{if } \delta(i-1) < \delta(i) \\ 1_{F\delta(i)}, & \text{if } \delta(i-1) = \delta(i). \end{cases}$$

We will denote the space of functors $\underline{n} \to G$ by $G^{\underline{n}}$.

We have now Segal's *nerve functor*

$$N : \mathrm{TOPCAT} \to \mathrm{TOP}^{\Delta^o}.$$

Namely, for each topological category G let

$$NG = \{G^{\underline{n}} | n \geq 0\}.$$

Every functor $\delta : \underline{m} \to \underline{n}$ then gives by composition a map $G^{\underline{n}} \to G^{\underline{m}}$, which gives the action map.

Among the topological categories are the *topological monoids*, where *Ob* G is a singleton; here $NG = \{G^n\}$. And among these are the *topological groups*, where there is in addition a continuous inverse function.

In summary up to this point, we have introduced the simplicial spaces, the objects of a category TOP^{Δ^o}. We have furnished a number of simplicial spaces to look at more fully, by means of two natural functors

$$\mathrm{TOP} \to \mathrm{TOP}^{\Delta^o}, \qquad \mathrm{TOPCAT} \to \mathrm{TOP}^{\Delta^o}.$$

We need ways of studying simplicial spaces, and in particular need Milnor's *realization* of a simplicial space.

The Realization $|X|$ of a Simplicial Space X

We are ready now to describe Milnor's process [2.9] for obtaining a k-space $|X|$ from any given simplicial space X. Namely, consider

$$\times_\Delta : \mathrm{TOP}^{\Delta^o} \times \mathrm{TOP}^{\Delta} \to \mathrm{Top},$$

fix the Δ-space ∇ and thus define

$$|X| = X \times_\Delta \nabla.$$

Then $|X|$ is a k-space called the *realization* of X, and we have the functor $|*| : \mathrm{TOP}^{\Delta^o} \to \mathrm{Top}$. Let $\pi : \coprod X(m) \times \nabla(m) \to |X|$ denote the quotient map sending each $(x, v) \in \coprod X(m) \times \nabla(m)$ into its equivalence class which is denoted by $x \times_\Delta v$.

Given a small category G, then we call a *generalized colimit* for G^o-spaces a functor $\mathrm{TOP}^{G^o} \to \mathrm{Top}$ of the form $X \to X \times_G V$, where V is a fixed left G-space with each $V(p)$ a contractible space. Thus $|X|$ is a generalized colimit of the Δ^o-space X. This example is the most important of the generalized colimits.

The Eilenberg-Zilber Analysis

For any Δ^o-space X, the inclusion $i : Epi\ \Delta \to \Delta$ induces the restriction $i^\# X$ which is the $(Epi\ \Delta)^o$-space obtained by restricting the structural category from Δ^o to $(Epi\ \Delta)^o$. We give here the Eilenberg-Zilber analysis [2.3] of $i^\# X$ for all X.

Fix a Δ^o- space X in TOP. If $\gamma : m \to n$ is an epimorphism in Δ, then the action map $\gamma^* : X(n) \to X(m)$ is a homeomorphism of $X(n)$ onto a closed subset of $X(m)$. For choose a section α of γ, and note that the composition

$$X(n) \xleftarrow{\alpha^*} X(m) \xleftarrow{\gamma^*} X(n)$$

is the identity, hence $\gamma^*\alpha^*$ is a retracting map of $X(m)$ onto the image of γ^*, and the assertion follows.

Call an epimorphism $\gamma : m \to n$ *proper* if γ is not the identity morphism of m. If X is in TOP^{Δ^o}, define $X^{deg}(m) \subset X(m)$ to be $X^{deg}(m) = \bigcup \gamma^* X(n)$ where the union is over all proper epimorphisms γ with source m, and call an element $x \in X = \coprod X(n)$ *nondegenerate* if it belongs to some $X(m) - X^{deg}(m)$. Note that $X^{deg}(m)$ is a finite union of closed subsets of $X(m)$, hence is closed in $X(m)$.

(2.5) *If X is in TOP^{Δ^o} and $x \in X(m)$, then there exists a unique nondegenerate element y of X such that for some epimorphism $\gamma : m \to n$ we have $y \in X(n)$ and $x = y\gamma$. The epimorphism γ is also uniquely determined by x.*

PROOF. It is easy to see that there is at least one triple n, γ, y satisfying all conditions except possibly uniqueness. Suppose there are two such triples: a nondegenerate $y_1 \in X(n_1)$ and an epimorphism $\gamma_1 : m \to n_1$ with $y_1\gamma_1 = x$, and a nondegenerate $y_2 \in X(n_2)$ and an epimorphism $\gamma_2 : m \to n_2$ with $y_2\gamma_2 = x$. We will consider the monomorphisms $\alpha_1 : n_1 \to m$ with $\gamma_1\alpha_1 = 1_{n_1}$, and the monomorphisms $\alpha_2 : n_2 \to m$ with $\gamma_2\alpha_2 = 1_{n_2}$. There exists at least one of each. Since $\gamma_1\alpha_1 = 1_{n_1}$, then

$$y_1 = y_1\gamma_1\alpha_1 = x\alpha_1$$

and similarly $y_2 = x\alpha_2$. Then

$$y_2\gamma_2\alpha_1 = x\alpha_1 = y_1$$

and similarly $y_1\gamma_1\alpha_2 = y_2$. We next see that both $\gamma_2\alpha_1$ and $\gamma_1\alpha_2$ are monomorphisms. Consider $\gamma_2\alpha_1$ as typical. Write $\gamma_2\alpha_1 = \alpha'\gamma'$ where α' is a monomorphism and γ' is an epimorphism. Then

$$y_2\alpha'\gamma' = y_1.$$

If γ' were a proper epimorphism, we would have a contradiction to the fact that y_1 is nondegenerate. Thus $\gamma_2\alpha_1$ is a monomorphism. Similarly, so is $\gamma_1\alpha_2$ a monomorphism. Since $\gamma_2\alpha_1 : n_1 \to n_2$ and $\gamma_1\alpha_2 : n_2 \to n_1$ are both monomorphisms then $n_1 = n_2$; call their common value n, and note that $\gamma_2\alpha_1 = 1_n$ and $\gamma_1\alpha_2 = 1_n$. Hence the epimorphisms γ_1 and γ_2 have precisely the same sections. Hence $\gamma_1 = \gamma_2$ by (2.2). Since γ_1^* is one-to-one, then $y_1 = y_2$. □

(2.6) *Consider the pushout diagram of epimorphisms in Δ*

$$\begin{tikzcd} m \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & n_1 \arrow[d, "\varphi_1"'] \\ n_2 \arrow[r, "\varphi_2"] & p \end{tikzcd}$$

and let $\tau = \varphi_1\gamma_1 = \varphi_2\gamma_2$. If X is a simplicial space, then

$$\begin{tikzcd} X(m) & X(n_1) \arrow[l, "\gamma_1^*"'] \\ X(n_2) \arrow[u, "\gamma_2^*"] & X(p) \arrow[l] \arrow[u] \end{tikzcd}$$

is a pullback diagram: i.e.

$$\gamma_1^*X(n_1) \cap \gamma_2^*X(n_2) = \tau^*X(p).$$

PROOF. Let $x \in \gamma_1^*X(n_1) \cap \gamma_2^*X(n_2)$. Then there exist $y_1 \in X(n_1)$ and $y_2 \in X(n_2)$ with

$$x = y_1\gamma_1 = y_2\gamma_2.$$

Choose by (2.5) the unique nondegenerate $y \in X(r)$ and the unique epi $\gamma : m \to r$ such that $x = y\gamma$. Similarly there are nondegenerate elements corresponding to both y_1 and y_2, but the uniqueness of the nondegenerate y corresponding to x implies that y serves also for y_1 and y_2. That is, there are epis $\rho_1 : n_1 \to r$ and $\rho_2 : n_2 \to r$ such that

$$y_1 = y\rho_1 \ , \ y_2 = y\rho_2.$$

Then $x = y\rho_1\gamma_1 = y\rho_2\gamma_2$ and from (2.5) we also get $\rho_1\gamma_1 = \rho_2\gamma_2$. Since τ is the pushout of γ_1 and γ_2, there is an epi $\mu : p \to r$ with $\mu\tau = \rho_1\gamma_1 = \rho_2\gamma_2$. Then $x = y\mu\tau$ and $x \in \tau^*\mu^*X(r)$ and the theorem follows. $\square$

The Filtration of $|X|$

We can give now Milnor's analysis of the equivalence relation $\sim$ on $\coprod X(m) \times \nabla(m)$. Here $\sim$ is the least equivalence relation such that given

$$x \in X(n), \qquad n \overset{\delta}{\leftarrow} m, \qquad v \in \nabla(m),$$

then $(x\delta, v) \sim (x, \delta v)$. Roughly speaking, $\sim$ turns out to be very nice because, by (2.5), X behaves well with respect to *Epi* Δ and, by (2.3), ∇ behaves well with respect to $Mono\,\Delta$.

(2.7) *For any $X \in TOP^{\Delta^o}$ and any $(x, v) \in \coprod X(m) \times \nabla(m)$, there exists a unique representative of the equivalence class $[(x, v)]$ of the form (y, w) where y and w are both nondegenerate. If $(y, w) \in X(p) \times \nabla(p)$ and $(x, v) \in X(m) \times \nabla(m)$ are distinct, then $p < m$.*

Proof. We define a retracting function Φ from $\coprod X(m) \times \nabla(m)$ onto its subset consisting of all (y, w) with y and w both nondegenerate. Given $(x, v) \in X(m) \times \nabla(m)$, choose the unique monomorphism $\alpha : r \to m$ and the unique nondegenerate element $u \in \nabla(r) - \partial\nabla(r)$ with $v = \alpha u$. Then

$$(x, v) = (x, \alpha u) \sim (x\alpha, u).$$

Using (2.5), there is a unique nondegenerate y in some $X(s)$ and epimorphism $\gamma : r \to s$ with $x\alpha = y\gamma$. Then

$$(x\alpha, u) = (y\gamma, u) \sim (y, \gamma u).$$

Define $\Phi(x, v) = (y, \gamma u)$ and note that y and γu are both nondegenerate. Note that $\Phi(x, v) \sim (x, v)$ and that if x and v are nondegenerate, then $\Phi(x, v) = (x, v)$.

We show that if δ is a morphism of Δ and if

$$x \in X(n), \qquad n \overset{\delta}{\leftarrow} m, \qquad v \in \nabla(m)$$

then

$$\Phi(x\delta, v) = \Phi(x, \delta v).$$

We can assume that

$$\Phi(x\delta, v) = (y_1, \gamma_1 u_1)\ ,\ \Phi(x, \delta v) = (y_2, \gamma_2 u_2),$$

where

$$v = \alpha_1 u_1\ ,\ x\delta\alpha_1 = y_1\gamma_1\ ,\ \delta v = \alpha_2 u_2\ , x\alpha_2 = y_2\gamma_2$$

as in the definition of Φ. Then

$$\delta\alpha_1 u_1 = \alpha_2 u_2.$$

Let $\delta\alpha_1 = \alpha_3\gamma_3$ be the factorization of $\delta\alpha_1$ into an epi γ_3 followed by a mono α_3. Then

$$\alpha_3\gamma_3 u_1 = \alpha_2 u_2\ ,\ \alpha_3 = \alpha_2\ ,\ \gamma_3 u_1 = u_2.$$

Substituting α_3 for α_2 in $x\alpha_2 = y_2\gamma_2$, we get

$$x\alpha_3\gamma_3 = y_2\gamma_2\gamma_3\ ,\ x\delta\alpha_1 = y_2\gamma_2\gamma_3\ ,\ y_1\gamma_1 = y_2\gamma_2\gamma_3$$

and $y_1 = y_2$, $\gamma_1 = \gamma_2\gamma_3$. Hence

$$\gamma_1 u_1 = \gamma_2\gamma_3 u_1 = \gamma_2 u_2$$

and $\Phi(x\delta, v) = \Phi(x, \delta v)$ for all (x, δ, v). Hence Φ is constant on each equivalence class and the result follows. $\square$

We are now ready to use this analysis. In our present setting with no homotopy as yet built into our methods, we use the following criterion for potential interest of a generalized colimit $\mathrm{TOP}^G \to \mathrm{Top}$. Namely, to be of potential interest it must take the G-spaces in TOP^G, or at least those that we care about, into TOP. The realization satisfies this in the strongest way, by taking every Δ^o-space in TOP into TOP.

Theorem 2.8 *For $X \in TOP^{\Delta^o}$ and $\pi : \coprod X(m) \times \nabla(m) \to |X|$ the natural quotient map, the realization $|X| = X \times_\Delta \nabla$ is a filtered k-space given by*

$$|X| = \bigcup |X|_n \text{ where } |X|_n = \pi(X(n) \times \nabla(n)).$$

For each n, π gives a relative homeomorphism

$$(X(n), X^{deg}(n)) \times (\nabla(n), \partial\nabla(n)) \to (|X|_n, |X|_{n-1})$$

mapping (x, v) into $x \times_\Delta v$. In particular, each point of $|X|_n - |X|_{n-1}$ has a unique representation as $x \times_\Delta v$ where $x \in X(n) - X^{deg}(n)$ and $v \in \nabla(n) - \partial\nabla(n)$. Each $|X|_n$ is compactly generated and $|X|$ is compactly generated. Thus we have the functor

$$|\diamond| : TOP^{\Delta^o} \to TOP.$$

Proof. We begin by letting

$$|X|_n = \pi(\coprod_{s \le n} X(s) \times \nabla(s)).$$

We then use (1.8) on

$$\begin{array}{ccc} \coprod_{s\le n} X(s) \times \nabla(s) & & \coprod X(m) \times \nabla(m) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ |X|_n & \xrightarrow{i} & |X|, \end{array}$$

guided by the function Φ above. Namely, consider all diagrams

$$\beta : s \xleftarrow{\gamma} p \xrightarrow{\alpha} m$$

where γ is an epimorphism, α is a monomorphism and $s \leq n$. There are only a finite number of such diagrams β for each m. For each β, there are the diagrams

$$X(s) \xrightarrow{\gamma^*} X(p) \xleftarrow{\alpha^*} X(m)\ ,\ \nabla(p) \xrightarrow{\alpha_*} \nabla(m),$$

where γ^* and α_* are homeomorphisms onto closed sets, and where α^* maps onto $X(p)$. Let $Z(\beta) \subset X(m) \times \nabla(m)$ be given by

$$Z(\beta) = (\alpha^*)^{-1}\gamma^* X(s) \times \alpha_* \nabla(p),$$

which is closed in $X(m) \times \nabla(m)$, and note that there are only a finite number for each m. The reader should check that $\pi^{-1} i(|X|_n) = \bigcup_\beta Z(\beta)$. For each β, define

$$r_\beta : Z(\beta) \to X(s) \times V(s)$$

by $r_\beta(x,v) = (y, \gamma u)$ where $v = \alpha u$ and $x\alpha = y\gamma$. The conditions of (1.8) are met, thus

$$\pi' : \coprod_{s \leq n} X(s) \times V(s) \to |X|_n$$

is a quotient map and $|X|_n$ is closed in $|X|$. If $A \subset |X|$ and all $A \cap |X|_n$ are closed, then $\pi^{-1}A$ meets each $X(s) \times \nabla(s)$ in a closed set, thus $\pi^{-1}(A)$ is closed and A is closed. That is, $|X| = \bigcup |X|_n$ is a filtration of $|X|$ in Top.

In order to show that $\pi'' : X(n) \times \nabla(n) \to |X|_n$ is also a quotient map, (1.8) can be applied to

$$\begin{array}{ccc} X(n) \times \nabla(n) & & \coprod_{s \leq n} X(s) \times \nabla(s) \\ {\scriptstyle \pi''} \big\downarrow & & {\scriptstyle \pi'} \big\downarrow \\ |X|_n & =\!= & |X|_n. \end{array}$$

Namely, let $\sigma_n : X(n) \times \nabla(n) \to X(n) \times \nabla(n)$ be the identity. For $s < n$, choose a monomorphism $\alpha : s \to n$ and an epimorphism $\gamma : n \to s$ with $\gamma\alpha = 1_s$ and define

$$\sigma_s : X(s) \times \nabla(s) \to X(n) \times \nabla(n)$$

by $\sigma_s(x,v) = (x\gamma, \alpha v)$. Then (1.8) shows that π'' is a quotient map and (2.7) shows that

$$\pi : (X(n), X^{deg}(n)) \times (\nabla(n), \partial\nabla(n)) \to (|X|_n, |X|_{n-1})$$

is a relative homeomorphism. It follows inductively that each $|X|_n$ is compactly generated and hence $|X|$ is compactly generated. $\square$

Note as a corollary that if X is a simplicial set, then $|X|$ is a CW-complex with an n-cell for each nondegenerate $x \in X(n)$; in fact, the open n-cell corresponding to x is all $x \times_\Delta v$ where $v \in \nabla(n) - \partial\nabla(n)$. For basic facts about CW-complexes, see Spanier's text [2.12] or that of Whitehead [1.6].

Milnor's Product Theorem

Theorem (2.9) *If X and Y are simplicial spaces in TOP, then there is in TOP^{Δ^o} the categorical two-fold product*

$$X \times_{Ob\,\Delta} Y = \{X(n) \times Y(n) | n \geq 0\},$$

where Δ acts diagonally on the right. The projection Δ^o-maps

$$X \xleftarrow{\pi_1} X \times_{Ob\,\Delta} Y \xrightarrow{\pi_2} Y$$

give maps

$$|X| \xleftarrow{\pi_{1*}} |X \times_{Ob\,\Delta} Y| \xrightarrow{\pi_{2*}} |Y|,$$

and the induced map

$$\pi_{1*} \times \pi_{2*} : |X \times_{Ob\,\Delta} Y| \to |X| \times |Y|$$

is a homeomorphism.

Proof. There is the product $\Delta \times \Delta$-space $X \times Y$, and the Δ- space ∇. If $i : \Delta \to \Delta \times \Delta$ is the diagonal embedding, we get from (1.22) and (2.4) that

$$i^{\#}(X \times Y) \times_\Delta \nabla \simeq (X \times Y) \times_{\Delta\times\Delta} (\nabla \times \nabla).$$

It is readily checked that

$$(X \times Y) \times_{\Delta\times\Delta} (\nabla \times \nabla) \simeq (X \times_\Delta \nabla) \times (Y \times_\Delta \nabla).$$

Moreover

$$i^{\#}(X \times Y) = X \times_{Ob\,\Delta} Y$$

and the theorem follows. $\square$

The above homeomorphism takes $(x, y) \times_\Delta v$ into $(x \times_\Delta v, y \times_\Delta v)$. Its inverse represents a point $(x \times_\Delta v, y \times_\Delta w)$ in the form $(x \times_\Delta \delta_1 u, y \times_\Delta \delta_2 u)$ and maps it into $(x\delta_1, y\delta_2) \times_\Delta u$.

The following generalized form of (2.9) is also useful. In it, we call a $\Delta^o \times \Delta^o$-space a *bisimplicial space*. If Z is a bisimplicial space, then its realization is defined by

$$||Z|| = Z \times_{\Delta\times\Delta} (\nabla \times \nabla).$$

If i is the diagonal inclusion of Δ into $\Delta\times\Delta$, the realization $||Z||$ is homeomorphic to $i^{\#}Z \times_\Delta \nabla = |i^{\#}Z|$.

(2.10) *Let W be a $G^o \times \Delta^o$-space and let Z be a $G \times \Delta^o$-space. We can consider W as a functor $G^o \to TOP^{\Delta^o}$ and thus obtain a G^o-space from the composition*

$$G^o \to TOP^{\Delta^o} \xrightarrow{|\diamond|} TOP,$$

which we denote by $|W| = \{|W(p, \diamond)|\}$. Similarly we obtain a G-space from Z which we denote by $|Z| = \{|Z(p, \diamond)|\}$. Alternatively for each $m, n \geq 0$ we obtain a *k-space* $W(\diamond, m) \times_G Z(\diamond, n)$*, thus obtaining a bisimplicial space which we denote by* $W \times_G Z$*. Then we have*

$$||W \times_G Z|| \simeq |i^{\#}(W \times_G Z)| \simeq |W| \times_G |Z|.$$

PROOF. From (2.9) we have the homeomorphism

$$|i^{\#}(W \times Z)| = |W \times_{Ob\,\Delta} Z| \to |W| \times |Z|$$

together with its inverse $|W| \times |Z| \to |W \times_{Ob\,\Delta} Z|$. Both of these are seen to preserve equivalences, so that one gets maps

$$|i^{\#}(W \times_G Z)| \to |W| \times_G |Z|, \qquad |W| \times_G |Z| \to |i^{\#}(W \times_G Z)|$$

which are seen to be inverse to each other. $\square$

When we use (2.10), one of the above (say Z) will only be a G-space. Then Z is regarded as a $G \times \Delta^o$-space trivially by letting $Z(p, n) = Z(p)$ and letting each δ act as the identity. In this case, the conclusion can be written as $|i^{\#}(W \times_G Z)| \simeq |W| \times_G Z$. If instead W is a G^o-space and Z is in $G \times \Delta^o$-space, then the conclusion becomes $|i^{\#}(W \times_G Z)| \simeq W \times_G |Z|$.

Segal's Homotopy Theorem

If $\alpha, \beta : G \to H$ are continuous functors joining topological categories, recall that a natural transformation $T : \alpha \to \beta$ assigns to each $p \in Ob\ G$ a morphism $Tp : \alpha p \to \beta p$ in H such that for any $g : p \to q$ in G there is commutativity in

$$\begin{array}{ccc} \alpha p & \xrightarrow{\alpha g} & \alpha q \\ {\scriptstyle Tp}\downarrow & & {\scriptstyle Tq}\downarrow \\ \beta p & \xrightarrow{\beta g} & \beta q. \end{array}$$

Then T is *continuous* if the function $T : Ob\ G \to Mor\ H$ is continuous. There is the following theorem of Segal [2.11].

2.11 *Let* $\alpha, \beta : G \to H$ *be continuous functors joining topological categories which are related by a continuous natural transformation* $T : \alpha \to \beta$*. Then the maps* $\alpha_*, \beta_* : |N(G)| \to |N(H)|$ *are homotopic.*

PROOF. Let $\mathcal{I}$ denote the category with two objects $\{0, 1\}$ and with precisely one nonidentity morphism, that being of the form $0 \to 1$. Then $N(\mathcal{I})$ has two nondegenerate elements in dimension 0 corresponding to the two objects, and one nondegenerate element in dimension one corresponding to the nonidentity morphism. That is, $|N(\mathcal{I})|$ is the unit interval. Consider now the given data, which is precisely what is required to give a continuous functor

$$\mathcal{I} \times G \to H.$$

There is the induced map

$$|N(\mathcal{I} \times G)| \simeq I \times |N(G)| \to |N(H)|,$$

which is precisely the desired homotopy. □

Having developed classic properties of the realization, at least those not including a deeper use of homotopy, we set up now its use in the study of TOP^G for G a small category.

The Functor $M_1 : \mathbf{TOP}^G \to \mathbf{TOPCAT}$

In order to exploit more fully the composition

$$\mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o} \xrightarrow{|\diamond|} \mathrm{TOP},$$

we note that for any small category G and any G-space Y in TOP we get a topological category M_1Y. This construction is due to Segal [2.11].

Define the topological category M_1Y to have

$$Ob\ M_1Y = Y = \coprod_{p \in Ob\ G} Y(p);$$

we assume the various $Y(p)$ already disjoint so that the objects are the various $y \in Y(p)$ for all $p \in Ob\ G$. Define the space of morphisms by

$$Mor\ M_1Y = G \times_{Ob\ G} Y,$$

so that morphisms are pairs (g, y) where $p \xleftarrow{g} q$ and $y \in Y(q)$. The structure functions of M_1Y are as follows:

(i) the source of the morphism $(g, y) \in G(p, q) \times Y(q)$ is y, and the target is gy; thus in arrow form we write

$$gy \xleftarrow{(g,y)} y, \qquad (g, y) \in G \times_{Ob\ G} Y;$$

two of the four variables in the arrow are easily derived from the other two, so that we also write the arrow as $\diamond \xleftarrow{(g,\diamond)} y$ where each diamond denotes here a variable to be filled in from the given variables;

(ii) the identity morphism 1_y is $y \xleftarrow{(1_q,y)} y$ for $y \in Y(q)$;

(iii) the composition

$$g'gy \xleftarrow{(g',gy)} gy \xleftarrow{(g,y)} y$$

is defined by

$$(g', gy)(g, y) = (g'g, y),$$

or in diamond notation as $(g', \diamond)(g, y) = (g'g, y)$.

It can then be checked that we have a functor

$$M_1 : \mathrm{TOP}^G \to \mathrm{TOPCAT}.$$

In passing, note that one easily computes the simplicial space which is the image of Y under

$$\mathrm{TOP}^G \xrightarrow{M_1} \mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o}.$$

In order to do so, one has to compute all functors $\underline{n} \to M_1 Y$. These are all the diagrams

$$\diamond \xleftarrow{(g_1,\diamond)} \diamond \xleftarrow{(g_2,\diamond)} \cdots \diamond \xleftarrow{(g_n,\diamond)} y, \qquad (g_1, \cdots, g_n, y) \in G^{\underline{n}} \times_{Ob\ G} Y.$$

Thus $NM_1Y = \{G^{\underline{n}} \times_{Ob\ G} Y\}$. The vertices of NM_1Y are the points of $Y = \coprod Y(p)$, for $n > 0$ the n-simplices are all $(g_1, \cdots, g_n, y)$ as above, and the non-degenerate simplices are those for which no g_i is an identity morphism.

The image of Y under the composition

$$\mathrm{TOP}^G \xrightarrow{M_1} \mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o} \xrightarrow{|*|} \mathrm{TOP}$$

is denoted by $B_G Y$. It is called the *standard homotopy colimit* of the G-space Y. Points of $B_G Y$ are of the form

$$(g_1, \cdots, g_n, y) \times_\Delta (t_0, \cdots, t_n),$$

and have a unique representation where no g_i is an identity morphism and $(t_0, \cdots, t_n) \in \nabla(n) - \partial\nabla(n)$.

The Functor M_0 : $\mathbf{TOP}^{G^o} \to \mathbf{TOPCAT}$

Fix a small category G and a right G-space X in TOP. It is easy to modify the above to construct a topological category $M_0 X$. Set

$$Ob\ M_0X = X = \coprod X(p), \qquad Mor\ M_0X = X \times_{Ob\ G} G$$

so that morphisms are written as

$$x \xleftarrow{(x,g)} xg, \qquad (x,g) \in X \times_{Ob\ G} G.$$

One obtains

$$M_0 : \mathrm{TOP}^{G^o} \to \mathrm{TOPCAT}.$$

The simplicial space NM_0X is given by

$$NM_0X = \{X \times_{Ob\ G} G^{\underline{n}} | n \geq 0\}.$$

Given a right G-space X, the Milnor realization $|NM_0X|$ is the *standard homotopy colimit* $B_{G^o}X$ of the G^o-space X.

May's Bifunctor $M : \mathbf{TOP}^{G^o} \times \mathbf{TOP}^G \to \mathbf{TOPCAT}$

There is a general setting due to May [2.6] which includes the above as special cases, and which we use in a basic way.

Fix the small category G, a G^o-space X in TOP, and a G-space Y in TOP. Following May, we define a two-sided topological category $M(X,Y)$ by

$$Ob\ M(X,Y) = \coprod X(p)\times Y(p) = X\times_{Ob\ G}Y,\ Mor\ M(X,Y) = X\times_{Ob\ G}G\times_{Ob\ G}Y.$$

The morphism $(x,g,y) \in X(p) \times G(p,q) \times Y(q)$ is written in arrow form as

$$(x, gy) \xleftarrow{(x,g,y)} (xg, y).$$

In arrow form, four of the seven entries are derived from the other three so that in simplified form the arrow can be written

$$(x, \diamond) \xleftarrow{(\diamond,g,\diamond)} (\diamond, y),$$

where the diamonds can be filled in from the other data.

Compositions exist only in the case

$$(x, g'gy) \xleftarrow{(x,g',gy)} (xg', gy) \xleftarrow{(xg',g,y)} (xg'g, y),$$

where $(x,g',gy)(xg',g,y) = (x,g'g,y)$. If only independent variables are displayed, the composition of

$$(x, \diamond) \xleftarrow{(\diamond,g',\diamond)} (\diamond, \diamond) \xleftarrow{(\diamond,g,\diamond)} (\diamond, y)$$

is $(x, g'g, y)$.

The simplicial space $NM(X,Y)$ is given by

$$NM(X,Y) = \{X \times_{Ob\ G} G^{\underline{n}} \times_{Ob\ G} Y\},$$

thus its elements can be abbreviated in the form

$$(x, g_1, \cdots, g_n, y) \in X \times_{Ob\ G} G^{\underline{n}} \times_{Ob\ G} Y.$$

The action of Δ^o can be readily computed; if $n \xleftarrow{\delta} m$, then

$$(x, g_1, \cdots, g_n, y)\delta = (xg_1 \cdots g_{\delta(0)}, \cdots, g_{\delta(i-1)+1} \cdots g_{\delta(i)}, \cdots, g_{\delta(m)+1} \cdots g_n y),$$

where if $\delta(0) = 0$ the 0th-term is x, if $\delta(i-1) = \delta(i)$ the ith-term is the appropriate identity element, and if $\delta(m) = n$ the last term is y.

From the above, one can compute the degenerate elements of the simplicial space $NM(X,Y)$. Namely, $(x, g_1, \cdots, g_n, y)$ is degenerate if and only if some g_i is an identity morphism.

The Bifunctor $\bigotimes_G : \mathbf{TOP}^{G^o} \times \mathbf{TOP}^G \to \mathbf{TOP}$

Denote by $\bigotimes_G$ the bifunctor which is the composition

$$\mathrm{TOP}^{G^o} \times \mathrm{TOP}^G \xrightarrow{M} \mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o} \xrightarrow{|*|} \mathrm{TOP}.$$

Think of this composition as assigning to each G^o-space X and each G-space Y an exploded version $X \bigotimes_G Y$ of the space $X \times_G Y$. The properties evident up to now constitute the following proposition.

(2.12) *For G a small category, for X in TOP^{G^o} and for Y in TOP^G, there is the compactly generated space $X \bigotimes_G Y$ whose points are of the form*

$$(x, g_1, \cdots, g_n, y) \times_\Delta (t_0, \cdots, t_n)$$

where

$$(x, g_1, \cdots, g_n, y) \in X \times_{Ob\ G} G^{\underline{n}} \times_{Ob\ G} Y.$$

There is a natural transformation of bifunctors, given as the maps $X \bigotimes_G Y \to X \times_G Y$ which send the above point into the point

$$xg_1 \cdots g_n \times_G y = x \times_G g_1 \cdots g_n y \in X \times_G Y.$$

There is a filtration $X \bigotimes_G Y = \bigcup (X \bigotimes_G Y)_n$ and $(X \bigotimes_G Y)_n - (X \bigotimes_G Y)_{n-1}$ consists of all

$$(x, g_1, \cdots, g_n, y) \times_\Delta (t_0, \cdots, t_n)$$

with no g_i an identity morphism and with $(t_0, \cdots, t_n)$ in $\nabla(n) - \partial\nabla(n)$.

The Functor $E_G(\diamond) : \mathbf{TOP}^G \to \mathbf{TOP}^G$

We now have yet another opportunity to obtain a functor from a bifunctor, by replacing X by the G^o-space $G(p, \diamond) = \{G(p, q)\}$.

Fix a small category G and a G-space Y. For each $p \in Ob\ G$, we can take the G^o-space X to be $G(p, \diamond) = \{G(p, q) | q \in Ob\ G\}$. The resulting compactly generated space $G(p, \diamond) \bigotimes_G Y$ has points

$$(g_0, g_1, \cdots, g_n, y) \times_\Delta (t_0, \cdots, t_n).$$

Denote by $E_G Y$ the resulting G-space

$$E_G Y = \{(E_G Y)(p) = G(p, \diamond) \bigotimes_G Y\},$$

where g acts on the above point by

$$g((g_0, g_1, \cdots, g_n, y) \times_\Delta (t_o, \cdots, t_n)) = (gg_0, g_1, \cdots, g_n, y) \times_\Delta (t_0, \cdots, t_n).$$

(2.13) *We have the functor $E_G(\diamond) : TOP^G \to TOP^G$, where $E_G Y(p) = G(p, \diamond) \bigotimes_G Y$ has as points all*

$$(g_0, g_1, \cdots, g_n, y) \times_\Delta (t_0, \cdots, t_n)$$

for which

$$p \xleftarrow{g_0} p_0 \xleftarrow{g_1} \cdots \xleftarrow{g_n} p_n, \qquad y \in Y(p_n).$$

There is a natural transformation $T : E_G(\diamond) \to 1$ of functors, where given the G-space Y, the G-map $T_Y : E_G Y \to Y$ is given by

$$T_Y((g_0, g_1, \cdots, g_n, y) \times_\Delta (t_0, \cdots, t_n)) = g_0 g_1 \cdots g_n y.$$

For each $p \in Ob\ G$, the resulting map

$$(E_G Y)(p) = G(p, \diamond) \bigotimes_G Y \to Y(p)$$

is a homotopy equivalence of spaces.

Proof. On one hand, there is the topological category H given by $M(G(p, \diamond), Y)$ whose objects are the pairs $(g_0, y) \in G(p, \diamond) \times_{Ob\ G} Y$ and whose morphisms are the triples (g_0, g, y) in $G(p, \diamond) \times_{Ob\ G} G \times_{Ob\ G} Y$. On the other hand, there is the topological category K whose space of objects is $Y(p)$ and whose morphisms 1_y are all identity morphisms. Moreover $|N(H)| = (E_G Y)(p)$ and $|N(K)| = Y(p)$. There is a continuous functor $F : H \to K$ taking the object (g_0, y) into the object $g_0 y$ and taking the morphism (g_0, g, y) into $1_{g_0 g y}$. There is the functor $F' : K \to H$ which takes an object $y \in Y(p)$ into the object $(1_p, y)$. The composition FF' is the identity. The composition $F'F$ sends an object (g_0, y) into $(1_p, g_0 y)$ and sends morphisms into identity morphisms. There is then a continuous natural transformation $S : 1 \to F'F$ given on any object (g_0, y) by

$$(1_p, g_0 y) \xleftarrow{(1_p, g_0, y)} (g_0, y).$$

Hence by (2.11) the composition

$$|N(H)| \xrightarrow{F_*} |N(K)| \xrightarrow{F'_*} |N(H)|$$

is homotopic to the identity. The composition $F_* F'_*$ is the identity, hence F_* is a homotopy equivalence. One has to check that it is the map in question. □

(2.14) *The colimit of the G-space $E_G Y$ is naturally identified with the standard homotopy colimit $B_G Y$ of the G-space Y.*

Proof. A direct proof can be given, or one can use (2.10). In the latter, let the W of (2.10) be the G-space $Ob\ G$ and let the Z of (2.10) be the $G \times \Delta^o$-space arising from $NM(G, Y)$. Then from (2.10) we get

$$colim\ E_G Y = (Ob\ G) \times_G |Z| \simeq |i^{\#}((Ob\ G) \times_G Z)| \simeq B_G Y. \quad \square$$

The Exploded Version EG of the $G \times G^o$-space G

If we apply May's construction to the example $X = G(p, \diamond)$ and $Y = G(\diamond, q)$, we receive for each $p, q \in Ob\ G$ the compactly generated space $G(p,\diamond) \bigotimes_G G(\diamond, q)$. Let

$$EG = G \bigotimes_G G = \{G(p,\diamond) \bigotimes_G G(\diamond, q) \mid p, q \in Ob\ G\}$$

denote the resulting $G \times G^o$-space, whose elements are all

$$(g_0, g_1, \cdots, g_n, g_{n+1}) \times_\Delta (t_0, \cdots, t_n), \qquad (g_0, g_1, \cdots, g_n, g_{n+1}) \in \underline{G^{n+2}}.$$

There is the natural $G \times G^o$-map $EG \to G$ taking the above point into $g_0 \cdots g_{n+1} \in G$.

(2.15) *For any G-space Y, we have*

$$E_G Y \simeq EG \times_G Y.$$

PROOF. The natural map $EG \times_G Y \to E_G Y$ is given by

$$((g_0, g_1, \cdots, g_n, g_{n+1}) \times_\Delta v) \times_G y \mapsto (g_0, g_1, \cdots, g_n, g_{n+1} y) \times_\Delta v.$$

One can apply (2.10) to show that it is a homeomorphism. $\square$

The Functor $E^G(\diamond) : \mathbf{TOP}^G \to \mathbf{TOP}^G$

We have been neglecting the mapping bifunctor. There is an analogue of the above explosion functor, namely a functor

$$E^G(\diamond) : \mathrm{TOP}^G \to \mathrm{TOP}^G$$

which assigns to each Y in TOP^G an exploded version $E^G Y$ in TOP^G. There is also a natural transformation $T' : 1 \to E^G(\diamond)$ of functors, assigning to each G-space Y a G-map $T' : Y \to E^G Y$. The G-space is given by

$$(E^G Y)(q) = (Y^{EG(\diamond, q)})_{\mathrm{TOP}^G},$$

i.e. $E^G Y(q)$ is all G-maps $EG(\diamond, q) \to Y$ where the action assigns to $g : q \to q'$ the map $E^G Y(q) \to E^G Y(q')$ as the composition $\phi \mapsto \phi g^*$ in

$$Y \leftarrow EG(\diamond, q) \xleftarrow{g^*} EG(\diamond, q').$$

The natural transformation is the G-map

$$T' = \tau^\# : Y \simeq (Y^G)_{\mathrm{TOP}^G} \to (Y^{EG})_{\mathrm{TOP}^G},$$

where $\tau : EG \to G$ is the natural $G \times G^o$-map $EG \to G$.

For each G-space Y and each $q \in Ob\ G$, the map $Y(q) \to (E^G Y)(q)$ is a homotopy equivalence of spaces. We need a little background in order to prove it.

Given a G-space Y, denote by $I \times Y$ the G-space

$$(I \times Y)(p) = I \times Y(p),$$

where G acts on $I \times Y$ by $g(t, y) = (t, gy)$. There are G-maps $\pi_0, \pi_1 : Y \to I \times Y$ given by $\pi_0(y) = (0, y)$ and $\pi_1(y) = (1, y)$. Define G-maps $\phi_0 : Y \to Y'$ and $\phi_1 : Y \to Y'$ to be *homotopic* in TOP^G if there exists a G-map $\Phi : I \times Y \to Y'$ such that $\phi_0 = \Phi\pi_0$ and $\phi_1 = \Phi\pi_1$. Define a G-map $\phi : Y \to Y'$ to be a homotopy equivalence in TOP^G if there exists a G-map $\theta : Y' \to Y$ such that $\phi\theta$ and $\theta\phi$ are homotopic to the identity in TOP^G.

We need the following generalization of (2.13).

(2.16) *Let Y be a $G \times G^o$-space, and let $E_G Y$ denote the $G \times G^o$-space*

$$E_G Y = G \bigotimes Y,$$

where the G^o-action is given by

$$((g_0, \cdots, g_n, y) \times_\Delta v)g = (g_0, \cdots, g_n, yg) \times_\Delta v.$$

There is the natural $G \times G^o$-map $E_G Y \to Y$ given by

$$(g_0, \cdots, g_n, y) \times_\Delta v \mapsto g_0 \cdots g_n y.$$

For each $p \in Ob\ G$, the G^o-map $E_G Y(p, \diamond) \to Y(p, \diamond)$ is a homotopy equivalence of G^o-spaces.

Proof. We need first to generalize the proof of (2.13). There is the topological category H with objects $(g_0, y) \in G(p, \diamond) \times_{Ob\ G} Y$ and morphisms

$$(g_0, gy) \xleftarrow{(g_0, g, y)} (g_0 g, y)$$

indexed by the elements of $G(p, \diamond) \times_{Ob\ G} G \times_{Ob\ G} Y$. The category H splits as a disjoint union of subcategories $H(q)$ where $H(q)$ is all objects with $y \in Y(\diamond, q)$ and all morphisms with $y \in Y(\diamond, q)$. Each $q \xleftarrow{g'} q'$ in G gives a functor $H(q) \to H(q')$ taking an object (g_0, y) into (g_0, yg') and a morphism (g_0, g, y) into (g_0, g, yg'). Thus we have a contravariant functor $G \to \mathrm{TOPCAT}$ taking q into $H(q)$ and g into the above functor. Then by composition we get the contravariant functor

$$G \to \mathrm{TOPCAT} \xrightarrow{|N\diamond|} \mathrm{TOP}$$

and thus we have $|NH| = \coprod |N(H(q)|$ and in addition we have that $\{|N(H(q))|\}$ is a right G-space.

As in the proof of (2.13), there is the functor $F'F : H \to H$ which takes each $H(q)$ into itself. Moreover $F'_* F_*$ is seen to be a G^o-map. The question is whether the homotopy from the identity to $F'_* F_*$ constructed in the proofs of (2.13) and (2.11) is then a homotopy of G^o-maps. One can do this by considering $\mathcal{I} \times H = \coprod \mathcal{I} \times H(q)$, by letting G^o act on $\mathcal{I} \times H$ as functors operating trivially on the $\mathcal{I}$-coordinate, and by showing that the induced functor $\mathcal{I} \times H \to H$ constructed in the above proofs commutes with the contravariant functors on the two categories. We leave the details to the reader, but one obtains that the identity is homotopic to $F'_* F_*$ as G^o-maps. □

We note that there is an opposite of (2.16), proved in a similar way. If X is a $G \times G^o$-space and if $p \in Ob\, G$, there is the G-space $X \bigotimes_G G(\diamond, q)$ and a natural G-map $X \bigotimes_G G(\diamond, q) \to X(\diamond, q)$, which is a homotopy equivalence of G-spaces.

(2.17) *For each $q \in Ob\, G$, the G-map $EG(\diamond, q) \to G(\diamond, q)$ is a homotopy equivalence in TOP^G. Similarly for each $p \in Ob\ G$ the G^o-map $EG(p, \diamond) \to G(p, \diamond)$ is a homotopy equivalence in TOP^{G^o}.*

This is a corollary to (2.16) and its opposite.

(2.18) *Let Y be a G-space, and let $q \in Ob\, G$. Then the G-map $Y \to E^G Y$ has $Y(q) \to (E^G Y)(q)$ a homotopy equivalence of spaces.*

This follows from (2.17).

The Category $0 \to 1$

Note an example, that in which G is the category $0 \to 1$ with two objects $\{0, 1\}$ and a single nonidentity morphism $0 \to 1$. That is, G is the category previously denoted in this chapter by $\mathcal{I}$. A G-space is then a map $g : Y_0 \to Y_1$ in TOP.

The $G \times G^o$-space EG can be computed. Up to natural homeomorphism it is

$$EG(\diamond, 0) \xleftarrow{\ g^*\ } EG(\diamond, 1)$$

$$\begin{tikzcd}
EG(0,\diamond) \arrow[d, "g_*"'] & 0 \arrow[d, "i"'] & \emptyset \arrow[l] \arrow[d] \\
EG(1,\diamond) & I & 1. \arrow[l, "j"']
\end{tikzcd}$$

Hence

$$(E_G Y)(0) \simeq EG(0, \diamond) \times_G Y \simeq 0 \times Y_0,$$
$$(E_G Y)(1) \simeq EG(1, \diamond) \times_G Y \simeq I \times Y_0 \cup_g Y_1.$$

Thus $E_G Y$ is up to natural isomorphism the inclusion

$$0 \times Y_0 \hookrightarrow I \times Y_0 \cup_g Y_1,$$

which is the standard model in homotopy theory for the cofibration associated with the map g.

Similarly $(E^G Y)(0)$ is all G-maps from $EG(\diamond, 0)$ into Y, hence

$$(E^G Y)(0) \subset Y_0 \times (Y_1)^I$$

consists of all $(y_0, \sigma) \in Y_0 \times (Y_1)^I$ with $gy_0 = \sigma(0)$. Moreover, $(E^G Y)(1)$ is all G-maps from $EG(\diamond, 1)$ into Y, hence $(E^G Y)(1) = Y_1$. Thus $E^G Y$ is the map

$$(E^G Y)(0) \to Y_1, \qquad (y_0, \sigma) \mapsto \sigma(1),$$

which is the standard model in homotopy theory for the fibration associated with the map g.

We review in Chapter 3 the general topology of cofibrations, fibrations, and homotopy equivalences, because these topics are deeply interwoven with the study of actions of small categories. After having done that, we can then complete in Chapter 4 the beginning we have made here.

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CHAPTER III

Cofibrations, Fibrations, and Homotopy Equivalences

We now take an interlude from development of simplicial spaces to review the basic homotopy of cofibrations, fibrations and homotopy equivalences. Our eventual purpose is to review the equivariant general topology which has been invented in order to understand homotopy theory better. That being the case, one will understand better if one starts with the simple examples which have to do with the role of cofibrations and fibrations in topology.

The reader who wishes to consult original sources will find an extensive literature. One could consult some of the older works such as those of Fox [3.3,1943], J.H.C. Whitehead [3.9,1949], and Hurewicz [3.4,1955]. In the next generation there are works such as those of Dold [3.2,1963], Spanier [2.12,1966], D. Puppe [3.7,1967] and Strom [3.8,1966-1968]. Besides Spanier's book, one should also note the books of tom Dieck, Kamps, and Puppe [3.1,1970], G.W. Whitehead [1.6,1978] and I.M. James [3.5,1984].

Cofibered Pairs

Let (X, A) be a closed pair of spaces in TOP. Then (X, A) is a *cofibered pair* if given a map $\phi : X \to Y$ in TOP and a homotopy $H_0 : I \times A \to Y$ with $H_0(0, a) = \phi(a)$ for all $a \in A$, then there exists a homotopy $H : I \times X \to Y$ with

$$H(t, a) = H_0(t, a),\ a \in A, \qquad H(0, x) = \phi(x),\ x \in X.$$

Equivalently, (X, A) is a cofibered pair if and only if there exists a retracting map of $I \times X$ onto $0 \times X \cup I \times A$.

It is easily checked that if $X \supset A \supset B$ and if (X, A) and (A, B) are cofibered pairs, then (X, B) is a cofibered pair.

Suppose (X, A) is a cofibered pair and that r is a retraction of $I \times X$ onto $0 \times X \cup I \times A$. Let the projection maps be denoted by

$$I \xleftarrow{\pi_1} I \times X \xrightarrow{\pi_2} X,$$

and define a function $u : X \to I$ by

$$u(x) = lub_{t \in I} |t - \pi_1 r(t, x)|.$$

It follows from (1.10) that $I \times X$ has the product topology of the category top, and one can readily show that u is continuous. It is also the case that $u(x) = 0$ if and only if $x \in A$. For if $x \notin A$, it can be seen that if t is a sufficiently small positive number then $\pi_1 r(t, x) = 0$ and hence $u(x) > 0$. We start a collection of properties of cofibered pairs (X, A) with this one.

(i) There exists a map $u : X \to I$ with $u^{-1}(0) = A$.

Continue to assume a cofibered pair (X, A) with a retracting map r of $I \times X$ onto $0 \times X \cup I \times A$. There is a homotopy $H : I \times X \to X$ defined by

$$H(t, x) = \pi_2 r(t, x),$$

and it has the following properties;

(ii) $H(0, x) = x$ for all $x \in X$;
(iii) $H(t, a) = a$ for all $a \in A$;
(iv) $H(t, x) \in A$ whenever $1 \geq t > u(x)$.

Conversely, if (X, A) is a pair in TOP such that there exists a map $u : X \to I$ and a homotopy $H : I \times X \to X$ satisfying (i)-(iv), then (X, A) is a cofibered pair (see Strom [3.8]). For one can then define the retracting map r by

$$r(t, x) = \begin{cases} (0, H(t, x)), & \text{for } t \leq u(x) \\ (t - u(x), H(u(x), x)), & \text{for } t \geq u(x). \end{cases}$$

If (X, A) and (Y, B) are cofibered pairs, then $(X \times Y, A \times Y \cup X \times B)$ is a cofibered pair. For if u, H satisfy (i)-(iv) for (X, A) and if v, K satisfy (i)-(iv) for (Y, B), then w, L satisfy (i)-(iv) for $(X \times Y, A \times Y \cup X \times B)$, where

$$w(x, y) = min\ (u(x), v(y)),\ \ L(t, x, y) = (H(min(t, v(y)), x), K(min(t, u(x)), y)).$$

If $\phi : (X, A) \to (Y, B)$ is a relative homeomorphism of closed pairs in TOP, then (X, A) a cofibered pair implies (Y, B) a cofibered pair. For let r be a retraction of $I \times X$ onto $0 \times X \cup I \times A$. The natural quotient map $X \sqcup B \to Y$ yields a quotient map $I \times (X \sqcup B) \to I \times Y$. We have a retraction

$$r \sqcup 1_{I \times B} : I \times X \sqcup I \times B \to 0 \times (X \sqcup B) \cup I \times (A \sqcup B).$$

The diagram

$$\begin{array}{ccc} I \times X \sqcup I \times B & \longrightarrow & 0 \times (X \sqcup B) \cup I \times (A \sqcup B) \\ \downarrow & & \downarrow \\ I \times Y & & 0 \times Y \cup I \times B \end{array}$$

induces a retraction $I \times Y \to 0 \times Y \cup I \times B$.

A *cofibered* filtered space $X = \bigcup X_n$ in TOP is a filtered space such that (X_n, X_{n-1}) is a cofibered pair for each $n > 0$. It is then the case that (X, X_0) is a cofibered pair. Fix a map $\phi : X \to Y$ and a homotopy $H_0 : I \times X_0 \to Y$ such that $H_0(0, x_0) = \phi(x_0)$ for all $x_0 \in X_0$. One can extend to a homotopy $H_1 : I \times X_1 \to Y$ such that

$$H_1(t, x_0) = H_0(t, x_0),\ x_0 \in X_0, \qquad H_1(0, x_1) = \phi(x_1),\ x_1 \in X_1.$$

Similarly, one obtains inductively a sequence $H_n : I \times X_n \to Y$ and obtains $H : I \times X \to Y$ as the common value of the H_n.

The Category TOP$\backslash A$ of Spaces Under A

Fix a compactly generated space A. The category TOP$\backslash A$ of *spaces under* A has as objects all maps $\nu : A \to X$ in TOP. We most often denote the object simply by X, regarding the map ν as implicit. The morphisms $\phi : X \to Y$ of TOP$\backslash A$ are the commutative diagrams

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\nu'} \\ X & \xrightarrow{\phi} & Y \end{array}$$

in TOP.

There is a homotopy relation $\sim$ in TOP$\backslash A$. Given $\phi_0, \phi_1 : X \to Y$ with

$$\phi_0 \nu = \nu', \qquad \phi_1 \nu = \nu',$$

then ϕ_0 is homotopic to ϕ_1 in TOP$\backslash A$ if there exists a homotopy $H : I \times X \to Y$ joining ϕ_0 to ϕ_1 such that

$$H(t, \nu(a)) = \nu'(a),\ a \in A,\ t \in I.$$

The homotopy category πTOP$\backslash A$ has as objects all spaces X under A and as morphisms $X \to Y$ all homotopy classes $[\phi]$ of morphisms $\phi : X \to Y$ in TOP$\backslash A$. A map $\phi : X \to Y$ in TOP$\backslash A$ is a *homotopy equivalence in* TOP$\backslash A$ if there exists a map $\theta : Y \to X$ in TOP$\backslash A$ with $\phi\theta$ and $\theta\phi$ homotopic to the identity in TOP$\backslash A$.

We call a map $\phi : X \to Y$ in TOP$\backslash A$ a *weak homotopy equivalence in* TOP$\backslash A$ if $\phi : X \to Y$ is a homotopy equivalence in TOP. Such a ϕ has a homotopy inverse in TOP but not necessarily in TOP$\backslash A$.

Cofibrations and TOP$\backslash A$

A map $\nu : A \to X$ is a *cofibration* if ν is an inclusion map onto a closed subset $\nu(A)$ of X, and if $(X, \nu(A))$ is a cofibered pair. Denote by COF$\backslash A$ the full subcategory of TOP$\backslash A$ whose objects are the cofibrations $\nu : A \to X$.

Let $\nu : A \to X$ be an arbitrary space under A. There is then the diagram

$$I \times A \xleftarrow{i} 1 \times A \simeq A \xrightarrow{\nu} X$$

and its pushout diagram

$$\begin{array}{ccc} 1 \times A & \xrightarrow{\nu} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ I \times A & \xrightarrow{\pi} & I \times A \cup_\nu X. \end{array}$$

We then have the relative homeomorphism of pairs

$$(I \times A, \partial I \times A) \to (I \times A \cup_\nu X, 0 \times A \sqcup X),$$

from which it follows that the latter pair is a cofibered pair. Since $(0\times A \sqcup X, 0\times A)$ is also a cofibered pair, then $(I \times A \cup_\nu X, 0 \times A)$ is a cofibered pair. Alternatively, the map

$$A \xrightarrow{\nu'} I \times A \cup_\nu X$$

which identifies $a \in A$ with $(0,a)$ is a cofibration. This is the mapping cylinder construction pointed out at the end of Chapter 2. There is the commutative diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\nu'"'] & A \arrow[d, "\nu"'] \\ I \times A \cup_\nu X \arrow[r, "\phi"] & X \end{tikzcd}$$

where ϕ is induced by the map $I \times A \sqcup X \to X$ which maps (t,a) into $\nu(a)$ and x into x. It can be checked that ϕ is a weak homotopy equivalence in TOP$\backslash A$. We often denote $I \times A \cup_\nu X$ by EX and ν' by $E\nu$.

Theorem 3.1 *Consider the commutative diagram*

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\nu"'] & A \arrow[d, "\nu'"'] \\ X \arrow[r, "\phi"] & Y \end{tikzcd}$$

in TOP, where ν and ν' are cofibrations. If ϕ is a weak homotopy equivalence in TOP$\backslash A$, then ϕ is a homotopy equivalence in TOP$\backslash A$.

Proof. Choose a map $\theta : Y \to X$ with $\theta\phi \sim 1_X$ in TOP. Let $H : I \times X \to X$ be a homotopy with

$$H(0,x) = \theta\phi(x), \qquad H(1,x) = x.$$

One next uses the fact that ν' is a cofibration. Define

$$K_0 : 0 \times Y \cup I \times \nu'(A) \to X$$

by

$$K_0(t, \nu' a) = H(t, \nu a), \qquad K_0(0,y) = \theta(y),$$

checking that the definitions coincide on $0 \times \nu' A$. Since ν' is a cofibration, there exists an extension $K : I \times Y \to X$.

Define $\mu : Y \to X$ by $\mu(y) = K(1,y)$. Then

$$\mu(\nu' a) = K(1, \nu' a) = H(1, \nu a) = \nu a,$$

and μ is a morphism of TOP$\backslash A$.

One next shows that $\mu\phi \sim 1_X$ in TOP$\backslash A$, using the cofibration condition on ν. Since $(X, \nu A)$ is a cofibered pair, so is

$$(I, \partial I) \times (I, 0) \times (X, \nu A).$$

Define

$$M_0 : \partial I \times I \times X \cup I \times 0 \times X \cup I \times I \times \nu A \to X$$

by

$$M_0(0,u,x) = K(u,\phi(x)), \qquad M_0(1,u,x) = H(u,x),$$

$$M_0(t,0,x) = \theta\phi(x), \qquad M_0(t,u,\nu a) = K(u,\nu' a) = H(u,\nu a),$$

and check that M_0 is a well defined continuous function. Extend M_0 to a map

$$M : I \times I \times X \to X,$$

and define $L : I \times X \to X$ by

$$L(t,x) = M(t,1,x).$$

Then check that L is a homotopy in TOP$\backslash A$ from $\mu\phi$ to 1_X.

Thus $\phi : X \to Y$ has a left homotopy inverse $\mu : Y \to X$ in TOP$\backslash A$. Similarly μ has a left homotopy inverse $\tau : X \to Y$ in TOP$\backslash A$, thus

$$\phi \sim \tau\mu\phi \sim \tau$$

in TOP$\backslash A$ and ϕ and μ are homotopy inverses in TOP$\backslash A$. $\square$

Corollary 3.2 *Let (X,A) be a cofibered pair. Then the inclusion $A \xrightarrow{i} X$ is a homotopy equivalence if and only if A is a strong deformation retract of X. In particular, (X,A) is a cofibered pair if and only if $0 \times X \cup I \times A$ is a strong deformation retract of $I \times X$.*

Proof. To obtain the first conclusion, apply (3.1) to

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "i"] \\ A \arrow[r, "i"] & X. \end{tikzcd}$$

To obtain the second, note that $0 \times X \cup I \times A \hookrightarrow I \times X$ is a homotopy equivalence. $\square$

Fibrations and the Category TOP/B of Spaces over B

Fix a compactly generated space B. The category TOP/B of *spaces over* B has as objects the maps $g : X \to B$ in TOP. As with spaces under A, we often denote the object $g : X \to B$ by X. The morphisms $\phi : X \to Y$ of TOP/B are the commutative diagrams

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

in TOP.

Given an object $g : X \to B$ in TOP/B there is the object $g\pi : I \times X \to B$, where $\pi : I \times X \to X$ is the projection. If $\theta_0, \theta_1 : X \to Y$ are maps in TOP/B, then θ_0 is *homotopic to* θ_1 *in* TOP/B, written $\theta_0 \sim \theta_1$, if there is a morphism $H : I \times X \to Y$ such that

$$H(0,x) = \theta_0(x), H(1,x) = \theta_1(x), x \in X.$$

A map $\phi : X \to Y$ in TOP/B is a *homotopy equivalence in* TOP/B if there is a map $\theta : Y \to X$ in TOP/B with $\phi\theta \sim 1_Y$ and $\theta\phi \sim 1_X$ in TOP/B. A homotopy equivalence in TOP/B is also called a *fiber homotopy equivalence.* A map $\theta : X \to Y$ in TOP/B is a *weak homotopy equivalence in* TOP/B if it is a homotopy equivalence in TOP.

A *fibration* in TOP is a map $g : Y \to B$ such that for any commutative diagram in TOP

$$\begin{array}{ccc} 0 \times X & \xrightarrow{\phi_0} & Y \\ {\scriptstyle i}\downarrow & & {\scriptstyle g}\downarrow \\ I \times X & \xrightarrow{\phi_1} & B \end{array}$$

there exists a *lifting*: a map $\theta : I \times X \to Y$ such that $\theta i = \phi_0$ and $g\theta = \phi_1$. We denote by FIB/B the full subcategory of TOPwith objects the fibrations $g : Y \to B$.

Immediate remarks then include the following:

(3.3;i) A composition of two fibrations is a fibration.

(3.3;ii) Every projection map $A \times Y \to Y$ in TOP is a fibration.

(3.3;iii) The maps $e_0 : B^I \to B$, $e_0(\sigma) = \sigma(0)$, and $e_{0,1} : B^I \to B \times B$, $e_{0,1}(\sigma) = (\sigma(0), \sigma(1))$, in TOP are fibrations.

(3.3;iv) If $B' \xrightarrow{\phi} B \xleftarrow{g} Y$ is a diagram in TOP where g is a fibration, and if

$$\begin{array}{ccc} Y' & \xrightarrow{\phi_0} & Y \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow \\ B' & \xrightarrow{\phi} & B \end{array}$$

denotes the pullback diagram, then g' is a fibration; when the context is clear, we will write it as $g' : \phi^* Y \to B'$.

(3.3;v) If $g : Y \to B$ is a fibration and if B' is a closed subset of B, then $g|Y' : Y' \to B'$ is a fibration, where $Y' = g^{-1}B'$.

(3.3;vi) Given a map $g : X \to B$ in TOP, denote by $E'X$ the subspace of $X \times B^I$ of all $[(x,\sigma) : g(x) = \sigma(0)]$ and define $E'g : E'X \to B$ by $E'g(x,\sigma) = \sigma(1)$. $E'X$ is the pullback of

$$X \xrightarrow{g} B \xleftarrow{e_0} B^I$$

and the map $E'g : E'X \to B$ is a fibration.

(3.3;vii) Given a space B in TOP and a cofibered pair (X, A) in TOP, then the restriction map $B^X \to B^A$ is a fibration.

The following theorem is due to Dold [3.2].

Theorem 3.4 *Consider the morphism $\phi : X \to Y$ in TOP/B, i.e. the commutative diagram in TOP*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle g}\downarrow & & {\scriptstyle g'}\downarrow \\ B & = & B \end{array}$$

and suppose that g and g' are both fibrations. If ϕ is a weak homotopy equivalence in TOP/B, then ϕ is a homotopy equivalence in TOP/B.

Proof. Let $\theta : Y \to X$ be a homotopy inverse for ϕ in TOP. Then $\phi\theta \sim 1_Y$, and there is a homotopy $H : I \times Y \to Y$ joining $\phi\theta$ to 1_Y. Hence $g'H$ is a homotopy joining $g\theta$ to g'. Since g is a fibration, there is a map $K : I \times Y \to X$ with $gK = g'H$ and $K(0, y) = \theta(y)$. Define $\mu : Y \to X$ by $\mu(y) = K(1, y)$ and note that μ is a morphism in TOP/B.

One next shows the existence of a homotopy $L : I \times Y \to Y$ joining $\phi\mu$ to 1_Y and with $g'L(t, y) = g'(y)$, i.e. that $\phi\mu \sim 1_Y$ in TOP/B. Define

$$M_0 : I \times 0 \times Y \cup \partial I \times I \times Y \to Y$$

by

$$M_0(0, t, y) = \phi K(t, y),$$

$$M_0(1, t, y) = H(t, y),$$

$$M_0(s, 0, y) = \phi\theta(y),$$

Define $M : I \times I \times Y \to B$ by

$$M(s, t, y) = gK(t, y) = g'H(t, y).$$

There is a homeomorphism $(I \times I, 0 \times I) \to (I \times I, I \times 0 \cup \partial I \times I)$ and hence a homeomorphism $(I \times I \times Y, 0 \times I \times Y) \to (I \times I \times Y, I \times 0 \times Y \cup \partial I \times I \times Y)$. That is, there is a commutative diagram

$$\begin{array}{ccc} 0 \times I \times Y & \xrightarrow{h_0} & I \times 0 \times Y \cup \partial I \times I \times Y \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow \\ I \times I \times Y & \xrightarrow{h} & I \times I \times Y \end{array}$$

in TOP with i, j inclusions and h_0, h homeomorphisms. Since g' is a fibration, the commutative diagram

$$\begin{array}{ccc} 0 \times I \times Y & \xrightarrow{M_0 h_0} & Y \\ {\scriptstyle i}\downarrow & & {\scriptstyle g'}\downarrow \\ I \times I \times Y & \xrightarrow{Mh} & B \end{array}$$

has a lifting $L' : I \times I \times Y \to Y$. Define $L(s, y) = L'h^{-1}(s, 1, y)$. Note that L is a homotopy between $\phi\mu$ and 1_Y in TOP/B.

Thus ϕ has a right homotopy inverse μ in TOP$/B$. Similarly μ has a right homotopy inverse τ, and

$$\phi \sim \phi\mu\tau \sim \tau.$$

Thus ϕ and μ are homotopy inverses in TOP$/B$. $\square$

Cofibrations, Fibrations, and Liftings

Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi_0} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ X & \xrightarrow{\phi_1} & B \end{array}$$

in TOP, a *lifting* of the diagram is a map $\theta : X \to Y$ in TOP such that $\theta f = \phi_0$ and $g\theta = \phi_1$. We are interested in conditions on f and g that ensure that the diagram has a lifting. We start with a theorem from Strom [3.8].

Theorem 3.5 *Let $g : Y \to B$ be a fibration and let (X, A) be a cofibered pair with the inclusion $i : A \to X$ a homotopy equivalence in TOP. Then every commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\phi_0} & Y \\ {\scriptstyle i}\downarrow & & {\scriptstyle g}\downarrow \\ X & \xrightarrow{\phi_1} & B \end{array}$$

has a lifting $\theta : X \to Y$.

Proof. Let $\mu : X \to I$ be a map with $\mu^{-1}(0) = A$. By (3.2) there is a strong deformation retraction of X onto A, in the form $H : I \times X \to X$ of a homotopy $\{H_t : 0 \le t \le 1\}$ with H_1 the identity map of X, with each H_t restricted to A the identity map of A, and with H_0 a retraction of X onto A. It is no restriction to suppose in addition that

$$H(\mu(x), x) = H(1, x)\ ,\ x \in X.$$

For otherwise one would replace H by H', where

$$H'(t, x) = \begin{cases} H(t/\mu(x), x), & \text{if } \mu(x) > 0 \text{ and } \mu(x) \ge t \ge 0 \\ x, & \text{if } \mu(x) = 0 = t \\ H(1, x), & \text{if } 1 \ge t \ge \mu(x). \end{cases}$$

Hence we suppose the added condition holds. Then the commutative diagram

$$\begin{array}{ccc} 0 \times X & \xrightarrow{\phi_0 H_0} & Y \\ {\scriptstyle j}\downarrow & & {\scriptstyle g}\downarrow \\ I \times X & \xrightarrow{\phi_1 H} & B \end{array}$$

has a lifting $K : I \times X \to Y$. Define $\theta : X \to E$ by $\theta(x) = K(\mu(x), x)$. □

Corollary 3.6 *Consider the commutative diagram in TOP*

$$\begin{array}{ccc} 0 \times X \cup I \times A & \xrightarrow{\phi_0} & Y \\ {\scriptstyle i}\downarrow & & {\scriptstyle g}\downarrow \\ I \times X & \xrightarrow{\phi_1} & B \end{array}$$

where (X, A) is a cofibered pair and g is a fibration. There is a lifting $\theta : I \times X \to Y$.

Proof. By (3.2), $(I \times X, 0 \times X \cup I \times A)$ is a cofibered pair with the inclusion a homotopy equivalence. □

Theorem 3.7 *Consider the commutative diagram in TOP*

$$\begin{array}{ccc} A & \xrightarrow{\phi_0} & Y \\ {\scriptstyle i}\downarrow & & {\scriptstyle g}\downarrow \\ X & \xrightarrow{\phi_1} & B \end{array}$$

where (X, A) is a cofibration and where g is a fibration and a homotopy equivalence in TOP. Then the diagram has a lifting.

Proof. There is the weak homotopy equivalence in TOP$/B$ given by

$$\begin{array}{ccc} Y & \xrightarrow{g} & B \\ {\scriptstyle g}\downarrow & & {\scriptstyle 1_B}\downarrow \\ B & = & B \end{array}$$

which is then by (3.4) a homotopy equivalence in TOP$/B$. Let $s : B \to Y$ denote a homotopy inverse in TOP$/B$. Then $gs = 1_B$, and there is a homotopy $H : I \times Y \to Y$ joining sg to 1_Y as a fiber homotopy. Consider the commutative diagram

$$\begin{array}{ccc} 0 \times X \cup I \times A & \xrightarrow{F_0} & Y \\ \downarrow & & {\scriptstyle g}\downarrow \\ I \times X & \xrightarrow{F} & B \end{array}$$

where

$$F(t, x) = \phi_1(x),$$

$$F_0(0, x) = s\phi_1(x),$$

$$F_0(t, a) = H(t, \phi_0(a)).$$

By (3.6), there is a lifting $G : I \times X \to Y$ and one can take $G(1, x)$ as the desired lifting. $\square$

Two Homotopy Lifting Theorems

Using the cofibration and fibration associated to a map in TOP, we next prove two homotopy lifting theorems by weakening the hypotheses in (3.5) and (3.7).

Given a map $\nu : A \to X$ in TOP, there is the associated cofibration $E\nu : A \to EX$ and the pushout diagram

$$\begin{tikzcd} 1 \times A \arrow[r, "\nu"] \arrow[d, "i"'] & X \arrow[d, "j"'] \\ I \times A \arrow[r, "\pi"] & EX. \end{tikzcd}$$

Suppose that we are given a homotopy commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi_0"] \arrow[d, "\nu"'] & Y \arrow[d, "g"'] \\ X \arrow[r, "\phi_1"] & B \end{tikzcd}$$

with homotopy $H_0 : I \times A \to B$ joining $g\phi_0$ and $\phi_1\nu$. The maps $H_0 : I \times A \to B$ and $\phi_1 : X \to B$ induce a map on the mapping cylinder $f : EX \to B$. There is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi_0"] \arrow[d, "E\nu"'] & Y \arrow[d, "g"'] \\ EX \arrow[r, "f"] & B. \end{tikzcd}$$

Conversely, suppose that we are given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi_0"] \arrow[d, "E\nu"'] & Y \arrow[d, "g"'] \\ EX \arrow[r, "f"] & B. \end{tikzcd}$$

The diagram

$$\begin{tikzcd} A \arrow[r, "\phi_0"] \arrow[d, "\nu"'] & Y \arrow[d, "g"'] \\ X \arrow[r, "fj"] & B \end{tikzcd}$$

is homotopy commutative with homotopy $f\pi$.

Homotopy commutative diagrams can also be interpreted in terms of associated fibrations. Given a map $g : Y \to B$ in TOP, there is the associated fibration $E'g : E'Y \to B$ and the pullback diagram

$$\begin{array}{ccc} E'Y & \xrightarrow{\tau} & B^I \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle e_0} \\ Y & \xrightarrow{g} & B. \end{array}$$

Suppose that we are given a homotopy commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi_0} & Y \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\phi_1} & B \end{array}$$

with homotopy $H_0 : I \times A \to B$ joining $g\phi_0$ and $\phi_1\nu$. By (1.14) there is the map $LH_0 : A \to B^I$. The maps ϕ_0 and LH_0 induce a map $f : A \to E'Y$. The diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & E'Y \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle E'g} \\ X & \xrightarrow{\phi_1} & B \end{array}$$

is commutative. Conversely, given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & E'Y \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle E'g} \\ X & \xrightarrow{\phi_1} & B, \end{array}$$

the diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi f} & Y \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\phi_1} & B \end{array}$$

is homotopy commutative with homotopy $L^{-1}(\tau f)$.

Theorem 3.8 *Suppose that $\nu : A \to X$ is a homotopy equivalence in TOP, and that $g : Y \to B$ is a fibration. Given a homotopy commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\phi_0} & Y \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\phi_1} & B \end{array}$$

in TOPand a homotopy $H_0 : I \times A \to B$ joining $g\phi_0$ to $\phi_1\nu$, there exists a map $\theta : X \to Y$ with $g\theta = \phi_1$ and a homotopy $H : I \times A \to Y$ joining ϕ_0 and $\theta\nu$ with

$H_0 = gH$.

Proof. There is the pushout diagram

$$\begin{array}{ccc} 1 \times A & \xrightarrow{\nu} & X \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow \\ I \times A & \xrightarrow{\pi} & EX \end{array}$$

and the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi_0} & Y \\ {\scriptstyle E\nu}\downarrow & & {\scriptstyle g}\downarrow \\ EX & \xrightarrow{f} & B \end{array}$$

described above. The map $E\nu : A \to EX$ is a cofibration and a homotopy equivalence in TOP. By (3.5) there is a lifting $h : EX \to Y$. Let $\theta = hj$ and $H = h\pi$. $\square$

The following is the homotopy extension lifting property, or as Boardman and Vogt [4.1] have called it, HELP.

Theorem 3.9 (HELP) *Suppose that $g : Y \to B$ is a homotopy equivalence in TOP, and that $\nu : A \to X$ is a cofibration. Suppose that*

$$\begin{array}{ccc} A & \xrightarrow{\phi_0} & Y \\ {\scriptstyle \nu}\downarrow & & {\scriptstyle g}\downarrow \\ X & \xrightarrow{\phi_1} & B \end{array}$$

is a homotopy commutative diagram and that we are given a homotopy $H_0 : I \times A \to B$ joining $g\phi_0$ to $\phi_1\nu$. Then there exists a map $\theta : X \to Y$ with $\theta\nu = \phi_0$ and a homotopy $H : I \times X \to B$ joining $g\theta$ to ϕ_1 with $H(1_I \times \nu) = H_0$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{k} & E'Y \\ {\scriptstyle g}\downarrow & & {\scriptstyle E'g}\downarrow \\ B & \xrightarrow{1_B} & B \end{array}$$

where $k(x) = (x, \sigma_x)$ and σ_x denotes the constant path at $g(x)$. Since g is a homotopy equivalence in TOP, $E'g$ is a homotopy equivalence in TOP. There is the pullback diagram

$$\begin{array}{ccc} E'Y & \xrightarrow{\tau} & B^I \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle e_0}\downarrow \\ Y & \xrightarrow{g} & B \end{array}$$

and the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & E'Y \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle E'g} \\ X & \xrightarrow{\phi_1} & B \end{array}$$

described above. By (3.7) there is a lifting $h : X \to E'Y$. Let $\theta = \pi h$ and let $H = L^{-1}(\tau h)$. $\square$

Homotopy Equivalences and Weak Homotopy Equivalences in $\mathbf{TOP}^G$

We now return to the mainstream of our topic by investigating further than at the end of Chapter 2 the category G with two objects 0 and 1 and one nonidentity morphism $0 \to 1$. The objects X of $\mathrm{TOP}^{0\to 1}$ are the maps $X_0 \xrightarrow{g} X_1$ and the morphisms $\phi : X \to Y$ are the G-maps, i.e. the commutative diagrams

$$\begin{array}{ccc} X_0 & \xrightarrow{\phi_0} & Y_0 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ X_1 & \xrightarrow{\phi_1} & Y_1. \end{array}$$

As introduced just prior to (2.16), there is a notion of homotopy in TOP^G for any small category G. Given the G-space X, there is the G-space $I \times X$, and there are the G-maps

$$\pi_0, \pi_1 : X \to I \times X.$$

Two G-maps $\phi, \theta : X \to Y$ are *homotopic* in TOP^G if there exists a G-map $H : I \times X \to Y$ with $\phi = H\pi_0$ and $\theta = H\pi_1$. This is an equivalence relation $\sim$ on the G-maps $X \to Y$. A G-map $\phi : X \to Y$ is a *homotopy equivalence in* TOP^G if there exists a G-map $\theta : Y \to X$ with $\theta\phi$ and $\phi\theta$ both homotopic in TOP^G to the identity G-map. Denote by HE the subcategory of TOP^G whose morphisms are the homotopy equivalences in TOP^G.

The homotopy category $\pi\mathrm{TOP}^G$ of TOP^G is the category whose objects are the G-spaces X and whose morphisms are the homotopy classes $[\phi]$ of G-maps $\phi : X \to Y$. There is the functor

$$F' : \mathrm{TOP}^G \to \pi\mathrm{TOP}^G$$

which is the identity on objects and which takes ϕ into $[\phi]$. The isomorphisms of $\pi\mathrm{TOP}^G$ are precisely the homotopy classes whose representatives are homotopy equivalences in TOP^G. Given any functor $F : \mathrm{TOP}^G \to \mathcal{C}$ such that whenever ϕ is a homotopy equivalence of TOP^G then $F\phi$ is an isomorphism of $\mathcal{C}$, there exists a unique functor $F'' : \pi\mathrm{TOP}^G \to \mathcal{C}$ with $F = F''F'$. To prove this, note for any G-space X that $\pi_0, \pi_1 : X \to I \times X$ are homotopy equivalences in TOP^G with common homotopy inverse the projection G-map $\nu : I \times X \to X$. Then $F\pi_0 = F\pi_1$ and if $\phi_0, \phi_1 : X \to Y$ are homotopic in TOP^G then $F\phi_0 = F\phi_1$.

In the language of Gabriel-Zisman [2.4], we thus have $\pi\mathrm{TOP}^G = \mathrm{TOP}^G[\mathrm{HE}^{-1}]$. That is, $\pi\mathrm{TOP}^G$ is the category obtained from TOP^G by inverting the homotopy equivalences in TOP^G.

We need also the *weak homotopy equivalences of* TOP^G, that is the G-maps $\phi : X \to Y$ such that for each $p \in Ob\ G$ the map $\phi_p : X(p) \to Y(p)$ is a homotopy equivalence in TOP. Denote by WHE the subcategory of TOP^G whose morphisms are the weak homotopy equivalences in TOP^G.

Cofibrations and Fibrations as Objects of $\mathrm{TOP}^{0\to 1}$

We return to $\mathrm{TOP}^{0\to 1}$. Denote by COF the full subcategory of $\mathrm{TOP}^{0\to 1}$ whose objects are all the cofibrations in $\mathrm{TOP}^{0\to 1}$, and by FIB the full subcategory of $\mathrm{TOP}^{0\to 1}$ whose objects are the fibrations in $\mathrm{TOP}^{0\to 1}$.

Theorem 3.10 *Let $v : A \to X$ and $\tau : B \to Y$ be cofibrations in TOP, and let $\phi : \nu \to \tau$ be a morphism in $TOP^{0\to 1}$ such that both $\phi_0 : A \to B$ and $\phi_1 : X \to Y$ are homotopy equivalences in TOP. That is, suppose there is a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\phi_0} & B \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\tau} \\ X & \xrightarrow{\phi_1} & Y \end{array}$$

with ϕ a weak homotopy equivalence in $TOP^{0\to 1}$. Then ϕ is a homotopy equivalence in $TOP^{0\to 1}$.

Proof. We will prove that ϕ has a right homotopy inverse θ, and the usual argument will then show that ϕ and θ are homotopy inverses. Start by picking any homotopy inverse $\theta_0 : B \to A$ and picking any homotopy $H_0 : I \times B \to B$ joining $\phi\theta_0$ to 1_B. Then use (3.9) on the diagram

$$\begin{array}{ccc} B & \xrightarrow{\nu\theta_0} & X \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\phi_1} \\ Y & \xrightarrow{1_Y} & Y \end{array}$$

and the homotopy τH_0 to obtain $\theta_1 : Y \to X$ and a homotopy $H_1 : I \times Y \to Y$ such that H_0 and H_1 give a homotopy joining $\phi\theta$ to 1.

The freedom to pick θ_0 and the homotopy H_0 arbitrarily will be useful in other contexts. □

Theorem 3.11 *Consider the commutative diagram in TOP*

$$\begin{array}{ccc} Y' & \xrightarrow{\phi_0} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ B' & \xrightarrow{\phi_1} & B \end{array}$$

where g' and g are fibrations and where ϕ_0 and ϕ_1 are homotopy equivalences in TOP. That is, consider the weak homotopy equivalence $\phi : \mathcal{E}' \to \mathcal{E}$ in $TOP^{0\to 1}$, where $\mathcal{E}'$ and $\mathcal{E}$ are both fibrations. Then ϕ is a homotopy equivalence in $TOP^{0\to 1}$.

Proof. It suffices to show that ϕ has a left homotopy inverse. Choose a left homotopy inverse $\theta_1 : B \to B'$ for ϕ_1 and a homotopy $H_1 : I \times B' \to B'$ joining $1_{B'}$ to $\theta_1\phi_1$. Now use (3.8) on the diagram

$$\begin{array}{ccc} Y' & \xrightarrow{1_Y} & Y' \\ {\scriptstyle \phi_0}\downarrow & & \downarrow{\scriptstyle g'} \\ Y & \xrightarrow{\theta_1 g} & B' \end{array}$$

and the homotopy $H_1(1_I \times g')$ to obtain a map $\theta_0 : Y \to Y'$ and a homotopy $H_0 : I \times Y' \to Y'$ joining 1_Y and $\theta_0\phi_0$. $\square$

Categories with Principal Models

Let C be a category with a homotopy relation on its morphisms, and thus an associated homotopy category πC. One assumes an equivalence relation $\sim_{p,q}$ on each set of morphisms $p \leftarrow q$, such that if $g, g' : q \to p$ have $g \sim_{p,q} g'$ then for any

$$s \xleftarrow{h} p, \quad q \xleftarrow{f} r$$

we have $hgf \sim_{s,r} hg'f$. Denote by HE the subcategory of C whose morphisms are the homotopy equivalences of C. Assume we are given a subcategory WHE of C containing HE, whose morphisms are called the *weak homotopy equivalences* of C. Let M be a subcategory of C, whose objects are called *models*. The objects of M are called *principal models* or *principal objects* if there is a functor $E : C \to C$ and a natural transformation $T : E \to 1$ which satisfy the following:

(1) For each object X of C, the object EX is a model.
(2) For each object X of C, $T_X : EX \to X$ is a weak homotopy equivalence.
(3) If X and Y are models and $\phi : X \to Y$ is a weak homotopy equivalence, then ϕ is a homotopy equivalence.
(4) If $\phi, \theta : X \to Y$ are homotopic morphisms then $E\phi, E\theta : EX \to EY$ are homotopic.
(5) If the morphism $\phi : X \to Y$ is a weak homotopy equivalence, then $E\phi : EX \to EY$ is a weak homotopy equivalence (and therefore a homotopy equivalence by (3)).

Theorem 3.12 *Suppose that C is a category with principal models as above, and consider the diagram in C*

$$A \xrightarrow{\phi} Y \xleftarrow{\theta} X$$

where A is a model and where θ is a weak homotopy equivalence. Then there is exactly one homotopy class of maps $\mu : A \to X$ such that $\phi \sim \theta\mu$.
PROOF. Consider the diagram

$$\begin{tikzcd} EA \arrow[r, "E\phi"] \arrow[d, "T_A"'] & EY \arrow[d, "T_Y"'] & EX \arrow[l, "E\theta"'] \arrow[d, "T_X"'] \\ A \arrow[r, "\phi"] & Y & X. \arrow[l, "\theta"'] \end{tikzcd}$$

It follows from our assumptions that T_A and $E\theta$ are homotopy equivalences, therefore invertible in the homotopy category. Hence there exists μ with the homotopy relation holding. We have now to show that it is unique up to homotopy. Otherwise there are nonhomotopic maps $\mu_1, \mu_2 : A \to X$ with $\theta\mu_1 \sim \theta\mu_2$. From the diagram

$$\begin{tikzcd} EA \arrow[r, "E\mu_i"] \arrow[d, "T_A"'] & EX \arrow[r, "E\theta"] \arrow[d, "T_X"'] & EY \arrow[d, "T_Y"'] \\ A \arrow[r, "\mu_i"] & X \arrow[r, "\theta"] & Y \end{tikzcd}$$

we see that the maps $E\mu_1, E\mu_2$ are not homotopic while $E(\theta)E(\mu_1) \sim E(\theta)E(\mu_2)$. But $E(\theta)$ is an isomorphism in the homotopy category, which furnishes a contradiction. □

Cofibrations as Principal Models in $\mathbf{TOP}^{0\to 1}$

Given a morphism ϕ in $\mathrm{TOP}^{0\to 1}$, that is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi_0"] \arrow[d, "\nu"'] & B \arrow[d, "\tau"'] \\ X \arrow[r, "\phi_1"] & Y, \end{tikzcd}$$

there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\theta_0"] \arrow[d, "E_\nu"'] & B \arrow[d, "E_\tau"'] \\ EX \arrow[r, "\theta_1"] & EY, \end{tikzcd}$$

where $\theta_0 = \phi_0$ and θ_1 is the map on quotients induced by the map $I \times A \sqcup X \to I \times B \sqcup Y$, $(t, a) \mapsto (t, \phi_0(a))$, $a \in A$, $t \in I$, $x \mapsto \phi_1(x)$, $x \in X$. This gives a functor $E : \mathrm{TOP}^{0\to 1} \to \mathrm{TOP}^{0\to 1}$ by $\nu \mapsto E\nu$, $\phi \mapsto \theta$. There is a natural transformation $T : E \to 1$ given by the diagram

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "E_\nu"'] & A \arrow[d, "\nu"'] \\ EX \arrow[r, "\phi'"] & X \end{tikzcd}$$

where $\phi'\pi_\nu(t,a) = \nu(a)$, $a \in A$, $\phi'\pi_\nu(x) = x$, $x \in X$.

The functor E and the natural transformation T exhibit COF as a subcategory of principal models in TOP$^{0\to 1}$. It is clear from the mapping cylinder construction that each E_ν is a model and each $T_\nu : E\nu \to \nu$ is a weak homotopy equivalence in TOP$^{0\to 1}$. By (3.10) a weak homotopy equivalence between models is a homotopy equivalence. The conditions (iv) and (v) will normally be automatic in our cases. For (iv), suppose that $\nu : A \to X$ and $\tau : B \to Y$ are objects in TOP$^{0\to 1}$ and ϕ,$\theta : \nu \to \tau$ are homotopic morphisms in TOP$^{0\to 1}$, with the homotopy being exhibited by the commutative diagram

$$\begin{array}{ccc} I \times A & \xrightarrow{H_0} & I \times B \\ {\scriptstyle 1_I \times \nu}\downarrow & & \downarrow{\scriptstyle 1_I \times \tau} \\ I \times X & \xrightarrow{H_1} & I \times Y. \end{array}$$

The functor E takes this diagram to the commutative diagram

$$\begin{array}{ccc} I \times A & \xrightarrow{G_0} & I \times B \\ {\scriptstyle 1_I \times E_\nu}\downarrow & & \downarrow{\scriptstyle 1_I \times E_\tau} \\ I \times EX & \xrightarrow{G_1} & I \times EY \end{array}$$

where $G_0 = H_0$ and G_1 is the map on quotients induced by the map $I \times I \times A \sqcup I \times X \to I \times I \times B \sqcup I \times Y$, $(t,s,a) \mapsto (t, H_0(s,a))$, $a \in A$, $(t,x) \mapsto H_1(t,x)$, $x \in X$. G exhibits a homotopy between $E\phi$ and $E\theta$.

The above argument also shows that, given an object A in TOP, COF$\backslash A$ is a subcategory of principal models in TOP$\backslash A$.

Another classic example of a category with principal models is as follows. Take as starting point the category top, and apply the simplicial apparatus of Chapter 2. There are functors

$$\text{top} \xrightarrow{(\diamond)^\nabla} \text{SET}^{\Delta^o} \xrightarrow{|\diamond|} \text{TOP},$$

where for any space A denote by A^∇ the simplicial set $\{A^{\nabla(n)}\}$ where the topology on $A^{\nabla(n)}$ is replaced by the discrete topology. We thus have the composite functor

$$E : \text{top} \to \text{top}$$

which sends A into $|A^\nabla|$. Moreover one has a natural transformation $T : E \to 1$ where given a space A then

$$T_A : |A^\nabla| \to A$$

sends each $\sigma \times_\Delta (t_0, \cdots, t_n)$ into $\sigma(t_0, \cdots, t_n)$. The principal models in this case are the CW-complexes; the weak homotopy equivalences are those maps in top which are always called the weak homotopy equivalences of top. There results, then, an example of what we have called a category with principal models. See Spanier [2.12] or Whitehead [1.6] for full treatment, perhaps in a different format.

Categories with Coprincipal Models

As for categories with principal models, let C be a category with a homotopy relation, associated homotopy category πC, and subcategories HE and WHE. Let M be a subcategory of C, whose objects are called *models*. The objects of M are called *coprincipal models* or *coprincipal objects* if there is a functor $E' : C \to C$ and a natural transformation $T : 1 \to E'$ which satisfy the following:

(i) For each object X of C, the object $E'X$ is a model.
(ii) Each $T_X : X \to E'X$, X an object of C, is a weak homotopy equivalence.
(iii) If X and Y are models and $\phi : X \to Y$ is a weak homotopy equivalence, then ϕ is a homotopy equivalence.
(iv) If $\phi, \theta : X \to Y$ are homotopic morphisms then $E'\phi, E'\theta : E'X \to E'Y$ are homotopic.
(v) If the morphism $\phi : X \to Y$ is a weak homotopy equivalence, then $E'\phi : E'X \to E'Y$ is a weak homotopy equivalence (and hence a homotopy equivalence).

Theorem 3.13 *Let C be a category with coprincipal models as above. If*

$$A \xleftarrow{\phi} X \xrightarrow{\mu} Y$$

is a diagram in C where A is a model and μ is a weak homotopy equivalence, then there is a unique homotopy class of morphisms $\theta : Y \to A$ with $\phi \sim \theta\mu$. In particular, given an object X of C, there is a model A and a weak homotopy equivalence $X \to A$ unique up to a homotopy equivalence.

The proof is an exercise.

Fibrations as Coprincipal Models in $\mathbf{TOP}^{0\to 1}$

Given an object $g : Y \to B$ in $\mathrm{TOP}^{0\to 1}$, by (3.3;vi) there is an associated fibration $E'g : E'Y \to B$. Given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\phi_0"] \arrow[d, "f"'] & Y \arrow[d, "g"'] \\ A \arrow[r, "\phi_1"] & B \end{tikzcd}$$

in TOP, there is a commutative diagram

$$\begin{tikzcd} E'X \arrow[r, "\theta_0"] \arrow[d, "E'f"'] & E'Y \arrow[d, "E'g"'] \\ A \arrow[r, "\theta_1"] & B, \end{tikzcd}$$

where $\theta_0(x, \sigma) = (\phi_0(x), \phi_1\sigma)$, $x \in X$, $\sigma \in B^I$, and $\theta_1 = \phi_1$. This gives a functor $E' : \mathrm{TOP}^{0\to 1} \to \mathrm{TOP}^{0\to 1}$ by $g \mapsto E'g$, $\phi \mapsto \theta$. There is a natural transformation

$T : 1 \to E'$ given by the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\phi_0} & E'Y \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle E'g} \\ B & \xrightarrow{1_B} & B, \end{array}$$

where $\phi_0(y) = (y, \sigma_{g(y)})$, $y \in Y$, and $\sigma_{g(y)}$ is the constant path at $g(y)$.

The functor E' and the natural transformation T exhibit FIB as a subcategory of coprincipal models in $\mathrm{TOP}^{0\to 1}$. Condition (iii) follows from (3.11), and the other conditions follow routinely as for cofibrations. FIB$/B$ is a subcategory of coprincipal models for TOP$/B$.

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CHAPTER IV

A Basic Model for Equivariant General Topology

The equivariant general topology of actions of small categories emerged as an established topic around 1970. Topological presentations were given by Boardman-Vogt [4.1], May [2.8], and Segal [4.4], and a semi-simplicial presentation was given by Bousfield-Kan [2.1]. Among other topics aimed at applications, each gave a supporting model for the equivariant general topology that was needed for the applications. Among new features were the consideration of equivariant topology as a subject encompassing actions of small categories, the presentation of homotopy colimits and homotopy limits as primary invariants of a G-space, and the more thorough incorporation of homotopy into the core of equivariant topology.

It is our goal in these first four chapters to give a beginning model for the supporting equivariant general topology, drawn from the above work of Boardman-Vogt, May, and Segal. In this chapter, we provide a beginning account of the homotopy colimits and the homotopy limits of G-spaces in TOP, where G is any small category. One needs a chosen way of producing for each G-space X in TOP standard models for these homotopy limits and homotopy colimits. We chose the most standard model for homotopy colimits in Chapter 2, the functor

$$B_G(\diamond) : \mathrm{TOP}^G \to \mathrm{TOP}, \qquad X \mapsto B_G X.$$

In this chapter, we have to explain the context in a meaningful way, i.e. the special role of the principal G-spaces and their duals, the coprincipal G-spaces. We want to end up for each small category G with a category of principal models and a category of coprincipal models as was done in a simpler setting in Chapter 3; the terminology "principal" used there was motivated by the examples of this chapter.

The material of this chapter is one of two approaches to the subject, that in which general operator domains acting on spaces are taken as given and spaces are used as invariants of the actions. In the other approach, spaces are taken as given and one seeks an equivariant framework to understand the spaces better; one seeks constructions which assign operator domains to a space. We wait until later chapters to introduce this aspect of the subject, in which Stasheff has been the leader. See his overview [4.5] for background on both sides.

The Subcategory $Id\,G$ of G and Extension $\mathbf{TOP}^{Id\,G} \to \mathbf{TOP}^G$

There is the subcategory $Id\ G$ of G whose morphisms are all the identity morphisms of G, and an $Id\ G$-space is precisely a collection $A = \{A(p)|p \in Ob\ G\}$ of spaces. Thus the category $\mathrm{TOP}^{Id\ G}$ has as objects the collections A of compactly generated spaces, and as morphisms $\phi : A \to A'$ all collections $\phi = \{\phi_p : A(p) \to A'(p)\}$ of maps.

If $i : Id\ G \to G$ denotes the inclusion functor, there is the extension functor

$$i_\# : \mathrm{TOP}^{Id\ G} \to \mathrm{Top}^G$$

of Chapter 1. Given a collection $A = \{A(p)\}$ of compactly generated spaces, then the G-space $i_\# A$ has

$$(i_\# A)(p) = \coprod_{q \in Ob\ G} G(p,q) \times A(q),$$

and the action of G assigns to $p' \xleftarrow{g} p$ and $(g', x) \in G(p,q) \times A(q)$ the element $g(g', x) = (gg', x)$. Thus we can denote the G-space $i_\# A$ by $G \times_{Ob\ G} A$. Clearly if each $A(p)$ is in TOP, then each $(i_\# A)(p)$ is in TOP, and we have the functor

$$i_\# : \mathrm{TOP}^{Id\ G} \to \mathrm{TOP}^G.$$

Note that we could equivalently consider $\mathrm{TOP}^{Id\ G}$ as the category $\mathrm{TOP}/Ob\ G$ of spaces over the discrete space $Ob\ G$.

These G-spaces $i_\# A$ will be fundamental building blocks for nice G-spaces. To see why, it is instructive to look first at the untopologized setting.

Free G-Sets

A G-set is a functor $X : G \to \mathrm{SET}$. It thus assigns to each $p \in Ob\ G$ a set $X(p)$ and to each $g : p \to q$ a function $g_* : X(p) \to X(q)$ sending $x \in X(p)$ into $g_* x = gx \in X(q)$, satisfying $g_* g'_* = (gg')_*$ and $1_{p*} = 1$. A G-set X is said to be *free* if there exists a collection $A = \{A(p)|p \in Ob\ G\}$, where each $A(p)$ is a subset of $X(p)$, such that given q and $x \in X(q)$ there is a unique $p \in Ob\ G$, $a \in A(p)$ and $g : p \to q$ such that $ga = x$. In the semi-simplicial presentation of our topic, the free G-sets are a basic notion. See Dror Farjun [4.2] for an exposition of the use made of this concept. In a topological presentation such as this one, the most basic G-spaces will be free as G-sets but the choice of generating set A has to be tightly connected to the topology. We will soon introduce these G-spaces precisely, and call them the *principal G-spaces*. They have been presented as we present them in the monoid case by Steenrod [4.6], and in the general case by Boardman-Vogt [4.1], although neither bothered to name them. The concept of principal G-space used here differs somewhat from that used classically for compact Lie groups, but we argue later in the chapter that it is the proper concept, since it gives the appropriate class of principal G-spaces for the case in which G is the category $0 \to 1$.

Consider the full subcategory SET^G of TOP^G whose objects are the discrete G-spaces, i.e. the G-sets. Consider also the full subcategory $\mathrm{SET}^{Id\,G}$ of $\mathrm{TOP}^{Id\,G}$ whose objects are the collections A of discrete $Id\ G$-spaces, i.e. the full subcategory whose objects A have each $A(p)$ a set, and whose morphisms $\phi : A(p) \to A'(p)$ have each ϕ_p a function from $A(p)$ to $A'(p)$. Then it is easy to present the free G-sets in terms of the extension functor. There is the extension functor $i_\# : \mathrm{SET}^{Id\,G} \to \mathrm{SET}^G$ sending an $Id\,G$-set A into the G-set

$$i_\# A = G \times_{Ob\,G} A,$$

and X is a free G-set if and only if X is isomorphic in SET^G to some $i_\# A = G \times_{Ob\,G} A$.

Principal G-Spaces

As above, the category $\mathrm{TOP}^{Id\,G}$ has as objects all collections $A = \{A(p) | p \in Ob\,G\}$ of compactly generated spaces, and as morphisms $\phi : B \to A$ all collections $\phi = \{\phi_p\}$ of maps $\phi_p : B(p) \to A(p)$. If each $B(p)$ is a closed subset of $A(p)$, then we speak of (A, B) as a closed pair in $\mathrm{TOP}^{Id\,G}$. If each $(A(p), B(p))$ is a cofibered pair in TOP, then we say that (A, B) is a cofibered pair in $\mathrm{TOP}^{Id\,G}$.

There is the above functor

$$i_\# : \mathrm{TOP}^{Id\,G} \to \mathrm{TOP}^G$$

sending an $Id\,G$-space A into the G-space

$$i_\# A = G \times_{Ob\,G} A = \{\coprod_{q \in Ob\,G} G(p, q) \times A(q)\}.$$

A G-space X is said to be a *principal G-space* if there exists a filtration $X = \bigcup_{n \geq 0} X_n$ of X in TOP^G such that:

(i) X_0 is homeomorphic in TOP^G to $i_\# A_0 = G \times_{Id\,G} A_0$ for some $Id\,G$-space A_0;

(ii) for each $n > 0$ there is a closed cofibered pair (A_n, B_n) in $\mathrm{TOP}^{Id\,G}$ and a relative homeomorphism

$$(G \times_{Ob\,G} A_n, G \times_{Ob\,G} B_n) \to (X_n, X_{n-1})$$

of G-spaces. Equivalently, there is a G-map $G \times_{Ob\,G} B_n \to X_{n-1}$ and a pushout diagram

$$\begin{array}{ccc} G \times_{Ob\,G} B_n & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ G \times_{Ob\,G} A_n & \longrightarrow & X_n. \end{array}$$

(4.1) *For every Y in TOP^G the G-space $E_G Y$ of Chapter 2 is a principal G-space. Similarly the $G \times G^o$-space EG of Chapter 2 is a principal $G \times G^o$-space.*

Proof. We must first produce a filtration of $E_G Y$ as a G-space. From (2.8), each of its constituent spaces $(E_G Y)(p)$ is filtered in TOP as $(E_G Y)(p) = \bigcup (G(p, \diamond) \bigotimes_G Y)_n$. From the fact that $(G(p, \diamond) \bigotimes_G Y)_n$ consists of all

$$(g_0, g_1, \cdots, g_n, y) \times_\Delta (t_0, \cdots, t_n)$$

it is clear that any $g : p \to p'$ in G maps $(G(p,\diamond) \bigotimes_G Y)_n$ into $(G(p',\diamond) \bigotimes_G Y)_n$. That is, $E_G Y$ is naturally filtered as a G-space, say as $E_G Y = \bigcup (E_G Y)_n$.

Clearly $(E_G Y)_0$ is identified with all (g_0, y) in $G \times_{Ob\, G} Y$, thus $(E_G Y)_0$ is of the desired form. More precisely, $(E_G Y)_0$ is $G \times_{Ob\, G} i^{\#} Y$.

Let $n > 0$. We must use the existing relative homeomorphism from (2.8)

$$(G \times_{Ob\, G} (G^{\underline{n}}, G^{\underline{n},deg}) \times_{Ob\, G} Y) \times_\Delta (\nabla(n), \partial\nabla(n)) \to ((E_G Y)_n, (E_G Y)_{n-1})$$

in TOP^G, where $G^{\underline{n},deg}$ denotes all $(g_1, \cdots, g_n) \in G^{\underline{n}}$ such that some g_i is an identity morphism. We can write the left hand side as

$$G \times_{Ob\, G} [((G^{\underline{n}}, G^{\underline{n},deg}) \times_{Ob\, G} Y) \times_\Delta (\nabla(n), \partial\nabla(n))]$$

which is of the desired form.

To see that EG is a principal $G \times G^o$-space one proceeds in an analogous way, arriving finally at the building blocks

$$G \times_{Ob\, G} [(G^{\underline{n}}, G^{\underline{n},deg}) \times_\Delta (\nabla(n), \partial\nabla(n)))] \times_{Ob\, G} G. \quad \square$$

Thus we have the functor $E_G(\diamond) : \mathrm{TOP}^G \to \mathrm{TOP}^G$ together with the natural transformation $T : E_G(\diamond) \to 1$, and each $E_G Y$ is a principal G-space. We denote by PRINC^G the full subcategory of TOP^G whose objects are the principal G-spaces.

(4.2) *If X is a principal G-space, then its colimit X/G is a compactly generated space. Thus we have the colimit functor*

$$colim : PRINC^G \to TOP.$$

Proof. Inductively one shows the filtration $X/G = \bigcup X_n/G$ in Top has each X_n/G compactly generated, from which it follows from (1.18) that X/G is compactly generated. To establish the induction, one needs that X_n/G is a pushout of

$$A_n \hookleftarrow B_n \to X_{n-1}/G$$

in TOP, after which one uses (1.20). $\square$

Homotopy Properties of Principal G-Spaces

We have adapted the following key theorem from Boardman-Vogt [4.1]. It is the remaining nontrivial need in showing that TOP^G is a category with principal objects in the sense of Chapter 3. A G-map $\phi : X \to Y$ is said to be a *weak homotopy equivalence in* TOP^G if each $\phi_p : X(p) \to Y(p)$ is a homotopy equivalence in TOP.

Theorem 4.3 *If X and Y are principal G-spaces and if $\phi : X \to Y$ is a weak homotopy equivalence in TOP^G, then ϕ is a homotopy equivalence in TOP^G.*

PROOF. Suppose X and Y are principal G-spaces, and that $\phi : X \to Y$ is a weak homotopy equivalence in TOP^G. Let $Y = \bigcup Y_n$ be a filtration for Y satisfying the properties for a principal G-space. Take $n > 0$ and make the following inductive assumption: there exists a G-map $\theta_{n-1} : Y_{n-1} \to X$ and a G-homotopy $H_{n-1} : I \times Y_{n-1} \to Y$ such that H_{n-1} joins $\phi\theta_{n-1}$ to the inclusion $i : Y_{n-1} \to Y$. The induction hypothesis will be established when we have extended θ_{n-1} to $\theta_n : Y_n \to X$ and H_{n-1} to $H_n : I \times Y_n \to Y$ so that the induction hypotheses hold at this next level.

Let (A, B) be a cofibered pair in $\mathrm{TOP}^{Id\, G}$ and let $f : (G \times_{Ob\, G} A, G \times_{Ob\, G} B) \to (Y_n, Y_{n-1})$ be a relative homeomorphism in TOP^G. Note that for each p in $Ob\, G$ there is a natural inclusion of $A(p)$ in $G \times_{Ob\, G} A$; simply identify $y \in Y(p)$ with $(1_p, y) \in G \times_{Ob\, G} Y$. Let $f' : A(p) \to Y_n(p)$ and $f'' : B(p) \to Y_{n-1}(p)$ denote the restriction of f to these subspaces. We then get a diagram of maps in TOP

$$\begin{array}{ccccc} B(p) & \xrightarrow{f''} & Y_{n-1}(p) & \xrightarrow{(\theta_{n-1})_p} & X(p) \\ {\scriptstyle j}\downarrow & & {\scriptstyle k}\downarrow & & {\scriptstyle \phi_p}\downarrow \\ A(p) & \xrightarrow{f'} & Y_n(p) & \xrightarrow{i} & Y(p) \end{array}$$

for each $p \in Ob\, G$.

Associated with the shortened diagram

$$\begin{array}{ccc} B(p) & \xrightarrow{(\theta_{n-1})_p f''} & X(p) \\ {\scriptstyle j}\downarrow & & {\scriptstyle \phi_p}\downarrow \\ A(p) & \xrightarrow{if'} & Y(p) \end{array}$$

we also have a homotopy $H'_0 : I \times B(p) \to Y(p)$ joining $\phi_p(\theta_{n-1})_p f''$ to $if'j$, obtained as the composition

$$I \times B(p) \xrightarrow{id \times f''} I \times Y_{n-1}(p) \xrightarrow{H_{n-1}} Y(p).$$

Thus we can apply (3.9) to obtain

$$\Theta' : A(p) \to X(p)$$

for all $p \in Ob\, G$ with $\Theta' j = (\theta_{n-1})_p f''$ as well as a homotopy

$$H' : I \times A(p) \to Y(p)$$

for all p extending H'_0 and joining $\phi_p \Theta'$ to if'.

We now have a $Id\, G$-map $\Theta' : A \to X$ and a $Id\, G$-homotopy $H' : I \times A \to Y$ joining $\phi\Theta'$ to if'. The $Id\, G$-map $\Theta' : A \to X$ has a unique extension to a G-map $\Theta : G \times_{Ob\, G} A \to X$ and the $Id\, G$-homotopy $H' : I \times A \to Y$ has a unique extension to a G-homotopy $H : I \times (G \times_{Ob\, G} A) \to Y$.

We then have a commutative diagram

$$\begin{array}{ccc} G \times_{Ob\,G} B & \longrightarrow & Y_{n-1} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \theta_{n-1}} \\ G \times_{Ob\,G} A & \xrightarrow{\Theta} & X \end{array}$$

in TOP^G, and hence a pushout G-map $\theta_n : Y_n \to X$.

Similarly there is a commutative diagram

$$\begin{array}{ccc} I \times (G \times_{Ob\,G} B) & \longrightarrow & I \times Y_{n-1} \\ \Big\downarrow & & \Big\downarrow \\ I \times (G \times_{Ob\,G} A) & \longrightarrow & Y \end{array}$$

in TOP^G and a pushout G-map $H_n : I \times Y_n \to Y$. The inductive assumption can now be checked at the next level, and it can also be checked at the first level as well.

Assuming the induction as having been established, there is a G-map $\theta : Y \to X$ and a G-homotopy $H : I \times Y \to Y$ joining $\phi\theta$ to 1. Hence θ is a right homotopy inverse for ϕ in TOP^G. Since X is also principal, θ in the same way has a right homotopy inverse ψ in TOP^G. Thus θ has both a right homotopy inverse ψ and a left homotopy inverse ϕ and is thus a homotopy equivalence in TOP^G. Hence ϕ is a homotopy equivalance in TOP^G. □

Corollary 4.4 *Consider the category TOP^G, together with the notion of homotopy in TOP^G and weak homotopy equivalence in TOP^G. Take also the functor*

$$E_G(\diamond) : TOP^G \to TOP^G$$

given in Chapter 2, together with the natural transformation $T : E_G(\diamond) \to 1$. Let the full subcategory of principal objects of TOP^G be the full subcategory $PRINC^G$ of principal G-spaces. With this given structure, TOP^G is a category with principal objects in the sense of Chapter 3.

The proof is an exercise.

A Homotopy Colimit BX of a G-Space X

Given a G-space X, a *principalization* of X is a pair $(EX, [\phi])$ consisting of a principal G-space EX and a homotopy class of G-maps $\phi : EX \to X$ each of which is a weak homotopy equivalence in TOP^G. The *standard* principalization of X is E_GX together with the homotopy class of its natural G-map $E_GX \to X$.

(4.5) *Given two principalizations represented by $\phi_0 : E_0X \to X$ and $\phi_1 : E_1X \to X$, there exists a unique homotopy class of G-maps $\theta : E_0X \to E_1X$ such that*

$$\begin{array}{ccc} E_0X & \xrightarrow{\theta} & E_1X \\ {\scriptstyle\phi_0}\big\downarrow & & {\scriptstyle\phi_1}\big\downarrow \\ X & = & X \end{array}$$

is homotopy commutative in TOP^G, and each θ is a homotopy equivalence in TOP^G.

This follows immediately from (3.12) and (4.4).

Let X be a G-space. A *homotopy colimit* BX of X is then a compactly generated space BX together with a given homotopy class of maps $BX \to B_GX$, each of which is a homotopy equivalence in TOP. The *standard* homotopy colimit B_GX of X is the space defined in Chapter 2 and used above; in this case the given homotopy class $B_GX \to B_GX$ is the class of maps homotopic to the identity.

Given two homotopy colimits B_0X and B_1X of X, we get a uniquely defined homotopy class of maps $B_0X \to B_1X$ in TOP, those for which the diagram

$$\begin{array}{ccc} B_0X & \longrightarrow & B_1X \\ \big\downarrow & & \big\downarrow \\ B_GX & = & B_GX \end{array}$$

is homotopy commutative.

Every principalization $(EX, [\phi])$ of a G-space X gives a homotopy colimit $BX = EX/G$ of X. This follows immediately from (4.5).

Consider for a moment where we are in categorical terms. We have a base category πTOP^G, the homotopy category of TOP^G. We have another category πPRINC^G, the homotopy category of principal G-spaces, a full subcategory of πTOP^G with an inclusion functor $S : \pi\mathrm{PRINC}^G \to \pi\mathrm{TOP}^G$. In the language of MacLane [1.2,p.58], given an object X of πTOP^G, a *universal arrow* for S at X is a principal G-space E and a homotopy class of G-maps $[\phi] : E \to X$ such that if E' is any principal G-space and $[\theta] : E' \to X$ is any homotopy class of G-maps then there exists a unique homotopy class $[\nu] : E' \to E$ with $[\theta] = [\phi][\nu]$. It can readily be seen that if $(E, [\phi])$ is a universal arrow for S at X, then ϕ is a weak homotopy equivalence in TOP^G. Thus the universal arrows are identical with the principalizations.

The Category TOP^G [WHE^{-1}] in which WHE's Are Inverted

Note that we have a model for TOP^G [WHE^{-1}] up to equivalence of categories. The composed functor

$$\mathrm{TOP}^G \xrightarrow{E_G(\diamond)} \mathrm{PRINC}^G \to \pi\mathrm{PRINC}^G$$

takes every weak homotopy equivalence into an isomorphism. Hence it induces a functor

$$\mathrm{TOP}^G\,[\mathrm{WHE}^{-1}] \to \pi\mathrm{PRINC}^G.$$

On the other hand there is composition

$$\pi\mathrm{PRINC}^G \hookrightarrow \pi\mathrm{TOP}^G \simeq \mathrm{TOP}^G\,[\mathrm{HE}^{-1}] \to \mathrm{TOP}^G\,[\mathrm{WHE}^{-1}].$$

By adjointness arguments (see Gabriel-Zisman [2.4]), these set up an equivalence of categories.

Theorem 4.6 *There is the above equivalence of categories*

$$TOP^G\,[WHE^{-1}] \sim \pi PRINC^G.$$

Alternatively we can take for a precise model for $TOP^G\,[WHE^{-1}]$ the category whose objects are the G-spaces and whose morphisms $X \to Y$ are the homotopy classes in TOP^G of G-maps $E_GX \to E_GY$.

From this point of view one can consider the standard homotopy colimit as the composition

$$\mathrm{TOP}^G\,[\mathrm{WHE}^{-1}] \to \pi\mathrm{PRINC}^G \xrightarrow{colim} \pi\mathrm{TOP}.$$

Note that it follows from (3.12) and (4.4) that in $\mathrm{TOP}^G\,[\mathrm{WHE}^{-1}]$ the morphisms $X \to Y$ are in one-to-one correspondence with the homotopy classes of G-maps $E_GX \to Y$.

Principal G-Spaces for the Category $0 \to 1$

When we wish to test the meaning of some proposed construct, we first examine its meaning when G is the category $0 \to 1$. Here the G-spaces X are the maps $\nu : X_0 \to X_1$. We seek a characterization of the principal G-spaces.

In this example, an Id G-space A is a pair $[A(0), A(1)]$ of compactly generated spaces, and $G \times_{Ob\,G} A$ is the inclusion $A(0) \hookrightarrow A(0) \sqcup A(1)$. These are the building blocks. If $X = \bigcup X_n$ is a principal G-space, then the G-space X_0 must be G-homeomorphic to some

$$A_0(0) \hookrightarrow A_0(0) \sqcup A_0(1).$$

Given a cofibered pair (Y, C) in TOP, we wish to show that the cofibration $C \hookrightarrow Y$ is a principal G-space $X = \bigcup X_n$. As a first stage, we take for X_0 the identity map $C \to C$, which is of the proper form.

The next stage X_1 is obtained as the pushout in TOP^G of a diagram

$$G \times_{Ob\,G} A \hookleftarrow G \times_{Ob\,G} B \to X_0.$$

For each $p \in Ob\ G$, this gives a rectangular pushout diagram. For $0 \to 1$, the total diagram specifying X_1 is therefore a cubical diagram. The proper choice for our present purpose is to take $A_1 = [\emptyset, Y]$ and for B_1 the pair $[\emptyset, C]$, so that we obtain the G-space X_1 as the vertical map $X_1(0) \to X_1(1)$ in the diagram below.

$$\begin{tikzcd}
B_1(0) = \emptyset \arrow[rr] \arrow[dr] \arrow[dd] & & X_0(0) = C \arrow[dr] \arrow[dd] & \\
& A_1(0) = \emptyset \arrow[rr] \arrow[dd] & & X_1(0) = C \arrow[dd] \\
B_1(0) \coprod B_1(1) = C \arrow[rr] \arrow[dr] & & X_0(1) = C \arrow[dr] & \\
& A_1(0) \coprod A_1(1) = Y \arrow[rr] & & X_1(1) = Y
\end{tikzcd}$$

Clearly X_1 is then just the given cofibration $C \hookrightarrow Y$, thus we have the following.

(4.7) *Every cofibration $\nu : C \to Y$ is a principal G-space, where G is the category $0 \to 1$.*

We will prove the converse. In order to do so, we take any principal G-space $X = \bigcup X_n$ where X_0 is of the form $A_0(0) \hookrightarrow A_0(0) \sqcup A_0(1)$ and where inductively X_n is read off the diagram below, it being assumed inductively that X_{n-1} is a cofibration.

$$\begin{tikzcd}
B_n(0) \arrow[rr] \arrow[dr] \arrow[dd] & & X_{n-1}(0) \arrow[dr] \arrow[dd] & \\
& A_n(0) \arrow[rr] \arrow[dd] & & X_n(0) \arrow[dd, "\nu_n"'] \\
B_n(0) \coprod B_n(1) \arrow[rr] \arrow[dr] & & X_{n-1}(1) \arrow[dr] & \\
& A_n(0) \coprod A_n(1) \arrow[rr] & & X_n(1)
\end{tikzcd}$$

The properties of the diagram are as follows.

(i) The top and bottom faces are pushout diagrams.
(ii) All maps in the right vertical face except possibly ν_n are cofibrations.
(iii) The left vertical face is a pullback diagram of cofibered inclusions.

In order to prove ν_n a cofibration, we use the following lemma.

(4.8) *Assume the following commutative diagram in TOP.*

$$
\begin{tikzcd}
B \arrow[rr, "\varphi"] \arrow[dr, "i"] \arrow[dd, "\kappa"'] & & X \arrow[dr, "j"] \arrow[dd, "\nu", near start] & \\
& A \arrow[rr, "\theta", near start] \arrow[dd, "\kappa'", near start] & & Y \arrow[dd, "\nu'"] \\
B' \arrow[rr, "\varphi'", near end] \arrow[dr, "i'"'] & & X' \arrow[dr, "j'"] & \\
& A' \arrow[rr, "\theta'"] & & Y'
\end{tikzcd}
$$

Assume also the following:

(i) the top face and the bottom face are pushout diagrams;
(ii) the left vertical face is a pullback diagram of cofibrations;
(iii) all maps in the right vertical face except possibly ν' are cofibrations.

Then ν' is a cofibration.

Proof. We may as well suppose in the left vertical face that all the cofibrations are inclusion maps of cofibered pairs. The pullback property is then that $A \cap B' = B$.

For any space Z in TOP, there is the fibration $\pi : Z^I \to Z$ which assigns to a path f in Z its first point $f(0)$. It is necessary and sufficient for $\nu' : Y \to Y'$ to be a cofibration that every commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\mu} & Z^I \\
{\scriptstyle \nu'}\downarrow & & {\scriptstyle \pi}\downarrow \\
Y' & \xrightarrow{\mu'} & Z
\end{array}
$$

have a lifting $\sigma : Y' \to Z^I$.

Since ν is a cofibration, and since π is both a fibration and a homotopy equivalence, one can apply (3.7) to

$$
\begin{array}{ccc}
X & \xrightarrow{\mu j} & Z^I \\
{\scriptstyle \nu}\downarrow & & {\scriptstyle \pi}\downarrow \\
X' & \xrightarrow{\mu' j'} & Z,
\end{array}
$$

obtaining $\lambda : X' \to Z^I$ with $\pi\lambda = \mu' j'$ and $\lambda\nu = \mu j$.

The map $\lambda\phi' : B' \to Z^I$ then has

$$\pi\lambda\phi' = \mu' j' \phi' = \mu' \theta' i'.$$

Consider the maps

$$\lambda\phi' : B' \to Z^I, \qquad \mu\theta : A \to Z^I.$$

It is seen that

$$\begin{array}{ccc} B = A \cap B' & \xrightarrow{i} & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \mu\theta} \\ B' & \xrightarrow{\lambda\phi'} & Z^I \end{array}$$

commutes, since

$$\lambda\phi' k = \lambda\nu\phi = \mu j\phi = \mu\theta i.$$

That is, we get a well defined map $B' \cup A \to Z^I$ which we denote by $\lambda\phi' \cup \mu\theta$.

We use without proof Lillig's theorem; see Lillig [3.6] or James [3.5]. Lillig's theorem asserts that if X is a compactly generated space and if A and B are closed subsets such that A, B and $A \cap B$ are cofibered subsets of X, then $A \cup B$ is a cofibered subset of X.

By Lillig's Theorem, $(A', B' \cup A)$ is a cofibered pair. Hence the commutative diagram

$$\begin{array}{ccc} B' \cup A & \xrightarrow{\lambda\phi' \cup \mu\theta} & Z^I \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ A' & \xrightarrow{\mu'\theta'} & Z \end{array}$$

has a lifting $\rho : A' \to Z^I$ with

$$\pi\rho = \mu'\theta', \quad \rho i' = \lambda\phi', \quad \rho k' = \mu\theta.$$

The commutative diagram

$$\begin{array}{ccc} B' & \xrightarrow{\phi'} & X' \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle \lambda} \\ A' & \xrightarrow{\rho} & Z^I \end{array}$$

gives, by the pushout property of the bottom face, a unique map $\sigma : Y' \to Z^I$ such that

$$\sigma\theta' = \rho, \qquad \sigma j' = \lambda.$$

One then checks that σ is a lifting of the original diagram. $\square$

Corollary 4.9 *For G the category $0 \to 1$, a G-space X of the form $X_0 \xrightarrow{\nu} X_1$ is a principal G-space if and only if the map ν is a cofibration in TOP.*

Relationship Between the Reduced Product and Mapping Bifunctors

We come back to the mapping bifunctor of Chapter 1

$$\mathrm{TOP}^G \times (\mathrm{Top}^{G\times H^o})^o \to \mathrm{TOP}^H,$$

where given X in TOP^G and Y in $\mathrm{Top}^{G\times H^o}$ we get $(X^Y)_{\mathrm{Top}^G}$ in Top^H defined by

$$(X^Y)_{\mathrm{Top}^G}(q) = (X^{Y(\diamond,q)})_{\mathrm{Top}^G}.$$

That is, $(X^Y)_{\mathrm{Top}^G}(q)$ is the space of all G-maps $Y(\diamond, q) \to X$. Here $(X^{Y(\diamond,q)})_{\mathrm{Top}^G}$ is a closed subset of $\prod_p X(p)^{Y(p,q)}$, which inherits the property of being compactly generated from $X(p)$, thus is in TOP^H as asserted.

We risk confusion by using X^Y in place of $(X^Y)_{\mathrm{Top}^G}$, and depend on the reader to supply the context.

Theorem 4.10 *Let X be in TOP^G, let Y be in $Top^{G\times H^o}$, and let Z be in $Top^{H\times K^o}$. We then have $Y\times_H Z$ in $Top^{G\times K^o}$, and from $TOP^G\times(Top^{G\times K^o})^o \to TOP^K$ get*

$$X^{Y\times_H Z}$$

in TOP^K. Alternatively, from $TOP^G \times (Top^{G\times H^o})^o \to TOP^H$ we get X^Y in TOP^H, whence from $TOP^H \times (Top^{H\times K^o})^o \to TOP^K$ we get

$$(X^Y)^Z$$

in TOP^K. There is a natural homeomorphism of K-spaces

$$X^{Y\times_H Z} \simeq (X^Y)^Z.$$

Proof. We assume the following about the basic mapping bifunctor $\mathrm{TOP} \times (\mathrm{Top})^o \to \mathrm{TOP}$.

(i) If M is closed in X, the natural map $M^Y \to X^Y$ is an inclusion map onto a closed subset.

(ii) From (1.15), if $\pi : Y \to Z$ is a quotient map, then $\pi^\# : X^Z \to X^Y$ is an inclusion map onto a closed subset.

(iii) $X^{\coprod_{q\in Q} Y(q)} \simeq \prod_{q\in Q} X^{Y(q)}$.

(iv) $(\prod_{p\in P} X(p))^Y \simeq \prod_{p\in P} X(p)^Y$.

(v) $\prod_{p,q} X(p,q)^{Y(p,q)} \simeq \prod_p(\prod_q X(p,q)^{Y(p,q)}) \simeq \prod_q(\prod_p X(p,q)^{Y(p,q)})$.

(vi) $X^{Y\times Z} \simeq (X^Y)^Z$.

Assuming these, the proof transforms the spaces $X^{Y\times_H Z}(r)$ and $[(X^Y)^Z](r)$ into closed subsets of the compactly generated space

$$\prod_{p,q} X(p)^{Y(p,q)\times Z(q,r)}.$$

From the quotient map

$$\coprod_q Y(p,q) \times Z(q,r) \to (Y \times_H Z)(p,r)$$

and from (ii), (iii), (v) we get an inclusion map onto a closed subset

$$X^{Y \times_H Z}(r) \hookrightarrow \prod_p \prod_q X(p)^{Y(p,q)\times Z(q,r)} \simeq \prod_{p,q} X(p)^{Y(p,q)\times Z(q,r)}.$$

From (i), (iv), (v), and (vi) we get a closed inclusion

$$[(X^Y)^Z](r) \hookrightarrow \prod_q (X^{Y(\diamond,q)})^{Z(q,r)} \hookrightarrow \prod_q (\prod_p X(p)^{Y(p,q)})^{Z(q,r)}$$

$$\simeq \prod_q \prod_p X(p)^{Y(p,q)\times Z(q,r)} \simeq \prod_{p,q} X(p,q)^{Y(p,q)\times Z(q,r)}.$$

It is seen that both of the images in

$$\prod_{p,q} X(p)^{Y(p,q)\times Z(q,r)}$$

consist of all

$$\phi_{p,q} : Y(p,q) \times Z(q,r) \to X(p)$$

such that if

$$p' \xleftarrow{g} p, \quad y \in Y(p,q), \quad z \in Z(q,r)$$

then $\phi_{p',q}(gy,z) = g\phi_{p,q}(y,z)$, and if

$$y \in Y(p,q'), \quad q' \xleftarrow{h} q, \quad z \in Z(q,r)$$

then $\phi_{p,q}(yh,z) = \phi_{p,q'}(y,hz)$. The theorem follows. $\square$

The Standard Homotopy Limit

Corollary 4.11 *Let X and Y be in TOP^G, and consider the mapping bifunctor $TOP^G \times (Top^G)^o \to TOP$. There is a homeomorphism $X^{E_G Y} \simeq (E^G X)^Y$ in TOP.*

Proof. From (2.15), we can take $E_G Y = EG \times_G Y$, while $E^G Y$ is defined to be X^{EG}, using the mapping bifunctor $\mathrm{TOP}^G \times (\mathrm{Top}^{G\times G^o})^o \to \mathrm{TOP}^G$. From (4.10) we get

$$X^{E_G Y} \simeq X^{EG\times_G Y} \simeq (X^{EG})^Y = (E^G X)^Y. \quad \square$$

We now use the fact that TOP^G and TOP^{G^o} have terminal objects. For example, TOP^G has as terminal object any G-space Z for which each $Z(p)$ is a singleton. The most natural such G-space Z has $Z(p) = \{p\}$, i.e. is $Z = Ob\ G$. So we use $Ob\ G$ to denote either a terminal G-space or a terminal G^o-space as well as the set of objects of G. If W is any G-space, its colimit can be taken to be $(Ob\ G) \times_G W$ where $Ob\ G$ denotes the terminal G^o-space, and its limit can be taken to be $(W^{Ob\ G})_{\mathrm{TOP}^G}$, where $Ob\ G$ denotes the terminal G-space. For any G^o-space W, similarly its colimit can be taken to be $W \times_G (Ob\ G)$.

The *standard universal G-space* E_G is $E_G Z$ where $Z = Ob\ G$ is the terminal G-space. Up to natural isomorphism, it is given by $E_G = EG \times_G (Ob\ G)$. Similarly, we take E_{G^o} to be the G^o-space $(Ob\ G) \times_G EG$. It is seen that E_G can be taken to consist of all

$$(g_0, g_1, \cdots, g_n) \times_\Delta (t_0, \cdots, t_n),$$

with $g((g_0, g_1, \cdots, g_n) \times_\Delta (t_0, \cdots, t_n)) = (gg_0, g_1, \cdots, g_n) \times_\Delta (t_0, \cdots, t_n)$, while E_{G^o} can be taken to consist of all

$$(g_1, \cdots, g_n, g_{n+1}) \times_\Delta (t_0, \cdots, t_n).$$

Denote by B_G the colimit of E_G and by B_{G^o} the colimit of E_{G^o}. From the associativity (1.21) of the reduced product, we have

$$B_G \simeq (Ob\ G) \times_G (EG \times_G (Ob\ G)) \simeq ((Ob\ G) \times_G EG) \times_G (Ob\ G) \simeq B_{G^o}.$$

In fact, we take for $B_G = B_{G^o}$ the space $|N(G)|$, thus the space whose points are of the form

$$(g_1, \cdots, g_n) \times_\Delta (t_0, \cdots, t_n).$$

The compactly generated space $B_G = B_{G^o}$ is called the *standard classifying space* of G or G^o.

We also summarize as follows.

(4.12) *For any small category G, we can take the standard classifying space B_G to be the Milnor realization $|NG|$ of the nerve of G. If p is an object of G, then E_G is given by taking $E_G(p)$ to be the classifying space $B_{G(p,\diamond)}$ of the category $G(p, \diamond)$ whose objects are the morphisms $p \xleftarrow{g} q$ of G, and with the morphisms $g' \leftarrow g$ of $G(p, \diamond)$ being all commutative diagrams*

$$\begin{tikzcd} r \arrow[d, "g'"'] & q \arrow[l, "g''"'] \arrow[d, "g"] \\ p \arrow[r, equal] & p \end{tikzcd}$$

in G.

For any G-space Y, the standard homotopy colimit $B_G Y$ of Y can be taken by (2.14) to be the colimit of $E_G Y$, thus

$$B_G Y \simeq (Ob\ G) \times_G E_G Y \simeq (Ob\ G) \times_G (EG \times_G Y) \simeq E_{G^o} \times_G Y,$$

and we have the standard model $E_{G^o} \times_G Y$ for $B_G Y$.

For any G-space X, define the *standard homotopy limit* $B^G X$ of X to be the limit of the G-space $E^G X$. Thus

$$B^G X = (E^G X)^{Ob\ G} \simeq (X^{EG})^{Ob\ G} \simeq X^{EG \times_G (Ob\ G)} \simeq X^{E_G}.$$

Thus the standard homotopy limit $B^G X$ can be taken to be the space of G-maps $E_G \to X$.

The Functor $\mathbf{TOP}^{Id\ G} \to \mathbf{TOP}^G$, $A \mapsto A^G$

The mapping bifunctor $\mathrm{TOP}^{Id\ G} \times (\mathrm{Top}^{(Id\ G)\times G^o})^o \to \mathrm{TOP}^G$ will be used. Recall that an object A in $\mathrm{TOP}^{Id\ G}$ is a collection $A = \{A(p)\}$ of compactly generated spaces. As a fixed $\mathrm{TOP}^{(Id\ G)\times G^o}$-space, take $\{G(p,q)\}$ where G acts on the right. The resulting G-space is then

$$(A^G)_{\mathrm{Top}^{Id\ G}}(q) = \prod_{p \in Ob\ G} A(p)^{G(p,q)},$$

together with its natural action of G. We denote this G-space by $(A^G)_{\mathrm{Top}^G}$, or by greater abuse of notation, by A^G. We then have the functor

$$\diamond^G : \mathrm{TOP}^{Id\ G} \to \mathrm{TOP}^G, \qquad A \mapsto A^G.$$

Consider (4.10) applied to A in $\mathrm{TOP}^{Id\ G}$, G in $\mathrm{TOP}^{(Id\ G)\times G^o}$, and Y in TOP^G. Then

$$A^{G\times_G Y} \simeq (A^G)^Y.$$

In the notation used above, $G \times_G Y$ results from the reduced product bifunctor

$$\mathrm{Top}^{(Id\ G^o)\times G} \times \mathrm{Top}^G \to \mathrm{Top}^{Id\ G}$$

and in fact the above $G \times_G Y$ is seen to be $i^\# Y$ where i denotes the inclusion $Id\ G \to G$. Thus the $(Id\ G)$-maps $i^\# Y \to A$ are naturally identified with the G-maps $Y \to A^G$.

A morphism $\phi : A \to A'$ in $\mathrm{TOP}^{Id\ G}$ is called a *fibration in* $\mathrm{TOP}^{Id\ G}$ if each $\phi_p : A(p) \to A'(p)$ is a fibration in TOP. If C is in $\mathrm{TOP}^{Id\ G}$ and if (A, B) is a closed cofibered pair in $\mathrm{TOP}^{Id\ G}$, then the natural map $C^A \to C^B$ is a fibration in $\mathrm{TOP}^{Id\ G}$.

Consideration of $(X, Y) \mapsto X^Y$ for X Fixed

Theorem 4.13 *Let $Y : H^o \to Top^{G\times G^o}$ be a functor, also interpreted as an H^o-diagram in $Top^{G\times G^o}$. Let colim Y denote the $G \times G^o$-space which is a colimit in $Top^{G\times G^o}$ of the H^o-diagram Y. For X fixed in TOP^G, there is the contravariant functor*

$$Top^{G\times G^o} \to TOP^G, \qquad W \mapsto X^W.$$

Thus we obtain the covariant functor $X^Y : H \to TOP^G$, which we interpret as an H-diagram in TOP^G. Then there is a natural homeomorphism

$$X^{colim\ Y} \simeq lim\ X^Y$$

in TOP^G.

Proof. From (4.10) we get

$$X^{colim\ Y} \simeq X^{Y\times_H (Ob\ H)} \simeq (X^Y)^{Ob\ H} \simeq lim\ X^Y. \quad \square$$

Thus for example if Y is the colimit in $\mathrm{Top}^{G\times G^o}$ of a diagram

$$Y_0 \to Y_1 \to \cdots \to Y_n \to \cdots,$$

then X^Y is the limit in TOP^G of the dual diagram

$$X^{Y_0} \leftarrow X^{Y_1} \leftarrow \cdots \leftarrow X^{Y_n} \leftarrow \cdots.$$

As a special case, it follows from (4.1) that EG is a filtered colimit $EG = \bigcup (EG)_n$ in $\mathrm{Top}^{G\times G^o}$, from which it follows that $E^G X = X^{EG}$ is the limit in TOP^G of the diagram

$$X^{(EG)_0} \leftarrow \cdots \leftarrow X^{(EG)_n} \leftarrow \cdots.$$

We must understand the inductive relationship between $X^{(EG)_{n-1}}$ and $X^{(EG)_n}$, as well as the nature of $X^{(EG)_0}$.

In order to do so, we recall first from (4.1) that there is a space A_0 in $\mathrm{TOP}^{Id\,(G\times G^o)}$ with

$$(EG)_0 = G \times_{Ob\,G} A_0 \times_{Ob\,G} G.$$

Then

$$X^{(EG)_0} \simeq (X^{G\times_{Ob\,G} A_0})^G \simeq ((i^\# X)^{A_0})^G.$$

If we let C_0 denote the $Id\,G$-space $(i^\# X)^{A_0}$, then we have $X^{(EG)_0}$ of the above form $(C_0)^G$ for some C_0 in $\mathrm{TOP}^{Id\,G}$.

It also follows from (4.1) that there is in $\mathrm{TOP}^{Id\,(G\times G^o)}$ a closed cofibered pair (A_n, B_n) such that $(EG)_n$ is the pushout in $\mathrm{TOP}^{G\times G^o}$ of a diagram

$$G \times_{Ob\,G} A_n \times_{Ob\,G} G \hookleftarrow G \times_{Ob\,G} B_n \times_{Ob\,G} G \to (EG)_{n-1}.$$

Hence $X^{(EG)_n}$ is the pullback of the diagram

$$((i^\# X)^{A_n})^G \to ((i^\# X)^{B_n})^G \leftarrow X^{(EG)_{n-1}}.$$

If we let $C_n = (i^\# X)^{A_n}$ and $D_n = (i^\# X)^{B_n}$ in $\mathrm{TOP}^{Id\,G}$, then we have that $X^{(EG)_n}$ is the pullback of a diagram

$$(C_n)^G \xrightarrow{(\pi_n)^G} (D_n)^G \leftarrow (EG)_{n-1}$$

in TOP^G. Moreover $\pi_n : C_n \to D_n$ is a fibration in $\mathrm{TOP}^{Id\,G}$, being the dual of a cofibration $B_n \hookrightarrow A_n$.

Coprincipal G-Spaces

Given X in TOP^G, then X is a *coprincipal G-space* if X is the limit in TOP^G of a diagram in TOP^G

$$X_0 \xleftarrow{\nu_1} X_1 \xleftarrow{\nu_2} \cdots \xleftarrow{\nu_n} X_n \xleftarrow{\nu_{n+1}} \cdots,$$

and

(i) there exists a space C_0 in $\mathrm{TOP}^{Id\,G}$ such that X_0 is homeomorphic in TOP^G to $(C_0)^G$,

(ii) for each $n > 0$ there exists a fibration $\pi_n : C_n \to D_n$ in $\mathrm{TOP}^{Id\,G}$ and a G-map $X_{n-1} \to (D_n)^G$ such that the diagram

$$(C_n)^G \xrightarrow{(\pi_n)^G} (D_n)^G \longleftarrow X_{n-1}$$

in TOP^G has pullback X_n.

(4.14) *For each G-space X in TOP, the G-space $E^G X$ is a coprincipal G-space.*

The proof was given above.

Theorem 4.15 *If $\phi : X \to Y$ is a weak homotopy equivalence in TOP^G, where X and Y are coprincipal G-spaces, then ϕ is a homotopy equivalence in TOP^G.*

Proof. Since X is coprincipal, we assume it the limit of a diagram

$$X_0 \xleftarrow{\nu_1} \cdots \xleftarrow{\nu_n} X_n \xleftarrow{\nu_{n+1}} \cdots$$

in TOP^G such that

(i) X_0 is G-homeomorphic to $(C_0)^G$ for some C_0 in $\mathrm{TOP}^{Id\,G}$,
(ii) for $n > 0$ there is a fibration $\pi_n : C_n \to D_n$ in $\mathrm{TOP}^{Id\,G}$ and a pullback diagram in TOP^G

$$\begin{array}{ccc} X_n & \xrightarrow{f'} & (C_n)^G \\ {\scriptstyle \nu_n}\big\downarrow & & \big\downarrow{\scriptstyle (\pi_n)^G} \\ X_{n-1} & \xrightarrow{f} & (D_n)^G. \end{array}$$

As in (4.3), one proceeds inductively. We assume a G-map $\theta_{n-1} : Y \to X_{n-1}$ together with a G-homotopy $H_{n-1} : I \times X \to X_{n-1}$ which joins $\theta_{n-1}\phi$ to the G-map $\mu_{n-1} : X \to X_{n-1}$ which comes with the limit structure of X.

In the inductive step, one uses the diagram

$$\begin{array}{ccccccc} X(p) & \xrightarrow{\mu_n} & X_n(p) & \xrightarrow{f'} & (C_n)^G(p) & \xrightarrow{\psi'} & C_n(p) \\ {\scriptstyle \phi_p}\big\downarrow & & {\scriptstyle \nu_n}\big\downarrow & & {\scriptstyle (\pi_n)^G}\big\downarrow & & {\scriptstyle \pi_n}\big\downarrow \\ Y(p) & \xrightarrow{\theta_{n-1}} & X_{n-1}(p) & \xrightarrow{f} & (D_n)^G(p) & \xrightarrow{\psi} & D_n(p) \end{array}$$

in its shortened form

$$\begin{array}{ccc} X(p) & \xrightarrow{\psi' f' \mu_n} & C_n(p) \\ {\scriptstyle \phi_p}\big\downarrow & & {\scriptstyle \pi_n}\big\downarrow \\ Y(p) & \xrightarrow{\psi f \theta_{n-1}} & D_n(p). \end{array}$$

The homotopy $H_{n-1} : I \times X(p) \to X_{n-1}(p)$ furnishes a homotopy commutativity for the diagram as the map $H_0 = \psi f H_{n-1}$.

Since ϕ_p is a homotopy equivalence and π_n is a fibration, we can apply (3.8). Thus there exists a map $\theta' : Y(p) \to C_n(p)$ and a homotopy $H' : I \times X(p) \to C_n(p)$ such that

(i) H' joins $\theta'\phi_p$ to $\psi' f' \mu_n$,
(ii) $\pi_n \theta' = \psi f \theta_{n-1}$,
(iii) $\pi_n H'$ extends H_0.

Thus we have an Id G-map $\theta' : Y \to C_n$ and an Id G-map $H' : I \times X \to C_n$. There are the associated unique G-maps $\theta : Y \to (C_n)^G$ and $H : I \times X \to (C_n)^G$. The commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\theta} & (C_n)^G \\ {\scriptstyle \theta_{n-1}} \big\downarrow & & \big\downarrow {\scriptstyle (\pi_n)^G} \\ X_{n-1} & \xrightarrow{f} & (D_n)^G \end{array}$$

gives $\theta_n : Y \to X_n$ by the pullback property. Similarly one gets the homotopy $H_n : I \times X \to X_n$. The induction is established, and the theorem follows readily. $\square$

Corollary 4.16 *Consider the category TOP^G, together with the notion of homotopy in TOP^G and of weak homotopy equivalence in TOP^G given in Chapter 3. Take also the functor*

$$E^G(\diamond) : TOP^G \to TOP^G$$

given in Chapter 2, together with the natural transformation $T' : 1 \to E^G(\diamond)$. Let the full subcategory of coprincipal objects be the full subcategory $COPRINC^G$ of coprincipal G-spaces. With this given structure, TOP^G is a category with coprincipal objects in the sense of Chapter 3.

The Homotopy Limits $B'X$ of a G-Space X

A *coprincipalization* of a G-space X in TOP is a pair consisting of a coprincipal G-space $E'X$ and a homotopy class of G-maps $\phi : X \to E'X$ each representative of which is a weak homotopy equivalence in TOP^G. Given two coprincipalizations $[\phi_0] : X \to E'_0X$ and $[\phi_1] : X \to E'_1X$, there exists a unique homotopy class of G-maps $\theta : E'_0X \to E'_1X$ such that

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle \phi_0} \big\downarrow & & \big\downarrow {\scriptstyle \phi_1} \\ E'_0X & \xrightarrow{\theta} & E'_1X \end{array}$$

is homotopy commutative in TOP^G, and each θ is a homotopy equivalence in TOP^G. This follows from (3.13).

For any G-space X, the above G-space E^GX is called the *standard coprincipalization* of X.

A *homotopy limit* $B'X = holimX$ of a G-space X in TOP is the pair $(E'X, \phi)$ consisting of a compactly generated space $B'X$ and a homotopy equivalence $\phi : B'X \to B^GX$ in TOP.

Given a coprincipalization $(E'X, \phi)$ of the G-space X, then the limit $B'X$ of the G-space $E'X$ is a homotopy colimit of X.

Define a (not necessarily standard) *universal* G-space E to be a principal G-space E such that each $E(p)$ is contractible. By (4.4), any two such are joined by a unique homotopy class of homotopy equivalences in TOP^G. In particular, E is joined to E_G by a unique homotopy class of homotopy equivalences in TOP^G. Thus the colimit E/G is naturally homotopy equivalent to B_G. We will call $B = E/G$ a (nonstandard) *classifying space* of G.

(4.17) *Let X be a G-space in TOP and let E' be a universal G^o-space. Then $E' \times_G X$ is a homotopy colimit of X. Let E be a universal G-space. Then the space $(X^E)_{TOP^G}$ is a homotopy limit of X.*

The proof is clear.

Coprincipal G-Spaces for the Category $0 \to 1$

For G the category $0 \to 1$, then $Id\ G$ can be considered as the set $\{0, 1\}$ with two objects, hence an object C of $\mathrm{TOP}^{Id\ G}$ is an ordered pair $C(0), C(1)$ of compactly generated spaces. Interpreting the objects of TOP^G as maps $\pi : X \to Y$, one must first interpret C^G. It is checked to be the projection map $C(0) \times C(1) \to C(1)$.

Let X be a coprincipal G-space, thus the limit in TOP^G of a diagram

$$X_0 \xleftarrow{\nu_1} \cdots \xleftarrow{\nu_n} X_n \xleftarrow{\nu_{n+1}} \cdots$$

such that

(i) X_0 is of the form $(C_0)^G$ for C_0 in $\mathrm{TOP}^{Id\ G}$,
(ii) for $n > 0$, X_n is determined by a pullback diagram

$$\begin{array}{ccc} X_n & \longrightarrow & (C_n)^G \\ {\scriptstyle \nu_n}\big\downarrow & & \big\downarrow{\scriptstyle (\pi_n)^G} \\ X_{n-1} & \longrightarrow & (D_n)^G \end{array}$$

in TOP^G for some fibration $\pi_n : C_n \to D_n$ in $\mathrm{TOP}^{Id\ G}$.

Inductively, if $X_{n-1}(0) \to X_{n-1}(1)$ is known, then $X_n(0) \to X_n(1)$ comes from the diagram below.

$$
\begin{tikzcd}
X_n(0) \arrow[rr] \arrow[dr] \arrow[dd] & & C_n(0) \times C_n(1) \arrow[dr] \arrow[dd] & \\
& X_{n-1}(0) \arrow[rr] \arrow[dd] & & D_n(0) \times D_n(1) \arrow[dd] \\
X_n(1) \arrow[rr] \arrow[dr] & & C_n(1) \arrow[dr] & \\
& X_{n-1}(1)) \arrow[rr] & & D_n(1)
\end{tikzcd}
$$

The top and bottom faces are pullback diagrams, the vertical maps of the right face are projections, and the horizontal maps of the right face are fibrations.

If $\pi : E \to B$ is a fibration in TOP, one can build a coprincipal G-space X which, when interpreted as simply a map, gives π. For example, take X_0 to be the identity map $B \to B$, and let X_1 be given by the choices represented in the diagram below.

$$
\begin{tikzcd}
E \arrow[rr, "1 \times \pi"] \arrow[dr, "\pi"] \arrow[dd, "\pi"'] & & E \times B \arrow[dr, "\pi \times 1"] \arrow[dd, "proj"] & \\
& B \arrow[rr, "diag"] \arrow[dd, "="'] & & B \times B \arrow[dd, "proj"] \\
B \arrow[rr, "="] \arrow[dr, "="'] & & B \arrow[dr, "="] & \\
& B \arrow[rr, "="] & & B
\end{tikzcd}
$$

Thus every fibration in TOP can be regarded as a coprincipal G-space for G the category $0 \to 1$.

The converse is also true.

Theorem 4.18 *For G the category $0 \to 1$, a G-space of the form $\pi : X \to Y$ is a coprincipal G-space if and only if π is a fibration in TOP.*

We do not bother to write out the details. The proof is based on an induction, as we would do it not precisely dual to that given for (4.8). The interested reader should be able to construct the inductive proof from the diagram.

$$\begin{tikzcd}
0 \times A \arrow[r] \arrow[dd] & X_n(0) \arrow[rr] \arrow[dr] \arrow[dd] & & C_n(0) \times C_n(1) \arrow[r] \arrow[dr] \arrow[dd] & C_n(0) \arrow[dr] & \\
& & X_{n-1}(0) \arrow[rr] \arrow[dd] & & D_n(0) \times D_n(1) \arrow[r] \arrow[dd] & D_n(0) \\
I \times A \arrow[r] & X_n(1) \arrow[rr] \arrow[dr] & & C_n(1) \arrow[dr] & & \\
& & X_{n-1}(1)) \arrow[rr] & & D_n(1) &
\end{tikzcd}$$

We regard this work as having up to this stage concentrated on giving a very basic model for equivariant general topology, that for arbitrary actions of any untopologized small category on compactly generated spaces, where the actions are assumed to be associative and have identities in the strictest sense. The major emphasis of this model is on homotopy colimits and homotopy limits. Topologists have used to advantage various models: relax somewhat the strict requirements on associativity and identities of the action, put topology on G, etc. We save any remarks on these until later chapters.

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CHAPTER V

The Construction of Principal G-Spaces

The basic tool of this chapter assigns to each functor $\theta : H \to G$ the *extension functor* $\theta_{\#} : \mathrm{Top}^H \to \mathrm{Top}^G$ which generalizes the construction of Chapter 1. This functor carries principal H-spaces into principal G-spaces.

For most small categories G the only principal G-spaces X that we thus far noted are the standard principal G-spaces $E_G X$ of Chapters 2 and 4, one for each X in TOP^G. Thus the only model we have noted for the homotopy colimit of a G-space X is the standard model $B_G X$. We cannot explore the richness of our subject with this handicap. In later chapters, nonstandard principal G-spaces will have to be developed for some of the small categories G at the core of topology. Some general principles for such constructions are available, and in this chapter we collect some of the theory that we will use as a guide in constructing nonstandard models case by case. This is collected from work of Steenrod, as given by Cooke-Finney [5.2], Cartan [5.1], Grothendieck [5.4], Gray [5.3], and Quillen [5.5].

Before doing so, we develop the capability for the systematic use of cell complexes so that we will not be confined as we are at this point to those complexes whose open cells and face relations are always modeled on open simplices. For this we need Steenrod's regular complexes with identifications, and in a categorical form. Steenrod has done most of the work for us, so we have only to reformulate it. There will then be at hand what we call the *cellular categories* Ψ. Every such Ψ has E_Ψ a regular cell complex and there is a family F of identifications on E_Ψ with $E_\Psi / F = B_\Psi$ through the natural quotient map

$$\pi : E_\Psi \to B_\Psi.$$

Then B_Ψ is a regular cell complex with identifications in the sense of Steenrod, with an open cell $D(p)$ for each object p of Ψ and with $D(q)$ a face of $D(p)$ for each morphism $q \to p$ of Ψ. For such categories, the homotopy colimit of any Ψ^o-space X is already well presented in its form as $X \times_\Psi E_\Psi$ and there is rarely genuine need for nonstandard models for them.

We next translate certain constructions of Cartan, as they occur in his 1954-1955 Cartan Notes [5.1], into topological terms. In topological terms, the goal is a codified presentation of particularly interesting principal G-spaces for the important small categories G. One gets for each cellular category Ψ and each functor $\theta : \Psi \to G$ the principal G-space $\theta_\# E_\Psi$, where

$$\theta_\# : \mathrm{Top}^\Psi \to \mathrm{Top}^G$$

denotes the extension functor generated by θ. As a rule of thumb, many geometrically interesting principal G-spaces can be presented in the form $\theta_\# E_\Psi$ where Ψ is a cellular category, E_Ψ is the standard universal Ψ-space, and $\theta : \Psi \to G$ is some functor. Here we have followed similar constructions of Quillen [5.5].

One can describe the above as a variant of constructions of Gray [5.3]. Namely, given a small category G there is the category CAT$/G$ of small categories over G. We then construct a functor

$$\mathrm{CAT}/G \to \mathrm{PRINC}^G,$$

made out of a functor

$$\mathrm{CAT}/G \to \mathrm{CAT}^G$$

of Grothendieck [5.4] and Gray [5.3].

As another rule of thumb, for each G one can present geometrically interesting nonstandard universal G-spaces E in the form $E = \theta_\# E_\Psi$. This is a topological form of Cartan's acyclic constructions, and is closely related to Quillen's Theorem A [5.5]. We call a *topological resolution* of the small category G a pair consisting of a cellular category Ψ and a functor $\theta : \Psi \to G$ such that $\theta_\# E_\Psi$ is a universal G-space. Every $\theta_\# E_\Psi$ is automatically a principal G-space and one here confines attention to those for which in addition $(\theta_\# E_\Psi)(p)$ is a contractible space for each object p of G. It is then automatically the case that $\theta_* : B_\Psi \to B_G$ is a homotopy equivalence. In fact, for every G-space X, it is then the case that $\theta^\# X \times_\Psi E_\Psi$ is a nonstandard model for the homotopy colimit of the G-space X. For those G which are not cellular, we can seek cellular categories Ψ and functors $\Psi \to G$ such that $\theta_\# E_\Psi$ is a topological resolution of G, and thus can be used as a replacement for the often difficult standard model E_G.

The only nontrivial topological resolution of a small category that we compute in this chapter is one for Δ, taken from Segal [4.4]. Let $i : Mono\,\Delta \hookrightarrow \Delta$ denote the inclusion of the subcategory $Mono\,\Delta$ of monos into Δ. Then we show that $i : Mono\,\Delta \to \Delta$ is a topological resolution of Δ. Thus in particular $i_\# E_{Mono\,\Delta}$ is a (nonstandard) universal space for Δ, and we obtain Segal's form

$$(i^\# X) \times_{Mono\,\Delta} \nabla$$

for the homotopy colimit of any Δ^o-space X in TOP. We take some care in exhibiting it clearly, since in the next chapter we have to thoroughly understand its properties. Our treatment requires joins of categories, and classifying spaces of such joins.

We are endebted to C.H. Giffin for much help in our efforts to understand this material.

Cellular Categories: the Poset Conditions

A small category Ψ will be called a cellular category if it satisfies three conditions, which we develop slowly.

Condition 1 *Given an object p of Ψ there are only a finite number of morphisms of Ψ of the form $\psi : q \to p$.*

It follows immediately from Condition 1 that for each object p there are only a finite number of diagrams of the form

$$r \xrightarrow{\psi'} q \xrightarrow{\psi} p.$$

Recall that we denote by $\Psi(p, \diamond)$ the category whose objects are the morphisms $q \xrightarrow{\psi} p$ of Ψ, with the morphisms $\psi \to \psi'$ in $\Psi(p, \diamond)$ being all commutative diagrams

$$\begin{tikzcd} q \arrow[r, "\psi''"] \arrow[d, "\psi"'] & r \arrow[d, "\psi'"'] \\ p \arrow[r, equal] & p \end{tikzcd}$$

in Ψ. Thus it follows from Condition 1 that each category $\Psi(p, \diamond)$ has only a finite number of objects and morphisms.

Recall from (4.12) that E_Ψ is determined by $E_\Psi(p) = B_{\Psi(p,\diamond)}$ for each object p of Ψ. We seek in the next two conditions increasingly to restrict each category $\Psi(p, \diamond)$, and thereby each $E_\Psi(p)$ and E_Ψ.

We need a prototype example. For this we pick $Mono\ \Delta$, the category whose objects are the nonnegative integers and whose morphisms $\delta : m \to n$ are the order preserving monos

$$\delta : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}.$$

Here the category $(Mono\ \Delta)(n, \diamond)$ can be equated with the category whose objects are the finite nonempty subsets of $\{0, \cdots, n\}$ and whose morphisms are inclusions. In particular, the objects of $(Mono\ \Delta)(n, \diamond)$ are naturally partially ordered, with the structure of the category that given by the partial ordering. This is the direction we go in Condition 2.

Condition 2 *The small category Ψ satisfies the following:*

(i) given morphisms $p \xrightarrow{\psi} q$ and $q \xrightarrow{\psi'} p$ in Ψ, then $p = q$ and $\psi = \psi' = 1_p$;

(ii) given a diagram $q \xrightarrow{\psi} p \xleftarrow{\psi'} r$ in Ψ, then there exists at most one morphism $\psi'' : q \to r$ in Ψ such that

$$\begin{tikzcd} q \arrow[r, "\psi''"] \arrow[d, "\psi"'] & r \arrow[d, "\psi'"'] \\ p \arrow[r, equal] & p \end{tikzcd}$$

commutes.

Now for each object p of Ψ, consider the set

$$\{\psi \in Mor\ \Psi \mid \psi : \diamond \to p\}$$

and on this set define $\psi \leq \psi'$ if and only if there exists a morphism ψ'' in Ψ such that $\psi = \psi'\psi''$. Then the reader should check that Condition 2 is equivalent to the condition that $\leq$ is a partial ordering on the above set for each object p of Ψ, together with the condition that $\Psi\ (p, \diamond)$ is the category associated with this poset. Note that our prototype $Mono\ \Delta$ satisfies Condition 2.

Thus far we have required that each $\Psi(p, \diamond)$ be the category associated with a finite poset. We now examine the benefits of this requirement. In order to do so, we need the concept of a *simplicial complex*.

Let V be a given set. Denote by $R(V)$ the real vector space generated by V; i.e. the vector space of all functions $t : V \to R$ such that $t(v) \neq 0$ for only a finite number of $v \in V$. Give $R(V)$ the topology induced by its finite dimensional subspaces. The *simplex* $\nabla(V)$ is then all $t \in R(V)$ with $t(v) \geq 0$ for all v and with

$$\sum_{v \in V} t(v) = 1.$$

A *finite dimensional subsimplex* is then a subset $\nabla\{v_0, v_1, \cdots, v_n\}$ for some nonempty finite subset $\{v_0, \cdots, v_n\}$ of V, consisting of all $t \in \nabla(V)$ with $t(v) \neq 0$ implying $v = v_i$ for some i. A *standard simplicial complex* is a subspace of some $\nabla(V)$ consisting of a union of finite dimensional subsimplices of it; its simplices are all $\nabla\{v_0, \cdots, v_n\}$ contained in it. A *simplicial complex* is a space A together with a homeomorphism h of some standard simplicial complex B onto A. The *simplices* of A are all images under h of simplices of B.

(5.1) *If G is a small category derived from a partial ordering on $Ob\, G$, then B_G is naturally a simplicial complex, with an n-simplex for every*

$$s_0 > s_1 > \cdots > s_n$$

in $Ob\, G$, equivalently for every diagram in G of the form

$$s_0 \xleftarrow{g_1} s_1 \xleftarrow{g_2} \cdots \xleftarrow{g_n} s_n$$

where no g_i is an identity morphism. The simplex consists of all

$$(g_1, \cdots, g_n) \times_\Delta (t_0, t_1, \cdots, t_n),$$

where $(t_0, \cdots, t_n) \in \nabla(n)$.

Proof. One has to first examine the nerve $X = NG$ in SET^{Δ^o}. The nerve has a nondegenerate element for each diagram

$$s_0 \xleftarrow{g_1} s_1 \xleftarrow{g_2} \cdots \xleftarrow{g_n} s_n$$

for which each g_i is a nonidentity morphism. Note that the s_i then determine the g_j. Each face of a nondegenerate element is therefore a nondegenerate element. In $|NG|$, we therefore get the points

$$(g_1, g_2, \cdots, g_n) \times_\Delta (t_0, t_1, \cdots, t_n).$$

Now take $\nabla(Ob\ G)$ and in it take the standard simplicial complex B which is the union of all $\nabla\{s_0, \cdots, s_n\}$ for all

$$s_0 > s_1 > \cdots > s_n.$$

Let $h : B \to B_G$ map a function $t : \nabla(Ob\ G) \to R$ in $\Delta\{s_0, \cdots, s_n\}$ with

$$t(s_i) \geq 0, \quad t(s) \neq 0 \text{ for } s \neq s_i, \quad \sum t(s_i) = 1$$

into

$$(g_1, \cdots, g_n) \times_\Delta (t(s_0), \cdots, t(s_n)).$$

The remark follows. $\square$

We return now to a small category Ψ which satisfies Conditions 1 and 2. Fix an object p of Ψ and consider the category $\Psi(p, \diamond)$. Then $E_\Psi(p) = B_{\Psi(p,\diamond)}$ is a finite simplicial complex which has an n-simplex consisting of all

$$(\psi_0, \psi_1, \cdots, \psi_n) \times_\Delta (t_0, t_1, \cdots, t_n)$$

for each diagram

$$p \xleftarrow{\psi_0} p_0 \xleftarrow{\psi_1} \cdots \xleftarrow{\psi_n} p_n$$

in Ψ with no ψ_i an identity morphism for $i > 0$. Denote this n-simplex in $E_\Psi(p)$ by

$$[\psi_0, \psi_1, \cdots, \psi_n].$$

Given a morphism $\psi : q \to p$ in Ψ, there is the induced map $\psi_* : E_\Psi(q) \to E_\Psi(p)$ and ψ_* maps each n-simplex

$$[\psi_0, \psi_1, \cdots, \psi_n]$$

homeomorphically onto the n-simplex

$$[\psi\psi_0, \psi_1, \cdots, \psi_n]$$

of $E_\Psi(p)$.

One can now check from Condition 2 that each $\psi : q \to p$ is a monomorphism in Ψ. That having been done, one can next check that each $\psi_* : E_\Psi(q) \to E_\Psi(p)$ is an inclusion map onto a subcomplex of the finite simplicial complex $E_\Psi(p)$.

For each p, let $E_\Psi^{deg}(p)$ denote the union of the images of all the

$$\psi_* : E_\Psi(q) \to E_\Psi(p)$$

corresponding to $\psi : q \to p$ with $q \neq p$. Then $E_\Psi^{deg}(p)$ consists of all simplices

$$[\psi_0, \psi_1, \cdots, \psi_n]$$

for which all ψ_i including ψ_0 are not identities. Given such a simplex, one can form the $(n+1)$-simplex

$$[1_p, \psi_0, \cdots, \psi_n]$$

in $E_\Psi(p)$ and thus check that $E_\Psi(p)$ is naturally the simplicial cone over $E_\Psi^{deg}(p)$.

Continue to assume that Ψ is a small category satisfying Conditions 1 and 2. We next consider for a given diagram $q \xrightarrow{\psi} p \xleftarrow{\psi'} r$ in Ψ the intersection

$$\psi_*(E_\Psi(q)) \cap \psi'_*(E_\Psi(r)).$$

Let $x = \psi_* y = \psi'_* z$ where $y \in E_\Psi(q)$ and $z \in E_\Psi(r)$. There is the unique representation

$$y = (\psi_0, \psi_1, \cdots, \psi_m) \times_\Delta (t_0, \cdots, t_m),$$

where no ψ_i is an identity morphism for $i > 0$ and where $t_i > 0$ for all i. There is the unique representation

$$z = (\psi'_0, \psi'_1, \cdots, \psi'_n) \times_\Delta (u_0, \cdots, u_n),$$

where no ψ'_j is an identity for $j > 0$ and where $u_j > 0$ for all j. Then

$$x = (\psi\psi_0, \psi_1, \cdots, \psi_m) \times_\Delta (t_0, \cdots, t_m),$$

$$x = (\psi'\psi'_0, \psi'_1, \cdots, \psi'_n) \times_\Delta (u_0, \cdots, u_n).$$

It follows from uniqueness that $m = n$, $\psi_i = \psi'_i$ for $i > 0$, $t_i = u_i$ for all i, and that $\psi\psi_0 = \psi'\psi'_0$. Thus $\psi_*(E_\Psi(q)) \cap \psi'_*(E_\Psi(r))$ is the union of all the simplices

$$[\psi\psi_0, \psi_1, \cdots, \psi_n] = [\psi'\psi'_0, \psi_1, \cdots, \psi_n]$$

over all the commutative diagrams

$$\begin{array}{ccc} s & \xrightarrow{\psi_0} & q \\ {\scriptstyle\psi'_0}\big\downarrow & & {\scriptstyle\psi}\big\downarrow \\ r & \xrightarrow{\psi'} & p. \end{array}$$

Thus either $\psi_*(E_\Psi(q)) \cap \psi'_*(E_\Psi(r)) = \emptyset$, or else for any

$$x \in \psi_*(E_\Psi(q)) \cap \psi'_*(E_\Psi(r))$$

and $y \in E_\Psi(q)$, $z \in E_\Psi(r)$ with $\psi_* y = x = \psi'_* z$, there exist a commutative diagram

$$\begin{array}{ccc} s & \xrightarrow{\psi_0} & q \\ {\scriptstyle\psi'_0}\big\downarrow & & {\scriptstyle\psi}\big\downarrow \\ r & \xrightarrow{\psi'} & p \end{array}$$

and $w \in E_\Psi(s)$ such that $\psi_{0*} w = y$ and $\psi'_{0*} w = z$.

Cellular Categories: the Combinatorial Condition

Conditions 1 and 2 are weaker than the final condition 3, which we first check in our prototype example and then state. This condition requires the language of combinatorial topology, for which our reference is the Tata Lectures of Stallings [5.6].

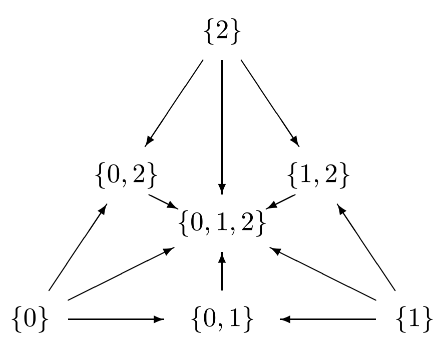
So return to the special case $\Psi = Mono\ \Delta$ for a moment. We observed that $(Mono\ \Delta)(n, \diamond)$ can be taken to be all nonempty subsets of

$$\{0, 1, \cdots, n\}$$

under inclusion, but we didn't compute the resulting finite simplicial complex, which has a vertex for each such subset S and a k-simplex for each strictly decreasing sequence

$$S_0 \supset S_1 \supset \cdots \supset S_k$$

of such subsets. One may as well take as a model for such a finite simplicial complex the barycentric subdivision $Sd\ \nabla(n)$ of the standard n-simplex. Thus we take $E_{Mono\ \Delta}(n) = Sd\ \nabla(n)$ and $E^{deg}_{Mono\ \Delta}(n)$ the barycentric subdivision of $\partial\nabla(n)$. This is the route we go in Condition 3.



$$E_{Mono\Delta}(2) = B_{Mono\Delta}(2, \diamond) \simeq Sd\ \nabla(2)$$

Condition 3 *For each object p of Ψ, the finite simplicial complex $E_\Psi(p)$ is combinatorially equivalent to a combinatorial cell C of some dimension $d(p)$, in such a way that the subcomplex $E^{deg}_\Psi(p)$ corresponds to the boundary sphere ∂C of C.*

If Conditions 1-3 are satisfied, we say that Ψ is a *cellular category.*

In our example $Mono\ \Delta$, note that if $i : Mono\ \Delta \to \Delta$ is inclusion, then simplicially we have $E_{Mono\ \Delta} = Sd\ i^{\#}\nabla$ and ignoring simplicial structure $E_{Mono\ \Delta} \simeq i^{\#}\nabla$.

The Standard Classifying Space of a Cellular Category

We first summarize the topological structure of E_Ψ.

(5.2) *Let Ψ be a cellular category. Denote by E_Ψ the disjoint union*

$$E_\Psi = \coprod_{p \in Ob\ \Psi} E_\Psi(p).$$

For each $\psi : q \to p$, let

$$C(\psi) = \psi_*(E_\Psi(q) - E_\Psi^{deg}(q)) \subset E_\Psi(p) \subset E_\Psi.$$

Then the $C(\psi)$ are pairwise disjoint and cover E_Ψ. There is the natural filtration $E_\Psi = \bigcup E_n$, where E_n is the union of all $C(\psi)$ where $\psi : q \to p$ and $d(q) \leq n$. Then E_Ψ is a CW-complex, with the above filtration, such that for each cell C there is an attaching map

$$f : (\nabla(n), \partial\nabla(n)) \to (\overline{C}, \overline{C} - C)$$

with $f : \nabla(n) \to \overline{C}$ a homeomorphism. With the simplicial structure taken into account, f can be taken to be a combinatorial equivalence.

The proof is an exercise.

Let $\pi : E_\Psi \to B_\Psi$ denote the natural quotient map, and $\sim$ the equivalence relation on E_Ψ for which $B_\Psi = E_\Psi / \sim$.

(5.3) *Let Ψ be a cellular category. If $x \in E_\Psi(q)$ and $y \in E_\Psi(r)$, then $x \sim y$ if and only if there exists a diagram*

$$q \xleftarrow{\omega} s \xrightarrow{\omega'} r$$

and an element $z \in E_\Psi(s)$ with $x = \omega_* z$ and $y = \omega'_* z$.

Proof. Let $\sim'$ be the relation described above. We must show that $\sim'$ is an equivalence relation. Suppose $x \in E_\Psi(p)$, $x' \in E_\Psi(p')$, and $x'' \in E_\Psi(p'')$ have $x \sim' x'$ and $x' \sim' x''$. There exist $p \xleftarrow{\psi} q \xrightarrow{\psi'} p'$ and $y \in E_\Psi(q)$ with $\psi_* y = x$ and $\psi'_* y = x'$. Moreover, there exist $p' \xleftarrow{\psi_0} r \xrightarrow{\psi'_0} p''$ and $z \in E_\Psi(r)$ with $\psi_{0*} z = x'$ and $\psi'_{0*} z = x''$. Since $q \xrightarrow{\psi'} p' \xleftarrow{\psi_0} r$ has $\psi'_* y = x' = \psi_{0*} z$, there exist a commutative diagram

$$\begin{array}{ccc} s & \xrightarrow{\omega} & q \\ {\scriptstyle \omega_0}\downarrow & & \downarrow{\scriptstyle \psi'} \\ r & \xrightarrow{\psi_0} & p' \end{array}$$

and $w \in E_\Psi(s)$ with $\omega_* w = y$ and $\omega_{0*} w = z$. One then has $p \xleftarrow{\psi\omega} s \xrightarrow{\psi'_0 \omega_0} p''$ with

$$\psi_* \omega_* w = x, \qquad \psi'_{0*} w = x'',$$

hence $x \sim' x''$. Thus $\sim'$ is an equivalence relation. It is then easy to see that $\sim = \sim'$. $\square$

(5.4) *Let Ψ be a cellular category. For each object p of Ψ, define $D(p) \subset B_\Psi$ by $D(p) = \pi(C(1_p))$ where*

$$C(1_p) = E_\Psi(p) - E_\Psi^{deg}(p).$$

Then the $D(p)$ are pairwise disjoint and fill up B_Ψ. Let $B_\Psi = \bigcup B_n$ where B_n is the union of all $D(p)$ for which $d(p) \leq n$. This is a filtration of B_Ψ, and $\pi : E_\Psi \to B_\Psi$ is a homeomorphism of each $C(\psi)$ onto $D(1_q)$ for each $\psi : q \to p$. Thus B_Ψ is a CW-complex and $\pi : E_\Psi \to B_\Psi$ is a CW-map which maps open cells homeomorphically onto open cells.

The proof is an exercise.

We should reexamine our prototype $Mono\ \Delta$. Note that as an example of (5.2), $E_{Mono\ \Delta}$ is a CW-complex whose cells are the various faces of the various simplices $\nabla(n)$. The barycentric subdivision is implicit only. The standard classifying space $B_{Mono\ \Delta}$ is given by

$$B_{Mono\ \Delta} = (\coprod \nabla(n))/\sim,$$

where $\sim$ is the least equivalence relation such that

$$(t_0, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_n) \sim (t_0, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n).$$

Thus $B_{Mono\ \Delta}$ is the *infinite dimensional dunce hat D* of Zeeman, with points

$$[t_0, t_1, \cdots, t_n]$$

where $(t_0, \cdots, t_n) \in \nabla(n)$ for some n and where any $t_i = 0$ can be deleted. As an example of (5.4), then $D = B_{Mono\ \Delta}$ has exactly one open cell $D(n)$ in each dimension, the set of

$$[t_0, t_1, \cdots, t_n]$$

for which all $t_i > 0$. The space D is contractible, as one can see by computing its fundamental group and its homology groups.

In summary, if Ψ is a cellular category then B_Ψ is a CW-complex with an open cell $D(p)$ of dimension $d(p)$ for each object p of Ψ, and $D(q)$ is a face of $D(p)$ for each morphism $\psi : q \to p$. This is really a matter of regular cell complexes with identification, as we now see.

Steenrod's Regular Complexes with Identifications

Steenrod in his lectures on cell complexes [5.2] defined a *regular complex* to be a CW-complex $K = \bigcup K_n$ such that for each open cell C there exists a homeomorphism $f : \nabla(n) \to \overline{C}$ mapping $\partial\nabla(n)$ onto $\overline{C} - C$. Thus (5.2) asserts that E_Ψ is a regular complex.

Steenrod goes on to define a *family F of identifications*. Here F is a category whose objects are the cells of K and whose morphisms $f : C \to C'$ are homeomorphisms $f : \overline{C} \simeq \overline{C'}$ mapping $\overline{C} - C$ onto $\overline{C'} - C'$, and such that

(i) compositions in F are compositions of maps, and identities in F are identity maps,
(ii) if f is in the family then so is f^{-1},
(iii) if $f : C \to C$ is in the family then f is the identity, and
(iv) if $f : C \to C'$ is in F and if D is a cell of K with $D \subset \overline{C}$, then the restriction $f|D : D \to fD$ is in F.

Then he defined a *regular complex with identifications* to be an ordered pair (K, F), where K is a regular complex and F is a family of identifications on K. Given (K, F), he defined a space K/F which is the space of the regular complex with identifications.

He also showed that if K is a regular complex, then there is a naturally induced regular complex $Sd\ K$, which is a simplicial complex, and a homeomorphism $Sd\,K \simeq K$.

We now change slightly Steenrod's regular complexes and hereafter assume this change made. Given a cell C of K, we add that there must be a homeomorphism $f : \nabla(n) \to \overline{C}$ which is a combinatorial equivalence when the simplicial subdivision is taken into account.

We can now state precisely the way in which we have translated Steenrod.

Theorem 5.5 *Let Ψ be a cellular category. Then $K = E_\Psi$ as given by (5.2) is a regular cell complex. For each diagram $p \xleftarrow{\psi} r \xrightarrow{\psi'} q$ in Ψ, let*

$$g : \overline{C(\psi)} \to \overline{C(\psi')}$$

be defined by $g(\psi x) = \psi' x$. Then the family F of all such g is a family of identifications, and $K/F \simeq B_\Psi$.

Conversely, suppose (K, F) is a regular complex with identifications. We first define a category Ψ. An object p of Ψ is an equivalence class of cells of K under the action of F, and we write $p = [C]$ if C is a representative of p. Consider the set of all ordered pairs (C, D) of cells such that $\overline{C} \supset D$, and on this set define $\sim$ to be the equivalence relation such that if we have (C, D) and an element $g : C \to C' = gC$ of F then

$$(C, D) \sim (gC, gD).$$

Define a morphism of Ψ to be an equivalence class of such pairs. If (C, D) represents a morphism, write the morphism as

$$\psi_{C,D} : [D] \to [C],$$

and define composition by $\psi_{C,D}\psi_{D,E} = \psi_{C,E}$ whenever $\overline{C} \supset D$ and $\overline{D} \supset E$. Then Ψ is a cellular category.

Proof. We consider the second part, thus fix the regular complex (K, F) with identifications. Suppose the category Ψ has been defined as above. Fix an object $[C]$ of Ψ, and consider the category $\Psi([C], \diamond)$. The objects of $\Psi([C], \diamond)$ are all the $\psi_{C,D}$ for which $\overline{C} \supset D$. This is seen to be a one-to-one correspondence, so we can take the objects of $\Psi([C],\diamond)$ to be the subcells of C. We next assume a commutative diagram

$$\begin{array}{ccc} [E] & \xrightarrow{\psi''} & [D] \\ {\scriptstyle\psi}\big\downarrow & & {\scriptstyle\psi'}\big\downarrow \\ [C] & = & [C] \end{array}$$

in Ψ. We may as well then pick D to be a subcell of C and ψ' to be $\psi_{C,D}$. Then we may as well pick E to be a subcell of D, and $\psi'' = \psi_{E,D}$. Then $\psi = \psi_{C,E}$. In short, $\Psi([C],\diamond)$ is the poset of all subcells of C under inclusion. Steenrod in his lectures worked out $B_{\Psi([C],\diamond)}$ in all but name; he shows

$$B_{\Psi([C],\diamond)} \simeq Sd\,\overline{C}.$$

We have assumed $Sd\,\overline{C}$ to be a combinatorial cell. Thus Ψ is cellular. $\square$

Cellular Functors Joining Cellular Categories

For a cellular category Ψ, both E_Ψ and B_Ψ are naturally CW-complexes. In passing, we denote here the nature of those functors $\phi : \Psi \to \Psi'$ which give CW-maps

$$B_\Psi \to B_{\Psi'}.$$

Given cellular categories Ψ and Ψ', define a functor $\phi : \Psi \to \Psi'$ to be *cellular* if given an object s of Ψ and a morphism $\psi' : \diamond \to \phi(s)$ in Ψ', then there exists a morphism $\psi : \diamond \to s$ in Ψ with $\phi(\psi) = \psi'$. The functor ϕ is *strictly cellular* if the morphism ψ above is always unique.

It is seen that if ϕ is cellular and r is a given object of Ψ, then for each diagram

$$\phi(r) \xleftarrow{\psi'_0} q_0 \xleftarrow{\psi'_1} \cdots \xleftarrow{\psi'_n} q_n$$

of Ψ' there is a diagram

$$r \xleftarrow{\psi_0} p_0 \xleftarrow{\psi_1} \cdots \xleftarrow{\psi_n} p_n$$

of Ψ which maps into it. In particular the map

$$\phi_* : E_\Psi(p) \to E_{\Psi'}(\phi(p))$$

is then an epi. It follows that the induced map $\phi_* : B_\Psi \to B_{\Psi'}$ is a CW-map.

If in the above ϕ is strictly cellular, then $\phi_* : E_\Psi(p) \to E_{\Psi'}(\phi(p))$ is a homeomorphism and the map $\phi_* : B_\Psi \to B_{\Psi'}$ maps each open cell $D(p)$ from (5.4) homeomorphically onto the open cell $D(\phi(p))$. The definitions of cellular and strictly cellular used above are meaningful even if the categories are not cellular. Thus we can speak of *cellular functors* and *strictly cellular functors* $\theta : H \to G$ generally.

Generalizations of the Restriction and Extension Functors

We need now to put the restriction and extension functors of Chapter 1 in their proper form. Suppose given a functor $\theta : H \to G$ joining small categories. The restriction functor $\theta^{\#} : \mathrm{Top}^G \to \mathrm{Top}^H$ is very easy to describe. Given a G-space X, then the H-space $\theta^{\#}X$ has $(\theta^{\#}X)(s) = X(\theta(s))$ and the action is given by

$$H(s', s) \times X(\theta(s)) \to X(\theta(s')), \quad (h, x) \mapsto (\theta(h))x.$$

It is clear that we have $\theta^{\#} : \mathrm{TOP}^G \to \mathrm{TOP}^H$.

We now describe the extension functor $\theta_{\#} : \mathrm{Top}^H \to \mathrm{Top}^G$. First, the functor

$$1 \times \theta : G \times H^o \to G \times G^o$$

induces

$$(1 \times \theta)^{\#} : \mathrm{Top}^{G \times G^o} \to \mathrm{Top}^{G \times H^o}.$$

There is the natural element G of $\mathrm{Top}^{G \times G^o}$ and thus one receives the $(G \times H^o)$-space

$$C_\theta = (1 \times \theta)^{\#} G.$$

The extension functor $\theta_{\#} : \mathrm{Top}^H \to \mathrm{Top}^G$ then sends an H-space Y into the G-space

$$\theta_{\#} Y = C_\theta \times_H Y,$$

using the bifunctor $\mathrm{Top}^{G \times H^o} \times \mathrm{Top}^H \to \mathrm{Top}^G$.

The $G \times H^o$-set C_θ can be written out as follows:

(i) for each object r of G and s of H, take $C_\theta(r, s)$ to be the set $G(r, \theta(s))$ of all morphisms $r \xleftarrow{g} \theta(s)$ in G; when we need a topology on C_θ, we take the discrete topology;

(ii) the left action of G on C_θ is given by

$$G(r', r) \times C_\theta(r, s) \to C_\theta(r', s), \qquad (g', g) \mapsto g'g;$$

the right action of H on C_θ is given by

$$C_\theta(r, s) \times H(s, s') \to C_\theta(r, s'), \qquad (g, h) \mapsto g\theta(h).$$

As a G-space, the $(G \times H^o)$-set C_θ is free in the sense of Chapter 2. Its generating set is just the set of all $1_{\theta(r)}$ for all objects r of H.

(5.6) *Suppose given a diagram $G \xleftarrow{\theta} H \xleftarrow{\phi} K$ of small categories and functors. Consider the $G \times H^o$-set C_θ and the $H \times K^o$-set C_ϕ. For each triple of objects r, s, t of G, H, K respectively there is the function*

$$C_\theta(r, s) \times C_\phi(s, t) \to C_{\theta\phi}(r, t)$$

taking a pair consisting of $r \xleftarrow{g} \theta(s)$ in G and $s \xleftarrow{h} \phi(t)$ in H into the composition $g(\theta(h))$ of

$$s \xleftarrow{g} \theta(s) \xleftarrow{\theta(h)} \theta\phi(r).$$

Using the functor $\times_H : Top^{G\times H^o} \times Top^{H\times K^o} \to Top^{G\times K^o}$ *to obtain* $C_\theta \times_H C_\phi$ *in* $Top^{G\times H^o}$*, we thus have from above a* $(G \times K^o)$*-map*

$$C_\theta \times_H C_\phi \to C_{\theta\phi}, \qquad (g, h) \mapsto g(\theta(h)).$$

Proof. Let $g \in C_\theta(r, s)$, let $s \overset{h}{\leftarrow} s'$ be a morphism of H, and let $h' \in C_\phi(s', t)$. The elements $(gh, h') = (g\theta(h), h')$ and (g, hh') map into the same element $g\theta(h)\theta(h')$ of $C_{\theta\phi}(r, t)$, thus giving a well defined map $C_\theta \times_H C_\phi \to C_{\theta\phi}$. □

(5.7) *In the above diagram of small categories and functors* $G \overset{\theta}{\leftarrow} H \overset{\phi}{\leftarrow} K$*, suppose that either* θ *or* ϕ *is an inclusion functor of a subcategory. Then*

$$C_\theta \times_H C_\phi \simeq C_{\theta\phi}.$$

Proof. If ϕ is an inclusion functor, then given $r \overset{g}{\leftarrow} \theta\phi(t)$ in $C_{\theta\phi}(r, t)$ since $\phi(t) = t$ we have $r \overset{g}{\leftarrow} \theta(t)$ in $C_\theta(r, t)$ and $g \times_H 1_t$ in $C_\theta \times_H C_\phi$ maps onto the element g of $C_{\theta\phi}$. Thus the function is onto and one checks that it is one-to-one. Similarly if θ is an inclusion functor. □

Given a functor $\theta : H \to G$ joining small categories, one can take the functor $\theta^\# : \mathrm{Top}^{G^o} \to \mathrm{Top}^{H^o}$ to be given by $\theta^\# X = X \times_G C_\theta$, which the reader should check.

If X is a G^o-space and Y is an H-space, then it follows from (1.21) that

$$\theta^\# X \times_H Y \simeq (X \times_G C_\theta) \times_H Y \simeq X \times_G (C_\theta \times_H Y) \simeq X \times_G \theta_\# Y,$$

and hence the generalized form of (1.22) holds.

One can use (5.6) with extension functors in the following way. Consider a diagram $G \overset{\theta}{\leftarrow} H \overset{\phi}{\leftarrow} K$ of small categories and functors, and let Y be a K-space. Then

$$\theta_\# \phi_\# Y \simeq (C_\theta \times_H C_\phi) \times_K Y$$

and there is from (5.6) a $(G \times K^o)$-map

$$\theta_\# \phi_\# Y \to C_{\theta\phi} \times_K Y = (\theta\phi)_\# Y.$$

Theorem 5.8 *Let* $\theta : H \to G$ *be a functor joining small categories, and consider the extension functor*

$$\theta_\# : Top^H \to Top^G.$$

If X *is a principal* H*-space in TOP, then* $\theta_\# X$ *is a principal* G*-space in TOP.*

Proof. Denote by $Id\ H$ the subcategory of H consisting of all the identity morphisms, and let $i : Id\ H \to H$ denote the inclusion functor. We first check the assertion for those principal H-spaces of the form $X = i_\# A$. If X is such a principal H-space, we must show that $\theta_\# X$ is a principal G-space of the building block type. For each collection $A = \{A_r\}$ of compactly generated spaces indexed by the objects of H, so that A is an $(Id\ H)$-space, consider

$$X = C_i \times_{Id\ H} A, \qquad X(p) = \coprod_{q \in Ob\ H} H(p,q) \times A(q).$$

Then we have

$$\theta_{\#} X = C_\theta \times_H (C_i \times_{Id\ H} A).$$

From (1.21) and (5.7) we get

$$\theta_{\#} X \simeq (C_\theta \times_H C_i) \times_{Id\ H} A \simeq C_{\theta i} \times_{Id\ H} A.$$

We can then also factor the composed functor θi as

$$Id\ H \xrightarrow{\theta'} Id\ G \xrightarrow{j} G,$$

thus using (5.7) again rewrite as

$$\theta_{\#} X \simeq C_j \times_{Id\ G} (C_{\theta'} \times_{Id\ H} A)$$

and check that $\theta_{\#} X$ is a principal G-space in TOP of the form $j_{\#} Y$.

One can pass to the general case by a typical adjointness argument. There are natural transformations

$$T_X : X \to \theta^{\#} \theta_{\#} X, \quad \theta_{\#} \theta^{\#} Y \to Y$$

for each H-space X in Top and each G-space Y in Top. The reader should write these down. From this one gets a standard adjointness fact that $\theta_{\#}$ preserves colimits of diagrams. Suppose we have a diagram of H-spaces X_p and H-maps $\phi_{p',p}$. Let Z denote the colimit of the diagram in Top^H. Then we have the diagram $\theta_{\#} X_p$ and its colimit Y in Top^G. Thus we have structure maps $\theta_{\#} X_p \to Y$. Thus we obtain the diagram

$$X_p \to \theta^{\#} \theta_{\#}(X_p) \to \theta^{\#} Y$$

and the composition $X_p \to \theta^{\#} Y$. Thus we have a natural H-map $Z \to \theta^{\#} Y$. Hence we get the G-maps

$$\theta_{\#} Z \to \theta_{\#} \theta^{\#} Y \to Y.$$

One then argues that $\theta_{\#} Z$ is in fact a colimit of $\theta_{\#} X_p$.

Hence one gets that $\theta_{\#}$ takes principal H-spaces of the form $i_{\#} A$ into principal G-spaces of the form $j_{\#} B$, it takes pushouts in Top^H into pushouts in Top^G, and filtered spaces in Top^H into filtered spaces in Top^G. One then checks that $\theta_{\#}$ takes principal H-spaces in TOP into principal G-spaces in TOP. $\square$

Categorical Description of $\theta_{\#}E_H$ for a Functor $\theta : H \to G$

If $\theta : H \to G$ is a functor joining small categories, we can present the principal G-space $\theta_{\#}E_H$ quite explicitly. We first present it in a categorical form. Construct C_θ in $\mathrm{SET}^{G\times H^o}$ as above. Recall from Chapter 1 that $\mathrm{SET}^{G\times H^o} \simeq (\mathrm{SET}^{H^o})^G$. Then from the functors

$$\mathrm{SET}^{H^o} \xrightarrow{M_0} \mathrm{CAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o} \xrightarrow{|*|} \mathrm{TOP}$$

of Chapter 2 we get

$$(\mathrm{SET}^{H^o})^G \xrightarrow{(M_0)^G} \mathrm{CAT}^G \xrightarrow{N^G} (\mathrm{TOP}^{\Delta^o})^G \xrightarrow{|*|^G} \mathrm{TOP}^G.$$

Thus we get from the element C_θ of $\mathrm{SET}^{G\times H^o}$ first an element of CAT^G and from this an element of TOP^G. Given $\theta : H \to G$ then denote the above element of CAT^G by $\mathcal{C}_\theta : G \longrightarrow \mathrm{CAT}$. For each object r of G we have the category $\mathcal{C}_\theta(r)$, and for each morphism $g : r \to r'$ of G we have a functor $g_* : \mathcal{C}_\theta(r) \to \mathcal{C}_\theta(r')$.

The functor $\mathcal{C}_\theta : G \longrightarrow \mathrm{CAT}$ can be readily displayed.

(i) The objects of $\mathcal{C}_\theta(r)$ are all the ordered pairs (g, s) where s is an object of H and where $r \xleftarrow{g} \theta(s)$ is a morphism of G.

(ii) For each morphism $h : s' \to s$ in H and each object (g, s) in $\mathcal{C}_\theta(r)$ we get a morphism $h : (g\theta(h), s') \to (g, s)$ in $\mathcal{C}_\theta(r)$.

(iii) For each morphism $g : r \to r'$ in G we get a functor $g_* : \mathcal{C}_\theta(r) \to \mathcal{C}_\theta(r')$ which sends an object (g', s) into the object (gg', s) and we get a morphism $h : (g'\theta(h), s') \to (g', s)$ of $\mathcal{C}_\theta(r)$ into the morphism $h : (gg'\theta(h), s') \to (gg', s)$ of $\mathcal{C}_\theta(r')$.

The point of defining $\mathcal{C}_\theta$ is that

$$(\theta_{\#}E_H)(r) \simeq B_{\mathcal{C}_\theta(r)}.$$

Moreover for any morphism $g : r \to r'$ in G we have the induced functor $\mathcal{C}_\theta(r) \to \mathcal{C}_\theta(r')$, and the induced map

$$B_{\mathcal{C}_\theta(r)} \to B_{\mathcal{C}_\theta(r')}$$

agrees with

$$g_* : (\theta_{\#}E_H)(r) \to (\theta_{\#}E_H)(r').$$

Thus $\theta_{\#}E_H$ is readily exhibited from $\mathcal{C}_\theta$, and we thus call $\mathcal{C}_\theta$ the *categorical form* of $\theta_{\#}E_H$.

In the following, we use the fact that every functor $\underline{n} \to \mathcal{C}_\theta(r)$ is of the form

$$(g, s_0) \xleftarrow{h_1} (g\theta(h_1), s_1) \xleftarrow{h_2} \cdots \xleftarrow{h_n} (g(\theta(h_1))\cdots(\theta(h_n)), s_n)$$

for some diagram

$$s_0 \xleftarrow{h_1} \cdots \xleftarrow{h_n} s_n$$

in H. Now fix an object r of G and consider the right H-space $C(r, \diamond)$. We get

$$(\theta_{\#}E_H)(r) \simeq C(r, \diamond) \times_H E_H$$

by the definition of $\theta_\#$. But the above computation of all functors $\underline{n} \to \mathcal{C}_\theta(r)$ shows that

$$C(r, \diamond) \times_H E_H \simeq B_{\mathcal{C}_\theta(r)}.$$

Hence we have the following.

(5.9) *Let $\theta : H \to G$ be a functor joining small categories. Consider the categorical form $\mathcal{C}_\theta : G \to CAT$ of θ. Then we have*

$$(\theta_\# E_H)(r) \simeq B_{\mathcal{C}_\theta(r)}.$$

Note also from the above that we can present $(\theta_\# E_\Psi)(r)$ more directly in terms of standard homotopy colimits as

$$(\theta_\# E_\Psi)(r) = B_{H^\circ}(C_\theta(r, \diamond)).$$

In the style of Gray [5.3], denote by CAT$/G$ the category of small categories over G. That is, an object of CAT$/G$ is a functor $\theta : H \to G$ and a morphism is a commutative diagram of functors

$$\begin{tikzcd} H_0 \arrow[r, "\theta''"] \arrow[d, "\theta"'] & H_1 \arrow[d, "\theta'"] \\ G \arrow[r, equal] & G. \end{tikzcd}$$

Then we have constructed the functor CAT$/G \to$ CAT^G and an associated functor

$$\text{CAT}/G \to \text{PRINC}^G.$$

Topological Resolutions of a Small Category G

A *topological resolution of a small category* G is a pair consisting of a cellular category Ψ and a functor $\theta : \Psi \to G$ such that for each object p of G, the space $(\theta_\# E_\Psi)(p)$ is contractible; alternatively, such that in the categorical form $\mathcal{C}_\theta$, each space $B_{\mathcal{C}_\theta(r)}$ is contractible.

(5.10) *Let $\theta : \Psi \to G$ be a topological resolution of G. Then the induced map*

$$B_\Psi \xrightarrow{\theta_*} B_G$$

is a homotopy equivalence in TOP.

Proof. On one hand, the colimit of the principal G-space $\theta_\# E_\Psi$ is given by

$$(Ob\ G) \times_G \theta_\# E_\Psi \simeq (Ob\ \Psi) \times_\Psi E_\Psi \simeq B_\Psi.$$

On the other hand, the principal G-space $\theta_\# E_\Psi$ is a universal G-space in the sense of (4.17) and its colimit is naturally homotopy equivalent to B_G. $\square$

(5.11) *Let $\theta : \Psi \to G$ be a topological resolution of the small category G. Then $\theta_\# E_\Psi$ is a universal G-space in TOP with $\theta_\# E_\Psi$ a regular cell complex and with* *each $(\theta_\# E_\Psi)(p)$ a subcomplex of $\theta_\# E_\Psi$. Moreover the colimit B of the G-space $\theta_\# E_\Psi$ is a regular cell complex with identifications. In particular, these spaces are all CW-complexes and the maps*

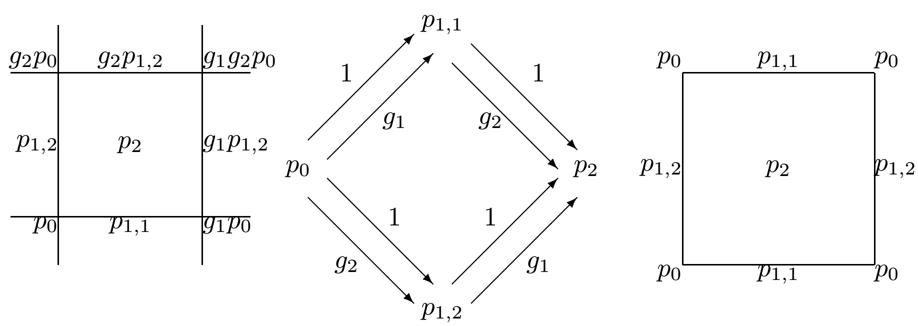
$$g_* : (\theta_\# E_\Psi)(p) \to (\theta_\# E_\Psi)(q)$$

corresponding to morphisms $g : p \to q$ as well as the natural map

$$\theta_\# E_\Psi \to (\theta_\# E_\Psi)/G \simeq B_\Psi$$

are all CW-maps which map open cells homeomorphically onto open cells.

We note two trivial examples of topological resolutions.



$E = \theta_\# E_\Psi$ $\qquad\qquad$ $\theta : \Psi \to \mathbb{Z} \times \mathbb{Z}$ $\qquad\qquad$ $B_\Psi = B_{\mathbb{Z}\times\mathbb{Z}}$

Consider the group $Z \times Z$ where Z is the infinite cyclic group. Call its generators g_1 and g_2, and write the group multiplicatively. Take the plane as a regular complex acted upon by $Z \times Z$ where

$$g_1(x, y) = (x + 1, y), \qquad g_2(x, y) = (x, y + 1).$$

Denote by p_0 the origin and take as 0-cells all its translates. Denote by $p_{1,1}$ the segment from $(0, 0)$ to $(1, 0)$, and by $p_{1,2}$ the segment from $(0, 0)$ to $(0, 1)$. The 1-cells are all translates of $p_{1,1}$ and $p_{1,2}$. Denote by p_2 the unit square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. The 2-cells are all translates of p_2. Denote an arbitrary choice among the cells p_0, $p_{1,1}$, $p_{1,2}$, p_2 by p_ω. These are the objects of Ψ. There is a morphism $p_\omega \leftarrow p_{\omega'}$ for each $p_\omega \supset gp_{\omega'}$, and composition is given by

$$(p_\omega \supset g(p_{\omega'}))(p_{\omega'} \supset g'(p_{\omega''})) = (p_\omega \supset gg'(p_{\omega''})).$$

These choices are as indicated in the latter part of (5.5). The functor $\theta : \Psi \to Z \times Z$ then sends each $p_\omega \supset g(p_{\omega'})$ into g. This translation of geometric information into categorical language is illustrated above, obtaining the torus as nonstandard classifying space for $Z \times Z$.

As a second easy example of this kind of translation of geometric examples into categorical terms, let G denote the category whose free generators are

$$0 \xleftarrow{g_0} 1 \xleftarrow{g_1} \cdots \xleftarrow{g_{n-1}} n \xleftarrow{g_n} \cdots .$$

The standard universal space can here be presented. Let $\nabla(\infty)$ denote the infinite dimensional standard simplex whose vertices are $\{v_i | 0 \leq i < \infty\}$. Let $\nabla^n(\infty)$ denote the infinite dimensional subsimplex spanned by all v_i with $n \leq i$. Then the standard universal G-space E_G can be checked to be

$$\nabla(\infty) \hookleftarrow \nabla^1(\infty) \hookleftarrow \cdots \hookleftarrow \nabla^n(\infty) \hookleftarrow \cdots .$$

Nevertheless, a classic easy presentation of homotopy colimits is Milnor's mapping telescope which amounts to computing homotopy colimits of G^o-spaces using a topological resolution of G. We present the nonstandard universal space and codify it in our style.

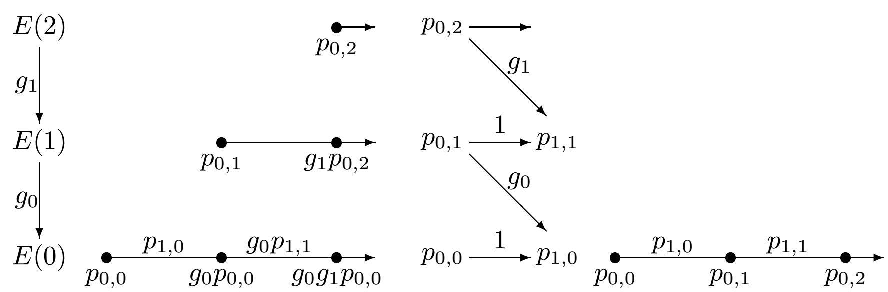
The nonstandard universal G-space E is the inverse system

$$E(0) \xleftarrow{g_0} E(1) \xleftarrow{g_1} \cdots \xleftarrow{g_{n-1}} E(n) \xleftarrow{g_n} \cdots ,$$

where $E(n) \subset R^2$ is all (x, n) with $x \geq n$ and where $g_n(x, n+1) = (x, n)$. The generating 0-cells are $p_{0,n} = (n, n)$ and the generating 1-cells are the intervals $p_{1,n}$ consisting of all (x, n) with $n \leq x \leq n+1$. The nonidentity morphisms of Ψ are the inclusions

$$p_{1,n} \supset p_{0,n}, \qquad p_{1,n} \supset g_n p_{0,n+1}.$$

Then we can present this in the form $\theta : \Psi \to G$ pictorially.



$E = \theta_\# E_\Psi$ $\qquad$ $\theta : \Psi \to G$ $\qquad$ $B_\Psi \sim B_G$

One should know the homotopy colimits generated by the above universal G-space E. A G^o-space X is a direct system

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \cdots .$$

A nonstandard homotopy colimit of X is then the space $X \times_G E$ which can be reinterpreted as

$$X \times_G E \simeq (\coprod_{n \geq 0} X_n \times [n, n+1]) / \sim,$$

where $\sim$ is the least equivalence relation with

$$(x, n+1) \sim (f_n(x), n+1)$$

for all $x \in X_n$. The homotopy colimit is then a bunch of mapping cylinders strung together along a line, i.e. is Milnor's mapping telescope.

The Cone cG over a Small Category G

We will eventually show that the inclusion functor $i : Mono\ \Delta \to \Delta$ is a topological resolution of Δ, following Segal [4.4]. We need to know about joins in order to state the form of $i_{\#}E_{Mono\ \Delta}$ most clearly. One can make joins out of cones and products, so we first need to know about cones. Given a topological category G, define the cone cG over G to be the topological category given by:

(i) $Ob\ cG = Ob\ G \sqcup \{-1\}$, where -1 has been picked arbitrarily as an object disjoint from the objects of G;

(ii) $Mor\ cG = Mor\ G \sqcup Ob\ G \sqcup \{-1\}$; there are then three subspaces of arrows; first, the already existing arrows of G; second, for each object p of G an arrow from -1 to p which is denoted by $(p, -1)$, the space of all such being topologized as $Ob\ G$; finally, the singleton identity morphism 1_{-1} of -1;

(iii) for any morphism $p' \xleftarrow{g} p$ of G, commutativity holds in

$$\begin{tikzcd} -1 \arrow[r, equal] \arrow[d, "{(p',-1)}"'] & -1 \arrow[d, "{(p,-1)}"'] \\ p' & p. \arrow[l, "g"'] \end{tikzcd}$$

There are also the standard cones over spaces. If A is a space, then the cone cA over A is given by

$$cA = (I \times A)/\sim,$$

where $\sim$ is the least equivalence relation on $I \times A$ such that $(0, a) \sim (0, a')$ for any $a, a' \in A$. One checks that if A is compactly generated then cA is compactly generated. For cA is a pushout of

$$I \times A \hookleftarrow 0 \times A \longrightarrow pt.$$

(5.12) *The standard classifying space $B_{cG} = |N(cG)|$ is naturally homeomorphic to the compactly generated space cB_G which is the cone in the sense of spaces over the space B_G.*

The proof is as an exercise for the reader.

The Join $*$ in TOPCAT

Given topological categories G_0 and G_1, we now consider the product category $cG_0 \times cG_1$. Its object space is the disjoint union

$$Ob\ G_0 \times Ob\ G_1 \sqcup Ob\ G_0 \times \{-1\} \sqcup \{-1\} \times Ob\ G_1 \sqcup \{-1\} \times \{-1\}.$$

Let $G_0 * G_1$ denote the full subcategory of $cG_0 \times cG_1$ whose object space is

$$Ob\ (G_0 \star G_1) = Ob\ G_0 \times Ob\ G_1 \sqcup Ob\ G_0 \times \{-1\} \sqcup \{-1\} \times Ob\ G_1.$$

We call the topological category $G_0 * G_1$ the *join* of G_0 and G_1.

(5.13) *There is a natural isomorphism $c(G_0 * G_1) \simeq cG_0 \times cG_1$ for all topological categories G_0 and G_1.*

Proof. Consider $c(G_0 * G_1)$, whose objects are those of $G_0 * G_1$ together with the object -1. Map the objects of $G_0 * G_1$ into objects of $cG_0 \times cG_1$ by inclusion. Map the singleton -1 into the objects of $cG_0 \times cG_1$ by sending -1 to $(-1, -1)$. Then finish by checking out morphisms. $\square$

One defines an $(n+1)$-fold join

$$G_0 * G_1 * \cdots * G_n$$

as the full subcategory of $cG_0 \times \cdots \times cG_n$ whose objects are all $(r_0, r_1, \cdots, r_n)$ such that each r_i is either an object of G_i or else $r_i = -1$, and such that for at least one i we have r_i an object of G_i. Secondly, one defines the morphisms

$$h = (h_0, h_1, \cdots, h_n) : (r_0, \cdots, r_n) \to (s_0, \cdots, s_n)$$

in $G_0 * \cdots * G_n$ by

(i) h_i is a morphism of G_i if both r_i and s_i are objects of G_i,
(ii) $h_i = 1_{-1}$ if $r_i = s_i = -1$,
(iii) if $r_i = -1$ and $s_i \in Ob\, G_i$ then $h_i = (s_i, -1)$, and
(iv) requiring that it cannot happen that $r_i \in Ob\, G_i$ and $s_i = -1$.

The reader should check associativity up to natural isomorphism. Finally, one should check that the join is commutative up to natural isomorphism.

Thus in summary we have the bifunctor

$$* : \text{TOPCAT} \times \text{TOPCAT} \to \text{TOPCAT}.$$

This bifunctor is associative up to natural isomorphism, and also commutative up to natural isomorphism. The empty category serves as neutral element of the multiplication. Oddly enough, there are other ways of defining joins. The computations which justify our choice are those of (5.16).

Joins of Compactly Generated Spaces

First there is the cone cA over a space A. By analogy with categories, given compactly generated spaces A and B we can form the spaces $cA \times B$ and $A \times cB$, which we suppose intersect only in their common closed subset $A \times B$. We can then take for the join $A * B$ the union $cA \times B \cup A \times cB$. For spaces, there are many ways to write joins but they are all naturally homeomorphic.

For example, write cA as the pushout, through a change of parameter, of

$$[1/2, 1] \times A \hookleftarrow 1 \times A \longrightarrow pt,$$

so that $cA \times B$ is then the pushout of

$$[1/2, 1] \times A \times B \hookleftarrow 1 \times A \times B \xrightarrow{proj} B.$$

Similarly, regard cB as the pushout of

$$[0, 1/2] \times B \hookleftarrow 0 \times B \longrightarrow pt,$$

so that $A \times cB$ is then the pushout of

$$[0,1/2] \times A \times B \hookleftarrow 0 \times A \times B \xrightarrow{proj} A.$$

Regard $A \times B$ as $1/2 \times A \times B$ in each of these. Then $A * B$ is exhibited as a pushout of

$$I \times A \times B \hookleftarrow \{0,1\} \times A \times B \xrightarrow{f} A \sqcup B,$$

where $f(0,a,b) = a$ and $f(1,a,b) = b$. This is a second form.

Let Ψ denote the category whose objects are the nonempty subsets of the doubleton $\{0,1\}$ and whose morphisms are inclusions. Then a diagram

$$A \xleftarrow{f} C \xrightarrow{g} B$$

is a Ψ^o-space; call it X. Then the standard homotopy colimit

$$B_{\Psi^o} X = X \times_{\Psi} E_{\Psi}$$

is called the *standard homotopy pushout* of the diagram. One can take E_Ψ to be the diagram

$$0 \hookrightarrow I \hookleftarrow 1.$$

Thus the homotopy pushout is

$$[A \sqcup I \times C \sqcup B]/\sim,$$

where $\sim$ is the least equivalence relation with $(0,c) \sim f(c)$ and $(1,c) \sim g(c)$. That is, the homotopy pushout is the two-sided mapping cylinder $A \cup_f I \times C \cup_g B$.

Thus the standard homotopy pushout of the diagram

$$A \leftarrow A \times B \rightarrow B,$$

in which the maps are projections, is homeomorphic to the join $A * B$. We take any one of the above as the join $A * B$.

The Classifying Space of the Join $G * H$

Theorem 5.14 *Let G and H be topological categories, and for any topological category K let $B_K = |NK|$. Then*

$$B_{G*H} \simeq B_G * B_H.$$

Proof. Consider the topological category $G * H$, which is a certain full subcategory of $cG \times cH$. Included in it are the full subcategories $cG \times H$ and $G \times cH$, and every object of $G * H$ is in $cG \times H$ or in $G \times cH$. Moreover $G \times H$ is a full subcategory of both of them, and is their intersection.

Let symbols r_i, s_j denote objects of cG, cH respectively. Let symbols p_i, q_j denote objects of G, H respectively. Then $cG \times H$ has objects (r_i, q_j) while $G \times cH$ has objects (p_i, s_j). Suppose we have a diagram

$$(r_0, s_0) \xleftarrow{k_1} (r_1, s_1) \xleftarrow{k_2} \cdots \xleftarrow{k_n} (r_n, s_n)$$

in $G * H$. If some object (r_i, s_i) is in $cG \times H$ but not in $G \times H$ then we must have $r_i = -1$. It must then be the case that $r_j = -1$ for all $j \geq i$. Hence in particular we have $r_n = -1$.

If also some (r_i, s_i) is in $G \times cH$ but not in $G \times H$, then by similar reasoning we have $s_n = -1$. If both were the case, then we would have $(r_n, s_n) = (-1, -1)$, which is not in $G * H$. Hence the above diagram is either entirely contained in $cG \times H$ or else is entirely contained in $G \times cH$. From this it follows that B_{G*H} is the union of $B_{cG\times H}$ and $B_{G\times cH}$. Next one has to make the straight-forward check that

$$B_{cG\times H} \cap B_{G\times cH} = B_{G\times H}.$$

Finally, one uses (5.12) and (2.9) to obtain that

$$B_{G*H} \simeq (cB_G) \times B_H \cup B_G \times (cB_H)$$

and thus $B_{G*H} \simeq B_G * B_H$. $\square$

We assume without proof the straight-forward fact that the join of compactly generated spaces is associative up to natural homeomorphism. It then follows that

$$B_{G_0*\cdots*G_n} \simeq B_{G_0} * \cdots * B_{G_n}.$$

The Topological Resolution $i : Mono\, \Delta \to \Delta$

The cone $c\Delta$ over Δ is called the *augmented simplicial category.* One can take as a model for it the category whose objects are the integers $n \geq -1$, and whose morphisms $\tau : m \to n$ are the order preserving functions

$$\tau : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}.$$

Here the object -1 is interpreted as the empty set.

We need the associative bifunctor

$$\oplus : c\Delta \times c\Delta \to c\Delta$$

corresponding to disjoint union. Given objects p and q of $c\Delta$, denote by

$$p \xrightarrow{\alpha_{p,q}} p + q + 1 \xleftarrow{\beta_{p,q}} q$$

the unique diagram of monos in $c\Delta$ where $\alpha_{p,q}$ maps onto an initial segment and $\beta_{p,q}$ maps onto a terminal segment. Given morphisms $\tau : p' \to p$ and $\gamma : q' \to q$ in $c\Delta$, there is then a unique morphism $\tau \oplus \gamma : p' + q' + 1 \to p + q + 1$ in $c\Delta$ with commutativity in

$$\begin{tikzcd}
p' \arrow[r, "\alpha_{p',q'}"] \arrow[d, "\tau"'] & p' + q' + 1 \arrow[d, "\tau\oplus\gamma"'] & q' \arrow[l, "\beta_{p',q'}"'] \arrow[d, "\gamma"'] \\
p \arrow[r, "\alpha_{p,q}"] & p + q + 1 & q. \arrow[l, "\beta_{p,q}"']
\end{tikzcd}$$

Then $\oplus$ is given on objects by $p \oplus q = p + q + 1$, and as above on morphisms. This bifunctor restricts to an associative bifunctor on Δ.

(5.15) *Given a morphism $\delta : m \to n$ in $c\Delta$ and given objects p and q of $c\Delta$ with $n = p + q + 1$, then there exist unique morphisms $\tau : p' \to p$ and $\gamma : q' \to q$ with*

$$\delta = \tau \oplus \gamma.$$

Moreover, $\tau \oplus \gamma$ is a mono if and only if τ and γ are both monos.

This is an exercise for the reader.

Theorem 5.16 *Consider the cellular category $Mono\,\Delta$, and the inclusion functor $i = \theta : Mono\,\Delta \to \Delta$. Let $\mathcal{C}_\theta$ denote the categorical form of θ. Then*

$$\mathcal{C}_\theta(0) \simeq Mono\,\Delta, \qquad \mathcal{C}_\theta(m) * \mathcal{C}_\theta(n) \simeq \mathcal{C}_\theta(m+n+1).$$

Since $B_{Mono\,\Delta}$ is contractible, then every $B_{\mathcal{C}_\theta(n)}$ is contractible. Hence θ is a topological resolution of Δ, and $\theta_\# E_{Mono\,\Delta}$ is a universal space for Δ. If X is any Δ^o-space in TOP, then

$$X \times_\Delta \theta_\# E_{Mono\,\Delta} \simeq (i^\# X) \times_{Mono\,\Delta} E_{Mono\,\Delta} \simeq (i^\# X) \times_{Mono\,\Delta} \nabla$$

is a homotopy colimit for the Δ^o-space X.

Proof. The category $\mathcal{C}_\theta(m)$ has as objects all morphisms $\delta : r \to m$ of Δ, and as morphisms $\alpha : \delta \to \delta'$ all commutative diagrams

$$\begin{array}{ccc} r & \xrightarrow{\alpha} & r' \\ {\scriptstyle\delta}\downarrow & & {\scriptstyle\delta'}\downarrow \\ m & = & m \end{array}$$

in Δ for which α is a mono.

For $m = 0$, then the objects of $\mathcal{C}_\theta(0)$ are in one-to-one correspondence with the objects $r \geq 0$ of $Mono\,\Delta$. Moreover, the morphisms of $\mathcal{C}_\theta(0)$ are clearly in one-to-one correspondence with the morphisms of $Mono\,\Delta$, thus $\mathcal{C}_\theta(0) \simeq Mono\,\Delta$.

Suppose $m, n \geq 0$, where we must write down an isomorphism

$$\phi : \mathcal{C}_\theta(m) * \mathcal{C}_\theta(n) \simeq \mathcal{C}_\theta(m+n+1).$$

The objects of $\mathcal{C}_\theta(m) * \mathcal{C}_\theta(n)$ can be taken to be the ordered pairs (δ_1, δ_2) where

$$\delta_1 : p \to m, \quad \delta_2 : q \to n, \quad p \geq 0 \text{ or } q \geq 0.$$

We can regard all this as in $c\Delta$, and define

$$\phi(\delta_1, \delta_2) = \delta_1 \oplus \delta_2,$$

which is a morphism in Δ from $p + q + 1$ to $m + n + 1$. It follows from (5.15) that this is an isomorphism of object sets.

Consider next the morphisms of $\mathcal{C}_\theta(m) * \mathcal{C}_\theta(n)$. These are in one-to-one correspondence with the pairs of commutative diagrams in $c\Delta$,

$$\begin{tikzcd} p \arrow[r, "\alpha"] \arrow[d, "\delta"'] & p' \arrow[d, "\delta'"'] \\ m \arrow[r, equal] & m \end{tikzcd} \qquad \begin{tikzcd} q \arrow[r, "\beta"] \arrow[d, "\tau"'] & q' \arrow[d, "\tau'"'] \\ n \arrow[r, equal] & n \end{tikzcd}$$

for which both α and β are monos, and where either both $p, p' \geq 0$ or both $q, q' \geq 0$. Then define

$$\phi(\alpha, \beta) = \alpha \oplus \beta$$

and check that ϕ is an isomorphism of morphism sets using (5.15). Thus $\mathcal{C}_\theta(m) * \mathcal{C}_\theta(n) \simeq \mathcal{C}_\theta(m+n+1)$.

Since $B_{Mono\,\Delta}$ has been previously checked as contractible, and since the join of contractible spaces is contractible, it follows from (5.14) that every $B_{\mathcal{C}_\theta(m)}$ is contractible. Hence $i : Mono\,\Delta \hookrightarrow \Delta$ is a topological resolution of Δ. $\square$

References

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CHAPTER VI

Homotopy Colimits of Simplicial Spaces

We are here again, as in Chapter 2, in the inner core of our subject. The simplicial category Δ has objects the non-negative integers and morphisms $\delta : m \to n$ the order preserving functions

$$\delta : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}.$$

The most primitive reason that TOP^{Δ^o} is at the inner core is because of the functors

$$\mathrm{TOPCAT} \to \mathrm{TOP}^{\Delta^o} \rightleftarrows \mathrm{TOP}$$

of Chapter 2. We will continue to stress the interplay between TOP and these two fattened versions of TOP which are useful in its study.

We have first to understand for which simplicial spaces X the Milnor realization $|X|$ is a (non-standard) homotopy colimit. One needs a judiciously chosen non-standard universal Δ-space E and an equivariant map $\phi : E \to \nabla$ such that one can analyze when the induced map

$$X \times_\Delta E \to X \times_\Delta \nabla = |X|$$

is a homotopy equivalence. Here we use $i_\# E_{Mono\,\Delta}$, set up in Chapter 5 for this purpose. Thus for any Δ^o-space X, one has the homotopy colimit $BX = X \times_{Mono\,\Delta} \nabla$ of X. It then turns out that $|X|$ is a homotopy colimit of X if for each epi $\gamma : n \to n-1$ the pair $(X(n), \gamma^* X(n-1))$ is a cofibered pair in TOP; our treatment has been adapted from Segal [4.4]. This is vital to our development of the subject, since it is one of the justifications for basing the entire study of homotopy colimits on the realization. In order to prove this, we have to continue the study of homotopy equivalences that was started in Chapter 3.

In Chapter 4 we developed the basic theory of homotopy colimits only for TOP^G with G a discretely topologized small category. We next develop the theory for TOP^G where G is a topological category, having finally enough information on homotopy colimits in TOP^{Δ^o} to do it properly. For any topological category, one has as in Chapter 2 the diagram of functors

$$\mathrm{TOP}^G \xrightarrow{M_1} \mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o}.$$

For topological categories, one takes as the standard model $\mathcal{B}_G(\diamond)$ for the homotopy colimit the composition

$$\mathrm{TOP}^G \xrightarrow{M_1} \mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o} \xrightarrow{B} \mathrm{TOP}$$

assigning to the G-space X the space $\mathcal{B}_G(X) = NM_1(X) \times_{Mono\,\Delta} \nabla$. If G satisfies a suitable cofibration condition, then the natural map $\mathcal{B}_G(X) \to B_G(X)$ is a homotopy equivalence and one can think of $B_G(X)$ as a small model for $\mathcal{B}_G(X)$. If G does not satisfy the cofibration condition, then one uses only $\mathcal{B}_G(X)$. One then uses as standard classifying space $\mathcal{B}_G$ the space $NG \times_{Mono\,\Delta} \nabla$.

We next consider a *topological monoid*, i.e. a topological category whose space of objects is a singleton. For a topological monoid G, one thus has a continuous and associative multiplication $G \times G \to G$ with a unit element $1 \in G$. There is then readily defined what it means for a topological monoid to have homotopy inverses. We then prove a classic theorem that if G is a topological monoid with homotopy inverses, then there is a natural homotopy equivalence of the space G with the loop space $\Omega\mathcal{B}_G$. The purpose of this is to open up for later chapters the problem of exhibiting models for loop spaces.

By way of specific models, we exhibit in this chapter only the James model for the loop space ΩSA of the reduced suspension SA of a compactly generated space A with given base point, where A is further constrained by

(i) A is path connected,
(ii) the base point is cofibered in A, and
(iii) A is of the homotopy type of a CW-complex.

The theorem of James [6.4,1955] then asserts that ΩSA is of the homotopy type of the free topological monoid JA whose generators are the points of A, with the base point being the unit element. As a special case, the loop space of S^n for $n > 1$ is then up to homotopy the free topological monoid generated by S^{n-1}.

One proves it by a homotopy colimit computation, i.e. by showing that if A is a compactly generated space with cofibered base point then $\mathcal{B}_{JA} \sim SA$ and then applying the theorem on topological monoids with homotopy inverses. In the conclusion of this chapter, JA is itself interpreted as a homotopy colimit. Thus a major intent of the chapter is to treat the most classical homotopy colimit problems connected with actions of small categories, those connected with Δ^o-spaces and the Milnor realization, with topological monoids which have homotopy inverses, and with the free topological monoids JA generated by spaces A with base point.

We assume in the course of the exposition some theorems of the 1960's about fibrations due to the German homotopy theorists. For proofs of these background facts, we refer either to Dold [3.2] or to tom Dieck, Kamps and Puppe [3.1]. We also assume without proof a theorem of Hastings [6.3] on fibrations.

The Universal Δ-Space $i_\# E_{Mono\,\Delta}$

In Chapter 5, we showed that if $Mono\,\Delta$ denotes the subcategory of Δ consisting of all monos in Δ and if $i : Mono\,\Delta \to \Delta$ denotes the inclusion functor, then $i_\# E_{Mono\,\Delta}$ is a universal Δ-space. Moreover, we can take

$$E_{Mono\,\Delta} = \coprod_{n \geq 0} \nabla(n)$$

with $Mono\ \Delta$ acting as the face operators (although this is not the simplicial structure on $E_{Mono\Delta}$ used there). We restate (5.16) as Segal's form [4.4] of a homotopy colimit BX for any simplicial space.

Theorem 6.1 *Let X be any simplicial space in TOP. Then we can take for a homotopy colimit BX of X the compactly generated space*

$$BX = X \times_\Delta i_\# E_{Mono\,\Delta} \simeq i^\# X \times_{Mono\,\Delta} \nabla \simeq (\coprod X(n) \times \nabla(n))/\sim,$$

where $\sim$ is the least equivalence relation such that if $\alpha : n-1 \to n$ is the mono in Δ whose image does not contain the vertex $v_{i,n}$, then

$$(x, (t_0, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_n)) \sim (x\alpha, (t_0, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n))$$

for all $x \in X(n)$.

Every point of BX has representatives $x \times_{Mono\,\Delta} (t_0, \cdots, t_n)$ for various n, where $x \in X(n)$. Moreover, any $t_i = 0$ can be deleted, replacing x by $x\alpha$ for the appropriate mono α. Hence every point of BX has a representation as $x \times_{Mono\,\Delta} (t_0, \cdots, t_n)$ where $t_i > 0$ for all i. We leave it to the reader to show as in (2.7) that each point of BX has a unique representation in this form.

There is the following easier version of (2.8), whose proof we leave to the reader.

(6.2) *Let X be any simplicial space in TOP, and let*

$$\tau : \coprod X(n) \times \nabla(n) \to BX$$

denote the natural quotient map onto the above homotopy colimit of X. Then BX is filtered in TOP as $BX = \bigcup (BX)_n$ where $(BX)_n = \tau(X(n) \times \nabla(n))$, and there is the relative homeomorphism

$$\tau' : X(n) \times (\nabla(n), \partial\nabla(n)) \to ((BX)_n, (BX)_{n-1}).$$

A Description of the Map $i_\# E_{Mono\,\Delta} \to \nabla$

Let $E = i_\# E_{Mono\,\Delta}$ abbreviate the universal Δ- space given by

$$E(n) = \Delta(n, \diamond) \times_{Mono\,\Delta} \nabla;$$

for each n there is the map

$$\phi_n : \Delta(n, \diamond) \times_{Mono\,\Delta} \nabla \to \Delta(n, \diamond) \times_\Delta \nabla \simeq \nabla(n).$$

There results a Δ-map $\phi : E \to \nabla$. For every Δ^o-space X we get the map

$$1 \times_\Delta \phi : BX = X \times_\Delta E \to X \times_\Delta \nabla = |X|.$$

In order to understand this map, we need another form of the join of spaces.

Given compactly generated spaces $X_0, X_1, \cdots, X_n$, the iterated join can be exhibited as

$$X_0 * \cdots * X_n = (\nabla(n) \times X_0 \times \cdots \times X_n)/\sim$$

where $\sim$ is the equivalence relation such that

$$((t_0, \cdots t_n), x_0, \cdots, x_n) \sim ((t_0, \cdots, t_n), y_0, \cdots, y_n)$$

if and only if $x_i = y_i$ whenever $t_i \neq 0$. Denote by

$$\pi : \nabla(n) \times X_0 \times \cdots \times X_n \to X_0 * \cdots * X_n$$

the natural quotient map.

The two-fold join $X_0 * X_1$ then becomes $X_0 * X_1 = (I \times X_0 \times X_1)/\sim$, where $\sim$ is the equivalence relation given above. This is a form of the homotopy pushout of the diagram

$$X_0 \xleftarrow{proj} X_0 \times X_1 \xrightarrow{proj} X_1,$$

i.e. is the two-sided mapping cylinder of this diagram. Hence the two-fold join as used here coincides up to homeomorphism with that of Chapter 5.

Let $\delta : m \to n$ be a morphism of Δ, i.e. an order preserving function

$$\delta : \{0, \cdots, m\} \to \{0, \cdots, n\}.$$

Then for each $0 \leq i \leq n$ there is the face of $\nabla(m)$, possibly empty, spanned by the vertices $v_{j,m}$ for which $\delta(j) = i$. Denote this face by $\nabla(m_i - 1)$, where m_i is the number of elements in $\delta^{-1}(i)$, with $\nabla(-1)$ the empty set. Then there is a homeomorphism

$$f : \nabla(m_0 - 1) * \cdots * \nabla(m_n - 1) \simeq \nabla(m).$$

One simply defines

$$f\pi((t_0, \cdots, t_n), (u_{0,0}, \cdots, u_{m_0-1,0}), \cdots, (u_{0,n}, \cdots, u_{m_n-1,n})) =$$

$$(t_0 u_{0,0}, \cdots, t_0 u_{m_0-1,0}, \cdots, t_n u_{0,n}, \cdots, t_n u_{m_n-1,n}),$$

and checks that f is a homeomorphism. If some $\delta^{-1}(i)$ is empty, the appropriate $u_{\diamond,i}$ terms will simply not appear. Out of it, one gets for fixed n, m a homeomorphism

$$F : \Delta(n, m) \times \nabla(m) \simeq \coprod_{m_0 + \cdots + m_n = m+1} (\nabla(m_0 - 1) * \cdots * \nabla(m_n - 1)).$$

Recall that following (5.4) we exhibited the infinite dimensional dunce hat

$$D = B_{Mono\,\Delta} = (\coprod \nabla(m))/\sim$$

as all points $[u_0, \cdots, u_m]$ where $(u_0, \cdots, u_m)$ is a point of $\nabla(m)$ and where any $u_i = 0$ can be deleted. Let $\rho : \coprod \nabla(m) \to D$ denote the natural quotient map. There is then the diagram

$$\begin{array}{ccc} \coprod \Delta(n,m) \times \nabla(m) & \xrightarrow{F} & \coprod_{m_0+\cdots+m_n=m+1} (\nabla(m_0-1) * \cdots * \nabla(m_n-1)) \\ \tau \Big\downarrow & & \Big\downarrow \rho*\cdots*\rho \\ E(n) & & D * \cdots * D \end{array}$$

and hence an induced map $E(n) \to D * \cdots * D$ which makes the diagram commutative. It is tedious but not difficult to see that this is a homeomorphism of $E(n)$ onto the $(n+1)$-fold iterated join

$$D^{n+1,*} = D * D * \cdots * D.$$

There is a natural map $D^{n+1,*} \to D$. On one hand, it can be taken as the map $E(n) \to E(0)$ induced by the unique morphism $n \to 0$ in Δ. On the other hand, it can be exhibited as

$$\pi((t_0, \cdots, t_n), [u_{0,0}, \cdots, u_{m_0-1,0}], \cdots, [u_{0,n}, \cdots, u_{m_n-1,n}]) \mapsto$$
$$[t_0 u_{0,0}, \cdots, t_0 u_{m_0-1,0}, \cdots, t_n u_{0,n}, \cdots, t_n u_{m_n-1,n}].$$

(6.3) *The above universal Δ-space $E = \coprod E(n)$ can be exhibited explicitly as follows. Take each $E(n)$ to be the $(n+1)$-fold iterated join $D^{n+1,*}$ of the dunce hat D. Given a morphism $\delta : m \to n$ in Δ, let m_i denote the number of points in $\delta^{-1}(i)$, consider the above maps*

$$D^{m_0,*} \to D, \cdots, D^{m_n,*} \to D$$

and form the iterated join map

$$D^{m+1,*} = D^{m_0,*} * \cdots * D^{m_n,*} \to D * \cdots * D = D^{n+1,*}.$$

This is the action map $\delta_* : D^{m+1,*} \to D^{n+1,*}$. The Δ-map $\phi : E \to \nabla$ is for each $n \geq 0$ the join of maps $D \to pt$ as given by

$$E(n) = D * \cdots * D \to pt * \cdots * pt = \nabla(n).$$

Relations Between Homotopy Equivalences and Posets

We need now an effective overview of the elementary relations between cofibrations and homotopy equivalences. These interconnections can be regarded in two ways. The first is as follows.

Let Ψ be a poset. Then Ψ *satisfies the finiteness condition* if for each object p of Ψ the set $\{q | q \leq p\}$ is finite. Define a *cofibered Ψ-filtered space* X in TOP to be a compactly generated space together with closed subsets $X(p)$ for each object p such that:

(i) $X = \bigcup_{p \in \Psi} X(p)$ and $A \subset X$ is closed in X if and only if each $A \cap X(p)$ is closed;

(ii) if $q \leq p$ then $X(q)$ is contained in $X(p)$ and $(X(p), X(q))$ is a cofibered pair;

(iii) if we have $q \leq p$ and $r \leq p$ and $x \in X(q) \cap X(r)$, then there exists s with $s \leq q$ and $s \leq r$ such that $x \in X(s)$.

We also regard the poset as a category, with a single morphism $q \to p$ whenever $q \leq p$ and no morphisms otherwise. This is the second way to look upon the interconnections. The two ways are related as follows.

Theorem 6.4 *Let Ψ be a poset which satisfies the finiteness condition, and let X be a cofibered Ψ-filtered space in TOP. If Ψ also denotes the category associated with the poset Ψ, then there is the Ψ-space which associates to each p the space $X(p)$ and to the morphism $\psi : q \to p$ corresponding to $q \leq p$ the inclusion $X(q) \hookrightarrow X(p)$. This is a principal Ψ-space and the total space X is its colimit.*

Proof. For each object p of Ψ, let $d(p)$ be the maximal non-negative integer n such that there exists a diagram

$$p_0 < p_1 < \cdots < p_n = p.$$

Check that $d(p)$ is well-defined, and that $d(p)$ is of the form $1 + d(q)$ where $d(q)$ is the maximum of the $d(q')$ with $q' < p$, whenever this set is non-empty. Let X_n be the Ψ-space with $X_n(p)$ all $x \in X(p)$ for which there exists $q \leq p$ with $d(q) \leq n$ and $x \in X(q)$. Let A_n be defined by

$$A_n(p) = \begin{cases} X(p), & \text{for } d(p) = n \\ \emptyset, & \text{for } d(p) \neq n. \end{cases}$$

For each object p let $X^{deg}(p)$ be the union of all the $X(q)$ with $q < p$. Note by (ii) and extended use of Lillig's Theorem that each $X^{deg}(p) \subset X(p)$ is a cofibered inclusion. Then define B_n by

$$B_n(p) = \begin{cases} X^{deg}(p), & \text{for } d(p) = n \\ \emptyset, & \text{for } d(p) \neq n. \end{cases}$$

The reader will then show, using (iii) and Lillig's Theorem, that $(A_n(p), B_n(p))$ is a cofibered pair, that there is a relative homeomorphism

$$(\Psi \times_{Ob\,\Psi} A_n, \Psi \times_{Ob\,\Psi} B_n) \to (X_n, X_{n-1}),$$

and that $X = \bigcup X_n$ is therefore a principal Ψ-space. It can be checked that the total space is its colimit. □

Let Ψ be a poset satisfying the finiteness condition, and let X and Y be cofibered, Ψ-filtered spaces in TOP. Then a *Ψ-filtered map* $f : X \to Y$ is a collection of maps $f_p : X(p) \to Y(p)$ such that whenever $q \leq p$ commutativity holds in

$$\begin{array}{ccc} X(q) & \xrightarrow{f_q} & Y(q) \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow \\ X(p) & \xrightarrow{f_p} & Y(p). \end{array}$$

There is then an induced map $X \to Y$ in TOP which on $X(p)$ agrees with f_p, and this map we also denote by f. This map f is the colimit of the collection $\{f_p\}$.

Theorem 6.5 *Let Ψ be a poset satisfying the finiteness condition. Let X and Y be cofibered, Ψ-filtered spaces in TOP, and suppose we have a Ψ-filtered map $f : X \to Y$ such that each $f_p : X(p) \to Y(p)$ is a homotopy equivalence in TOP. Then the induced map $f : X \to Y$ is a homotopy equivalence in TOP.*

This is an immediate corollary of (6.4) and (4.3).

Example 1. Let Ψ be the poset with elements $\{0, 1, 2\}$ and partial order $1 > 0 < 2$. Then a cofibered Ψ-filtered space X in TOP is a compactly generated space X together with two closed subsets X_1 and X_2 such that $X = X_1 \cup X_2$ and such that $X_0 = X_1 \cap X_2$ is cofibered in both X_1 and X_2. Let Y be another cofibered, Ψ-filtered space; i.e. we have closed subsets Y_1 and Y_2 of Y with $Y = Y_1 \cup Y_2$ and with $Y_1 \cap Y_2$ cofibered in both Y_1 and Y_2. Suppose we have maps

$$f_1 : X_1 \to Y_1, \quad f_2 : X_2 \to Y_2$$

such that if $x \in X_1 \cap X_2$ then $f_1(x) = f_2(x)$. Let $f_0 : X_1 \cap X_2 \to Y_1 \cap Y_2$ denote the map defined by $f_0(x) = f_1(x) = f_2(x)$. Then if f_0, f_1 and f_2 are homotopy equivalences in TOP, so is the induced map $f : X \to Y$ a homotopy equivalence in TOP.

Example 2. Let Ψ be the poset

$$0 < 1 < \cdots < n < \cdots .$$

Then a cofibered, Ψ-filtered space in TOP is a compactly generated space X together with a cofibered filtration $X = \bigcup_{n \geq 0} X_n$ in TOP. Given two such, a Ψ-filtered map $f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ such that each f_n is an extension of f_{n-1}. There is then the induced map $f : X \to Y$ in TOP. If each f_n is a homotopy equivalence in TOP, then so is $f : X \to Y$ a homotopy equivalence in TOP.

Example 3. Let Ψ be the poset of proper subsets $\omega = \{i_0, \cdots, i_k\}$ of $\{0, 1, \cdots, n\}$, under inclusion. Then a cofibered, Ψ-filtered space in TOP is a space X in TOP together with closed subsets $X(\omega)$ such that:

(i) $X = \bigcup X(\omega)$;
(ii) if $\omega' \subset \omega$ then $X(\omega')$ is a cofibered subset of $X(\omega)$;
(iii) $X(\omega) \cap X(\omega') = X(\omega \cap \omega')$; we will assume in addition that $X(\emptyset) = \emptyset$.

Given two such, a Ψ-filtered map is a collection of maps $f_\omega : X(\omega) \to Y(\omega)$ satisfying the consistency condition. These induce a single map $f : X \to Y$ of the total spaces. If each f_ω is a homotopy equivalence, then so is f.

We need one more fact about homotopy equivalences.

(6.6) *Consider the commutative diagram in TOP*

$$\begin{tikzcd} X \arrow[d, "h_0"'] & A \arrow[l, "i"'] \arrow[r, "g"] \arrow[d, "h_1"'] & Y \arrow[d, "h_2"'] \\ X' & A' \arrow[l, "i'"'] \arrow[r, "g'"] & Y' \end{tikzcd}$$

in which (X, A) and (X', A') are cofibered pairs and h_0, h_1, h_2 are homotopy equivalences. Then the induced map $X \cup_g Y \to X' \cup_{g'} Y'$ is a homotopy equivalence.

Proof. Each row in the above diagram has a homotopy pushout, and these homotopy pushouts are two-sided mapping cylinders. Denote them by

$$X \cup_i I \times A \cup_g Y, \qquad X' \cup_{i'} I \times A' \cup_{g'} Y'$$

respectively. We have an induced map F of homotopy colimits and by (4.3) this induced map will itself be a homotopy equivalence in TOP. The added step needed is that each pushout is naturally homotopy equivalent to the homotopy pushout. Look at the first row. There is the commutative diagram

$$\begin{tikzcd} 1 \times A = A \arrow[r, equal] \arrow[d] & A \arrow[d, "i"'] \\ X \cup_i I \times A \arrow[r] & X \end{tikzcd}$$

which by (3.1) represents a homotopy equivalence in TOP$\backslash A$. Hence the pushouts of the diagrams

$$X \xleftarrow{i} A \xrightarrow{g} Y, \qquad X \cup_i I \times A \xleftarrow{j} A \xrightarrow{g} Y$$

are naturally homotopy equivalent. The pushout of the second of these is the homotopy pushout of the first. $\square$

The Cofibration Condition for Simplicial Spaces

In order to use effectively the cofibration results above, one considers simplicial spaces X which satisfy a cofibration condition. Call a morphism δ of Δ an *elementary degeneracy* if δ is an epi of the form $\delta : n \to n-1$ for some $n > 0$. As in Chapter 2, $\delta^* : X(n-1) \to X(n)$ is an inclusion map onto a closed subset $\delta^* X(n-1)$ of $X(n)$ for every elementary degeneracy. Say that X *satisfies the cofibration condition for simplicial spaces* if $(X(n), \delta^* X(n-1))$ is a cofibered pair for every elementary degeneracy.

Fix a simplicial space X which satisfies the cofibration condition. The reader can show that every epi in Δ is a composition of elementary degeneracies, and consequently that if $\delta : n \to m$ is an epi in Δ then $(X(n), \delta^* X(m))$ is a cofibered pair. Hence for every $n \geq 0$, the pair $(X(n), X^{deg}(n))$ is cofibered, using Lillig's Theorem.

We finally have more than one cosimplicial space that concerns us, having both ∇ and $E = i_{\#}E_{Mono\,\Delta}$ to consider. We need a temporary name for a cosimplicial space Y which satisfies the conditions of (2.3) as well as having that if $\alpha : n \to m$ is a mono in Δ, then the pair $(Y(m), \alpha_* Y(n))$ is a cofibered pair. We call such a cosimplicial space in TOP *nicely cofibered.* Thus for such a cosimplicial space Y, we have:

(i) if $\alpha : n \to m$ is a mono, then α_* maps $Y(n)$ homeomorphically onto a closed subset $\alpha_* Y(n)$ of $Y(m)$ and $(Y(m), \alpha_* Y(n))$ is a cofibered pair;
(ii) given $y \in Y(m)$, there exists a unique triple consisting of $n \leq m$, a mono $\alpha : n \to m$, and a non-degenerate $u \in Y(n)$ with $y = \alpha u$;
(iii) if the diagram $n_1 \xrightarrow{\alpha_1} m \xleftarrow{\alpha_2} n_2$ of monos has no pullback in $Mono\,\Delta$, then

$$\alpha_{1*}Y(n_1) \cap \alpha_{2*}Y(n_2) = \emptyset;$$

(iv) if the above diagram of monos has a pullback diagram of monos

$$\begin{array}{ccc} r & \xrightarrow{\rho_1} & n_1 \\ {\scriptstyle \rho_2}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ n_2 & \xrightarrow{\alpha_2} & m \end{array}$$

and if $\alpha_1\rho_1 = \alpha_2\rho_2 = \beta$, then

$$\alpha_{1*}Y(n_1) \cap \alpha_{2*}Y(n_2) = \beta_* Y(r)$$

.

(v) if $\gamma : m \to n$ is an epi and if $y \in Y(m)$ is non-degenerate, then $\gamma y \in Y(n)$ is nondegenerate.

The set $Y^{deg}(m)$ of degenerate elements of a nicely cofibered cosimplicial space is the union of all the closed, cofibered subsets $\alpha_* Y(n)$ over all monos $\alpha : n \to m$ with $n < m$. It follows from Lillig's Theorem that $(Y(m), Y^{deg}(m))$ is a cofibered pair.

(6.7) *Let X be a simplicial space in TOP which satisfies the cofibration condition for simplicial spaces, and let Y be a nicely cofibered cosimplicial space in TOP. Let $\sim$ denote the equivalence relation on $\coprod X(n) \times Y(n)$ such that*

$$X \times_\Delta Y = (\coprod X(n) \times Y(n))/\sim .$$

Then the analogues of (2.7) and (2.8) hold.

In fact, their proofs as given used only the above conditions. We have that $X \times_\Delta Y$ is filtered as

$$X \times_\Delta Y = \bigcup (X \times_\Delta Y)_n,$$

where if $\pi : \coprod X(n) \times Y(n) \to X \times_\Delta Y$ denotes the natural quotient map, then

$$(X \times_\Delta Y)_n = \pi(X(n) \times Y(n)).$$

Moreover, there is the relative homeomorphism

$$\pi' : (X(n), X^{deg}(n)) \times (Y(n), Y^{deg}(n)) \to ((X \times_\Delta Y)_n, (X \times_\Delta Y)_{n-1}).$$

The space $X \times_\Delta Y$ is then a cofibered filtered space.

Theorem 6.8 *Let X be a simplicial space in TOP which satisfies the cofibration condition for simplicial spaces, let Y and Z be nicely cofibered cosimplicial spaces, and let $\phi : Y \to Z$ be a Δ-map which is also a weak homotopy equivalence in TOP^Δ. Then the induced map*

$$1 \times_\Delta \phi : X \times_\Delta Y \to X \times_\Delta Z$$

is a homotopy equivalence in TOP.

Proof. The proof is an extended exercise in the paragraphs above on the relationship between cofibrations and homotopy equivalences. Let f denote the map $1 \times_\Delta \phi$ and let

$$f_n : (X \times_\Delta Y)_n \to (X \times_\Delta Z)_n$$

denote the restricted maps. By Example 2, it suffices to show each f_n a homotopy equivalence in TOP. This will be by induction on n.

We have first to look at $\phi_n : Y(n) \to Z(n)$, to note that this restricts to a map $\theta : Y^{deg}(n) \to Z^{deg}(n)$, and we have to prove that θ is a homotopy equivalence in TOP. This follows from Example 3 above.

We next have to consider the commutative diagram

$$\begin{array}{ccccc} X(n) \times Y(n) & \longleftarrow & X^{deg}(n) \times Y(n) \cup X(n) \times Y^{deg}(n) & \longrightarrow & (X \times_\Delta Y)_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ X(n) \times Z(n) & \longleftarrow & X^{deg}(n) \times Z(n) \cup X(n) \times Z^{deg}(n) & \longrightarrow & (X \times_\Delta Z)_{n-1}, \end{array}$$

and to check that the vertical maps are homotopy equivalences. One then uses (6.6) to establish the induction. $\square$

Corollary 6.9 *The cosimplicial spaces $E = i_\# E_{Mono\,\Delta}$ and ∇ are both nicely cofibered cosimplicial spaces Y such that each $Y(n)$ is a contractible space. The Δ-map $\phi : E \to \nabla$ defined earlier is automatically a weak homotopy equivalence in TOP^Δ. Hence for each simplicial space X satisfying the cofibration condition, the map*

$$1 \times_\Delta \phi : BX = X \times_\Delta E \to X \times_\Delta \nabla = |X|$$

is a homotopy equivalence in TOP. Hence for such X, $|X|$ is a homotopy colimit of X.

Easy Examples of $|X|$ as a Homotopy Colimit

Note as a first corollary that all the simplicial sets X satisfy the cofibration condition for simplicial spaces, where a simplicial set $X = \coprod X(n)$ is considered as a simplicial space by giving each $X(n)$ the discrete topology. Every subset of a discrete space is clearly both closed and cofibered. Hence for simplicial sets X, the realization $|X|$ is a homotopy colimit.

As a second corollary, fix a small category G and consider the functors

$$\mathrm{TOP}^{G^o} \times \mathrm{TOP}^{G} \xrightarrow{M} \mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o}$$

of Chapter 2. Fix a G^o-space X in TOP and a G-space Y in TOP. Then $NM(X, Y)$ is the simplicial space $Z = \coprod Z(n)$ where

$$Z(n) = X \times_{Ob\ G} G^{\underline{n}} \times_{Ob\ G} Y.$$

Then $Z(n)$ breaks up into open and closed subsets $Z(g_1, \cdots, g_n)$, one for each diagram

$$p_0 \xleftarrow{g_1} \cdots \xleftarrow{g_n} p_n$$

in the small category G. Here

$$Z(g_1, \cdots, g_n) = \{(x, g_1, \cdots, g_n, y)\}$$

with $(x, y) \in X(p_0) \times Y(p_n)$. There are two cases:

(i) some g_i is an identity morphism and so $Z(g_1, \cdots, g_n)$ is contained in $Z^{deg}(n)$;

(ii) no g_i is an identity morphism and so no points of $Z(g_1, \cdots, g_n)$ are degenerate.

Thus $Z(n)$ then breaks up into a disjoint union of open and closed subsets, and we consider $\delta^* Z(n-1)$ where $\delta : n \to n-1$ is an elementary degeneracy. It is seen that $\delta^* Z(n-1)$ consists of all

$$Z(g_1, \cdots, g_{i-1}, 1_p, g_{i+1}, \cdots, g_n)$$

so that $\delta^* N(n-1)$ is a disjoint union of some of the open and closed subsets into which $Z(n)$ is broken. Thus $(Z(n), \delta^* Z(n-1))$ is a cofibered pair in TOP and Z satisfies the cofibration condition for simplicial spaces. It then follows from (6.9) that $|NM(X, Y)|$ is a homotopy colimit of $NM(X, Y)$. Thus the constructs $B_G X$, $E_G X$, EG, and $X \bigotimes_G Y$ of Chapter 2 all are included in the case in which the simplicial space used in the construction satisfies the cofibration condition, so that the realization in all these cases is a homotopy colimit of the simplicial space.

The Classifying Spaces $\mathcal{B}_G$ of a Topological Category

We can now treat topological categories properly. Start with the question of how one makes classifying spaces of a topological category G. We continue to take the nerve functor

$$N : \mathrm{TOPCAT} \to \mathrm{TOP}^{\Delta^o}$$

as a given start. The question is how to follow it with a functor

$$F : \mathrm{TOP}^{\Delta^o} \to \mathrm{TOP}.$$

An answer is with something taking the nerve NG of the given topological category into a homotopy colimit of the simplicial space NG. Any two choices will then automatically be naturally homotopy equivalent.

Having given above a specific homotopy colimit functor

$$B : \mathrm{TOP}^{\Delta^o} \to \mathrm{TOP}$$

for all simplicial spaces, one can then take as classifying space functor $\mathcal{B}_\diamond$: TOPCAT $\to$ TOP the composition

$$\text{TOPCAT} \xrightarrow{N} \text{TOP}^{\Delta^o} \xrightarrow{B} \text{TOP}.$$

Thus one has an answer for all G. How does it compare with the standard B_G that we have used for small categories? For G a small category, the nerve $NG = \coprod G_{\underline{n}}$ satisfies the cofibration condition for simplicial spaces, thus there is the homotopy equivalence $\mathcal{B}_G \to B_G$. Which topological categories G have the simplicial space NG satisfying the cofibration condition, so that we can then use $|NG|$ as a classifying space?

Topological Categories Satisfying a Cofibration Condition

Before answering this, we have to generalize the notion of cofibration. We have to consider the meaning of cofibration in the category TOP$/P$ of compactly generated spaces over a compactly generated space P.

Take an object $\nu : X \to P$ of TOP$/P$. Then $I \times X$ denotes the object of TOP$/P$ whose structural map is given by $\nu'(t, x) = \nu(x)$. Projection of $I \times X$ onto X is a map in TOP$/P$. So are the maps $X \rightrightarrows I \times X$ which send x into $(0, x)$ and $(1, x)$ respectively. Thus there is the homotopy category of TOP$/P$, already considered in Chapter 3.

Closed pairs in TOP$/P$ are easy. Given X in TOP$/P$, and any closed subset A of X, then A inherits the structure of a space over P from X and (X, A) becomes a closed pair in TOP$/P$.

Thus one has the *cofibered closed pairs* (X, A) *in* TOP$/P$, where given a map $\phi : X \to Y$ in TOP$/P$ and a homotopy $H_0 : I \times A \to Y$ in TOP$/P$ with $H_0(0, a) = \phi(a)$ for all $a \in A$, then there exists a homotopy $H : I \times X \to Y$ in TOP$/P$ which extends H_0 and has $H(0, x) = \phi(x)$ for all $x \in X$.

Consider now a topological category G. Then $Mor\ G$ and $Ob\ G$ are compactly generated spaces, and among the structural maps of G is the map

$$Mor\ G \to Ob\ G \times Ob\ G, \quad (p \xleftarrow{g} q) \mapsto (p, q).$$

There is contained in $Mor\ G$ the closed subset $Id\ G$ of all identity morphisms of G.

Say that the topological category G *satisfies the cofibration condition* if the above closed pair $(Mor\ G, Id\ G)$, considered as a closed pair in TOP$/(Ob\ G \times Ob\ G)$, is a cofibered pair in TOP$/(Ob\ G \times Ob\ G)$.

Suppose for example that G is a topological monoid, so that $Ob\ G$ is a singleton, and we can identify G with the space of morphisms $Mor\ G$. Then G satisfies the cofibration condition if and only if $(G, \{1\})$ is a cofibered pair in TOP.

Theorem 6.10 *Let G be a topological category which satisfies the cofibration condition for topological categories. Then the simplicial space NG satisfies the cofibration condition for simplicial spaces. Hence the realization $|NG|$ is naturally homotopy equivalent to the homotopy colimit of NG and thus to $\mathcal{B}_G$.*

TOP/($P \times Q$) and the Proof of the Preceding Theorem

Consider TOP/($P \times Q$) where P and Q are compactly generated spaces. Regard the structural maps of a space over $P \times Q$ as the diagram

$$P \xleftarrow{\nu_1} X \xrightarrow{\nu_2} Q$$

in TOP, and a morphism $X \to Y$ as a commutative diagram

$$\begin{tikzcd} P \arrow[d, equal] & X \arrow[l, "\nu_1"'] \arrow[r, "\nu_2"] \arrow[d, "\phi"'] & Q \arrow[d, equal] \\ P & Y \arrow[l, "\nu_1'"'] \arrow[r, "\nu_2'"] & Q. \end{tikzcd}$$

If G is a topological category, then as above we have

$$Ob\ G \xleftarrow{\nu_1} Mor\ G \xrightarrow{\nu_2} Ob\ G,$$

where ν_1 and ν_2 take a morphism into its target and source respectively. Denote this space over $Ob\ G \times Ob\ G$ simply by G. As above, we have the closed pair $(G, Id\ G)$ in TOP/($Ob\ G \times Ob\ G$).

If P, Q, and R are compactly generated spaces, there is a bifunctor

$$\times_Q : \mathrm{TOP}/(P \times Q) \times \mathrm{TOP}/(Q \times R) \to \mathrm{TOP}/(P \times R).$$

Namely, given X in TOP/($P \times Q$) with structural maps

$$P \xleftarrow{\nu_1} X \xrightarrow{\nu_2} Q$$

and given Y in TOP/($Q \times R$) with structural maps

$$Q \xleftarrow{\nu_1'} Y \xrightarrow{\nu_2'} R,$$

there are the maps $X \times Y \rightrightarrows Q$ taking (x, y) into $\nu_2(x)$ and $\nu_1'(y)$ respectively. Denote by $X \times_Q Y$ the closed subset of $X \times Y$ consisting of all (x, y) with $\nu_2(x) = \nu_1'(y)$. Then $X \times_Q Y$ is closed in $X \times Y$, and hence is compactly generated. Moreover, there are the structural maps

$$P \xleftarrow{\nu_1''} X \times_Q Y \xrightarrow{\nu_2''} R,$$

where $\nu_1''(x, y) = \nu_1(x)$ and $\nu_2''(x, y) = \nu_2'(y)$. One completes the functor by assigning to $f : X \to Y$ in TOP/($P \times Q$) and $f' : X' \to Y'$ in TOP/($Q \times R$) an induced map

$$f \times_Q f' : X \times_Q Y \to X' \times_Q Y'.$$

This functor is easily extended to multifunctors such as

$$\mathrm{TOP}/(P \times Q) \times \mathrm{TOP}/(Q \times R) \times \mathrm{TOP}/(R \times S) \to \mathrm{TOP}/(P \times S).$$

Thus we regard the above multifunctor as associative.

If (X, A) is a closed pair in TOP/($P \times Q$) and (Y, B) is a closed pair in TOP/($Q \times R$), one receives a closed pair

$$(X, A) \times_Q (Y, B) = (X \times_Q Y, A \times_Q Y \cup X \times_Q B)$$

in TOP$/(P \times R)$. We need the following fact. If (X, A) is a closed cofibered pair in TOP$/(P \times Q)$ and (Y, B) is a closed cofibered pair in TOP$/(Q \times R)$, then $(X, A) \times_Q (Y, B)$ is a closed cofibered pair in TOP$/(P \times R)$. The reader will have to check that relevant portions of a simpler case at the beginning of Chapter 3 extend without change.

Consider finally a topological category G satisfying the cofibration condition. There is the simplicial space NG, and

$$(NG)(n) = G \times_{Ob\ G} \cdots \times_{Ob\ G} G$$

is the n-fold reduced product Moreover, if $\delta : n \to n-1$ is an elementary degeneracy then $\delta^* NG(n-1)$ is a reduced product

$$G \times_{Ob\ G} \cdots \times_{Ob\ G} G \times_{Ob\ G} Id\ G \times_{Ob\ G} G \times_{Ob\ G} \cdots \times_{Ob\ G} G$$

and $(NG(n), \delta^* NG(n-1))$ can be expressed as

$$G \times_{Ob\ G} \cdots G \times_{Ob\ G} (G, Id\ G) \times_{Ob\ G} G \times_{Ob\ G} \cdots \times_{Ob\ G} G.$$

By multiple use of the preceding paragraph, this is cofibered in TOP$/(Ob\ G \times Ob\ G)$ and thus in TOP. The theorem follows.

$\mathbf{TOP}^G$ and $\mathbf{TOP}^{G^o}$ for G a Topological Category

Out of the bifunctor $\times_Q :$ TOP$/(P \times Q) \times$ TOP$/(Q \times R) \to$ TOP$/(P \times R)$, we can take $P = pt$ or $R = pt$ or both and get special cases such as

$$\times_Q : \mathrm{TOP}/(P \times Q) \times \mathrm{TOP}/Q \to \mathrm{TOP}/P.$$

We assume these special cases, but leave details to the reader.

If G is a topological category, a *G-space X in* TOP consists of

(i) a space X in TOP$/Ob\ G$ with structure map $\nu : X \to Ob\ G$;
(ii) a morphism $G \times_{Ob\ G} X \to X$ in TOP$/Ob\ G$, $(g, x) \mapsto gx$; this is the *action map*;
(iii) for each $(g, g', x) \in G \times_{Ob\ G} G \times_{Ob\ G} X$, it is required that $g(g'x) = (gg')x$;
(iv) it is required for each $(1_p, x) \in Id\ G \times_{Ob\ G} X$ that $1_p x = x$.

Given G-spaces X and X' in TOP, a *G-map* $\phi : X \to X'$ is a map in TOP$/Ob\ G$ such that commutativity holds in the diagram of maps in TOP$/Ob\ G$

$$\begin{array}{ccc} G \times_{Ob\ G} X & \xrightarrow{act} & X \\ {\scriptstyle 1 \times_{Ob\ G} \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ G \times_{Ob\ G} X' & \xrightarrow{act} & X'. \end{array}$$

We thus have the category TOP^G whose objects are the G-spaces in TOP, and whose morphisms are the G-maps.

There is similarly the category TOP$^{G^o}$ of *right* G-spaces and *G-maps*.

If G and H are topological categories, the $G \times H^o$-spaces X in TOP merit special attention. First, X is a space over $Ob\ G \times Ob\ H$ and as such has structure maps
$$Ob\ G \xleftarrow{\nu_1} X \xrightarrow{\nu_2} Ob\ H.$$
Second, we take the action map as a map
$$G \times_{Ob\ G} X \times_{Ob\ H} H \to X \quad (g, x, g') \mapsto (gx)g' = g(xg') = gxg'$$
in TOP$/(Ob\ G \times Ob\ H)$, with the appropriate extra conditions. As in Chapter 2, there is the functor
$$M : \mathrm{TOP}^{G \times H^o} \times \mathrm{TOP}^{H \times K^o} \to \mathrm{TOPCAT},$$
where $Ob\ M(X, Y) = X \times_{Ob\ H} Y$, and $Mor\ M(X, Y) = X \times_{Ob\ H} H \times_{Ob\ H} Y$. There are natural maps
$$Ob\ M(X, Y) \to Ob\ G \times Ob\ K, Mor\ M(X, Y) \to Ob\ G \times Ob\ K,$$
which amount to a continuous functor
$$M(X, Y) \to Ob\ G \times Ob\ K$$
into the topological category whose morphisms are all identity morphisms.

The functor $N : \mathrm{TOPCAT} \to \mathrm{TOP}^{\Delta^o}$ then sends $M(X, Y)$ into an $NM(X, Y)$ for which there is for each n a map
$$(NM(X, Y))(n) \to Ob\ G \times Ob\ K$$
commuting with the action maps coming from morphisms $\delta : m \to n$ of Δ. In short, the composition
$$\mathrm{TOP}^{G \times H^o} \times \mathrm{TOP}^{H \times K^o} \xrightarrow{M} \mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^o}$$
maps each (X, Y) into $(\mathrm{TOP}/Ob\ G \times Ob\ K)^{\Delta^o}$. The realization $|\diamond|$ can then be seen to provide a commutative diagram
$$\begin{array}{ccc} (\mathrm{TOP}/(Ob\ G \times Ob\ K))^{\Delta^o} & \xrightarrow{|\diamond|} & \mathrm{TOP}/(Ob\ G \times Ob\ K) \\ \downarrow & & \downarrow \\ \mathrm{TOP}^{\Delta^o} & \xrightarrow{|\diamond|} & \mathrm{TOP}. \end{array}$$

(6.11) *If G is a topological category which satisfies the cofibration condition, then there is the functor*
$$TOP^G \to TOP^{\Delta^o}$$
sending X into the simplicial space $NM(G, X)$. This is a simplicial space satisfying the cofibration condition for simplicial spaces. Thus $|NM(G, X)|$ is a homotopy colimit for $NM(G, X)$. Define $E_G X = |NM(G, X)|$. Then $E_G X$ is in $TOP/Ob\ G$ and also in TOP^G, there is a natural G-map $T : E_G X \to X$, and T is a homotopy equivalence in $TOP/Ob\ G$.

PROOF. Let $\delta : n \to n-1$ be an elementary degeneracy, and abbreviate the compactly generated space $Ob\ G$ by P. Then $(NM(G,X))(n), \delta^*(NM(G,X))(n-1))$ is of the form

$$G \times_P \cdots \times_P G \times_P (G, Id\ G) \times_P G \times_P \cdots \times_P G \times_P X$$

and is thus a cofibered pair in TOP/P, where the structure map is given by

$$(g_0, g_1, \cdots, g_n, x) \mapsto \ target\ g_0.$$

It is then clear that $NM(G,X)$ satisfies the cofibration condition for simplicial spaces.

We must next see that $|NM(G,X)|$ is in TOP/P. This follows from the remarks above.

One next has to see that $|NM(G,X)|$, together with $|NM(G,X)| \to P$, its structure map given above, is in TOP^G. The action function is clear, it is just a matter of seeing that it is an action map in TOP/P. We only give the outline. One needs besides $NM(G,X)$ also $NM(G \times_P G, X)$, both in $(\mathrm{TOP}/P)^{\Delta^o}$. There is a natural morphism in $(\mathrm{TOP}/P)^{\Delta^o}$ from $NM(G \times_P G, X)$ to $NM(G,X)$ which follows from the morphism in $\mathrm{TOP}/(P \times P)$

$$G \times_P G \to G, \qquad (g, g') \mapsto gg'.$$

This natural morphism induces a morphism

$$|NM(G \times_P G, X)| \to |NM(G,X)|$$

in TOP/P. Finally,

$$|NM(G \times_P G, X)| \simeq G \times_P |NM(G,X)|.$$

Thus, $|NM(G,X)|$ can be considered a G-space.

The final part of the proof consists in generalizing (2.11) and (2.13). The generalization of (2.11) takes the following form. Let P denote a compactly generated space, and let also P denote the topological category whose object space is P, and whose morphisms are all identity morphisms. Denote by TOPCAT/P the category whose objects are pairs consisting of a topological category G and a continuous functor $G \to P$, and whose morphisms are all commutative diagrams

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G' \arrow[d] \\ P \arrow[r, equal] & P. \end{tikzcd}$$

Then (2.11) takes the following form. Suppose G and H are in TOPCAT/P and that $\phi, \theta : G \to H$ are morphisms of TOPCAT/P such that there exists a continuous natural transformation $T : \phi \to \theta$. Then the maps

$$|\phi|, |\theta| : |NG| \to |NH|$$

in TOP/P are homotopic in TOP/P.

Then the proof of (2.13) is repeated verbatim for the rest of the theorem. □

Principal G-Spaces for a Topological Category

The main arena of this tract is with actions of small categories. Nevertheless, we want to leave enough traces for the case of a topological category so that the interested reader can reconstruct this extended case. Besides, it gives good experience in the myriad possible extensions of the earlier chapters. We are ready now to outline the categories with principal models in the case of a topological category.

Fix a topological category G. There is then a functor $\mathrm{TOP}/(\mathit{Ob}\ G) \to \mathrm{TOP}/(\mathit{Ob}\ G)$ sending A into $G \times_{\mathit{Ob}\ G} A$. Moreover there is a map in $\mathrm{TOP}/(\mathit{Ob}\ G)$,

$$G \times_{\mathit{Ob}\ G} G \times_{\mathit{Ob}\ G} A \to G \times_{\mathit{Ob}\ G} A,$$

which sends (g, g', a) into (gg', a). Thus $G \times_{\mathit{Ob}\ G} A$ is a G-space and we have the functor

$$i_{\#} : \mathrm{TOP}/(\mathit{Ob}\ G) \to \mathrm{TOP}^G, \qquad A \mapsto G \times_{\mathit{Ob}\ G} A.$$

A *principal G-space* X in TOP is a G-space for which there exists a filtration $X = \bigcup_{n \geq 0} X_n$ in TOP^G such that

(i) there exists a space A_0 in $\mathrm{TOP}/(\mathit{Ob}\ G)$ such that

$$X_0 \simeq G \times_{\mathit{Ob}\ G} A_0$$

in TOP^G, and

(ii) for each $n \geq 0$ there exists a closed cofibered pair (A_n, B_n) in $\mathrm{TOP}/(\mathit{Ob}\ G)$ and a pushout diagram

$$\begin{tikzcd} G \times_{\mathit{Ob}\ G} B_n \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ G \times_{\mathit{Ob}\ G} A_n \arrow[r] & X_n \end{tikzcd}$$

in TOP^G.

If G is a topological category satisfying the cofibration condition, then each $E_G X$ is a principal G-space; i.e. the generalization of (4.1) then holds. Recall also from (6.11) that $E_G X \to X$ is then a G-map and a homotopy equivalence in $\mathrm{TOP}/(\mathit{Ob}\ G)$. For the case of a topological category G, these will be defined to be the *weak homotopy equivalences in* TOP^G; i.e. they are the G-maps which are also homotopy equivalences in $\mathrm{TOP}/(\mathit{Ob}\ G)$.

The reader who checks all details of these paragraphs will have to start with a burdensome extension of (3.9) from TOP to TOP/P, where P is a compactly generated space. Namely, consider the diagram

$$\begin{tikzcd} A \arrow[r, "\phi_0"] \arrow[d, "\nu"'] & Y \arrow[d, "g"'] \\ X \arrow[r, "\phi_1"] & B \end{tikzcd}$$

in TOP/P where ν is a cofibration in TOP/P and where g is a homotopy equivalence in TOP/P. Suppose we are given a homotopy $H_0 : I \times A \to B$ in TOP/P

joining $g\phi_0$ to $\phi_1\nu$. Then there exist a map $\theta : X \to Y$ in TOP$/P$ and a homotopy $H : I \times X \to B$ in TOP$/P$ joining $g\theta$ to ϕ_1 and with $H(1_I \times \nu) = H_0$. The appropriate supporting lemmas and proof will work as before.

With this extended form of (3.9). the reader can check that the proof of (4.3) continues to hold; i.e. if G is a topological category and if $\phi : X \to Y$ is a weak homotopy equivalence in TOP^G joining the principal G-space X to the principal G-space Y, then ϕ is a homotopy equivalence in TOP^G.

Theorem 6.12 *Let G be a topological category satisfying the cofibration condition. Consider the category TOP^G, together with the notion of homotopy in TOP^G and of weak homotopy equivalence in TOP^G. Take also the functor*

$$E_G(\diamond) : TOP^G \to TOP^G$$

of (6.9), together with the natural transformation $T : E_G(\diamond) \to 1$. Continue to denote by $PRINC^G$ the full subcategory of principal G-spaces. With this structure, TOP^G is a category with principal objects in the sense of Chapter 3.

The point of the above is that it is a direct extension of (4.4). How does one handle the case in which the topological category G does not satisfy the cofibration condition? The answer is that one can then substitute a functor $\mathcal{E}_G(\diamond) : \mathrm{TOP}^G \to \mathrm{TOP}^G$ for the above, based on the homotopy colimit functor $B : \mathrm{TOP}^{\Delta^o} \to \mathrm{TOP}$. Namely, take $NM(G, X)$ in TOP^{Δ^o} and follow with the functor B.

Theorem 6.13 *Let G be a topological category. Assume the structures of (6.12) except replace $E_G(\diamond)$ by $\mathcal{E}_G(\diamond)$ and a corresponding natural transformation*

$$T' : \mathcal{E}_G(\diamond) \to 1.$$

With this given structure, TOP^G is a category with principal objects in the sense of Chapter 3.

Homotopy Pullback Diagrams

We have found occasion to use already the standard homotopy pushout of a diagram

$$X_1 \xleftarrow{f} X_0 \xrightarrow{\phi_0} Y_0$$

in TOP, namely the two-sided mapping cylinder

$$C = X_1 \cup_f I \times X_0 \cup_{\phi_0} Y_0.$$

A commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\phi_0} & Y_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X_1 & \xrightarrow{\phi_1} & Y_1 \end{array}$$

in TOP is said to be a *homotopy pushout diagram*, or to be *homotopy cocartesian*, if the natural map

$$X_1 \cup_f \; I \times X_0 \; \cup_{\phi_0} Y_0 \to Y_1$$

is a homotopy equivalence in TOP.

We need now the dual notion. Let Ψ denote the small category

$$0 \hookrightarrow 2 \hookleftarrow 1,$$

and consider the Ψ-space

$$X_1 \xrightarrow{\phi_1} Y_1 \xleftarrow{g} Y_0.$$

The *standard homotopy limit*, or *standard homotopy pullback*, of this Ψ-space is the compactly generated space

$$L = X_1 \times_{Y_1} (Y_1)^I \times_{Y_1} Y_0,$$

consisting of triples

$$x \in X_1, \qquad \sigma : I \to Y_1, \qquad y \in Y_0$$

such that $\sigma(0) = \phi_1(x)$ and $\sigma(1) = g(y)$.

A commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\phi_0} & Y_0 \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ X_1 & \xrightarrow{\phi_1} & Y_1 \end{array}$$

in TOP is said to be a *homotopy pullback diagram*, or to be *homotopy cartesian*, if the natural map

$$X_0 \to L = X_1 \times_{\phi_1} (Y_1)^I \times_g Y_0$$

is a homotopy equivalence in TOP. We come back to such diagrams after setting up a circumstance in which they are needed.

Specializing the Topological Category to a Topological Monoid

A *topological monoid* is a topological category G for which the space *Ob* G is a singleton. Thus it is a compactly generated space G together with an associative multiplication

$$G \times G \to G, \qquad (g_1, g_2) \mapsto g_1 g_2,$$

for which there is a two-sided identity $1 \in G$. A *homomorphism* $f : G \to G'$ joining topological monoids is a map with

$$f(g_1 g_2) = f(g_1) f(g_2), \qquad f(1) = 1.$$

Denote by TOP MON the category of topological monoids and homomorphisms.

The topological monoid G is said to have *homotopy inverses* if there exists a map $\theta : G \to G$ such that the map

$$G \to G, \qquad g \mapsto g(\theta(g))$$

is homotopic to the constant map sending each g into 1. Technically this requires only a right homotopy inverse, but it follows that the right homotopy inverse is also a left homotopy inverse. Let $\mu(g) = (\theta(g))g$. Then the map $g \mapsto (\mu(g))(\mu(g))$ is homotopic to $g \mapsto \mu(g)$. Operating with $\theta(\mu(g))$ on the right we get that μ is homotopic to the constant map.

Return now to the problem at hand; i.e. properties of the classifying space of a topological monoid. Consider first the case in which the base point $\{1\}$ is cofibered in G. We can then take the standard universal G-space E_G and the standard classifying space B_G to be

$$E_G = |NM_1(G)|, \qquad B_G = |N(G)|$$

as in Chapter 2. Since $(B_G)_0 = Ob\ G$ is a singleton, then B_G has a natural cofibered base point b_0. Since $Ob\ G$ is a singleton, also E_G is a single compactly generated space upon which the monoid acts continuously. Moreover B_G is the space of orbits and there is the natural quotient map $\pi : E_G \to B_G$. It is readily checked that $\pi^{-1}(b_0) \simeq G$. There is then the natural commutative diagram

$$\begin{tikzcd} \pi^{-1}(b_0) \simeq G \arrow[r] \arrow[d] & E_G \arrow[d, "\pi"'] \\ b_0 \arrow[r] & B_G. \end{tikzcd}$$

We assume for the moment the following theorem, whose proof will occupy us for the rest of the chapter. Such theorems go back to Dold-Lashof [6.1] with weak homotopy equivalences replacing homotopy equivalences. In the present generality, the theorem is due to M. Fuchs [6.2]. Segal [4.4] has generalized such theorems; we use the language and methods of Segal and of V. Puppe [6.5].

Theorem 6.14 *Let G be a topological monoid with homotopy inverses, such that the base point $1 \in G$ is cofibered in G. Then the above commutative diagram*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & E_G \arrow[d, "\pi"'] \\ b_0 \arrow[r] & B_G \end{tikzcd}$$

is a homotopy pullback diagram in TOP.

If the base point is not cofibered in G, then we must use (6.13) instead of (6.12). We then have the G-space $\mathcal{E}_G$ instead of E_G, and the compactly generated space $\mathcal{B}_G$ instead of B_G, where

$$\mathcal{B}_G = NG \times_{Mono\,\Delta} \nabla.$$

Theorem 6.15 *Let G be a topological monoid with homotopy inverses. Then the commutative diagram*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \mathcal{E}_G \arrow[d] \\ b_0 \arrow[r] & \mathcal{B}_G \end{tikzcd}$$

is a homotopy pullback diagram in TOP.

We will begin shortly the proof of (6.15), but will first restate the theorems in terms of the natural maps

$$G \hookrightarrow \Omega B_G, \qquad G \hookrightarrow \Omega \mathcal{B}_G.$$

Corollary 6.16 *Let G be a topological monoid with homotopy inverses. Then the natural map*

$$G \to \Omega \mathcal{B}_G,$$

which takes $g \in G$ into the loop $t \mapsto g \times_{Mono\,\Delta} (1-t,t)$ of $\mathcal{B}_G$, is a homotopy equivalence in TOP. If also the base point $1 \in G$ is cofibered in G, then $G \to \Omega B_G$ is a homotopy equivalence in TOP.

Proof. Since $\mathcal{E}_G$ is contractible, the unique map $\mathcal{E}_G \to b_0$ is a homotopy equivalence. We have the commutative diagram

$$\begin{tikzcd} G \arrow[r, equal] \arrow[d] & G \arrow[d] \\ b_0 \times_{\mathcal{B}_G} (\mathcal{B}_G)^I \times_{\mathcal{B}_G} \mathcal{E}_G \arrow[r] & b_0 \times_{\mathcal{B}_G} (\mathcal{B}_G)^I \times_{\mathcal{B}_G} b_0 \simeq \Omega \mathcal{B}_G. \end{tikzcd}$$

The left hand map and the bottom map are homotopy equivalences, hence the righthand map is a homotopy equivalence. □

We have taken the following from Fuchs [6.2].

Theorem 6.17 *Let G be a topological monoid. Denote by $\pi_0 G$ the monoid of path components of G, with base point the path component containing $1 \in G$. If $\pi_0 G$ is a group and if G is of the homotopy type of a CW-complex, then G has homotopy inverses.*

Proof. The proof rests on consideration of the commutative diagram

$$\begin{tikzcd} G \times G \arrow[r, "\theta"] \arrow[d, "f"'] & G \times G \arrow[d, "f"'] \\ G \arrow[r, equal] & G, \end{tikzcd}$$

where $f(g_1, g_2) = g_1$ and $\theta(g_1, g_2) = (g_1, g_1 g_2)$. Now f is a fibration, and the fibers are the ordered pairs with fixed first coordinate. Then θ maps each fiber into itself. Since $\pi_0 G$ is a group, then θ maps each fiber into itself by a homotopy equivalence. If the base space G is of the homotopy type of a CW-complex, it then follows from a remarkable theorem of Dold [3.2] that θ is a fiber homotopy equivalence in the category of spaces over G. See also tom Dieck, Kamps and Puppe [3.1] for a proof. Now let $\phi : G \times G \to G \times G$ be a fiber homotopy inverse. Since $\theta\phi$ is homotopy equivalent to the identity in spaces over G, it is checked that $g_1 \mapsto \phi(g_1, 1)$ is a right homotopy inverse. □

The James Model for ΩSA

We continue to motivate the later work involving homotopy pullback diagrams with examples from their history, and thus now give an account of early, influential constructions of James [6.4]; see also tom Dieck, Kamps and Puppe [3.1].

Let A be a compactly generated space with cofibered base point a_0. We need the *free topological monoid* JA generated by the points of A, where the base point a_0 becomes the identity element of JA. As with the rest of this example, this is due to James. Define $JA = (\coprod_{n\geq 1} A^n)/\sim$, where $\sim$ is the least equivalence relation such that

$$(a_1, \cdots, a_n) \sim (a_1, \cdots, a_{i-1}, a_0, a_i, \cdots, a_n)$$

and where A^0 is a_0. There is the natural quotient map $\pi : \coprod A^n \to JA$. Let $(JA)_n = \pi A^n$, and show that $JA = \bigcup (JA)_n$ is a cofibered filtered space in TOP whose term $(JA)_0$ is the singleton $1 = \pi(a_0)$. For each positive n, there is the relative homeomorphism

$$\pi : (A^n, A^{n,deg}) \to ((JA)_n, (JA)_{n-1}),$$

where $A^{n,deg}$ consists of all $(a_1, \cdots, a_n)$ with at least one $a_i = a_0$.

The natural homeomorphisms $A^m \times A^n \simeq A^{m+n}$ give a well defined map

$$[(\coprod A^m)/\sim] \times [(\coprod A^n)/\sim] \to (\coprod A^r)/\sim,$$

thus we have the natural map $JA \times JA \to JA$, which makes JA a topological monoid. It is also the case that since a_0 is cofibered in A, then 1 is cofibered in JA. The monoid JA is filtered in the sense that the multiplication sends $(JA)_m \times (JA)_n$ into $(JA)_{m+n}$. Moreover $(JA)_1$ can be taken to be A. Thus we have the free topological monoid generated by A.

The following construction of James is then the prototype for numerous variations later. Denote by CA the reduced cone over A; that is,

$$CA = I \times A/0 \times A \cup I \times a_0.$$

Identify A as a closed subspace of CA by identifying A with its copy $1 \times A$. Thus we have

$$A = (JA)_1 \subset CA, \qquad JA \times A \subset JA \times CA.$$

We then have also the composition

$$JA \times A = JA \times (JA)_1 \to JA$$

where the last map is multiplication. Thus we have the diagram

$$JA \times CA \hookleftarrow JA \times A \to JA.$$

Denote the pushout of this diagram by E.

We see next that E is naturally filtered as $E = \coprod_{n\geq 0} E_n$. For $n = 0$ take E_0 as the natural base point, alternatively the pushout of

$$\emptyset \leftarrow \emptyset \to 1$$

and for $n > 0$ as the pushout of

$$(JA)_{n-1} \times CA \hookleftarrow (JA)_{n-1} \times A \to (JA)_n.$$

It is next checked that for $n \geq 1$ there is a relative homeomorphism

$$((JA)_{n-1}, (JA)_{n-2}) \times (CA, e) \to (E_n, E_{n-1}),$$

where e is the base point of CA. Thus (E_n, E_{n-1}) is a cofibered pair. Since e is a strong deformation retract of CA, then E_{n-1} is a strong deformation retract of E_n. It then follows that the base point E_0 is a strong deformation retract of E. In particular, the space E is contractible.

One next proceeds to the commutative diagram

$$\begin{array}{ccccc} JA \times CA & \xleftarrow{\ i\ } & JA \times A & \xrightarrow{\ m\ } & JA \\ {\scriptstyle p}\downarrow & & {\scriptstyle p_1}\downarrow & & \downarrow \\ CA & \xleftarrow{\ j\ } & A & \longrightarrow & pt, \end{array}$$

where p and p_1 are projection maps. The pushout of the top line is the contractible space E, the pushout of the bottom line is the reduced suspension $SA = CA/A$, and we have a map $q : E \to SA$ of pushouts. Thus we have a commutative diagram

$$\begin{array}{ccc} JA & \longrightarrow & E \\ \downarrow & & {\scriptstyle q}\downarrow \\ pt & \longrightarrow & SA. \end{array}$$

Whenever this diagram is a homotopy pullback diagram, then we will have a natural homotopy equivalence $JA \to \Omega SA$.

We will later prove the following theorem about homotopy pullbacks.

Theorem 6.18 *Suppose given the commutative diagram*

$$\begin{array}{ccccc} X_0 & \xleftarrow{\ \phi_0\ } & Y_0 & \xrightarrow{\ \theta_0\ } & Z_0 \\ {\scriptstyle \nu}\downarrow & & {\scriptstyle \nu'}\downarrow & & {\scriptstyle \nu''}\downarrow \\ X_1 & \xleftarrow{\ \phi_1\ } & Y_1 & \xrightarrow{\ \theta_1\ } & Z_1, \end{array}$$

where ϕ_0 and ϕ_1 are cofibrations and where both rectangles are homotopy pullback diagrams. Then both rectangles of the commutative diagram

$$\begin{array}{ccccc} X_0 & \longrightarrow & X_0 \cup_{\theta_0} Z_0 & \longleftarrow & Z_0 \\ {\scriptstyle \nu}\downarrow & & {\scriptstyle \nu'''}\downarrow & & {\scriptstyle \nu''}\downarrow \\ X_1 & \longrightarrow & X_1 \cup_{\theta_1} Z_1 & \longleftarrow & Z_1 \end{array}$$

are homotopy pullback diagrams, where ν''' is the induced map of pushouts.

For the moment we assume the theorem in order to obtain from it the following classic theorem of James [6.4].

Theorem 6.19 *Let A be a path connected space with cofibered base point a_0, and suppose A is of the homotopy type of a CW-complex. Then there is the homotopy pullback diagram*

$$\begin{array}{ccc} JA & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle q} \\ pt & \longrightarrow & SA \end{array}$$

and hence we have a homotopy equivalence $JA \sim \Omega SA$.

Proof. The theorem will follow from (6.18) if we can show that each rectangle of

$$\begin{array}{ccccc} JA \times CA & \xleftarrow{\ i\ } & JA \times A & \xrightarrow{\ m\ } & JA \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p_1}\big\downarrow & & \big\downarrow \\ CA & \xleftarrow{\ j\ } & A & \longrightarrow & pt \end{array}$$

is a homotopy pullback diagram.

The first rectangle is a pullback diagram of fibrations, hence is a homotopy pullback diagram.

The second rectangle is a homotopy pullback diagram if and only if the map

$$\mu : JA \times A \to JA \times A, \qquad (w, a) \mapsto (wa, a)$$

is a homotopy equivalence. This map $JA \times A \to JA \times A$ can be regarded as a map of spaces over A, and the proof of (6.17) can be applied. For each fixed a, μ is a homotopy equivalence of the fiber over a, since A and therefore JA are path connected. Since A has been assumed of the homotopy type of a CW-complex, then μ is a homotopy equivalence as in (6.17) by Dold's Theorem. □

The Category PULL of Homotopy Pullback Diagrams

Denote by MAP the category whose objects X are all maps

$$\nu = \nu_X : X_0 \to X_1$$

in TOP, and whose morphisms $\phi : X \to Y$ are the commutative diagrams

$$\begin{array}{ccc} X_0 & \xrightarrow{\phi_0} & Y_0 \\ {\scriptstyle \nu}\big\downarrow & & {\scriptstyle \nu'}\big\downarrow \\ X_1 & \xrightarrow{\phi_1} & Y_1 \end{array}$$

in TOP. This is the category we have previously denoted by $\mathrm{TOP}^{0\to 1}$. Recall that we denote by WHE the subcategory consisting of all $\phi : X \to Y$ such that both ϕ_0 and ϕ_1 are homotopy equivalences in TOP. Recall also that the coprincipal objects of MAP are those X for which $\nu : X_0 \to X_1$ is a fibration.

Denote by PULL the subcategory whose morphisms are all homotopy pullback diagrams. Then for each $\phi : X \to Y$ in PULL, we have the homotopy equivalence

$$X_0 \sim L(\phi) = X_1 \times_{Y_1} (Y_1)^I \times_{Y_1} Y_0.$$

For each object Y in MAP, there is the functorial factorization

$$Y_0 \hookrightarrow E'Y_0 \xrightarrow{E'\nu'} Y_1$$

of the map $\nu' : Y_0 \to Y_1$, as in (3.3;vi). Here $E'Y_0 = (Y_1)^I \times_{Y_1} Y_0$. Then the homotopy pullback of the original diagram can be checked to be the pullback of the diagram

$$X_1 \xrightarrow{\phi_1} Y_1 \xleftarrow{E'\nu'} E'Y_0.$$

Thus we have the pullback diagram

$$\begin{tikzcd} \phi_1^* E'Y_0 \arrow[r] \arrow[d] & E'Y_0 \arrow[d, "E'\nu'"'] \\ X_1 \arrow[r, "\phi_1"] & Y_1, \end{tikzcd}$$

and ϕ is in PULL if and only if the natural map $X_0 \to \phi_1^* E'Y_0$ is a homotopy equivalence in TOP. Equivalently, ϕ is in PULL if and only if the natural map

$$E'X_0 \to \phi_1^* E'Y_0$$

is a homotopy equivalence. It then follows from (3.4) that $E'X_0 \to \phi_1^* E'Y_0$ is a fiber homotopy equivalence in TOP$/X_1$.

If $\phi : X \to Y$ and $\theta : Y \to Z$ are in PULL, then $\theta\phi : X \to Z$ is in PULL

For since θ is in PULL, then the map

$$E'Y_0 \to \theta_1^* E'Z_0$$

is a fiber homotopy equivalence as spaces over Y_1. It then follows that

$$\phi_1^* E'Y_0 \to \phi_1^* \theta_1^* E'Z_0$$

is a fiber homotopy equivalence as spaces over X_1. Since ϕ is in PULL, then

$$E'X_0 \to \phi_1^* E'Y_0$$

is a homotopy equivalence. The remark follows by combining these.

WHE is Contained in PULL

For consider the commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "\phi_0"'] & \phi_1^* E' Y_0 \arrow[r] \arrow[d, "f"'] & X_1 \arrow[d, "\phi_1"'] \\
Y_0 \arrow[r, "i"] & E' Y_0 \arrow[r] & Y_1 .
\end{tikzcd}$$

We at this time assume without proof a standard theorem of fibrations: namely, since the above map ϕ_1 is a homotopy equivalence, then the above map $f : \phi_1^* E' Y_0 \to E' Y_0$ is a homotopy equivalence. For a proof, see tom Dieck, Kamps and Puppe [3.1,p.137]. Since ϕ_0, i and f are homotopy equivalences, then

$$X_0 \to \phi_1^* E' Y_0$$

is a homotopy equivalence.

Note the special case that all identity morphisms of MAP are in PULL.

If $\theta\phi$ is in PULL and one of ϕ, θ is in WHE, then the other is in PULL

Suppose for example that ϕ is in WHE, and that $\theta\phi$ is in PULL. Using again the above cited theorem concerning fibrations induced by homotopy equivalences, we get the diagram

$$\begin{tikzcd}
E' X_0 \arrow[r, "\sim"] & \phi_1^* E' Y_0 \arrow[r] \arrow[d, "\sim"'] & \phi_1^* \theta_1^* E' Z_0 \arrow[d, "\sim"'] \\
 & E' Y_0 \arrow[r] & \theta_1^* E' Z_0 .
\end{tikzcd}$$

It follows readily from the hypotheses that

$$\phi_1^* E' Y_0 \to \phi_1^* \theta_1^* E' Z_0$$

is a homotopy equivalence, from which it follows that

$$E' Y_0 \to \theta_1^* E' Z_0$$

is a homotopy equivalence.

The following is taken from tom Dieck, Kamps and Puppe [3.1].

Theorem 6.20 *Assume the commutative diagram in TOP,*

$$\begin{tikzcd}
X_0 \arrow[rr] \arrow[dr, "\sim"] \arrow[dd] & & Y_0 \arrow[dr, "\sim"] \arrow[dd] & \\
 & X_0' \arrow[rr] \arrow[dd] & & Y_0' \arrow[dd] \\
X_1 \arrow[rr] \arrow[dr, "\sim"] & & Y_1 \arrow[dr, "\sim"] & \\
 & X_1' \arrow[rr] & & Y_1'
\end{tikzcd}$$

where the four designated maps are homotopy equivalences in TOP. Then the back face is a homotopy pullback diagram if and only if the front face is a homotopy pullback diagram.

The proof is left as an exercise; use the preceding proposition.

The above are superficial properties of the category PULL. There are less superficial properties, taken from Segal[4.4] and V. Puppe [6.5], which do things like prove (6.18). We proceed slowly with these, and start by trying to understand better fiber homotopy equivalences.

A Characterization of Fiber Homotopy Equivalences

Given a compactly generated space B, denote by TOP$/B$ the category whose objects are the maps $\nu : E \to B$ and whose morphisms are the commutative diagrams

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\nu"'] & E' \arrow[d, "\nu'"] \\ B \arrow[r, equal] & B. \end{tikzcd}$$

A *Dold fibration* $\nu : E \to B$ is an object of TOP$/B$ which is homotopy equivalent in TOP$/B$ to a fibration $\nu' : E' \to B$.

We assume basic properties of Dold fibrations; see Dold [3.2] or tom Dieck, Kamps and Puppe [3.1]. Among these is the local characterization: if $\nu : E \to B$ is a map in TOP such that there exists a numerable covering $\{V_j | j \in J\}$ of B with each $\nu_j : \nu^{-1} V_j \to V_j$ a Dold fibration, then ν is a Dold fibration [3.1,p.157].

DEFINITION. Consider the class of fibrations $\nu : E \to B$ for B fixed. Define the fibrations $\nu_0 : E_0 \to B$ and $\nu_1 : E_1 \to B$ to be equivalent if there exists a Dold fibration $\pi : E \to I \times B$ and for $\epsilon > 0$ sufficiently small pullback diagrams

$$\begin{tikzcd} {[0,\epsilon] \times E_0} \arrow[r] \arrow[d, "1\times\nu_0"'] & E \arrow[d, "\pi"'] & {[1-\epsilon, 1] \times E_1} \arrow[l] \arrow[d, "1\times\nu_1"'] \\ {[0,\epsilon] \times B} \arrow[r, "i"] & I \times B & {[1-\epsilon, 1] \times B.} \arrow[l, "j"'] \end{tikzcd}$$

Write $\nu_0 \sim \nu_1$ if this condition is satisfied.

The relation $\sim$ is an equivalence relation. Transitivity follows readily from the local characterization of Dold fibrations.

If ν_0 and ν_1 are equivalent as above, then they are fiber homotopy equivalent. This is again a version of a standard theorem of fibrations. See tom Dieck, Kamps and Puppe [3.1,p.132].

In fact, it follows from methods of Dold that fibrations ν_0 and ν_1 are equivalent as above if and only if they are fiber homotopy equivalent. We proceed to set up the machinery needed, assuming fibrations ν_0 and ν_1 and a fiber homotopy equivalence $\phi : E_0 \to E_1$.

Assume first the following straight-forward special case of a theorem of Hastings [6.3]: the map

$$p : (E_1)^I \to E_1 \times_B B^I \times_B E_1$$

sending a path σ into the triple $(\sigma(0), \nu_1\sigma, \sigma(1))$ is a fibration.

Next consider the inclusion map

$$i : E_1 \times_B E_1 \hookrightarrow E_1 \times_B B^I \times_B E_1$$

sending the pair (e, e') with $\nu_1(e) = \nu_1(e')$ into the triple (e, σ, e') where σ is the constant path at $\nu_1(e)$. Take the pullback fibration

$$q : i^*(E_1)^I \to E_1 \times_B E_1$$

whose total space consists of all triples

$$e \in E_1, \qquad \sigma : I \to E_1, \qquad e' \in E_1$$

such that $\nu_1(e) = \nu_1(e')$ and $\nu_1\sigma$ is a constant path in B; i.e. such that the path σ lies in a fiber of ν_1.

Next factor into the above the fiber homotopy equivalence ϕ. Specifically, take the homotopy equivalence

$$\phi \times_B 1 : E_0 \times_B E_1 \to E_1 \times_B E_1$$

and pull back the fibration q, obtaining the pullback diagram

$$\begin{tikzcd} W(\phi) \arrow[r] \arrow[d, "\nu'"'] & i^*(E_1)^I \arrow[d, "q"] \\ E_0 \times_B E_1 \arrow[r] & E_1 \times_B E_1. \end{tikzcd}$$

Since q is a fibration, then ν' is a fibration.

It is readily checked that projection $E_0 \times_B E_1 \to E_1$ is a fibration, hence the composition

$$W(\phi) \xrightarrow{\nu'} E_0 \times_B E_1 \to E_1$$

gives a fibration $\nu : W(\phi) \to E_1$ due to Dold [3.2]. Here $W(\phi)$ consists of all triples

$$e_0 \in E_0, \qquad \sigma : I \to E_1, \qquad e_1 \in E_1$$

with σ contained in a single fiber of ν_1 and with $\phi(e_0) = \sigma(0)$ and $\sigma(1) = e_1$. Moreover ν maps (e_0, σ, e_1) into e_1.

There is the natural inclusion of E_0 in $W(\phi)$, sending e_0 into the triple $(e_0, \sigma, \phi(e_0))$ where σ is the constant path at $\phi(e_0)$. This is clearly a homotopy equivalence. Thus we have Dold's factorization of the fiber homotopy equivalence ϕ,

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d, "\nu_0"'] & W(\phi) \arrow[r, "\nu"] \arrow[d, "\nu''"'] & E_1 \arrow[d, "\nu_1"'] \\ B \arrow[r, equal] & B \arrow[r, equal] & B, \end{tikzcd}$$

where ν, ν_0, ν'', ν_1 are fibrations, all the top maps are homotopy equivalences, and where ν'' maps (e_0, σ, e_1) into $\nu_0 e_0 = \nu_1 e_1$.

Theorem 6.21 *Let B be a compactly generated space, and let ν_0 and ν_1 be fibrations over B. Then $\nu_0 \sim \nu_1$ in the above sense if and only if ν_0 and ν_1 are* *fiber homotopically equivalent.*

PROOF. Suppose ν_0 and ν_1 are fiber homotopically equivalent, via $\phi : E_0 \to E_1$. We have to prove in the above construction that $\nu_0 \sim \nu''$ and $\nu'' \sim \nu_1$.

In order to prove $\nu_0 \sim \nu''$, we need that

$$[0, 1/2] \times E_0 \cup_{E_0} [1/2, 1] \times W(\phi) \to I \times B$$

is a Dold fibration. We leave it as an exercise for the reader that it is fiber homotopy equivalent to the fibration $1 \times \nu_0 : I \times E_0 \to I \times B$.

The heart of the matter is proving that $\nu'' \sim \nu_1$. Since $\nu : W(\phi) \to E_1$ is both a fibration and a homotopy equivalence, we can use (3.8) on the commutative diagram

$$\begin{tikzcd} W(\phi) \arrow[r, equal] \arrow[d, "\nu"'] & W(\phi) \arrow[d, "\nu"'] \\ E_1 \arrow[r, equal] & E_1 \end{tikzcd}$$

to obtain a map $s : E_1 \to W(\phi)$ with $\nu s = 1$ and a homotopy $D : I \times W(\phi) \to W(\phi)$ joining 1 to $s\nu$ such that for each $w \in W(\phi)$ all the $D(t, w)$ lie in a single fiber of ν. This having been noted, we leave it as an exercise to show that

$$[0, 1/2] \times W(\phi) \cup_{E_1} [1/2, 1] \times E_1 \to I \times B$$

is fiber homotopically equivalent to $1 \times \nu_1 : I \times E_1 \to I \times B$. It follows that $\nu'' \sim \nu_1$. $\square$

Corollary 6.22 *Consider the fibrations $\nu_0 : E_0 \to B$ and $\nu_1 : E_1 \to B$, and let $\phi : E_0 \to E_1$ be a fiber homotopy equivalence in FIB/B. Then $\nu_0 \sim \nu_1$ by means of the Dold fibration $\pi : E(\phi) \to I \times B$ where*

$$E(\phi) = [0, 1/3] \times E_0 \cup_{W(\phi)} [1/3, 2/3] \times W(\phi) \cup_{E_1} [2/3, 1] \times E_1,$$

with the map π the natural map.

We also interpret the above as follows. Suppose we are given the fiber homotopy equivalence $E_0 \xrightarrow{\phi} E_1$ of fibrations over B. Replace it by the commutative diagram

$$\begin{tikzcd} E_0 \arrow[r, "i"] \arrow[d, "\nu_0"'] & W(\phi) \arrow[r, "\nu"] \arrow[d, "\nu''"'] & E_1 \arrow[r, "1"] \arrow[d, "\nu_1"'] & E_1 \arrow[d, "\nu_1"'] \\ B \arrow[r, equal] & B \arrow[r, equal] & B \arrow[r, equal] & B. \end{tikzcd}$$

Denote by T_0 the mapping telescope of the top line, and by T_1 the mapping telescope $T_1 = I \times B$ of the bottom line. Then the induced map $\rho : T_0 \to I \times B$ is a Dold fibration.

Extension of the Above Methods to PULL

Consider now a commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\phi_0} & Y_0 \\ {\scriptstyle\nu}\big\downarrow & & {\scriptstyle\nu'}\big\downarrow \\ X_1 & \xrightarrow{\phi_1} & Y_1, \end{array}$$

in TOP, where ν and ν' are fibrations. We wish to characterize when ϕ is a homotopy pullback diagram in terms of Dold fibrations.

Suppose first that ϕ is in PULL. There is then the fiber homotopy equivalence

$$\begin{array}{ccc} X_0 & \xrightarrow{\phi_0'} & \phi_1^* Y_0 \\ {\scriptstyle\nu}\big\downarrow & & \big\downarrow \\ X_1 & = & X_1, \end{array}$$

hence there is the Dold fibration $\pi : E(\phi_0') \to I \times X_1$ of (6.22).

There is also a fibration

$$\pi' : E'(\phi) \to [1, 3/2] \times X_1 \cup_{\phi_1} [3/2, 2] \times Y_1$$

at hand. Namely there is the natural map

$$[1, 3/2] \times X_1 \cup_{\phi_1} [3/2, 2] \times Y_1 \to Y_1$$

and one can pull back the fibration ν'.

There is in both $E(\phi_0')$ and $E'(\phi)$ the subset $1 \times \phi_1^* Y_0$ so that we can form the union

$$D(\phi) = E(\phi_1') \cup_{1 \times \phi_1^* Y_0} E'(\phi)$$

and obtain a union map

$$\tau : D(\phi) \to [0, 3/2] \times X_1 \cup_{\phi_1} [3/2, 2] \times Y_1$$

which is checked to be a Dold fibration. The base space of τ is a variation of the mapping cylinder of ϕ_1; denote it by $M(\phi_1)$ so that we have the Dold fibration $\tau : D(\phi) \to M(\phi_1)$.

There is the similar variation $M(\phi_0) = [0, 3/2] \times X_0 \cup_{\phi_0} [3/2, 2] \times Y_0$ of the mapping cylinder of ϕ_0, and a natural homotopy equivalence

$$D(\phi) \sim M(\phi_0).$$

Thus from the homotopy pullback diagram we have obtained a commutative diagram

$$\begin{array}{ccc} D(\phi) & \xrightarrow{\rho} & M(\phi_0) \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow \\ M(\phi_1) & = & M(\phi_1), \end{array}$$

where τ is a Dold fibration and ρ is a homotopy equivalence.

We restate the construction. We have considered the diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & \phi_1^* Y_0 \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d] & Y_0 \arrow[d] \\
X_1 \arrow[r] & X_1 \arrow[r] & X_1 \arrow[r] & Y_1 \arrow[r] & Y_1
\end{tikzcd}$$

and have taken the mapping telescopes T_0 and T_1 of top and bottom, as well as the induced map $\tau : T_0 \to T_1$. This has turned out to be a Dold fibration, and $T_0 \simeq D(\phi)$.

For simple linear diagrams in PULL, one can fit together the various $D(\phi)$. Consider first a commutative diagram

$$\begin{tikzcd}
Y_0 \arrow[d, "\nu"'] & X_0 \arrow[l, "\phi_0"'] \arrow[r, "\theta_0"] \arrow[d, "\nu'"'] & Z_0 \arrow[d, "\nu''"'] \\
Y_1 & X_1 \arrow[l, "\phi_1"'] \arrow[r, "\theta"] & Z_1
\end{tikzcd}$$

in PULL. One then has the commutative diagram

$$\begin{tikzcd}
D(\phi) \arrow[d, "\tau"'] & X_0 \arrow[l] \arrow[r] \arrow[d, "\nu'"'] & D(\theta) \arrow[d, "\tau'"'] \\
M(\phi_1) & X_1 \arrow[l] \arrow[r] & M(\theta_1)
\end{tikzcd}$$

and thus can form the union

$$\tau \cup_{X_1} \tau' : D(\phi) \cup_{X_0} D(\theta) \to M(\phi_1) \cup_{X_1} M(\theta_1).$$

Denote the domain by D, and denote the range by $M(1)$. There is a similar $M(0) = M(\phi_0) \cup_{X_0} M(\theta_0)$ and a commutative diagram

$$\begin{tikzcd}
D \arrow[r, "\sim"] \arrow[d, "\tau''"'] & M(0) \arrow[d] \\
M(1) \arrow[r, equal] & M(1)
\end{tikzcd}$$

where τ'' is a Dold fibration.

The payoff then comes as follows. We have a commutative diagram

$$\begin{tikzcd}
Y_0 \arrow[r] \arrow[d, "\nu"'] & D \arrow[d, "\tau''"'] & Z_0 \arrow[l] \arrow[d, "\nu''"'] \\
Y_1 \arrow[r] & M(1) & Z_1 \arrow[l]
\end{tikzcd}$$

which is a diagram of Dold fibrations and pullback diagrams. That is, this diagram is a diagram in PULL. It then follows from (6.20) that

$$\begin{tikzcd}
Y_0 \arrow[r] \arrow[d] & M(0) \arrow[d] & Z_0 \arrow[l] \arrow[d] \\
Y_1 \arrow[r] & M(1) & Z_1 \arrow[l]
\end{tikzcd}$$

is a diagram in PULL. But $M(0)$ is just a form of the homotopy pushout of

$$Y_0 \xleftarrow{\phi_0} X_0 \xrightarrow{\theta_0} Z_0,$$

$M(1)$ is the corresponding form of the homotopy pushout of the base maps, and the connecting map is that induced on homotopy pushouts.

We have proved a basic theorem of V. Puppe [6.5].

Theorem 6.23 *Suppose given the diagram*

$$\begin{tikzcd} Y_0 \arrow[d, "\nu"'] & X_0 \arrow[l, "\phi_0"'] \arrow[r, "\theta_0"] \arrow[d, "\nu'"'] & Z_0 \arrow[d, "\nu''"'] \\ Y_1 & X_1 \arrow[l, "\phi_1"'] \arrow[r, "\theta_1"] & Z_1 \end{tikzcd}$$

of homotopy pullback diagrams. Let M_0 and M_1 denote the standard homotopy pushouts of the top row and bottom row respectively, and let $\nu''' : M_0 \to M_1$ denote the induced map. Then the rectangles of

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & M_0 \arrow[d] & Z_0 \arrow[l] \arrow[d] \\ Y_1 \arrow[r] & M_1 & Z_1 \arrow[l] \end{tikzcd}$$

are homotopy pullback diagrams.

Theorem 6.18 follows as a corollary from (6.23), using (6.20) as an aid.

We leave the entirely similar following proposition to the reader. One must first prove it in terms of mapping telescopes, and then convert to cofibered filtered spaces with (6.20).

Theorem 6.24 *Suppose $X = \bigcup X_n$ and $Y = \bigcup Y_n$ are cofibered filtered spaces, and that we have the following diagram of maps*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] & X_1 \arrow[r] \arrow[d, "f_1"'] & X_2 \arrow[r] \arrow[d, "f_2"'] & {} \\ Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & {} \end{tikzcd}$$

where each rectangle is a homotopy pullback diagram. Let $f : X \to Y$ be the induced map of total spaces. Then

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] & X \arrow[d, "f"] \\ Y_0 \arrow[r] & Y \end{tikzcd}$$

is a homotopy pullback diagram.

Final Remarks on PULL

Let Ψ be a poset satisfying the finiteness condition as in (6.5). Let X and Y be Ψ-filtered spaces and suppose we have a Ψ-filtered map $f : X \to Y$, inducing the restriction maps $f_p : X(p) \to Y(p)$. Suppose in addition one has a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g_0} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g_1} & W \end{array}$$

such that for each p the composed diagram

$$\begin{array}{ccc} X(p) & \longrightarrow & Z \\ {\scriptstyle f_p}\downarrow & & \downarrow{\scriptstyle h} \\ Y(p) & \longrightarrow & W \end{array}$$

is a homotopy pullback diagram. We may as well assume h a fibration, and we do so. We then prove that the diagram

$$\begin{array}{ccc} X & \xrightarrow{g_0} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g_1} & W \end{array}$$

is a homotopy pullback diagram.

Let E be the pullback $g_1^* Z$, so that we have the fibration $h' : E \to Y$. Then the natural map $X(p) \to {h'}^{-1} Y(p)$ is a homotopy equivalence for each p. Since $Y(p)$ is a cofibered subset of Y, then ${h'}^{-1} Y(p)$ is cofibered in E by a result of Strom [3.8]. Thus E is the total space of a Ψ-filtered space. We can then apply (6.5) to obtain that the natural map $X \to E$ is a homotopy equivalence. Thus the result follows.

Proof of (6.15)

Let G be a topological monoid with homotopy inverses. Let X denote the simplicial space with $X(n) = G^{n+1}$, for which

$$\mathcal{E}_G = X \times_{Mono\,\Delta} \nabla = ||X||.$$

Similarly let $Y = NG$ denote the simplicial space with $Y(n) = G^n$, for which

$$\mathcal{B}_G = Y \times_{Mono\,\Delta} \nabla = ||Y||.$$

There is the Δ^0-map $f : X \to Y$ given by

$$f(g_0, g_1, \cdots, g_n) = (g_1, \cdots, g_n),$$

which induces the natural map $\mathcal{E}_G \to \mathcal{B}_G$.

The spaces $||X||$ and $||Y||$ are filtered as $||X|| = \bigcup ||X||_n$ and $||Y|| = \bigcup ||Y||_n$, and there are relative homeomorphisms

$$G^n \times (\nabla(n), \partial\nabla(n)) \to (||Y||_n, ||Y||_{n-1})$$

and similarly for X. At the first stage we have $||X||_0 = G$ and $||Y||_0 = pt$. To prove (6.15), it suffices to prove that each diagram

$$\begin{tikzcd} ||X||_{n-1} \arrow[r] \arrow[d] & ||X||_n \arrow[d] \\ ||Y||_{n-1} \arrow[r] & ||Y||n \end{tikzcd}$$

is a homotopy pullback diagram. This will follow from (6.18) if we can prove that each rectangle of

$$\begin{tikzcd} G^{n+1} \times \nabla(n) \arrow[d] & G^{n+1} \times \partial\nabla(n) \arrow[l] \arrow[r] \arrow[d] & ||X||_{n-1} \arrow[d] \\ G^n \times \nabla(n) & G^n \times \partial\nabla(n) \arrow[l] \arrow[r] & ||Y||_{n-1} \end{tikzcd}$$

is a homotopy pullback diagram.

The only difficulty is with the right hand rectangle above. Thus we confine our attention to

$$\begin{tikzcd} G^{n+1} \times \partial\nabla(n) \arrow[r] \arrow[d] & ||X||_{n-1} \arrow[d] \\ G^n \times \partial\nabla(n) \arrow[r] & ||Y||_{n-1}. \end{tikzcd}$$

Let Ψ be the poset of non-empty, proper subsets of $\underline{n}$. Then for each ω in Ψ we have subsets $A(\omega)$ of $G^{n+1} \times \partial\nabla(n)$ and $B(\omega)$ of $G^n \times \partial\nabla(n)$; that is, the two spaces are Ψ-filtered and we have the Ψ-filtered map $f : A \to B$.

For each ω, there is a unique mono δ in Δ with range n, such that the image of $\delta : \underline{m} \to \underline{n}$ is precisely ω. If we let $\nabla(\omega)$ denote the image of $\delta : \nabla(m) \to \nabla(n)$ then we get a commutative diagram

$$\begin{tikzcd} G^{n+1} \times \nabla(\omega) \arrow[r] \arrow[d] & G^{m+1} \times \nabla(m) \arrow[d] \\ G^n \times \nabla(\omega) \arrow[r] & G^m \times \nabla(m). \end{tikzcd}$$

Here a fundamental computation enters, which we leave to the reader. Namely, for any ω this is a homotopy pullback diagram. The various $\nabla(\omega)$ and $\nabla(m)$ do not effect this outcome, and for starters one can delete them all. The horizontal maps such as $G^{n+1} \to G^{m+1}$ are then the induced maps $\delta^* : X(n) \to X(m)$ and one proceeds.

With an inductive hypothesis, one can then take the composition of

$$G^{n+1} \times \nabla(\omega) \to G^{m+1} \times \nabla(m) \to ||X||_m \to ||X||_{n-1}$$

and similarly with $G^n \times \nabla(\omega)$ to obtain a homotopy pushout diagram

$$\begin{array}{ccc} G^{n+1} \times \nabla(\omega) & \longrightarrow & ||X||_{n-1} \\ \big\downarrow & & \big\downarrow \\ G^n \times \nabla(\omega) & \longrightarrow & ||Y||_{n-1}. \end{array}$$

One then has to take the above remarks on Ψ-filtered spaces to obtain that

$$\begin{array}{ccc} G^{n+1} \times \partial\nabla(n) & \longrightarrow & ||X||_{n-1} \\ \big\downarrow & & \big\downarrow \\ G^n \times \partial\nabla(n) & \longrightarrow & ||Y||_{n-1} \end{array}$$

is homotopy pullback. This establishes the induction and (6.15).

Turn finally to the proof of (6.14). There is the commutative diagram

$$\begin{array}{ccccc} G & \longrightarrow & \mathcal{E}_G & \xrightarrow{\sim} & E_G \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ pt & \longrightarrow & \mathcal{B}_G & \xrightarrow{\sim} & B_G. \end{array}$$

By (6.15), the left hand diagram is in PULL. The right hand diagram is in WHE and therefore in PULL. Hence the composite diagram is in PULL.

A Categorical Interpretation of the Theorem of James

In this work on homotopy colimits, it is appropriate to point out that the James space JA is the colimit of a certain G-space A^∞, that if the base point is cofibered then JA is also a homotopy colimit of A^∞, and that the theorem (6.19) of James thus yields another homotopy colimit ΩSA for A^∞ when A satisfies the conditions of (6.19).

We first replace the augmented simplicial category $c\Delta$ by an isomorphic copy Δ_+. In this version of $c\Delta$, the objects of Δ_+ are the non-negative integers and the morphisms $\delta : m \to n$ are the order preserving functions

$$\delta : \{1, \cdots, m\} \to \{1, \cdots, n\}.$$

The bifunctor $\oplus : \Delta_+ \times \Delta_+ \to \Delta_+$ corresponding to disjoint union then has $m \oplus n = m + n$. Denote by $Mono\ \Delta_+$ the subcategory whose morphisms are the order preserving monos.

Denote by TOP_* the category whose objects are compactly generated spaces A with base point a_0, and whose morphisms are base point preserving maps. There is a functor

$$\diamond^\infty : \mathrm{TOP}_* \to \mathrm{TOP}^{Mono\ \Delta_+}, \quad A \mapsto A^\infty = \coprod_{n \geq 0} A^n,$$

where if $\delta : m \to n$ then

$$\delta(a_1, \cdots, a_m) = (b_1, \cdots, b_n)$$

with $b_j = a_{\delta^{-1}(j)}$ whenever $\delta^{-1}(j)$ is non-empty, and $b_j = a_0$ whenever $\delta^{-1}(j)$ is empty.

If A is a compactly generated space with cofibered base point, it is readily checked that A^∞ is a principal $Mono\,\Delta_+$-space. For as free generators one can take all $(a_1, \cdots, a_n)$ for $n > 0$ with each $a_i \neq a_0$, and the empty set for $n = 0$.

The James space JA is precisely the colimit of A^∞. If the base point is cofibered, so that A^∞ is principal, then JA is also a homotopy colimit of A^∞.

Thus one can restate (6.19) as the computation of a homotopy colimit.

Theorem 6.25 *Let A be a compactly generated space with cofibered base point, and let $A^\infty = \coprod A^n$ be the $Mono\,\Delta_+$-space above. If A is path connected and of the homotopy type of a CW-complex, then ΩSA is a homotopy colimit of A^∞.*

Proof. It follows from (6.19) that for such an A we have a natural homotopy class of homotopy equivalences $JA \to \Omega SA$. Since JA is a homotopy colimit of A^∞, then so is ΩSA. $\square$

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CHAPTER VII

The Gabriel-Zisman Category $\wr\Lambda$ and the Construction of Topological Categories

In this chapter we review colimit constructions of small categories and homotopy colimit constructions of topological categories. We start by outlining the additional structure required of the small category G and the G^o-space X in order that colimits and homotopy colimits have the needed extra structure.

A *strictly monoidal* small category G is a small category G for which one has given a bifunctor $\oplus : G \times G \to G$, with functional values denoted by $(p, q) \mapsto p \oplus q$ and $(g, h) \mapsto g \oplus h$, such that associativity holds and such that there is a given object 0 such that $p \oplus 0 = p$ and $0 \oplus p = p$ for all objects p, as well as $g \oplus 1_0 = g$ and $1_0 \oplus g = g$ for all morphisms g. Thus both $Ob\ G$ and $Mor\ G$ are then monoids.

Let G be a strictly monoidal small category, and let X be a G^o- space. The functor $\oplus : G \times G \to G$ then induces $\oplus^{\#} : \mathrm{TOP}^{G^o} \to \mathrm{TOP}^{G^o \times G^o}$. We obtain the $G^o \times G^o$-space $\oplus^{\#} X$, given by

$$(\oplus^{\#} X)(p, q) = X(p \oplus q)$$

with its natural right action of $G \times G$. There is also the $G^o \times G^o$-space $X \times X$, given by

$$(X \times X)(p, q) = X(p) \times X(q)$$

with its natural action.

We say that a G^o-space X is *comultiplicative* if there is given a $G^o \times G^o$-map $\theta : \oplus^{\#} X \to X \times X$, i.e. an equivariant collection of maps

$$X(p \oplus q) \to X(p) \times X(q),$$

such that associativity holds, and for each object p the compositions

$$X(p) = X(p \oplus 0) \to X(p) \times X(0) \xrightarrow{proj} X(p),$$
$$X(p) = X(0 \oplus p) \to X(0) \times X(p) \xrightarrow{proj} X(p)$$

are both the identity map of $X(p)$.

In the above, we can project $X(p) \times X(0)$ and $X(0) \times X(p)$ onto $X(0)$, and thus assign to each $x \in X(p)$ two elements of $X(0)$, which we call the *source* and *target* of x.

If X is a comultiplicative G^o-space, for each pair (p, q) of objects we thus have the map

$$\theta_{p,q} : X(p \oplus q) \longrightarrow X(p) \times_{X(0)} X(q).$$

The comultiplicative G-space X is said to be *strictly comultiplicative* if each of these maps $\theta_{p,q}$ is a homeomorphism.

Denote by COMULT TOP$^{G^o}$ the category whose objects are the comultiplicative G^o-spaces X, and whose morphisms $X \to X'$ are the G^o-maps $\mu : X \to X'$ which preserve the comultiplicative structures. Denote by STR COMULT TOP$^{G^o}$ the full subcategory of COMULT TOP$^{G^o}$ whose objects are the strictly comultiplicative G^o-spaces.

If G is a strictly monoidal small category whose neutral object 0 is a terminal object, then there is a functor

$$\text{STR COMULT TOP}^{G^o} \to \text{CAT}$$

which assigns to each strictly multiplicative G^o-space X a small category whose set of objects is $X(0)$ and whose set of morphisms is the colimit of the G^o-space X. Here the topology on the colimit is ignored because it may not be weakly Hausdorff.

To obtain a reliable topology, one needs to use homotopy colimits. If G is a strictly monoidal small category whose neutral object 0 is a terminal object, then there is a functor

$$W : \text{STR COMULT TOP}^{G^o} \to \text{TOPCAT}$$

which assigns to each strictly comultiplicative G^o-space X a topological category whose space of objects is $X(0)$, and whose space of morphisms is the standard homotopy colimit $B_{G^o}X$.

We next present two basic strictly monoidal small categories Λ and $\wr\Lambda$ which are closely related to Δ. First of all, for each Δ^o-space X we need the Eilenberg-Moore maps on X [7.2,p.218], a family of maps

$$\theta_{m,n} : X(m+n) \to X(m) \times X(n)$$

which sends an element $x \in X(m+n)$ into the pair (x', x'') where x' is the front m-face of x and x'' is the back n-face.

This makes full sense categorically only when one restricts the structural category from Δ^o to Λ^o, where Λ is a certain subcategory of Δ. The objects of Λ are the non-negative integers and the morphisms $\lambda : m \to n$ are the order preserving functions

$$\lambda : \underline{m} = \{0, 1, \cdots, m\} \to \underline{n} = \{0, 1, \cdots, n\}$$

for which $\lambda(0) = 0$ and $\lambda(m) = n$. By identifying the last point of $\underline{m}$ with the first point of $\underline{n}$, we obtain a bifunctor $\oplus : \Lambda \times \Lambda \to \Lambda$, which on objects has $m \oplus n = m + n$.

Then Λ is strictly monoidal with 0 a terminal object, thus one can consider comultiplicative Λ^o-spaces and can put in a natural setting the properties of the Eilenberg-Moore maps on each simplicial space X. There is the functor

$$\mathrm{TOP}^{\Delta^o} \to \mathrm{COMULT\ TOP}^{\Lambda^o}, \quad X \mapsto i^{\#}X,$$

which sends X into the result $i^{\#}X$ of restricting the structural category, and from using the Eilenberg-Moore maps $X(m+n) \to X(m) \times X(n)$. This is an equivalence of categories.

If G is a topological category, one then has the simplicial space NG. The resulting Λ^o-space is strictly comultiplicative, and this characterizes the topological categories up to natural isomorphism in TOP^{Λ^o}. Thus one gets a composition

$$\mathrm{TOPCAT} \to \mathrm{STR\ COMULT\ TOP}^{\Lambda^o} \to \mathrm{TOPCAT}.$$

We conjecture that this functor is the Boardman-Vogt functor [4.1] $W : \mathrm{TOPCAT} \to \mathrm{TOPCAT}$ which assigns to each topological category G an exploded topological category WG using the Boardman-Vogt trees.

We now enlarge Λ to a category $\wr\Lambda$, which we call the *Gabriel-Zisman category.*

An object of $\wr\Lambda$ is a subset A of some $\underline{m} = \{0, 1, \cdots, m\}$ such that $0 \in A$ and $m \in A$. Given such subsets A of $\underline{m}$ and B of $\underline{n}$, there is a morphism $\lambda : A \to B$ in $\wr\Lambda$ for each morphism $\lambda : m \to n$ in Λ for which $\lambda(A) \supset B$. The objects can also be considered as ordered partitions $(m_1, \cdots, m_k)$ of m into positive integers, where the partition yields the subset

$$A = \{0, m_1, m_1 + m_2, \cdots, m_1 + \cdots + m_k\}.$$

The category $\wr\Lambda$ is strictly monoidal with a natural functor $\oplus$ given on objects by

$$(m_1, \cdots, m_j) \oplus (n_1, \cdots, n_k) = (m_1, \cdots, m_j, n_1, \cdots, n_k).$$

Moreover, the neutral object of $\wr\Lambda$ is also a terminal object.

The point of $\wr\Lambda$ is that there is a functor

$$\mathrm{TOP}^{\Delta^o} \to \mathrm{STR\ COMULT\ TOP}^{(\wr\Lambda)^o}$$

which assigns to the Δ^o-space X a strictly comultiplicative $(\wr\Lambda)^o$-space $\wr X$ which has

$$(\wr X)(m_1, \cdots, m_k) = X(m_1) \times_{X(0)} \cdots \times_{X(0)} X(m_k)$$

and whose action is natural. This functor is an equivalence of categories.

By composing functors, we get a functor

$$\mathrm{TOP}^{\Delta^o} \to \mathrm{STR\ COMULT\ TOP}^{(\wr\Lambda)^o} \to \mathrm{CAT}$$

which we call the Gabriel-Zisman construction [2.4], since it gives their functor $\mathrm{SET}^{\Delta^o} \to \mathrm{CAT}$ which is adjoint to the nerve functor. The composition

$$\mathrm{TOP} \to \mathrm{TOP}^{\Delta^o} \to \mathrm{CAT}$$

yields the fundamental groupoid functor.

We also get a composition

$$W : \mathrm{TOP}^{\Delta^o} \to \mathrm{STR\ COMULT\ TOP}^{(\wr\Lambda)^o} \to \mathrm{TOPCAT}$$

which can be regarded as either an extended Gabriel-Zisman construction [2.4], or an extended Boardman-Vogt construction [4.1].

The functor $W : \mathrm{TOP}^{\Delta^o} \to \mathrm{TOPCAT}$ completes the diagram of basic functors to

$$\mathrm{TOPCAT} \leftrightarrows \mathrm{TOP}^{\Delta^o} \rightleftarrows \mathrm{TOP}.$$

We can then use the methods of Segal to compute up to homotopy the space of morphisms of WX, at least for an appropriate generalization of Segal's special Δ^o-spaces X [4.4]. Roughly speaking, we call a Δ^o-space X *special* if each $X(n)$ is suitably determined up to homotopy by $X(0)$ and $X(1)$. More precisely, call the Δ^o-space X *special* if each of the Eilenberg-Moore maps

$$X(m+n) \to X(m) \times_{X(0)} X(n)$$

is a homotopy equivalence in $\mathrm{TOP}/X(0) \times X(0)$.

We then prove a theorem of the Segal type [4.4]. Namely, for a special Δ^o-space X, the homotopy colimit of $\wr X$ is $X(1)$, up to homotopy in $\mathrm{TOP}/X(0) \times X(0)$. Thus for some purposes a special Δ^o-space X can be replaced by the topological category WX which has $X(0)$ as space of objects and has space of morphisms $X(1)$ up to homotopy equivalence.

In the last part of the chapter, in the fashion of Stasheff [7.6] we show that $\wr\Lambda$ is related to cubes as Δ is related to simplices. Thus we can also regard $\wr\Lambda$ as the *cubical category*. Here one can obtain small models $\mathcal{W}X$ for WX, generalizing a construction of Vogt [4.7].

Simplicial Spaces as Comultiplicative Λ^o-Spaces

For each Δ^o-space X we have the Eilenberg-Moore maps on X [7.2,p.218], a family of maps

$$\theta_{m,n} : X(m+n) \to X(m) \times X(n)$$

which send an element $x \in X(m+n)$ into the pair (x', x'') where x' is the front m-face of x and x'' is the back n-face.

As in the introduction, this makes full sense categorically only when one restricts the structural category from Δ^o to Λ^o, where Λ is the subcategory of Δ whose objects are the non-negative integers and whose morphisms $\lambda : m \to n$ are the order preserving functions

$$\lambda : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}$$

for which $\lambda(0) = 0$ and $\lambda(m) = n$.

There is an associated bifunctor $\oplus : \Lambda \times \Lambda \to \Lambda$ which on objects has $m \oplus n = m + n$; if $\lambda : m \to m'$ and $\lambda' : n \to n'$ then $\lambda \oplus \lambda' : m + n \to m' + n'$ is defined by

$$(\lambda \oplus \lambda')(i) = \begin{cases} \lambda(i), & \text{for } 0 \le i \le m \\ m' + \lambda'(i - m), & \text{for } m \le i \le m + n. \end{cases}$$

Then Λ is a strictly monoidal category.

Other properties of Λ include the fact that 0 is a terminal object, that the only morphism with source 0 is the identity morphism of 0, and that given a morphism $\lambda : m + n \to r$ then there exist unique morphisms $\lambda' : m \to p$ and

$\lambda'' : n \to q$ for which $\lambda = \lambda' \oplus \lambda''$. There is the initial object 1 of Λ. If one denotes by ϵ_n the unique morphism $1 \to n$ then every morphism $\lambda : m \to n$ except 1_0 has a unique representation as

$$\lambda = \epsilon_{n_1} \oplus \cdots \oplus \epsilon_{n_m}$$

where $n_1 + \cdots + n_m = n$. That is, the morphisms of Λ constitute a free monoid, as do the objects.

Because Λ is strictly monoidal, one has the comultiplicative Λ^o-spaces and can put in a natural setting the properties of the Eilenberg-Moore maps on simplicial spaces.

Theorem 7.1 *There is the functor*

$$TOP^{\Delta^o} \to COMULT\ TOP^{\Lambda^o}, \quad X \mapsto i^{\#}X,$$

which sends X into the result $i^{\#}X$ of restricting the structural category, and from using the Eilenberg-Moore maps $X(m+n) \to X(m) \times X(n)$. This is an equivalence of categories.

Proof. We take the first sentence as clear. Suppose X' is any comultiplicative Λ^o-space; we must define a corresponding Δ^o-space X. Take $X(n) = X'(n)$. We must then use the comultiplication on X' to extend the action of the structural category from Λ^o to Δ^o. Every morphism $\delta : m \to n$ of Δ can be uniquely factored as a morphism $\lambda : m \to q$ of Λ, followed by a mono $\alpha : q \to n$ of Δ whose image is a full subinterval of $\{0, \cdots, n\}$. The question is then how to define

$$\alpha_* : X'(n) \to X'(q).$$

Here one can write $n = p + q + r$ for appropriate p, r and take the composition

$$X'(p+q+r) \to X'(p) \times X'(q) \times X'(r) \xrightarrow{proj} X'(q). \quad \square$$

Thus we have a first reformulation of Δ^o-spaces as the comultiplicative Λ^o-spaces.

The small category Λ has another form, related to the other strictly monoidal category associated with the simplicial category Δ, namely the augmented simplicial category Δ_+ of Chapter 6, whose objects are the non-negative integers, and whose morphisms $\delta : m \to n$ are the order preserving functions

$$\delta : \{1, \cdots, m\} \to \{1, \cdots, n\},$$

where the object 0 corresponds to the empty set. Then Δ_+ is a strictly monoidal category whose neutral element 0 is an initial object, and the only morphism into 0 is the identity morphism 1_0. Thus Δ_+ is a natural strictly monoidal category which contains a copy of Δ as a full subcategory.

(7.2) *The categories Δ_+ and Λ are related by an isomorphism*

$$(\Delta_+)^o \simeq \Lambda.$$

The isomorphism interprets the ordered set $\{0, 1, \cdots, n\}$ as the set of initial intervals of $\{1, \cdots, n\}$. If $\delta : m \to n$ is a morphism in Δ_+, then the isomorphism assigns to it the morphism $\delta^{-1} : n \to m$ of Λ which assigns to an initial interval J its inverse $\delta^{-1}J$.

We have completed the interpretation of simplicial spaces as comultiplicative Λ^o-spaces. Thus one can rewrite a simplicial space by restricting the structural category to $\Lambda^o \simeq \Delta_+$ and incorporating the Eilenberg-Moore maps.

The category COMULT TOP^{Λ^o} can only be an interim setting for us. Because Λ has an initial object 1, then Λ^o has a terminal object. Hence a Λ^o-space X has $X(1)$ as colimit and (non-standard) homotopy colimit. We need a larger category for which colimits and homotopy colimits are more non-trivial. The clues are in the Gabriel-Zisman construction [2.4] of a functor $\mathrm{SET}^{\Delta^o} \to \mathrm{CAT}$ adjoint to the nerve functor. We thus call the resulting category the *Gabriel-Zisman category.*

The Gabriel-Zisman Category $\wr\Lambda$

From one point of view, $\wr\Lambda$ has as objects all the ordered partitions $\omega = (m_1, \cdots, m_k)$ of a positive integer m into positive integers, together with the single partition of $m = 0$ that we denote by $\omega = \emptyset$.

In this notation, every $\omega = (m_1, \cdots, m_k)$ determines a unique object even if we only assume $m_i \geq 0$. One simply deletes all $m_i = 0$ and takes the object which results.

From another point of view, an object of $\wr\Lambda$ is a subset A of some $\underline{n} = \{0, 1, \cdots, m\}$ such that $0 \in A$ and $m \in A$. Given $\omega = (m_1, \cdots, m_k)$ then we can take

$$A = \{0, m_1, m_1 + m_2, \cdots, m_1 + \cdots + m_k\}.$$

If the object A of $\wr\Lambda$ is presented as a subset of $\underline{m}$ with $0, m \in A$, and if the object B is presented as a subset of $\underline{n}$ with $0, n \in B$, then there is a *morphism* $\lambda : A \to B$ in $\wr\Lambda$ for each morphism $\lambda : m \to n$ in Λ for which $\lambda(A) \supset B$. The identity morphism 1_A is obtained by taking $\lambda = 1_m$ and $A = B$. Compositions are obtained using the composition in Λ.

One can also regard the objects of $\wr\Lambda$ as the monos ω of Λ, with a morphism $\omega \to \omega'$ for each commutative diagram

$$\begin{tikzcd} k \arrow[r] \arrow[d, "\omega"'] & k' \arrow[d] & j \arrow[l] \arrow[d, "\omega'"] \\ m \arrow[r, "\lambda"] & n \arrow[r, equal] & n \end{tikzcd}$$

in Λ, all of whose morphisms except possibly λ are monos in Λ.

There is a natural inclusion functor $j : \Lambda \to \wr\Lambda$, namely as the full subcategory whose objects are the empty partition and the singleton partitions. There is also a natural functor $\mu : \wr\Lambda \to \Lambda$ sending $(m_1, \cdots, m_k)$ into $m_1 + \cdots + m_k$ and λ into λ. In terms of objects A, Λ is the full subcategory of $\wr\Lambda$ whose objects are all subsets $\{0, m\}$ of $\underline{m}$ for all $m \geq 0$, and there is the retracting functor $\mu : \wr\Lambda \to \Lambda$.

The category $\wr\Lambda$ is strictly monoidal, with a bifunctor $\oplus$ such that on objects

$$(m_1, \cdots, m_j) \oplus (n_1, \cdots, n_k) = (m_1, \cdots, m_j, n_1, \cdots, n_k).$$

The object $\emptyset$ is both a neutral object and a terminal object of $\wr\Lambda$. The value of $\oplus$ on morphisms of $\wr\Lambda$ is readily written out in terms of the value of $\oplus$ on morphisms of Λ.

If one wishes, one can write out $\wr\Lambda$ as a Grothendieck construction in the fashion of Thomason [7.7]. Here one displays a functor $F : \Lambda \to \mathrm{CAT}$ for which the Grothendieck construction $\Lambda \wr F$ is the Gabriel-Zisman category. Let $F(m)$ be the category whose objects A are all $\{0, m\} \subset A \subset \underline{m}$ and which has a morphism $A \to B$ whenever $A \supset B$. For each $\lambda : m \to n$ in Λ, take the functor $\lambda_* : F(m) \to F(n)$ given on objects by $\lambda_*(A) = \lambda(A)$.

(7.3) *One can specify generators for the morphisms of $\wr\Lambda$ as follows.*

- *(i) For singleton objects (m) and (n) with $m, n > 0$, and each morphism $\lambda : m \to n$ in Λ, one gets a morphism $(m) \to (n)$ of $\wr\Lambda$ for each $\lambda : m \to n$ of Λ.*
- *(ii) For each $n > 0$, there is a unique morphism $(n) \to \emptyset$ corresponding to the unique morphism $n \to 0$ of Λ.*
- *(iii) For each doubleton $\omega = (m, n)$ with $m, n > 0$, there is a morphism $(m, n) \to (m+n)$ corresponding to the morphism $1_{m+n} : m+n \to m+n$.*
- *(iv) One gets the morphisms obtained as $\rho_1 \oplus \cdots \oplus \rho_k$ where each term is one of the above and where $\oplus$ is the bifunctor for $\wr\Lambda$.*

Every morphism of $\wr\Lambda$ is a composition of such.

No doubt if one is persistent enough, one can write down an explicit proof in terms of the list of morphisms given above. A little such effort will convince one that the proposition is clear.

Theorem 7.4 *Consider a comultiplicative Λ^o-space X. Consider the inclusion $j : \Lambda \to \wr\Lambda$ noted above. The composition*

$$\mathit{STR\ COMULT\ TOP}^{(\wr\Lambda)^o} \to \mathit{TOP}^{(\wr\Lambda)^o} \xrightarrow{j^\#} \mathit{TOP}^{\Lambda^o}$$

maps $\mathit{STR\ COMULT\ TOP}^{(\wr\Lambda)^o}$ into $\mathit{COMULT\ TOP}^{\Lambda^o}$ and provides an equivalence of categories

$$\mathit{STR\ COMULT\ TOP}^{(\wr\Lambda)^o} \sim \mathit{COMULT\ TOP}^{\Lambda^o}.$$

Alternatively, there is a functor

$$\mathit{COMULT\ TOP}^{\Lambda^o} \to \mathit{STR\ COMULT\ TOP}^{(\wr\Lambda)^o}$$

which assigns to the comultiplicative Λ^o-space X the unique strictly comultiplicative $(\wr\Lambda)^o$-space $\wr X$ which has

$$(\wr X)(m_1, \cdots, m_k) = X(m_1) \times_{X(0)} \cdots \times_{X(0)} X(m_k)$$

and whose action on the generators of (7.3) are the natural actions, using the comultiplication for generators of type (iii).

PROOF. We outline the first assertion of the theorem. Suppose that Y is a strictly comultiplicative $(\wr\Lambda)^o$-space. Then the Λ^o-space $X = j^{\#}Y$ constructed above has $X(0) = Y(\emptyset)$ and $X(n) = Y((n))$ for $n > 0$. For $m, n > 0$ the generator $(m, n) \to (m + n)$ of type (iii) in (7.3) and the strict comultiplication gives a composition

$$\begin{array}{rcl} X(m+n) = Y((m+n)) \to Y((m,n)) = Y((m) \oplus (n)) & \simeq & Y((m)) \times Y((n)) \\ & = & X(m) \times X(n); \end{array}$$

one has to consider also the cases $m = 0$ and $n = 0$, and convince oneself that the result is a comultiplication on X.

We also merely outline the existence of the functor

$$\wr : \text{COMULT TOP}^{\Lambda^o} \to \text{STR COMULT TOP}^{(\wr\Lambda)^o}$$

which has $(\wr X)(m_1, \cdots, m_k) = X(m_1) \times_{X(0)} \cdots \times_{X(0)} X(m_k)$. Take as model for $\wr\Lambda$ the category with objects A and morphisms $\lambda : A \to B$ as described above. Let H denote the subcategory of $\wr\Lambda$ whose morphisms are all $\lambda : A \to B = \lambda(A)$. Then the generators of type (i), (ii) and (iv) generate H and one can convince oneself that H^o acts because no generators of type (iii) are included.

Let K denote the subcategory of $\wr\Lambda$ whose morphisms consist of all morphisms $A \to B$ where

$$\{0, n\} \subset B \subset A \subset \underline{n}$$

and where the given morphism of Λ is $1_n : n \to n$. The associativity of the comultiplication then gives an action of K^o.

Finally, every morphism of $\wr\Lambda$ can be written as a morphism of H followed by a morphism of K. Since each of H^o and K^o acts, one gets a candidate for an action of $(\wr\Lambda)^o$. One has to convince oneself that it is an action. $\square$

Corollary 7.5 *There is a functor*

$$\wr : TOP^{\Delta^o} \to STR\ COMULT\ TOP^{(\wr\Lambda)^o}$$

which assigns to a Δ^o-space X the $(\wr\Lambda)^o$-space $\wr X$ which has

$$(\wr X)(m_1, \cdots, m_k) = X(m_1) \times_{X(0)} \cdots \times_{X(0)} X(m_k),$$

and whose action is naturally given on the generators of (7.3) using the Eilenberg-Moore comultiplication for generators of type (iii). This functor is an equivalence of categories.

The Gabriel-Zisman Construction

We can easily construct small categories from strictly comultiplicative G^o-spaces.

(7.6) *Let G be a strictly monoidal small category whose neutral object* 0 *is also a terminal object of G. There is a functor*

$$STR\ COMULT\ TOP^{G^o} \to CAT$$

which assigns to a strictly comultiplicative G-space X the small category whose set of objects is $X(0)$, whose set of morphisms is the colimit of the G^o-space X, whose identity morphisms are given as the composition

$$X(0) \to X(p) \hookrightarrow X \to colim\ X,$$

and whose composition is induced by the comultiplication. Here we have not bothered with the topology on the small category because colim X may not be compactly generated.

PROOF. The comultiplication yields the maps $X(p) \to X(0) \times X(0)$ which assign a source and target to each $x \in X(p)$. These maps are natural, and thus induce

$$colim\ X \to X(0) \times X(0),$$

and source and target functions for the category. The unique morphisms $p \to 0$ in G yield compositions

$$X(0) \to X(p) \hookrightarrow X \to colim\ X,$$

which gives the identity morphisms of the category. The strict comultiplication yields natural homeomorphisms

$$X(p) \times_{X(0)} X(q) \simeq X(p+q),$$

which gives a well defined

$$colim\ X \times_{X(0)} colim\ X \to colim\ X$$

and composition in the category. $\square$

The following is a form of the Gabriel-Zisman construction.

Corollary 7.7 *The composition*

$$SET^{\Delta^o} \hookrightarrow TOP^{\Delta^o} \xrightarrow{\wr} STR\ COMULT\ TOP^{(\wr\Lambda)^o} \to CAT$$

is the Gabriel-Zisman functor $SET^{\Delta^o} \to CAT$ adjoint to the nerve functor $N : CAT \to SET^{\Delta^o}$.

It is worth while to write the above construction out explicitly. Fix a Δ^o-space X. There is then the small category whose objects are all $x \in X(0)$, i.e. all vertices of X. The morphisms can be described as follows. Take all k-tuples $(x_1, \cdots, x_k)$ where each x_i is in some $X(m_i)$ and where the last vertex of x_{i-1} coincides with the first vertex of x_i. Put an equivalence relation on this set as follows:

(i) any $x_i \in X(0)$ can be deleted if $k > 0$;
(ii) if $\lambda : n_i \to m_i$ and if $x_i\lambda = y_i$, then x_i can be replaced by y_i;
(iii) if $m_i = m + n$ then x_i can be replaced by x'_i, x''_i where x'_i is the front m-face and x''_i is the back n- face of x_i.

The result is the category defined as the image of X under the composition

$$\mathrm{TOP}^{\Delta^o} \to \mathrm{STR\ COMULT\ TOP}^{(\wr\Lambda)^o} \to \mathrm{CAT}.$$

One gets then as did Gabriel-Zisman that the composition of

$$\mathrm{TOP} \xrightarrow{\diamond^\nabla} \mathrm{TOP}^{\Delta^o} \to \mathrm{CAT}$$

is the fundamental groupoid functor, assigning to the space A the category whose objects are the points of A and whose morphisms from x_1 to x_0 are the path homotopy classes of paths in A from x_0 to x_1.

The Extended Gabriel-Zisman Construction and the Extended Boardman-Vogt Construction

Let G be a strictly monoidal small category. Say that a G-space Y is *multiplicative* if one is given a $G \times G$-map $\phi : Y \times Y \to \oplus^{\#} Y$, i.e. equivariant maps

$$\phi_{p,q} : Y(p) \times Y(q) \to Y(p \oplus q), \quad (y, y') \mapsto yy',$$

such that

(i) associativity holds, so that there are uniquely defined maps

$$\phi_{p,q,r} : Y(p) \times Y(q) \times Y(r) \to Y(p \oplus q \oplus r),$$

(ii) and there exists an element ϵ in $Y(0)$ such that $y\epsilon = y$ and $\epsilon y = y$ for all y.

The G-space Y is *strictly multiplicative* if it is multiplicative, if $Y(0)$ is the singleton ϵ, and if each $\phi_{p,q} : Y(p) \times Y(q) \to Y(p \oplus q)$ is a homeomorphism.

(7.8) *For a strictly monoidal category G, there is a natural multiplicative G-space, namely the standard universal G-space E_G with multiplication*

$$\oplus_* : E_G(p) \times E_G(q) \to E_G(p \oplus q).$$

The bifunctor $\oplus : G \times G \to$ provides the multiplication

$$E_G \times E_G \simeq E_{G \times G} \longrightarrow E_G.$$

Theorem 7.9 *Consider a strictly monoidal small category G whose neutral object 0 is a terminal object. Then from each strictly comultiplicative G^o-space X we get a topological category whose space of objects is $X(0)$ and whose space of morphisms is the homotopy colimit*

$$B_{G^o} X = X \times_G E_G.$$

PROOF. Since 0 was required to be a final object, for each p the unique morphism $p \to 0$ gives a uniquely defined map $X(0) \to X(p)$. Thus for each object p we have the natural diagram

$$X(0) \to X(p) \to X(0) \times X(0)$$

whose composition sends x_0 into (x_0, x_0).

We now examine $X \times_G E_G$ for the structure required of a space of morphisms. In the first place, for every $x \times_G e$ in $X \times_G E_G$, the element x has a source and a target in $X(0)$ and these are independent of the representation. There is also a natural copy of $X(0)$ in the homotopy colimit. For each $x_0 \in X(0)$, there is the element $x_0 \times_G \epsilon$ in $X \times_G E_G$. We have finally to check that compositions are defined in $X \times_G E_G$. Let

$$x \times_G e, \quad x' \times_G e'$$

be elements for which the source of x equals the target of x'. Then we have that $(x, x') \in X(p) \times_{X(0)} X(q)$ and we can take the element

$$x'' = \theta_{p,q}^{-1}(x, x') \in X(p \oplus q)$$

and define the composition

$$(x \times_G e)(x' \times_G e') = (\theta_{p,q}^{-1})(x, x') \times_G ee'. \quad \square$$

As an application, recall that we get from a topological category G its nerve and then from

$$i^{\#} : \mathrm{TOP}^{\Delta^o} \to \mathrm{TOP}^{\Lambda^o}$$

we get a strictly comultiplicative $\mathcal{N}G$ in STR COMULT TOP^{Λ^o}. We can then apply the above functor

$$\text{STR COMULT } \mathrm{TOP}^{\Lambda^o} \to \mathrm{TOPCAT},$$

thus obtaining from $\mathcal{N}G$ the topological category WG, whose space of objects is $Ob\, G$ and whose space of morphisms is

$$\mathcal{N}G \times_\Lambda E_\Lambda.$$

Note that there is a natural functor $WG \to G$ which is the identity on objects and which sends a morphism $(g_1, \cdots, g_n) \times_\Lambda e$ of WG into the morphism $g_1 \cdots g_n$ of G. We call this construction the Boardman-Vogt construction [4.1], although we have not checked that it coincides precisely with their construction. Such a homotopy colimit form of a Boardman-Vogt construction has been given by Shea [7.5].

We get an extended form of this construction.

Corollary 7.10 *Given a Δ^o-space X there is the topological category WX whose space of objects is $X(0)$ and whose space of morphisms is the homotopy colimit*

$$B_{(\wr\Lambda)^o}(\wr X) = \wr X \times_{\wr\Lambda} E_{\wr\Lambda}$$

of the strictly comultiplicative $(\wr\Lambda)^o$-space $\wr X$. This gives a functor

$$W : TOP^{\Delta^o} \to TOPCAT.$$

We can regard this construction as an extended Gabriel-Zisman construction [2.4], or an extended Boardman-Vogt construction [4.1].

For later purposes, we state the following mild generalization of (7.9), whose proof is precisely that of (7.9).

(7.11) *Let G be a strictly monoidal small category whose neutral object 0 is also a terminal object. Given a strictly comultiplicative G^o-space X and a multiplicative G-space Y, we get a small category whose set of objects is $X(0)$ and whose set of morphisms is $X \times_G Y$. If $X \times_G Y$ is weakly Hausdorff, this is a topological category.*

Special Δ^o-Spaces of the Type of Segal

Generalizing slightly Segal's definition [4.4], a Δ^o-space X is *special* if each of the Eilenberg-Moore maps

$$\theta_{m,n} : X(m+n) \to X(m) \times_{X(0)} X(n)$$

is a homotopy equivalence in $\mathrm{TOP}/X(0) \times X(0)$. Then for any $m > 0$ we have the homotopy equivalence

$$X(m) \sim X(1) \times_{X(0)} \cdots \times_{X(0)} X(1)$$

in $\mathrm{TOP}/X(0) \times X(0)$, thus we regard the special Δ^o-spaces as those Δ^o-spaces in which $X(0)$ and $X(1)$ determine each $X(m)$ up to homotopy. If X is special, we also say that the Λ^o-space $i^\# X$ is *strictly comultiplicative up to homotopy.*

The simple Δ^o-spaces that we use here are all special. For starters, the Δ^o-spaces in the image of $\mathrm{TOPCAT} \to \mathrm{TOP}^{\Delta^o}$ have the above maps natural homeomorphisms and are thus special. Next, the Δ^o-spaces in the image of $\diamond^\nabla : \mathrm{TOP} \to \mathrm{TOP}^{\Delta^o}$ all have $X(m) = A^{\nabla(m)}$ naturally homotopy equivalent to A and thus are very special in the sense that $X(0)$ determines each $X(m)$ up to homotopy. There is next Segal's variant of $\diamond^\nabla$, presented as a functor

$$\mathrm{TOP}_* \to \mathrm{TOP}^{\Delta^o}$$

where TOP_* is the category of compactly generated spaces A with base point a_0. Here $\nabla_0(m)$ denotes the subset of $\nabla(m)$ consisting of all vertices, and for each m we take all maps of pairs

$$\sigma : (\nabla(m), \nabla_0(m)) \to (A, a_0).$$

Thus one considers singular simplices of A all of whose vertices are at a_0. For the $(m+n)$-simplex $\nabla(m+n)$, the union of the front m-face and the back n-face is a strong deformation retract containing all vertices. Hence this construction yields special Δ^o-spaces.

We now have a homotopy colimit theorem, generalizing a theorem of Segal [4.4].

Theorem 7.12 *For any Δ^o-space X, there is a compactly generated space Y in $TOP/X(0) \times X(0)$ and a natural diagram of maps*

$$X(1) \xleftarrow{\phi} Y \xrightarrow{\theta} hocolim \wr X$$

in $TOP/X(0) \times X(0)$, with ϕ a homotopy equivalence in $TOP/X(0) \times X(0)$. If X is a special Δ^o-space, then θ is also a homotopy equivalence in $TOP/X(0) \times X(0)$. Thus if X is a special Δ^0-space, then $X(1)$ is a homotopy colimit of $\wr X$ in $(TOP/X(0) \times X(0))^{(\wr\Lambda)^o}$.

Proof. Consider any Δ^o-space X and the $(\wr\Lambda)^o$-space $\wr X$. The natural functors

$$\wr\Lambda \xrightarrow{\mu} \Lambda \xrightarrow{i} \Delta,$$

where $\mu(m_1, \cdots, m_k) = m_1 + \cdots + m_k$, give restriction functors

$$\mathrm{TOP}^{\Delta^o} \xrightarrow{i^\#} \mathrm{TOP}^{\Lambda^o} \xrightarrow{\mu^\#} \mathrm{TOP}^{(\wr\Lambda)^o}.$$

We thus have for each X the $(\wr\Lambda)^o$-space $\wr X$ with

$$(\wr X)(m_1, \cdots, m_k) = X(m_1) \times_{X(0)} \cdots \times_{X(0)} X(m_k)$$

and the $(\wr\Lambda)^o$-space $\mu^\# i^\# X$ given by

$$(\mu^\# i^\# X)(m_1, \cdots, m_k) = X(m_1 + \cdots + m_k).$$

Moreover, there is the $(\wr\Lambda)^o$-map $\mu^\# i^\# X \to \wr X$ given by the Eilenberg-Moore maps. Hence we get the map

$$\theta : Y = hocolim\ \mu^\# i^\# X \to hocolim\ \wr X$$

of the theorem.

For the special Δ^o-spaces this is a weak homotopy equivalence in $(\mathrm{TOP}/X(0)\times X(0))^{(\wr\Lambda)^o}$, and thus induces a homotopy equivalence of homotopy colimits in $\mathrm{TOP}/X(0) \times X(0)$. Here we have assumed that the basic results on homotopy colimits in TOP^G extend to $(\mathrm{TOP}/Q)^G$ for any compactly generated space Q.

We next have to compute the homotopy colimit of $\mu^\# i^\# X$. In order to do so, we compute the categorical form of the Λ-space $\mu_\# E_{\wr\Lambda}$ as in Chapter 5, in terms of the small categories $\mathcal{C}_r$ for each non-negative integer r. Here the objects of $\mathcal{C}_r$ are the ordered pairs $(\lambda, (m_1, \cdots, m_k))$ where $\lambda : m_1 + \cdots + m_k \to r$ is a morphism of Λ. If we consider the singleton object (r) of $\wr\Lambda$, then the objects of $\mathcal{C}_r$ are also precisely the morphisms

$$(m_1, \cdots, m_k) \to (r)$$

of $\wr\Lambda$. Thus we obtain by the methods of Chapter 5 that

$$(\mu_\# E_{\wr\Lambda})(r) \simeq E_{\wr\Lambda}((r)),$$

hence that this space is contractible, hence that $\mu_{\#}E_{\wr\Lambda}$ is a universal Λ-space. Hence

$$hocolim\ \mu^{\#}i^{\#}X = \mu^{\#}i^{\#}X \times_{\wr\Lambda} E_{\wr\Lambda} \simeq i^{\#}X \times_{\Lambda} \mu_{\#}E_{\wr\Lambda} \sim hocolim\ i^{\#}X.$$

Finally we note that Λ^o has 1 as terminal object, thus that $i^{\#}X$ has homotopy colimit $X(1)$. Thus we get the homotopy equivalence $X(1) \xleftarrow{\phi} Y$ of the theorem as the composition

$$hocolim\ \mu^{\#}i^{\#}X \to hocolim\ i^{\#}X \to X(1)$$

of homotopy equivalences. Here we have also assumed that the treatment of homotopy colimits of Chapters 4 and 5 extends routinely from TOP^G to $(\mathrm{TOP}/Q)^G$ for any compactly generated space Q. $\square$

Among the consequences is the following corollary yielding for each compactly generated space A with base point a_0 a very large topological monoid of the homotopy type of the loop space ΩA.

Corollary 7.13 *Denote by TOP_* the category of compactly generated spaces with base point, and by*

$$TOP_* \to TOP^{\Delta^o}$$

the functor assigning to A the simplicial space of all singular simplices in A all of whose vertices are a_0. Then the composition

$$TOP_* \to TOP^{\Delta^o} \to TOP^{(\wr\Lambda)^o} \xrightarrow{hocolim} TOP$$

assigns to (A, a_0) a topological monoid of the homotopy type of ΩA.

We need a little more than (7.12) and (7.13); we need the relationship between a special Δ^o-space X and the Δ^o-space $NW(X)$. We turn to this now.

A Special Δ^o-Space X Is Isomorphic to $NW(X)$ in $\mathbf{TOP}^{\Delta^o}[\mathbf{WHE}^{-1}]$

It is convenient to denote by TopCat the category analogous to TOPCAT except that there is only a k-space topology required on spaces of morphisms. If G is a strictly monoidal category whose neutral object is also a terminal object, we can then write (7.11) as a functor

$$\text{STR COMULT TOP}^{G^o} \times \text{MULT TOP}^G \to \text{TopCat}, \quad (X, Y) \mapsto X \times_G Y.$$

(7.14) *Let G be a strictly monoidal category. Consider the categories G^n for each non-negative integer n, where G^o is the category with one object 0 and one morphism. Given a comultiplicative G^o-space X, let X_n for $n > 0$ denote the G^n-space given by*

$$X_n(p_1, \cdots, p_n) = X(p_1 \oplus \cdots \oplus p_n),$$

with its natural action, and let X_0 denote the space $X(0)$. Given a multiplicative G-space Y, let Y^n for $n > 0$ denote the G^n-space given by

$$Y^n(p_1, \cdots, p_n) = Y(p_1) \times \cdots \times Y(p_n),$$

and let Y^0 denote the singleton ϵ. Then there is a functor

$$COMULT\ TOP^{G^o} \times MULT\ TOP^G \to Top^{\Delta^o}, \quad (X,Y) \mapsto \{X_n \times_{G^n} Y^n | n \geq 0\},$$

where the action map is given in the proof.

If 0 is a terminal object of G and if X is strictly comultiplicative, this Δ^o-space coincides with the nerve of the object of TopCat given by (7.11).

PROOF. Let $\delta : m \to n$ be a morphism of Δ. There is generated a functor $\delta^* : G^n \to G^m$ given on morphisms by

$$\delta^*(g_1, \cdots, g_n) = (g_{\delta(0)+1} \oplus \cdots \oplus g_{\delta(1)}, \cdots, g_{\delta(m-1)+1} \oplus \cdots \oplus g_{\delta(m)}),$$

and similarly on objects. If $\delta(i-1) = \delta(i)$, then the ith-coordinate is taken as 1_0 in the morphism case, or as 0 in the object case. If $m = 0$, the functor is unique anyway; if $n = 0$, the object maps into $(0, \cdots, 0)$ and the morphism into $(1_0, \cdots, 1_0)$.

Fix a morphism $\delta : m \to n$ of Δ, an object $(p_1, \cdots, p_n)$ of G^n and the image object $(p'_1, \cdots, p'_m)$ as given above. We then get $\delta^* : Y^n(p_1, \cdots, p_n) \to Y^m(p'_1, \cdots, p'_m)$ by

$$\delta^*(y_1, \cdots, y_n) = (y_{\delta(0)+1} \cdots y_{\delta(1)}, \cdots, y_{\delta(m-1)+1} \cdots y_{\delta(m)}).$$

The cases $n = 0$ and $m = 0$ also give well-defined maps.

One also gets a well defined map

$$\delta^* : X_n(p_1, \cdots, p_n) \to X_m(p'_1, \cdots, p'_m).$$

This requires a map

$$X(p_1 \oplus \cdots \oplus p_n) \to X(p_{\delta(0)+1} \oplus \cdots \oplus p_{\delta(m)}),$$

which follows from the comultiplication. The cases $m = 0$ and $n = 0$ are also covered.

Thus for each $\delta : m \to n$ one gets a map

$$X_n \times_{G^n} Y^n \to X_m \times_{G^m} Y^m, \quad a \times_{G^n} p \mapsto \delta^*(a) \times_{G^m} \delta^*(b).$$

This can be checked to be well defined and an action. The last sentence can also be checked, using

$$X(p_1 \oplus \cdots \oplus p_n) \simeq X(p_1) \times_{X(0)} \cdots \times_{X(0)} X(p_n). \quad \square$$

Theorem 7.15 *Let X be a Δ^o-space. Then X is special if and only if there exists a diagram*

$$X \xleftarrow{\phi} Y \xrightarrow{\theta} NW(X)$$

of Δ^o-spaces such that

(i) $Y(0) = X(0)$ and ϕ and θ are the identity on vertices,

(ii) the maps $\phi : Y(n) \to X(n)$ and $\theta : Y(n) \to (NW(X))(n)$ are homotopy equivalences in $TOP/X(0) \times X(0)$.

PROOF. It is easy to see that if X satisfies (i) and (ii) then X is special. For $NW(X)$ is special, and the conditions then imply X special.

Suppose now that X is special. We must make changes in the proof of (7.12). Most importantly, we must define the Δ^o-space Y. From (7.14), we have the functor

$$\text{COMULT TOP}^{\Lambda^o} \times \text{MULT TOP}^{\Lambda} \to \text{Top}^{\Delta^o}.$$

We have the comultiplicative Λ^o-space $i^{\#}X$. From the multiplicative space $E_{\wr\Lambda}$ we have constructed in (7.12) the Λ-space $\mu_{\#}E_{\wr\Lambda}$, and one checks that it is multiplicative. Thus one applies (7.14) to obtain a Δ^o-space

$$Y = \{(i^{\#}X)_n \times_{(\Lambda)^n} (\mu_{\#}E_{\wr\Lambda})^n\}.$$

The spaces involved are compactly generated, thus Y is in TOP^{Δ^o}. With this start, one can proceed to prove the theorem in the style of (7.12). □

One can now obtain from (6.14) and (6.17) the following theorem of Segal [4.4]. In it, one considers a special Δ^o-space X which has $X(0) = pt$. Then one has in each $X(n)$ a natural base point, hence one can consider the simplicial set $\{\pi_0(X(n)\}$. From the homotopy equivalence $X(m+n) \to X(m) \times_{X(0)} X(n)$ one then gets

$$\pi_0(X(m+n)) \simeq \pi_0(X(m)) \times \pi_0(X(n)),$$

hence the simplicial set is the nerve of a monoid up to natural isomorphism. Thus $\pi_0(X(1))$ is then naturally a monoid.

Corollary 7.16 *Let X be a special Δ^o-space which also has*

(i) $X(0)$ is a singleton,
(ii) each $X(n)$ is of the homotopy type of a CW-complex, and
(iii) the monoid $\pi_0(X(1))$ is a group.

Let $B : TOP^{\Delta^o} \to TOP$ be the homotopy colimit functor used in Chapter 6. Then the natural inclusion $X(1) \hookrightarrow \Omega BX$ is a homotopy equivalence. Alternatively, if X also satisfies the cofibration condition for simplicial sets, then the natural inclusion $X(1) \hookrightarrow \Omega|X|$ is a homotopy equivalence.

The Stasheff Realization of $(\wr\Lambda)^o$-Spaces

For Δ^o-spaces X which satisfy the cofibration condition, there is a smaller model for the homotopy colimit of $\wr X$ which is constructed in a fashion similar to the Milnor realization, i.e. by fixing a $\wr\Lambda$-space $\square$ which is much smaller than $E_{\wr\Lambda}$ and considering $\wr X \times_{\wr\Lambda} \square$ rather than $\wr X \times_{\wr\Lambda} E_{\wr\Lambda}$. We call this the *Stasheff realization* of X, since our definition of $\square$ grows out of constructions of Stasheff [7.6], who was the first to use cubes categorically.

We first relate cubes to Λ in what at first sight will appear ad hoc. Put a monoid structure on I, say by choosing the multiplication

$$s * t = max(s, t).$$

Then 0 is the unit element, and there is the useful element 1 with $1 * t = 1$ for all t. In any case, we then have the Λ-space $\Box$, where

$$\Box(n) \subset I^{n+1}, \quad \Box(n) = \{(t_0, t_1, \cdots, t_n) \in I^{n+1} | t_0 = 1, t_1 = 1\}.$$

The action map assigns to $\lambda : m \to n$ the map

$$\lambda_* : \Box(m) \to \Box(n)$$

given by

$$\lambda_*(t_0, \cdots, t_m) = (max\{t_i | \lambda(i) = 0\}, \cdots, max\{t_i | \lambda(i) = j\}, \cdots, max\{t_i | \lambda(i) = n\}).$$

Thus to check that $\Box$ is a multiplicative Λ-space requires appropriate maps $\Box(m) \times \Box(n) \to \Box(m+n)$, which are given by

$$\phi_{m,n}((1, t_1, \cdots, t_{m-1}, 1), (1, u_1, \cdots, u_{n-1}, 1) = (1, t_1, \cdots, t_{m-1}, 1, u_1, \cdots, u_{n-1}, 1).$$

We can obtain from the strictly multiplicative Λ-space $\Box$ a strictly multiplicative $\wr\Lambda$-space, which we also denote by $\Box$.

(7.17) *There is the natural $\wr\Lambda$-space $\Box$ given by*

$$\Box(A) = \{(t_0, \cdots, t_n) \in I^{n+1} | t_i = 1 \; for \; i \in A\}.$$

Here if A and B are objects of $\wr\Lambda$ as above and if $\lambda : A \to B$ then

$$\lambda_*(t_0, \cdots, t_m) = (u_0, \cdots, u_n)$$

where u_j is the max of all t_i for which $\lambda(i) = j$, or is zero if $\lambda^{-1}(j)$ is empty.

We then have the natural homeomorphism

$$\Box(A) \times \Box(B) \simeq \Box(A \oplus B),$$

$$((1, t_1, \cdots, t_{m-1}, 1), (1, u_1, \cdots, u_{n-1}, 1)) \mapsto (1, t_1, \cdots, t_{m-1}, 1, u_1, \cdots, u_{n-1}, 1).$$

The $\wr\Lambda$-space $\Box$ is strictly multiplicative.

We can now define the second of the realizations of Δ^o- spaces as the functor

$$| \diamond |_{\wr\Lambda} : \mathrm{TOP}^{\Delta^o} \to \mathrm{TOP}$$

defined by $|X|_{\wr\Lambda} = \wr X \times_{\wr\Lambda} \Box$. Having set the historic pattern for realizations in our earlier consideration of the Milnor realization, we have only to note that the pattern continues to hold.

Properties of the Stasheff Realization

The reader should first check that Λ has a unique splitting of its morphisms into epimorphisms followed by monomorphisms.

For $\{0,m\} \subset A \subset \{0,1,\cdots,m\}$ and $\{0,n\} \subset B \subset \{0,1,\cdots,n\}$, define a morphism $\lambda : A \to B$ in $\wr\Lambda$ to be a *mono* in $\wr\Lambda$ if λ is a mono in Λ. Similarly, define a morphism $\lambda : A \to B$ in $\wr\Lambda$ to be an *epi* in $\wr\Lambda$ if λ is an epi in Λ and if $B = \lambda(A)$.

(7.18) *The category $\wr\Lambda$ satisfies the conclusions of (2.1). The subcategory Epi $\wr\Lambda$ has pushouts. The subcategory $Mono \wr \Lambda$ has the restricted pullback condition.*

We can now define what it means for a $\wr\Lambda$-space to be *nicely cofibered*, simply by repeating the definition preceding (6.7) with the name of the category changed. If Y is a $\wr\Lambda$-space, then an element $y \in Y(A)$ is *degenerate* if there exists a mono $\lambda : B \to A$ in $\wr\Lambda$ which is not the identity and a $y' \in Y(B)$ such that $\lambda y' = y$.

7.19 *The $\wr\Lambda$-spaces $E_{\wr\Lambda}$ and $\square$ are both nicely cofibered. For any nicely cofibered $\wr\Lambda$-space Y, we have that each $(Y(A), Y^{deg}(A))$ is a cofibered pair in TOP.*

PROOF. A point of $E_{\wr\Lambda}$ can be written uniquely in the form

$$e = (\tau_0, \tau_1, \cdots, \tau_n) \times_\Delta (t_0, \cdots, t_n),$$

where the τ_i are morphisms of $\wr\Lambda$ such that the composition exists and for $i > 0$ no τ_i is the identity, and where $t_i > 0$ and $t_0 + \cdots + t_n = 1$. The action of $\wr\Lambda$ is given by

$$\tau((\tau_0, \tau_1, \cdots, \tau_n) \times_\Delta (t_0, \cdots, t_n)) = (\tau\tau_0, \tau_1, \cdots, \tau_n) \times_\Delta (t_0, \cdots, t_n).$$

That $E_{\wr\Lambda}$ is nicely cofibered is readily checked. Similarly for $\square$. $\square$

Let now X be a Δ^o-space, and consider the associated $(\wr\Lambda)^o$-space $\wr X$. Then we can say that $x \in (\wr X)(A)$ is *degenerate* if there exists an epi $\mu : A \to B$ in $\wr\Lambda$ which is not an identity morphism, and a $y \in (\wr X)(B)$ with $x = y\mu$. Otherwise, x is *nondegenerate*.

We will say that the simplicial space X satisfies the *strong cofibration condition* if for each epi $\mu : m \to n$ in Δ the pair $(X(m), \mu^* X(n))$ is cofibered in TOP$/X(0) \times X(0)$. If $X^{deg}(m)$ denotes the union of all $\mu^* X(n)$ for all proper epis μ, then the pair $(X(m), X^{deg}(m))$ is cofibered in TOP$/X(0) \times X(0)$.

(7.20) *Let X be in TOP^{Δ^o} and consider the associated $(\wr\Lambda)^o$-space $\wr X$. Let Y be a nicely cofibered $\wr\Lambda$-space. Then*

(1) *the analogue of (2.5) holds for $\wr X$,*
(2) *if $\sim$ denotes the equivalence relation on $\coprod (\wr X)(A) \times Y(A)$ for which*

$$\wr X \times_{\wr\Lambda} Y = (\coprod (\wr X)(A) \times Y(A))/\sim,$$

then the analogues of (2.7) and (2.8) hold, and

(3) considering $Ob\ \wr\Lambda$ as being a poset with $B \leq A$ the least relation such that $B \leq A$ if either there is a mono $B \to A$ in $\wr\Lambda$ or else there is an epi $A \to B$ in $\wr\Lambda$, then $\wr X \times_{\wr\Lambda} Y$ is a filtered $Ob\ \wr\Lambda$-space

$$\wr X \times_{\wr\Lambda} Y = \bigcup_{A \in Ob\ \wr\Lambda} (\wr X \times_{\wr\Lambda} Y)_A$$

and there is a relative homeomorphism in $TOP/X(0) \times X(0)$

$$\begin{gathered}((\wr X)(A), (\wr X)^{deg}(A)) \times (Y(A), Y^{deg}(A)) \\ \to ((\wr X \times_{\wr\Lambda} Y)_A, \textstyle\bigcup_{B<A} (\wr X \times_{\wr\Lambda} Y)_B).\end{gathered}$$

Note in the above that if $(p_1, \cdots, p_k)$ is an object of $\wr\Lambda$, then it is greater than any object either obtained from $(p_1, \cdots, p_k)$ by reducing the size of some p_i, or eliminating some p_i, or by replacing some singleton term p_i by a doubleton p, q where $p + q = p_i$. If X satisfies the cofibration condition for simplicial spaces, then $\wr X \times_{\wr\Lambda} Y$ is a cofibered, $\wr\Lambda$-filtered space in the sense preceding (6.4).

$|X|_{\wr\Lambda}$ is a Homotopy Colimit When X Satisfies the Strong Cofibration Condition

We now need to understand $E_{Mono\ \wr\Lambda}$. First of all, there is an action of $\wr\Lambda$ on $E_{Mono\ \wr\Lambda}$ given by the following general proposition.

Theorem 7.21 *Suppose that G is a small category with subcategories H and K such that*

$$Ob\ H = Ob\ K = Ob\ G$$

and such that every morphism g of G has a unique factorization $g = hk$ where h is a morphism of H and k is a morphism of K. Then G acts on E_H by

$$g((h_0, h_1, \cdots, h_n) \times_\Delta (t_0, \cdots, t_n)) = (h'_0, h'_1, \cdots, h'_n) \times_{\Delta^o} (t_0, \cdots, t_n),$$

where

$$gh_0 = h'_0 k_0, k_0 h_1 = h'_1 k_1, \cdots, k_{n-1} h_n = h'_n \mu_n$$

as in the commutative diagram

$$\begin{tikzcd}
p \arrow[d, "g"'] & p_0 \arrow[l, "h_0"'] \arrow[d, "k_0"'] & \cdots \arrow[l, "h_1"'] & p_n \arrow[l, "h_n"'] \arrow[d, "k_n"'] \\
q & q_0 \arrow[l, "h'_0"'] & \cdots \arrow[l, "h'_1"'] & q_n. \arrow[l, "h'_n"']
\end{tikzcd}$$

We also get from (7.21) a natural G-map $E_G \to E_H$. Namely, send

$$(g_0, g_1, \cdots, g_n) \times_\Delta (t_0, \cdots, t_n)$$

into

$$(h_0, h_1, \cdots, h_n) \times_\Delta (t_0, \cdots, t_n)$$

where the h_i are from the commutative diagram

$$\begin{tikzcd}
p \arrow[d, "g"'] & p_0 \arrow[l, "g_0"'] \arrow[d, "k_0"'] & \cdots \arrow[l, "g_1"'] & p_n \arrow[l, "g_n"'] \arrow[d, "k_n"'] \\
q & q_0 \arrow[l, "h_0"'] & \cdots \arrow[l, "h_1"'] & q_n . \arrow[l, "h_n"']
\end{tikzcd}$$

As a particular case, we get a natural $\wr\Lambda$-map $E_{\wr\Lambda} \to E_{Mono\,\wr\Lambda}$.

We now compute $E_{Mono\,\wr\Lambda}$. Let

$$\{0, n\} \subset A \subset \underline{n}$$

be an object of $\wr\Lambda$. Then $(Mono\ \wr\Lambda)(A, \diamond)$ is naturally the poset whose elements are all

$$\{0, n\} \subset A \subset B \subset C \subset \underline{n};$$

if we denote this object by (B, C) then there is a morphism

$$(B, C) \to (B', C')$$

whenever

$$A \subset B' \subset B \subset C \subset C'.$$

This is just a poset P_A, and its classifying space can be computed to be the subcube of I^{n+1} consisting of all $(1, t_1, \cdots, t_{n-1}, 1)$ such that $t_i = 1$ whenever $i \in A$. Each (B, C) contributes to this cube the face which has $t_i = 0$ whenever $i \notin C$ and $t_i = 1$ whenever $i \in B$.

$$\begin{tikzcd}
\{0,2,3\} \subset \{0,2,3\} \arrow[r] \arrow[d] \arrow[dr] & \{0,2,3\} \subset \{0,1,2,3\} \arrow[d] & \{0,1,2,3\} \subset \{0,1,2,3\} \arrow[l] \arrow[d] \arrow[dl] \\
\{0,3\} \subset \{0,2,3\} \arrow[r] & \{0,3\} \subset \{0,1,2,3\} & \{0,1,3\} \subset \{0,1,2,3\} \arrow[l] \\
\{0,3\} \subset \{0,3\} \arrow[r] \arrow[u] \arrow[ur] & \{0,3\} \subset \{0,1,3\} \arrow[u] & \{0,1,3\} \subset \{0,1,3\} \arrow[l] \arrow[u] \arrow[ul]
\end{tikzcd}$$

The Cube $E_{Mono\,\wr\Lambda}(\{0, 3\})$

The left action of $\wr\Lambda$ on $E_{Mono\,\wr\Lambda}$, which is assured by (7.21), can be written out explicitly in poset terms. The above poset can thus be checked to have classifying space B_{P_A} the subcube of I^{n+1} consisting of all $(1, t_1, \cdots, t_{n-1}, 1)$ for which $t_i = 1$ whenever $i \in A$, and the boundary of the cube is just the set of degenerate elements. That is, $Mono\ \wr\Lambda$ is a cellular category.

(7.22) *Suppose one takes $E_{Mono\,\wr\Lambda}$ together with its action by $\wr\Lambda$ given by (7.19). This is precisely the $\wr\Lambda$-space $\square$ and we then have from (7.19) the natural $\wr\Lambda$-map $E_{\wr\Lambda} \to \square$. It then follows, using (7.19) and a variant of (6.8), that for any Δ^o-space X satisfying the strong cofibration condition the induced map*

$$\wr X \times_{\wr\Lambda} E_{\wr\Lambda} \to \wr X \times_{\wr\Lambda} \square$$

is a homotopy equivalence in $TOP/X(0) \times X(0)$. Hence $|X|_{\wr\Lambda}$ is a homotopy colimit for $\wr X$ as a $(\wr\Lambda)^o$-space in $TOP/X(0) \times X(0)$.

The Topological Category $\mathcal{W}X = |X|_{\wr\Lambda}$ for X a Δ^o-Space

The following theorem gives an extended Vogt construction [4.7].

(7.23) *For every Δ^o-space X there is the topological category $\mathcal{W}X$ whose space of objects is $X(0)$ and whose space of morphisms is*

$$Mor\ \mathcal{W}X = \wr X \times_{\wr\Lambda} \square.$$

The generators of the morphisms are all

$$x \times_{\wr\Lambda} (1, t_1, \cdots, t_{n-1}, 1)$$

for $x \in X(n)$ and $(1, t_1, \cdots, t_{n-1}, 1) \in I^{n-1}$. Every generator can be written uniquely as a finite composition of those for which x is non-degenerate and $0 < t_i < 1$ for all i. The relations are generated by the following. If $t_i = 0$, this element is equal to the element obtained by deleting t_i and replacing x by its face opposite the ith vertex. If $t_i = 1$, this element is equal to the composition

$$(x' \times_{\wr\Lambda} (1, t_1, \cdots, t_{i-1}, 1))(x'' \times_{\wr\Lambda} (1, t_{i+1}, \cdots, t_{n-1}, 1))$$

where x' is the front i-face of x and x'' is the back $(n-i)$-face. If x is a degenerate element so that $x = y\delta$ for some epi δ and some nondegenerate y, then the above element is equal to $y \times_{\wr\Lambda} \delta_*(1, t_1, \cdots, t_{n-1}, 1)$.

If X satisfies the strong cofibration condition, then the continuous functor $WX \to \mathcal{W}X$ given by (7.19) is the identity on objects and a homotopy equivalence on morphisms, where the homotopy equivalence can be taken in $TOP/X(0)\times X(0)$.

Thus if X satisfies the strong cofibration condition, then $\mathcal{W}X$ can be used to give a homotopy model for WX. This assumes its simplest form for simplicial sets X. In particular, if A is any space with base point a_0, then one can apply the above to the simplicial set $X = (A, a_0)^{(\nabla, \nabla_0)}$ where topology is ignored. The above category $\mathcal{W}X$ may well be a topological form of the cobar construction of Adams [7.1], and $|X|_{\wr\Lambda}$ is then a topological monoid which should be weakly homotopy equivalent to the loop space ΩA, although we have not checked in this non-topologized form.

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CHAPTER VIII

Loop Space Models for the Homotopy Category of Based, Connected CW-Complexes

Consider the category $\mathcal{D}$ whose objects (A, a_0) are path connected spaces A with base point a_0, where A has a CW-structure in which a_0 is a vertex, and whose morphisms are maps of pairs. This category has the homotopy relation of maps of pairs, and one thus has the homotopy category $\mathcal{D}$ $[\mathrm{HE}^{-1}]$. In this chapter we present a little of a substantial body of work which presents categories equivalent to $\mathcal{D}$ $[\mathrm{HE}^{-1}]$, constructed in one way or another from the loop spaces of (A, a_0) and from operations on the loop spaces. There is an extensive historical background not covered in the body of this chapter.

In 1956, Milnor [8.1,8.2] introduced the category $\mathcal{C}$ whose objects are the CW-groups, and whose morphisms are the homomorphisms. There is on this category the weak homotopy equivalences, which are the homomorphisms of CW-groups which are also homotopy equivalences of spaces. The basic result is that $\mathcal{C}$ $[\mathrm{WHE}^{-1}]$ is equivalent to $\mathcal{D}$ $[\mathrm{HE}^{-1}]$. His account required countability assumptions on both the complexes and the CW-groups, but these can be eliminated using later work on k-space topologies. Milnor's methods are simplicial, and use the natural piecewise linear paths on a simplicial complex. We leave it to the reader to consult his papers.

If one thinks of the objects of $\mathcal{C}$ as spaces made out of loop spaces together with their H-space operations, the subject thus starts with this tightest of H-space structures, that of a group.

Later there was Stasheff's work [7.6] which sought the loosest of structure; this structure of his on a space A with base point a_0 has come to be called a strongly homotopy associative H-space structure on (A, a_0). Later work of Boardman-Vogt [4.1] and May [2.8] gave in passing alternate presentations, similarly with loose structure. See for example Thomason [7.8] as well as the work of Stasheff and May already cited.

One may as well take for $\mathcal{D}$ the full subcategory of the category TOP_* of spaces with base point whose objects are the pairs (A, a_0) which are homotopy equivalent in TOP_* to path connected CW-complexes modulo a vertex; for this category is equivalent to the model used above.

We only give an account of Segal's treatment [4.4] of this subject, which works in terms of even looser structure. One can take as a starting point the functor

$$R : \mathcal{D} \to \mathrm{TOP}^{\Delta^o}$$

which assigns to (A, a_0) the Δ^o-space X whose n-simplices are the singular n-simplices in A all of whose vertices are at a_0. At first sight, X might appear to have a far more complicated structure than the loop space ΩA. But on closer examination, one has

(i) $X(0) = pt$,
(ii) $X(1) = \Omega A$, and
(iii) for each $n > 1$ one has the fibration and homotopy equivalence $X(n) \to (\Omega A)^n$ which assigns to a singular n-simplex σ its sequence of n edges from vertex to successive vertex. In particular, X is a special Δ^o-space in the sense of Chapter 7, and moreover $X(0) = pt$.

One begins then by examining the special Δ^o-spaces X which have $X(0) = pt$. Each $X(n)$ then has a natural base point, and thus $\pi_0(X(n))$ is well defined as a set with base point. In fact, $\coprod \pi_0(X(n))$ then becomes a simplicial set, and the fact that X is special implies that $\coprod \pi_0(X(n))$ is strictly comultiplicative. Thus $\pi_0(X(1))$ receives in a natural way a monoid structure. In the above construction, $\pi_0(X(1)) = \pi_0(\Omega A)$ is a group. Thus we add this condition as well. Finally, we let $\mathcal{C}$ denote the full subcategory of TOP^{Δ^o} whose objects X are such that

(i) $X(0) = pt$,
(ii) X is a special Δ^o-space,
(iii) the monoid $\pi_0(X(1))$ is a group, and
(iv) each $X(n)$ is of the homotopy type of a CW-complex.

Then we prove Segal's theorem that $\mathcal{C}\,[\mathrm{WHE}^{-1}]$ is equivalent to $\mathcal{D}\,[\mathrm{HE}^{-1}]$.

We give one example of a corollary. Consider the category TOP MON of topological monoids and homomorphisms. Let WHE denote the subcategory whose morphisms are the homomorphisms $f : G \to G'$ which are also homotopy equivalences of spaces. There is the full subcategory $\mathcal{C}'$ of TOP MON whose objects G are such that

(i) G is of the homotopy type of a CW-complex, and
(ii) the monoid $\pi_0(G)$ is a group.

Then the category $\mathcal{C}'\,[\mathrm{WHE}^{-1}]$ is equivalent to $\mathcal{D}\,[\mathrm{HE}^{-1}]$.

The Functors $\mathbf{TOP}^{\Delta^o} \leftrightarrows \mathbf{TOP}$

We look for structures of the following type. Choose as the starting point a category $\mathcal{D}$ of topological objects, endowed with a natural homotopy relation and a subcategory HE of homotopy equivalences. One seeks a category $\mathcal{C}$ whose objects and morphisms have an appropriate equivariant interpretation, endowed both with homotopy and with weak homotopy equivalences. Along with $\mathcal{D}$ one seeks a functor $R : \mathcal{D} \to \mathcal{C}$, with the goal of interpreting $\mathcal{D}[\mathrm{HE}^{-1}]$ by use of R. As an auxiliary, one seeks a functor $L : \mathcal{C} \to \mathcal{D}$ so that one has a diagram

$$\mathcal{C} \leftrightarrows \mathcal{D}.$$

Also R should take homotopy equivalences into weak homotopy equivalences and L should take weak homotopy equivalences into homotopy equivalences. There should be a natural transformation $S : 1 \to RL$ and a natural isomorphism $T : LR \to 1$ for which the derived diagram

$$\mathcal{C}\ [\text{WHE}^{-1}] \leftrightarrows \mathcal{D}\ [\text{HE}^{-1}]$$

is an adjoint diagram. It will then be the case that the homotopy category $\mathcal{D}\ [\text{HE}^{-1}]$ is equivalent to the full subcategory of $\mathcal{C}\ [\text{WHE}^{-1}]$ whose objects are all the RA's. If one can get this far, then one will have a good start on a redescription of $\mathcal{D}\ [\text{HE}^{-1}]$. Such a structure we call informally a *partial model* for $\mathcal{D}\ [\text{HE}^{-1}]$, where *partial* refers to the fact that we may not have an independent description in terms of $\mathcal{C}$ alone of the full subcategory whose objects are isomorphic in $\mathcal{C}\ [\text{WHE}^{-1}]$ to some RA. A *full model* for $\mathcal{D}\ [\text{HE}^{-1}]$ will have been attained if one is in addition able to describe gracefully the full subcategory $\mathcal{C}'$ of $\mathcal{C}$ whose objects are isomorphic to some RA in $\mathcal{C}\ [\text{WHE}^{-1}]$. In this event, one has an equivalence

$$\mathcal{C}'\ [\text{WHE}^{-1}] \sim \mathcal{D}\ [\text{HE}^{-1}].$$

As an example for starting purposes, let $\mathcal{D}$ be the category TOP. Choose $\mathcal{C}$ to be the category TOP^{Δ^o} with its standard choice of weak homotopy equivalences. Let $E = \coprod E(n)$ be the universal Δ-space used in Chapters 5 and 6. There is the functor $R : \mathcal{D} \to \mathcal{C}$ defined by $RA = A^E = \coprod A^{E(n)}$, and the functor $L : \mathcal{C} \to \mathcal{D}$ defined by $LX = X \times_\Delta E$. These provide an adjoint diagram

$$\mathcal{C} \leftrightarrows \mathcal{D}.$$

There is in Δ the terminal Δ-space $Ob\ \Delta$ and the weak homotopy equivalence $E \to Ob\ \Delta$ of Δ-spaces. There is thus a weak homotopy equivalence $A^{Ob\ \Delta} \to A^E$ of Δ^o-spaces and an induced homotopy equivalence $A^{Ob\ \Delta} \times_\Delta E \to A^E \times_\Delta E$ of spaces. But $A^{Ob\ \Delta}$ is the constant Δ^o-space which assigns to each non-negative integer the space A and to each morphism of Δ the identity map. We then have that

$$A^{Ob\ \Delta} \times_\Delta E \simeq A \times B_{Mono\ \Delta} \sim A,$$

where $B_{Mono\ \Delta}$ is the contractible infinite dimensional dunce hat D. Thus each $T : LR(A) \to A$ is a homotopy equivalence.

We can now exhibit the full subcategory $\mathcal{C}'$ as the full subcategory of TOP^{Δ^o} whose objects X are such that for each $\delta : n \to m$ in Δ the map $\delta^* : X(m) \to X(n)$ is a homotopy equivalence in TOP. Suppose X is an object of $\mathcal{C}'$. One can then consider X' in TOP^{Δ^o} where for each n, $X'(n) \subset X(n)$ is the image $\delta^*(X(0))$ for the unique morphism $\delta : n \to 0$. Inclusion $X' \to X$ is then a weak homotopy equivalence in TOP^{Δ^o}, hence we have a weak homotopy equivalence $RL(X') \to RL(X)$ in TOP^{Δ^o}. But the natural transformation $S : X' \to RL(X')$ is readily checked to be a weak homotopy equivalence in TOP^{Δ}, and we have checked all the conditions for this model.

Theorem 8.1 *There is the adjoint diagram*

$$TOP^{\Delta^o} \leftrightarrows TOP$$

where $L(X) = X \times_\Delta E$ and $R(A) = A^E$. Let $\mathcal{C}'$ be the full subcategory of TOP^{Δ^o} whose objects X are such that $\delta^* : X(m) \to X(n)$ is a homotopy equivalence in TOP for each $\delta : n \to m$ in Δ. Then we have the equivalence of categories

$$\mathcal{C}'\,[WHE^{-1}] \sim TOP\,[HE^{-1}].$$

In this full model for TOP $[HE^{-1}]$, one can equally well use $R(A) = A^\nabla$, or even let $R(A)$ be the Δ^o-space X which assigns to each n the space A and to each $\delta : m \to n$ the identity map of A.

Note that the above $\mathcal{C}'$ can be thought of as the full subcategory of TOP^{Δ^o} consisting of all simplicial spaces X for which all $X(n)$ are determined up to homotopy by $X(0)$.

The Functors $R : \mathbf{TOP}^{\Delta^o} \leftrightarrows$ PAIR TOP

One generalize R above by constructing a simplicial space $X = R(A, A_0)$ from each closed pair (A, A_0) of compactly generated spaces such that A_0 intersects every path component of A. Denote the category with these as objects and maps of pairs as morphisms by PAIR TOP. Let us proceed with an attempt to use this as $\mathcal{D}$, subject to alterations.

There is functorial choice for a functor R : PAIR TOP $\to \mathrm{TOP}^{\Delta^o}$ and a slightly less functorial model, amounting to the same in the end. If one chooses the slightly less functorial Δ-space ∇, then one replaces each $\nabla(n)$ by the closed pair $(\nabla(n), \nabla_0(n))$, where $\nabla_0(n)$ is the 0-skeleton of $\nabla(n)$. Then take

$$X = R(A, A_0) = (A, A_0)^{(\nabla, \nabla_0)} = \coprod (A, A_0)^{(\nabla(n), \nabla_0(n))}.$$

If one considers only pairs (A, A), then one retrieves the case of the preceding paragraphs.

We usually will make the more functorial choice of E rather than ∇. Then the Δ-space E can be replaced by the pair (E, E_0) of Δ-spaces, where $E_0 = \coprod E_0(n)$ and $E_0(n)$ is the union of all $\delta_* E(0)$ over the morphisms $\delta : 0 \to n$ in Δ. Then one defines the more functorial

$$R : \text{PAIR TOP} \to \mathrm{TOP}^{\Delta^o}$$

so that

$$RA = (A, A_0)^{(E, E_0)} = \coprod (A, A_0)^{(E(n), E_0(n))}.$$

The natural map $(E, E_0) \to (\nabla, \nabla_0)$ shows the two choices for R to be naturally isomorphic in TOP^{Δ^o} $[\mathrm{WHE}^{-1}]$.

A choice for $L : \mathrm{TOP}^{\Delta^o} \to$ PAIR TOP is readily at hand. Namely, one can take

$$LX = (X \times_\Delta E, X \times_\Delta E_0).$$

It is checked that $X \times_\Delta E_0$ is precisely the space $X(0) \times E(0)$ where $E(0) = D$ is the infinite dimensional dunce hat. Since D is contractible, this is homotopy equivalent to $X(0)$.

Theorem 8.2 *Let PAIR TOP denote the category whose objects are the closed pairs (A, A_0) of compactly generated spaces such that A_0 intersects every path* *component of A, and whose morphisms are the maps of pairs. There are then adjoint functors*

$$TOP^{\Delta^o} \leftrightarrows PAIR\ TOP,$$

defined by

$$LX = (X \times_\Delta E, X \times_\Delta E_0), \quad R(A, A_0) = (A, A_0)^{(E, E_0)}.$$

Up to weak homotopy equivalence in TOP^{Δ^o}, one can use

$$R(A, A_0) = (A, A_0)^{(\nabla, \nabla_0)}.$$

The Map $LR(A, A_0) \to (A, A_0)$

For a pair (A, A_0) in PAIRTOP, we need to understand better the above map

$$LR(A, A_0) \to (A, A_0).$$

We use the slightly less functorial choice (∇, ∇_0). Let X denote the simplicial space $(A, A_0)^{(\nabla, \nabla_0)}$; then we must analyze the pair

$$(X \times_\Delta E, X \times_\Delta E_0) = (X \times_{Mono\ \Delta} \nabla, X \times_{Mono\ \Delta} \nabla_0).$$

The subspace $X \times_\Delta E_0$ is trivial to analyze; since $X(0) \simeq A_0$, it is precisely the space $A_0 \times E(0)$ and since $E(0)$ is contractible the projection map is a homotopy equivalence onto A. Thus we have only to understand $X \times_\Delta E$.

We need a companion Δ^o-space Y. Let $Y(n)$ be the space of all maps $\rho : I \times \nabla(n) \to A$ such that

$$\rho(0 \times \nabla(n)) = const, \quad \rho(1 \times \nabla_0(n)) \subset A_0.$$

For each $\delta : m \to n$ in Δ, define the action map by

$$(\delta^* \rho)(t, u) = \rho(t, \delta^*(u)).$$

There is a variant presentation Y' of Y, which follows by collapsing $0 \times \nabla(n)$ to a point, thus obtaining $\nabla(n+1)$ and a natural map $I \times \nabla(n) \to \nabla(n+1)$. In this variant, $Y'(n)$ becomes all maps $\rho' : (\nabla(n+1), \nabla_{0,+}(n+1)) \to (A, A_0)$ where $\nabla_{0,+}(n+1)$ denotes the union of all vertices of $\nabla(n+1)$ except the first one. We leave it to the reader to supply the action of Δ on Y'. There is then the natural map $Y \to Y'$ in TOP^{Δ^o}, a homotopy equivalence in TOP^{Δ^o}.

There is a natural Δ^o-map $\mu : Y \to X$. For each n, send ρ into the restriction of ρ to $1 \times \nabla(n)$. Alternatively, there is the natural Δ^o-map $Y' \to X$. It is readily checked that this latter is a weak homotopy equivalence in TOP^{Δ^o}, hence so also is $Y \to X$. Thus the natural map

$$Y \times_\Delta E \to X \times_\Delta E$$

is a homotopy equivalence. Thus we can understand $X \times_\Delta E$ up to homotopy equivalence by understanding $Y \times_\Delta E$.

There is a natural map $\pi : Y \times_\Delta E \to A$ which maps each point $\rho \times_\Delta e$ into $\rho(0 \times \nabla(n))$. Our next problem is to present an extensive array of cases in which

π is a homotopy equivalence of spaces. As a first assumption, it is helpful to have π onto, and in order to achieve this we assume that A_0 intersects each path component of A. To obtain that π is a homotopy equivalence of spaces, we need also that (A, A_0) is a CW-pair.

Theorem 8.3 *Let (A, A_0) be a CW-pair such that A_0 intersects each path component of A. Then the natural map*

$$(A, A_0)^{(E,E_0)} \times_\Delta E \to A$$

of (8.2) is a homotopy equivalence of spaces.

PROOF. The proof rests on the map $\pi : Y \times_\Delta E \to A$ given above.

We prove first that π is a Dold fibration. Given a path γ of A from a to b, there is a map $\gamma_* : \pi^{-1}(b) \to \pi^{-1}(a)$. Denote by Y_b all ρ such that $\rho(0 \times \nabla(n)) = b$, so that we have

$$\pi^{-1}(b) = Y_b \times_\Delta E.$$

We then have the Δ^o-map $\gamma_\# : Y_b \to Y_a$ sending ρ into $\gamma * \rho$ where

$$(\gamma * \rho)(t, u) = \begin{cases} \gamma(2t), & \text{for } 0 \le t \le 1/2 \\ \rho(2t-1, u), & \text{for } 1/2 \le t \le 1. \end{cases}$$

We then get an induced map

$$\gamma_* : \pi^{-1}(b) = Y_b \times_\Delta E \to Y_a \times_\Delta E = \pi^{-1}(a).$$

Suppose we are given a path homotopy $\{\gamma_t | 0 \le t \le 1\}$ of paths from a to b. The Δ^o-maps

$$\gamma_{0\#}, \gamma_{1\#} : Y_b \to Y_a$$

are readily seen to be homotopic as Δ^o-maps and hence

$$\gamma_{0*}, \gamma_{1*} : \pi^{-1}(b) \to \pi^{-1}(a)$$

are homotopic.

If we denote by $A^I \times_A Y$ all (γ, ρ) with $\gamma(1) = \rho(0 \times \nabla(n))$, then we can regard the above as giving a Δ^o-map

$$A^I \times_A Y \to Y$$

and hence a commutative diagram of maps

$$\begin{array}{ccc} A^I \times_A (Y \times_\Delta E) & \longrightarrow & Y \times_\Delta E \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ A^I & \xrightarrow{p} & A, \end{array}$$

where $p(\gamma) = \gamma(0)$. That is, continuity of the action is readily checked.

Let now U be an open set of the CW-complex A which has some $u_0 \in U$ as strong deformation retract. We then have for each $u \in U$ a natural path γ_u from u to u_0 determined by the deformation. We also have the map

$$U \times \pi^{-1}(u_0) \to \pi^{-1}(U), \quad (u, x) \mapsto \gamma_{u*}(x)$$

which is seen to be a homotopy equivalence of maps over U. Hence π is a Dold fibration over U. Since A is a CW-complex, it then follows from results of Dold [3.2] that π is a Dold fibration, since the open sets U which strongly contract onto a point then cover A.

We have now to prove that each fiber $\pi^{-1}(a)$ is contractible. It suffices to prove this for each $a_0 \in A_0$. This requires an argument about simplicial path spaces, implying that each fiber of $\pi : \pi^{-1}(A_0) \to A_0$ is contractible. Thus with this theorem we will have the diagram of homotopy equivalences

$$A \xleftarrow{\pi} Y \times_\Delta E \longrightarrow X \times_\Delta E$$

and the theorem will follow.

Given the above simplicial space X, denote by PX the simplicial space with $(PX)(n) = X(n+1)$. An n-simplex of PX is then a map

$$\sigma : (\nabla(n+1), \nabla_0(n+1)) \to (A, A_0).$$

Given $\delta : m \to n$ in Δ, there is the morphism $m+1 \to n+1$ in Δ given by $0 \mapsto 0$ and $i \mapsto \delta(i-1)+1$ for $i > 0$. Denote this morphism by $1_0 \oplus \delta : m+1 \to n+1$. There results the action of Δ^o on PX.

The above space $\pi^{-1}(A_0)$ can then be taken to be

$$\pi^{-1}(A_0) = PX \times_{Mono\,\Delta} \nabla,$$

and we designate a point of it as $\sigma_{n+1} \times_{Mono\,\Delta} t$ where $\sigma_{n+1} : (\nabla(n+1), \nabla_0(n+1)) \to (A, A_0)$ and where $t \in \nabla(n)$. The map $\pi : \pi^{-1}(A_0) \to A_0$ can be taken as

$$\pi(\sigma_{n+1} \times_{Mono\,\Delta} t) = \sigma(v_{0,n+1}).$$

For each $n \geq 0$, there is the morphism $\rho_{n+1} : n+2 \to n+1$ in Δ given by

$$\rho_{n+1}(0) = \rho_{n+1}(1) = 0, \quad \rho_{n+1}(i) = i-1$$

for $2 \leq i \leq n+1$. If $\delta : m \to n$ in any morphism in Δ, there is the commutative diagram

$$\begin{array}{ccc} m+2 & \xrightarrow{1_1 \oplus \delta} & n+2 \\ {\scriptstyle \rho_{m+1}} \big\downarrow & & {\scriptstyle \rho_{n+1}} \big\downarrow \\ m+1 & \xrightarrow{1_0 \oplus \delta} & n+1. \end{array}$$

One can write down in an elementary fashion a homotopy

$$H : I \times \pi^{-1}(A_0) \to \pi^{-1}(A_0)$$

which is a strong deformation retraction of $\pi^{-1}(A_0)$ onto a closed set intersecting each fiber in a point, and which takes a fiber into itself at every stage of the homotopy. One simply defines

$$H(u, \sigma_{n+1} \times_{Mono\,\Delta} t) = \sigma_{n+1}\rho_{n+1} \times_{Mono\,\Delta} (u, (1-u)t). \quad \square$$

A Partial Model for PAIR CW

We now pass from the above to a precise model. Think of PAIR TOP as the full subcategory of $\mathrm{TOP}^{0\to 1}$ whose objects are the closed inclusions $A_0 \hookrightarrow A_1$, for convenience adding the condition that A_0 intersects each path component of A. Take for homotopy in this category that of $\mathrm{TOP}^{0\to 1}$. Thus $f_0, f_1 : (A, A_0) \to (B, B_0)$ are homotopic if there is an appropriate $H : (I \times A, I \times A_0) \to (B, B_0)$.

Denote by PAIR CW the full subcategory of PAIR TOP whose objects are those which are homotopy equivalent to some (C, C_0) where C is a CW-complex and C_0 is a subcomplex intersecting every path component of C.

It is readily seen that if $f : (A, A_0) \to (B, B_0)$ is a morphism in PAIR CW such that f and f_0 are homotopy equivalences in

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "i"'] & B_0 \arrow[d, "j"'] \\ A \arrow[r, "f"] & B, \end{tikzcd}$$

then f is a homotopy equivalence in PAIR CW. This category has been defined so that homotopy equivalence and weak homotopy equivalence coincide. The equivariant categories $\mathcal{C}$ that we consider will have both homotopy and weak homotopy, while the categories $\mathcal{D}$ of topological objects and morphisms that we consider will have only homotopy. Thus we have passed to a category PAIR CW for which the possible notion of weak homotopy coincides with homotopy.

There is the usual functor $c : \mathrm{TOP} \to \mathrm{TOP}$ which assigns to each A a CW-complex. Namely let $cA = A^{\nabla} \times_{\Delta} \nabla$ where A^{∇} is given the discrete topology. If $f : A \to B$ is a map, then $cf : cA \to cB$ is a CW-map.

There is then the induced functor

$$c : \text{PAIR CW} \to \text{PAIR CW}, \quad (A, A_0) \to (cA, cA_0)$$

and a natural homotopy equivalence $c(A, A_0) \to (A, A_0)$.

Denote by CW^{Δ^o} the full subcategory of TOP^{Δ^o} whose objects are the simplicial spaces X such that each X has the homotopy type of a CW-complex. Then CW^{Δ^o} inherits homotopy and weak homotopy from TOP^{Δ^o}. Given a simplicial space X, there is the simplicial space cX, defined as the composition

$$\Delta^o \xrightarrow{X} \mathrm{TOP} \xrightarrow{c} \mathrm{TOP}.$$

Clearly cX is always in CW^{Δ^o}. If X is also in CW^{Δ^o}, then the natural Δ^o-map $cX \to X$ is a weak homotopy equivalence in CW^{Δ^o}, and in $\mathrm{CW}^{\Delta^o}[\mathrm{WHE}^{-1}]$ we have that cX and X are naturally isomorphic.

Now cX is a very nice model for a simplicial space. Each $(cX)(n)$ has a standard structure as a CW-complex, the maps $\delta^* : (cX)(n) \to (cX)(m)$ are cellular, hence the subsets $(cX)^{deg}(n) \subset (cX)(n)$ are CW-subcomplexes, hence $|cX|$ is a CW-complex. Moreover, cX automatically has the cofibration property so that $cX \times_{\Delta} E$ is homotopy equivalent to $|cX|$, thus has the homotopy type of a CW-complex. The following theorem then summarizes where we are, in terms of having constructed a partial model for PAIR CW $[\mathrm{HE}^{-1}]$.

Theorem 8.4 *We have adjoint functors*

$$CW^{\Delta^o} \leftrightarrows PAIR\ CW,$$

where

$$R : PAIR\ CW \to CW^{\Delta^o}, \quad (A, A_0) \mapsto (A, A_0)^{(E,E_0)},$$

and where

$$L : CW^{\Delta^o} \to PAIR\ CW, \quad X \mapsto (X \times_\Delta E, X \times_\Delta E_0).$$

The natural transformation $LR \to 1$ is always a homotopy equivalence in PAIR CW. Thus we have a partial model for PAIR CW $[HE^{-1}]$.

It is entirely possible that one could turn this into a full model for PAIR CW, but it is easier to handle Segal's case in which one presents a full model for $\mathcal{D}'$ $[\mathrm{HE}^{-1}]$ where $\mathcal{D}'$ is the full subcategory of PAIR TOP whose objects are homotopy equivalent in PAIR TOP to (C, C_0) where C is a path connected CW-complex and C_0 is a contractible subcomplex.

Segal's Full Model for the Case A_0 Contractible

Thus we now pass to the full subcategory $\mathcal{D}'$ of PAIR CW whose objects are of the form (A, A_0) where A is path connected and where A_0 is contractible. There is the equivalent category $\mathcal{D}$ whose objects are all (A, a_0) in PAIR CW for which A is path connected. We use these interchangeably, since they amount to the same in the end. Because it is slightly more convenient, we start with the case A_0 contractible.

There is the functor

$$R : \mathcal{D}' \to \mathrm{CW}^{\Delta^o}, \quad (A, A_0) \mapsto (A, A_0)^{(\nabla,\nabla_0)}$$

obtained by restricting the functor used earlier.

We have to make a selection of a full subcategory $\mathcal{C}$ of CW^{Δ^o} in order to obtain a full model for $\mathcal{D}'$ $[\mathrm{HE}^{-1}]$. At a minimum we need to work within the full subcategory of all X in TOP^{Δ^o} for which $X(0)$ is contractible. We may as well cut down further at the start by using whatever properties have already been noted for all $R(A, A_0)$ as (A, A_0) ranges over $\mathcal{D}'$. Here we have first the fact that each $R(A, A_0)$ is a special Δ^o-space.

There is a further reduction that is natural. For every Δ^o-space X with $X(0)$ contractible, there is a copy of $X(0)$ in each $X(n)$ and hence for each n, the set of path components of $X(n)$ has a natural base point; i.e. we can consider $\pi_0(X(n))$ as a set with base point. Hence we can consider $\coprod \pi_0(X(n))$ as a simplicial set. However, the fact that X has comultiplication strict up to homotopy implies that $\coprod \pi_0(X(n))$ is strictly comultiplicative, which implies that $\pi_0(X(1))$ is naturally a monoid. The fact then is that each $R(A, A_0)$ for A_0 contractible can be checked to have this associated monoid a group.

Theorem 8.5 *Let $\mathcal{D}'$ denote the full subcategory of PAIR CW consisting of all (A, A_0) with A path connected and A_0 contractible; as in the introduction, let $\mathcal{D}$ denote the full subcategory of PAIR CW whose objects are all (A, a_0) where A*

is path connected. Let $\mathcal{C}$ be the full subcategory of TOP^{Δ^o} whose objects are the Δ^o-spaces X such that

(i) each $X(n)$ is of the homotopy type of a CW-complex,
(ii) $X(0)$ is contractible,
(iii) X is a special Δ^o-space, and
(iv) $\pi_0(X(1))$ is a group in its natural monoid structure.

Then we have the equivalences of categories

$$\mathcal{C}\,[WHE^{-1}] \sim \mathcal{D}'\,[HE^{-1}] \sim \mathcal{D}[HE^{-1}].$$

On the left hand side of this equivalence, one can use instead of condition (ii) the condition that $X(0)$ is a singleton.

Suppose we use in the above the condition that $X(0)$ is a singleton. Then we can use Chapter 7 to give another full model for $\mathcal{D}$ [HE^{-1}]. One has functors

$$\mathrm{TOP}^{\Delta^o} \xrightarrow{W} \mathrm{TOPCAT} \xrightarrow{N} \mathrm{TOP}^{\Delta^0}.$$

If one takes $\mathcal{C}$ as in (8.5) with the option $X(0) = pt$, then one has

$$\mathcal{C} \xrightarrow{W} \mathrm{TOP}\ MON \xrightarrow{N} \mathcal{C}$$

and one can use (7.12) and additional work to show the following.

Theorem 8.6 *Suppose we let $\mathcal{C}'$ be the full subcategory of TOP MON whose objects are all topological monoids G such that*

(i) G is of the homotopy type of a CW-complex, and
(ii) $\pi_0(G)$ is a group.

Define a weak homotopy equivalence $\phi : G \to G'$ in TOP MON to be a homomorphism which is also a homotopy equivalence of spaces. Continue to denote by $\mathcal{D}$ the full subcategory of PAIR CW whose objects are all (A, a_0) where A is path connected. Then we have an equivalence of categories

$$\mathcal{C}'\,[WHE^{-1}] \sim \mathcal{D}\,[HE^{-1}].$$

References

(8.1) J. Milnor, *Construction of universal bundles, I*, Ann. of Math. **63** (1956), 272–284.

(8.2) J. Milnor, *Construction of universal bundles, II*, Ann. of Math. **63** (1956), 430–436.

CHAPTER IX

The Infinite Symmetric Product as a Source of Models of Spectra

We have thus far confined ourselves to those classical small categories of topology based in one way or another on order preserving functions from one finite ordered set to another. We must finally take some cognizance of those for which it is no longer required that they preserve order.

We interpret the beginning as being with the category $Mono\,\Sigma$ whose objects are the non-negative integers, and whose morphisms $\sigma : m \to n$ are all the monos

$$\sigma : \{1, \cdots, m\} \to \{1, \cdots, n\}.$$

Recall that TOP_* denotes the category of compactly generated spaces A with base point a_0. There is the functor

$$\mathrm{TOP}_* \to \mathrm{TOP}^{Mono\,\Sigma}, \quad A \mapsto A^\infty = \coprod A^n,$$

where the action map assigns to the morphism $\sigma : m \to n$ of $Mono\,\Sigma$ the map $\sigma_* : A^m \to A^n$ given by

$$\sigma_*(a_1, \cdots, a_m) = (a'_1, \cdots, a'_n)$$

where $a'_j = a_{\sigma^{-1}(j)}$ if $\sigma^{-1}(j) \neq \emptyset$ and $a'_j = a_0$ if $\sigma^{-1}(j) = \emptyset$.

The composition

$$\mathrm{TOP}_* \to \mathrm{TOP}^{Mono\,\Sigma} \xrightarrow{colim} \mathrm{Top}$$

then assigns to each compactly generated space A with base point a_0 a compactly generated space due to Dold-Thom [9.2,1958], which they called the *infinite symmetric product of* (A, a_0) and denoted by $SP^\infty(A)$. Points of $SP^\infty(A)$ can be written as $[a_1, \cdots, a_n]$, where $a_i \in A$, where any $a_i = a_0$ can be deleted, and where the point is unchanged if any permutation of its coordinates is performed. Then $SP^\infty(A)$ is an abelian monoid, with operation

$$[a_1, \cdots, a_m][a'_1, \cdots, a'_n] = [a_1, \cdots, a_m, a'_1, \cdots, a'_n].$$

Thus we start with this Dold-Thom functor

$$SP^\infty : \mathrm{TOP}_* \to \mathrm{AB\ TOP\ MON}.$$

If SA denotes the reduced suspension of A, then

$$SP^\infty(SA) \simeq B_{SP^\infty(A)}.$$

Since $SP^\infty(S^1)$ is of the homotopy type of S^1, the Dold-Thom theorem follows, that $SP^\infty(S^n)$ is a $K(Z,n)$ for $n > 0$.

A *spectrum* is a sequence $\{A_n, f_n | n \geq 0\}$ where each A_n is a compactly generated space with base point and where f_n is a base point preserving map

$$f_n : A_n \to \Omega A_{n+1}.$$

A spectrum is an Ω-*spectrum* if each f_n is a homotopy equivalence of pairs. Thus Dold-Thom constructed the spectrum $\{SP^\infty(S^n)\}$ with natural maps f_n. It is almost an Ω-spectrum; all but f_0 are homotopy equivalences. It can also be described without the infinite symmetric product language in classifying space terms. If G is an abelian topological monoid, then so is B_G. Given any abelian topological monoid G, one thus constructs a spectrum B^n_G whose 0th-term is the given G, and where thereafter a term is the classifying space of the preceding term. Dold-Thom thus gave an explicit construction for the spectrum described iteratively as starting with Z_+, the discrete abelian monoid of non-negative integers.

We next give McCord's generalization [1.3] of the infinite symmetric product construction to a functor

$$SP^\infty : \mathrm{TOP}_* \times \mathrm{AB\ TOP\ MON} \to \mathrm{AB\ TOP\ MON}, \quad (A, G) \mapsto SP^\infty(A; G).$$

To define it, we introduce Segal's category Γ [4.4] whose objects are the non-negative integers, and whose morphisms $\gamma : m \to n$ are all functions

$$\gamma : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}$$

for which $\gamma(0) = 0$. Note that one can interpret the category Γ as having objects certain finite sets with base point and as having morphisms all base point preserving maps joining them. That is, one has a natural functor $\Gamma \to \mathrm{TOP}_*$.

There is a functor

$$\mathrm{TOP}_* \to \mathrm{TOP}^{\Gamma^o}, \quad A \mapsto A^\infty = \coprod A^n,$$

where A^n is interpreted as all base point preserving maps

$$\{0, 1, \cdots, n\} \to A,$$

so that there is a natural right action of Γ. By abuse of notation, we have used $\diamond^\infty$ as the name of a functor $\mathrm{TOP}_* \to \mathrm{TOP}^{Mono\ \Sigma}$ and a functor $\mathrm{TOP}_* \to \mathrm{TOP}^{\Gamma^o}$. This is not as bad as it might be; at least they are interconnected by a natural functor $Mono\ \Sigma \to \Gamma^o$ which sends an object n into n, and which sends a morphism $\sigma : m \to n$ of $Mono\ \Sigma$ into the morphism $\gamma : n \to m$ of Γ given by $\gamma(j) = i$ whenever $\sigma(i) = j$ and $\gamma(j) = 0$ whenever $j = 0$ or $\sigma^{-1}(j) = \emptyset$.

One also gets a functor

$$\text{AB TOP MON} \to \text{TOP}^\Gamma, \quad G \mapsto \coprod G^n,$$

where $\gamma : m \to n$ induces $\gamma_* : G^m \to G^n$ given by

$$\gamma_*(g_1, \cdots, g_m) = (g'_1, \cdots, g'_n),$$

with

$$g'_j = \begin{cases} g_{k_1} + \cdots + g_{k_j}, & \text{for } \gamma^{-1}(j) = \{k_1, \cdots, k_j\} \\ 0, & \text{for } \gamma^{-1}(j) = \emptyset. \end{cases}$$

Then the reduced product bifunctor for Γ yields the McCord functor

$$\text{TOP}_* \times \text{AB TOP MON} \longrightarrow \text{TOP}, \quad (A, G) \mapsto SP^\infty(A; G) = A^\infty \times_\Gamma (\coprod G^n).$$

It is again important that $SP^\infty(A; G)$ has more structure than that of a space, it is in fact an abelian topological monoid, so that we have

$$SP^\infty : \text{TOP}_* \times \text{AB TOP MON} \longrightarrow \text{AB TOP MON}.$$

We give McCord's generalization of the Dold-Thom facts, namely that

$$SP^\infty(A; SP^\infty(B; G)) \simeq SP^\infty(A \wedge B; G),$$

from which one gets

$$SP^\infty(S^{n+1}; G) \simeq SP^\infty(S^1; SP^\infty(S^n; G)) \simeq B_{SP^\infty(S^n;G)}.$$

If G is a discrete abelian group, then $SP^\infty(S^n; G)$ is a $K(G, n)$. Thus one has a functor

$$\text{AB GP} \to \Omega - \text{SPECTRA}$$

assigning to each discrete abelian group an Ω-spectrum $\{SP^\infty(S^n; G)\}$, with its alternative description as the spectrum of iterated classifying spaces which starts with G.

In the notation of Chapter 7, every Γ-space is comultiplicative. Every strictly comultiplicative Γ-space Y with $Y(0) = pt$ is naturally homeomorphic to one associated with an abelian topological monoid as above.

Segal observed that there were interesting Γ-spaces Y in addition to those derived from an abelian topological monoid, and that the interesting Γ-spaces Y are those for which $Y(0) = pt$ (or equivalently $Y(0)$ contractible) and which are also strictly comultiplicative up to homotopy, i.e. have each $Y(m+n) \to Y(m) \times Y(n)$ a homotopy equivalence in TOP.

As an example, we consider the composition

$$\Gamma \to \text{TOP}_* \to \text{TOP}^{Mono\ \Sigma} \xrightarrow{hocolim} \text{TOP},$$

thus obtaining a version of Segal's most basic example of a Γ-space. Explicitly, this Γ-space assigns to n the homotopy colimit $Y(n)$ of the $Mono$ Σ-space $\{0, 1, \cdots, n\}^\infty$. We show that the space $Y(1)$ is homotopy equivalent to $\coprod B_{\Sigma(n)}$, where $\Sigma(n)$ is the symmetric group on n letters. In the next chapter, we present the small model for this Γ-space and clarify its connections with stable homotopy.

A third level of generality in the infinite symmetric product is due to Segal [4.4]. Namely, $\mathrm{TOP}^{\Gamma^o} \times \mathrm{TOP}^{\Gamma} \to \mathrm{TOP}$ yields a functor

$$\mathrm{TOP}_* \times \mathrm{TOP}^{\Gamma} \longrightarrow \mathrm{TOP}, \quad (A, Y) \mapsto SP^{\infty}(A; Y) = A^{\infty} \times_{\Gamma} Y.$$

There is the natural question of whether the added generality of using any Γ-space Y instead of an abelian topological monoid leads to extraordinary homology theories; this is Segal's topic.

In order to carry the program out, Segal had to extend the infinite symmetric product construction to a fourth level of generality, in order to display any additional structure $SP^{\infty}(A; Y)$ would have in addition to being a space. We follow Segal in constructing a functor

$$\mathcal{SP}^{\infty} : \mathrm{TOP}_* \times \mathrm{TOP}^{\Gamma} \to \mathrm{TOP}^{\Gamma}, \quad (A, Y) \mapsto \mathcal{SP}^{\infty}(A; Y)$$

such that when one fixes the object $n = 1$ of Γ one obtains $SP^{\infty}(A; Y)$. Thus the additional structure becomes the action of Γ on this Γ-space. Moreover, one has

$$SP^{\infty}(A; \mathcal{SP}^{\infty}(B; Y)) \simeq SP^{\infty}(A \wedge B; Y)$$

so that for any Γ-space Y with $Y(0) = pt$ one gets a spectrum $\{SP^{\infty}(S^n; Y)\}$ and an extraordinary homology theory associated with it. Naturally special attention is given to putting conditions on Y which make this an Ω-spectrum. Having done it for $Y(0) = pt$, one readily generalizes to the case $Y(0)$ contractible.

In summary, we introduce in this chapter consideration of $\mathrm{TOP}^{Mono\,\Sigma}$ and TOP^{Γ} as of basic importance to topology, concentrating on TOP^{Γ} as a source of spectra by means of a bifunctor

$$\mathrm{TOP}_* \times \mathrm{TOP}^{\Gamma} \to \mathrm{TOP}^{\Gamma}$$

which generalizes the classic infinite symmetric product

$$\mathrm{TOP}_* \times \mathrm{AB\ TOP\ MON} \to \mathrm{AB\ TOP\ MON}.$$

There is an the extensive body of material associated with producing spectra categorically, and analyzing the result in important special cases. We cover here the merest introduction, and from only one point of view. May is a pioneer in this field and one should see his works, for example [7.4]. A small sample of his work is in the next chapter.

The Infinite Symmetric Products of Dold-Thom

Denote by Σ the category whose objects are the non-negative integers, and whose morphisms $\sigma : m \to n$ are all the functions

$$\sigma : \{1, \cdots, m\} \to \{1, \cdots, n\}.$$

Note that the augmented simplicial category Δ_+ is a subcategory of Σ. There is also the subcategory $Iso\ \Sigma$ of all isomorphisms in Σ. For each $n \geq 0$ the isomorphisms $n \to n$ constitute the symmetric group $\Sigma(n)$, thus $Iso\ \Sigma = \coprod_{n \geq 0} \Sigma(n)$ where $\Sigma(0)$ denotes the category whose object is the empty set and which has just one morphism.

Any finite set of objects of Σ has a coproduct. For an ordered pair m, n of objects, choose a standard model $m \oplus n$ for the coproduct by the diagram

$$m \xrightarrow{\alpha_{m,n}} m + n \xleftarrow{\beta_{m,n}} n,$$

where $\alpha_{m,n}(i) = i$ and $\beta_{m,n}(i) = m + i$. It is then readily checked that Σ becomes naturally strictly monoidal. For given $\sigma_1 : m_1 \to n_1$ and $\sigma_2 : m_2 \to n_2$, the diagram

$$\begin{tikzcd} m_1 \arrow[r, "\alpha_{m_1,m_2}"] \arrow[d, "\sigma_1"'] & m_1 + m_2 & m_2 \arrow[l, "\beta_{m_1,m_2}"'] \arrow[d, "\sigma_2"] \\ n_1 \arrow[r, "\alpha_{n_1,n_2}"] & n_1 + n_2 & n_2 \arrow[l, "\beta_{n_1,n_2}"'] \end{tikzcd}$$

gives a well-defined $\sigma_1 \oplus \sigma_2 : m_1 + m_2 \to n_1 + n_2$.

It is clear that Σ is somewhat trivial from the point of view of the colimit and homotopy colimit of a Σ-space Y. Since 1 is a terminal object of Σ, then $Y(1)$ is the colimit of Y. Then $(E_\Sigma Y)(1)$ must be the homotopy colimit of Y. But the weak homotopy equivalence $E_\Sigma Y \to Y$ in TOP^Σ implies the homotopy equivalence $(E_\Sigma Y)(1) \to Y(1)$, hence $Y(1)$ is also a non-standard homotopy colimit of Y.

It is different with the subcategory $Mono\ \Sigma$. Dold-Thom [9.2] pointed out early the interest of colimits in this setting. Consider the category TOP_* of compactly generated spaces with base point, and define a functor

$$\mathrm{TOP}_* \to \mathrm{TOP}^{Mono\ \Sigma}, \qquad A \mapsto A^\infty = \coprod_{n \geq 0} A^n,$$

assigning to each mono $\sigma : m \to n$ in Σ the action map $\sigma_* : A^m \to A^n$ given by

$$\sigma_*(a_1, \cdots, a_m) = (b_1, \cdots, b_n)$$

where

$$b_j = \begin{cases} a_{\sigma^{-1}(j)}, & \text{for } \sigma^{-1}(j) \neq \emptyset \\ a_0, & \text{for } \sigma^{-1}(j) = \emptyset. \end{cases}$$

The colimit of A^∞ they called the *infinite symmetric product* of A and denoted by $SP^\infty A$.

(9.1) *Consider the functor*

$$TOP_* \to TOP^{Mono\ \Sigma}, \qquad A \mapsto A^\infty,$$

and denote by $SP^\infty A$ the colimit of A^∞. Then $SP^\infty A$ is compactly generated and has a natural filtration $SP^\infty A = \bigcup SP^p A$.

Proof. We first filter A^∞ as $A^\infty = \bigcup A^{\infty,p}$ in $\mathrm{TOP}^{Mono\ \Sigma}$. Here $A^{\infty,p} = \coprod A^{n,p}$ where $A^{n,p} \subset A^n$ consists of all $(a_1, \cdots, a_n)$ with $a_i \neq a_0$ for at most p values of i. Denote by $\pi : A^\infty \to SP^\infty A$ the natural quotient map. Each $A^{\infty,p}$ is readily seen to be a full inverse set. Since $A^{\infty,p}$ is closed in A^∞, then $\pi(A^{\infty,p})$ is closed in $SP^\infty A$; let

$$SP^p A = \pi A^{\infty,p}.$$

Clearly $SP^\infty A = \bigcup SP^p A$ is then a filtration in Top.

One next uses (1.8) to show that $A^p \to SP^p A$ is a quotient map. Consider the diagram

$$\begin{tikzcd} A^p \arrow[d] & \coprod A^{n,p} \arrow[d, "\pi'"'] \\ SP^p A \arrow[r, "="] & SP^p A, \end{tikzcd}$$

where π' is a quotient map. For each $S \subset \{1, \cdots, n\}$ having p points, let $A^\infty(S)$ denote all $(a_1, \cdots a_n)$ such that $a_i = a_0$ whenever $i \notin S$. If the points of S are enumerated in order as

$$S = \{i_1, \cdots, i_p\},$$

there is the map $\mu_S : A^\infty(S) \to A^p$ sending $(a_1, \cdots, a_n)$ into $(a_{i_1}, \cdots, a_{i_p})$, and one applies (1.8) to obtain the quotient map $A^p \to SP^p A$.

We then have the map of pairs

$$\pi'' : (A, a_0) \times \cdots \times (A, a_0) \to (SP^p A, SP^{p-1} A),$$

and this is a quotient map as well. We next have to consider the natural right action of the symmetric group $\Sigma(p)$ on A^p.

The orbit space $A^p/\Sigma(p)$ of this action is compactly generated. This follows for any action of a finite group on a compactly generated space from (1.19).

Proceeding thusly, one gets shortly a relative homeomorphism

$$((A, a_0) \times \cdots \times (A, a_0))/\Sigma(p) \to (SP^p A, SP^{p-1} A)$$

from which it follows inductively that $SP^p A$ is weakly Hausdorff by (1.20), thus $SP^\infty A$ is weakly Hausdorff by (1.18). $\square$

Cofibration Properties of $SP^\infty A$

(9.2) *Suppose A is a compactly generated space with cofibered base point a_0. Then the pair*

$$((A, a_0) \times \cdots \times (A, a_0))/\Sigma(p)$$

is a cofibered pair. Hence $SP^\infty A = \bigcup SP^p A$ is a cofibered filtered space.

Proof. Since (A, a_0) is cofibered, there exists a map $u : A \to I$ and a homotopy $H : I \times A \to A$ such that

(i) $u^{-1}(0) = \{a_0\}$;
(ii) $H(0, a) = a$ for all $a \in A$;
(iii) $H(t, a_0) = a_0$ for all $t \in I$;
(iv) $H(t, a) = a_0$ whenever $1 \geq t \geq u(a)$.

One considers next the pair $(A^p, B) = (A, a_0) \times \cdots \times (A, a_0)$. Define $v : A^p \to I$ by

$$v(a_1, \cdots, a_p) = min(u(a_1), \cdots, u(a_p))$$

and define $K : I \times A^p \to A^p$ by

$$K(t, a_1, \cdots, a_p) =$$

$$(H(min(t, u(a_2), \cdots, u(a_p)), a_1), \cdots, H(min(u(a_1), \cdots, u(a_{p-1}), t), a_p)).$$

One checks that (i)-(iv) hold for the pair (A^p, B) and the maps v, K. Hence (A^p, B) is a cofibered pair. Next one observes that the action of $\Sigma(p)$ on A^p relates well to v and K. For v takes the same value on an entire orbit, hence induces a map $w : A^p/\Sigma(p) \to I$. Moreover $K : I \times A^p \to A^p$ is equivariant, hence induces a map

$$L : I \times (A^p/\Sigma(p)) \to A^p/\Sigma(p).$$

The resulting maps w and L satisfy (i)-(iv) for the pair $(A^p/\Sigma(p), B/\Sigma(p))$, and the result follows. $\square$

The Classifying Space of the Abelian Topological Monoid $SP^\infty A$

The $Mono$ Σ-spaces A^∞ are strictly comultiplicative in the terms of Chapter 7. That is, there are functorial homeomorphisms $A^{m+n} \to A^m \times A^n$. Since $A^0 = pt$, it follows automatically that the colimit of A^∞ is a monoid. The bifunctor on $Mono$ Σ derives from a coproduct on Σ, hence there is the isomorphism $\sigma : m \oplus n \simeq n \oplus m$ which identifies two different coproducts of the unordered pair m, n. Hence $SP^\infty(A)$ is abelian.

Thus $SP^\infty A$ is an abelian topological monoid. We stop a moment for a few generalities about abelian topological monoids. Let G be such, where we assume the identity element 1 of G cofibered in G, so that we can use the standard B_G as a classifying space. Considering G as a category with one object and composition $G \times G \to G$, one sees that the abelian hypothesis makes composition $G \times G \to G$ a functor, so that we get

$$B_G \times B_G \simeq B_{G \times G} \to B_G.$$

It is easy to check that this is an associative operation on B_G. Now $(B_G)_0$ is a singleton, corresponding to the fact that G has a single object $*$. This element of B_G is seen to be an identity element for the operation on B_G, for the diagram

$$* \xleftarrow{1} \cdots \xleftarrow{1} *$$

leads to the representation $* = (1, \cdots, 1) \times_\Delta (t_0, \cdots, t_n)$, which is clearly the identity element of the monoid B_G.

(9.3) *If G is an abelian topological monoid with cofibered base point, then the standard classifying space B_G has the same property. If G is an abelian topological group with cofibered base point, so also is B_G.*

Proof. We have left only to show that if G is a topological group, then so is B_G. Let

$$x = (g_1, \cdots, g_n) \times_\Delta (t_0, \cdots, t_n)$$

be an element of B_G. Then

$$y = (g_1^{-1}, \cdots, g_n^{-1}) \times_\Delta (t_0, \cdots, t_n)$$

is seen to have $xy = 1$. $\square$

Theorem 9.4 *Let A be a compactly generated space with cofibered base point, and let SA denote the reduced suspension of A. Then there is an isomorphism*

$$B_{SP^\infty A} \simeq SP^\infty(SA)$$

of abelian topological monoids.

PROOF. We need a map

$$f : SP^\infty A \wedge (I/\partial I) \to SP^\infty(SA),$$

where SA denotes the reduced suspension $(I/\partial I) \wedge A$. We exhibit the map as a collection of base point preserving maps

$$f_t : SP^\infty A \to SP^\infty(SA)$$

such that f_0 and f_1 are constant maps into the base point, leaving it to the reader to check continuity. Regard A as a subspace of $SP^\infty A$, and use the property that every base point preserving map of A into an abelian topological monoid G can be extended uniquely to a morphism $SP^\infty A \to G$ of abelian topological monoids. Then for each t there is the map $A \to SP^\infty(SA)$ sending $a \in A$ into $t \wedge a \in SA \subset SP^\infty(SA)$, and denote by $f_t : SP^\infty A \to SP^\infty(SA)$ the unique extension to a morphism of abelian topological monoids.

We next wish to write down a map $F : B_{SP^\infty A} \to SP^\infty(SA)$. First it is convenient to shift models for $\nabla(n)$. Denote by $\nabla(n)$ the revised standard simplex

$$\nabla(n) = \{(u_1, \cdots, u_n) \in I^n | 0 \le u_1 \le \cdots \le u_n \le 1\}.$$

These coordinates are related to the standard coordinates $(t_0, \cdots, t_n)$ by

$$u_1 = t_0, \quad u_2 = t_0 + t_1, \cdots, u_n = t_0 \cdots + t_{n-1}.$$

Thus points of $B_{SP^\infty A}$ are now written as

$$x = (x_1, \cdots, x_n) \times_\Delta (u_1, \cdots, u_n)$$

where $x_i \in SP^\infty A$ and $0 \le u_1 \le \cdots \le u_n \le 1$. Then define

$$F((x_1, \cdots, x_n) \times_\Delta (u_1, \cdots, u_n)) = f_{u_1}(x_1) \cdots f_{u_n}(x_n),$$

where the right hand side uses the product in the abelian monoid $SP^\infty(SA)$.

Given two elements x and y of $B_{SP^\infty A}$, they can always be rewritten so as to have the same $u_1, \cdots, u_n$ coordinates, from which it follows that F preserves the product.

We next need a base point preserving map $h : SA \to B_{SP^\infty A}$. The 1-skeleton of $B_{SP^\infty A}$ is $S(SP^\infty A)$, which contains SA. Take this inclusion as the map $h : SA \to B_{SP^\infty A}$. There is then the unique extension to a morphism

$$H : SP^\infty(SA) \to B_{SP^\infty A}$$

of abelian topological monoids. The maps F and H are inverse to each other, and the theorem follows. □

In order to apply (6.17), we point out the following.

(9.5) *If A is a simplicial complex with a_0 a vertex, then $SP^{\infty}A$ is a simplicial complex. If A has cofibered base point and is of the homotopy type of a CW-complex, then $SP^{\infty}A$ is of the homotopy type of a CW-complex.*

Proof. First let A be a simplicial complex, with a_0 a vertex. Let A^n first be taken as a polyhedral cell complex whose cells are all the product cells, and then replace this polyhedral cell complex by its second barycentric subdivision, so that A^n is a simplicial complex. The symmetic group then operates simplicially on each A^p, and the orbit space $A^p/\Sigma(p)$ receives a simplicial subdivision in which each $\pi' : A^p \to A^p/\Sigma(p)$ is simplicial. Then inductively one receives a simplicial decomposition of each SP^pA in which $SP^{p-1}A$ is a subcomplex. Then $SP^{\infty}A$ is a simplicial complex.

Suppose next that A has cofibered base point, and is of the homotopy type of a CW-complex. Take then the simplicial set A^{∇} of singular simplices in A, with the topology of $A^{\nabla(n)}$ ignored, and take the homotopy colimit

$$B = A^{\nabla} \times_{Mono\,\Delta} \nabla.$$

First of all, there is the natural map $B \to A$ and this map is a homotopy equivalence. Next, there is a natural base point b_0 for B and the homotopy equivalence $B \to A$ gives by a cofibration fact a homotopy equivalence

$$(B, b_0) \to (A, a_0)$$

of pairs. Hence $SP^{\infty}A$ and $SP^{\infty}B$ are homotopy equivalent.

There is a choice to be made in the cellular structure on B, depending on whether one uses $\nabla(n)$ or $Sd\,\nabla(n)$ or $Sd^2\,\nabla(n)$. For present purposes, the best choice is $Sd^2\,\nabla(n)$, for then B is a simplicial complex. Then $SP^{\infty}B$ is a simplicial complex. The result follows. □

Corollary 9.6 *Let A be a path connected, compactly generated space with cofibered base point, with A of the homotopy type of a CW-complex. Then we have the natural homotopy equivalence*

$$SP^{\infty}A \sim \Omega B_{SP^{\infty}A}$$

of (6.16), and using (9.4) we have the homotopy equivalence

$$SP^{\infty}A \sim \Omega SP^{\infty}(SA).$$

Corollary 9.7 *We have that $SP^{\infty}S^0 = Z_+$, where Z_+ denotes the abelian monoid of non-negative integers, and for $n > 0$ we have that $SP^{\infty}S^n$ is a*

$K(Z, n)$.

PROOF. We have only to show that $SP^\infty S^1$ is a $K(Z, 1)$, which will follow if it is of the homotopy type of a circle. Since $SP^\infty S^1 \simeq B_{Z_+}$, it suffices to demonstrate a non-standard model for B_{Z_+} which is precisely S^1. This is easy, using the non-negative reals for a universal space with Z_+ operating by the translations through positive integers. $\square$

The Homotopy Colimit of the $Mono\,\Sigma$-Space $(S^0)^\infty = \coprod (S^0)^n$

It is natural to enquire about the homotopy colimits of the $Mono\,\Sigma$-space A^∞. We wait until the next chapter to consider this problem generally. Here we consider a very special case, the example $A = S^0$ where S^0 consists of two points, say with a as the only non-base point. The functor

$$M_1 : \mathrm{TOP}^{Mono\,\Sigma} \to \mathrm{TOPCAT}$$

of Chapter 2 then converts $(S^0)^\infty$ into a small category $\mathcal{C}(1)$ and the homotopy colimit of $(S^0)^\infty$ is the classifying space $B_{\mathcal{C}(1)}$. We need a precise model for $\mathcal{C}(1)$.

The objects of $\mathcal{C}(1)$ are just the points of the various $(S^0)^p$. Each such point is characterized by specifying which of the p coordinates are a, thus we can take the objects of $\mathcal{C}(1)$ to be all subsets

$$S \subset \{1, \cdots, p\}$$

for all $p \geq 0$. We will have a morphism in $\mathcal{C}(1)$ with domain S for each choice of a mono $\sigma : p \to q$ in Σ. The target of the morphism is seen to be precisely $\sigma(S)$, thus for each mono $\sigma : p \to q$ and each $S \subset \{1, \cdots, p\}$ we get a morphism

$$\sigma : S \to \sigma(S)$$

in $\mathcal{C}(1)$.

Our problem is to compute $B_{\mathcal{C}(1)}$ up to homotopy. In order to do so we exhibit functors

$$\phi : \mathcal{C}(1) \to Iso\,\Sigma, \quad \theta : Iso\,\Sigma \to \mathcal{C}(1)$$

for which $\phi\theta = 1$ and for which there is a natural transformation

$$T : \theta\phi \to 1.$$

It will then follow that $B_{\mathcal{C}(1)} \sim B_{Iso\,\Sigma} = \coprod_{p \geq 0} B_{\Sigma(p)}$.

Recall that $Mono\,\Delta_+$ is a subcategory of $Mono\,\Sigma$, where Δ_+ is the augmented simplicial category of Chapters 6 and 7. Given an object S of $\mathcal{C}(1)$, there is then a unique mono $\delta_S : k \to p$ in Δ_+ with

$$\delta_S\{1, \cdots, k\} = S.$$

Define $\phi(S) = k$. Given a morphism $\sigma : S \to \sigma(S)$ in $\mathcal{C}(1)$, let $\phi(\sigma)$ be such that

$$\begin{array}{ccc} k & \xrightarrow{\phi(\sigma)} & k \\ \delta_S \big\downarrow & & \delta_{\sigma(S)} \big\downarrow \\ p & \xrightarrow{\sigma} & q \end{array}$$

commutes. Thus we have ϕ.

There is the subcategory of $\mathcal{C}(1)$ whose objects are all the

$$S = \{1, \cdots, p\},$$

and whose morphisms are in natural correspondence with $Iso\ \Sigma$. Let θ be the inclusion map onto this subcategory.

We have left only to define $T : \theta\phi \to 1$. The morphisms $\delta_S : k \to p$ in $Mono\ \Delta_+$ perform this function, because of the commutative diagram

$$\begin{array}{ccc} k = \theta\phi(S) & \xrightarrow{\theta\phi(\sigma)} & k = \theta\phi(\sigma) \\ \delta_S \big\downarrow & & \delta_{\sigma(S)} \big\downarrow \\ S & \xrightarrow{\sigma} & \sigma(S) \end{array}$$

for each morphism $\sigma : S \to \sigma(S)$ of $\mathcal{C}(1)$.

We have thus proved the following theorem.

Theorem 9.8 *The $Mono\ \Sigma$-space $(S^0)^\infty$ has as a non-standard homotopy colimit*

$$B_{Iso\ \Sigma} = \coprod_{p \geq 0} B_{\Sigma(p)}.$$

We should formalize a little better the strictly monoidal category $Iso\ \Sigma$, as a subcategory of the strictly monoidal category $Mono\ \Sigma$. Its objects are the non-negative integers; its morphisms are the isomorphisms of Σ; the bifunctor

$$\oplus : Iso\ \Sigma \times Iso\ \Sigma \to Iso\ \Sigma$$

is obtained by restricting the bifunctor of $Mono\ \Sigma$.

Generalization to the Finite Sets $\{0, 1, \cdots, n\}$ with Base Point 0

We now follow Segal through the generalization of (9.8) resulting from replacing S^0 by any finite set with base point. It is just as well to restrict ourselves to the finite sets $\{0, 1, \cdots, n\}$ with base point 0. Let Γ denote the category whose objects are the non-negative integers, and whose morphisms $\gamma : m \to n$ are the base point preserving functions

$$\gamma : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}.$$

Beware: Segal denoted this category by Γ^o.

We then have the composite functor

$$\Gamma \hookrightarrow \mathrm{TOP}_* \to \mathrm{TOP}^{Mono\ \Sigma} \xrightarrow{hocolim} \mathrm{TOP},$$

giving a Γ-space which we want to compute up to homotopy. More precisely, we want to find a Γ-space which to equivalent to this one in TOP^Γ [WHE^{-1}]. This will be our form of a basic Γ-space of Segal [4.4]. As above, for each n we let $\mathcal{C}(n)$ denote the category resulting by applying

$$M_1 : \mathrm{TOP}^{Mono\ \Sigma} \to \mathrm{TOPCAT}$$

to the $Mono\ \Sigma$-space $\{0, 1, \cdots, n\}^\infty$. For each $\sigma : m \to n$ in Γ, we also compute the resulting functor

$$\gamma_* : \mathcal{C}(m) \to \mathcal{C}(n);$$

i.e., we have to start with the functor $\mathcal{C}$ which is the composition

$$\Gamma \to \mathrm{TOP}^{Mono\ \Sigma} \xrightarrow{M_1} \mathrm{TOPCAT}$$

from which one obtains the associated composite functor

$$\Gamma \xrightarrow{\mathcal{C}} \mathrm{TOPCAT} \xrightarrow{B_\diamond} \mathrm{TOP},$$

i.e. the Γ-space $\coprod_{n \geq 0} B_{\mathcal{C}(n)}$. The following theorem of Segal then exhibits homotopy models for $B_{\mathcal{C}(n)}$.

Theorem 9.9 *Consider the Γ-space $Y = \coprod_{n\geq 0} Y(n)$ where $Y(n)$ is the homotopy colimit of the $Mono\ \Sigma$-space $\{0, 1, \cdots, n\}^\infty$. Then $Y(0)$ is contractible and $Y(n)$ for $n > 0$ is naturally homotopy equivalent to the classifying space*

$$B_{Iso\ \Sigma \times \cdots \times Iso\ \Sigma} = B_{Iso\ \Sigma} \times \cdots \times B_{Iso\ \Sigma},$$

where the products are n-fold products.

Proof. We show first that $B_{\mathcal{C}(0)}$ is contractible. It is readily seen that $Y(0)$ is precisely $B_{Mono\ \Sigma}$. Now the element 0 of $Mono\ \Sigma$ is an initial object such that the only morphism $n \to 0$ is the identity morphism of 0. Hence in the language of Chapter 5, $Mono\ \Sigma$ is the cone over the full subcategory $\mathcal{D}$ whose objects are the positive integers. By (5.12), $B_{Mono\ \Sigma}$ is then the cone over $B_{\mathcal{D}}$ and is therefore contractible.

Next take $n > 0$. We can then take for the objects of $\mathcal{C}(n)$ all of the disjoint n-tuples of subsets

$$(S_1, \cdots, S_n) \subset \{1, \cdots, m\};$$

given another object

$$(T_1, \cdots, T_n) \subset \{1, \cdots, p\},$$

we get a morphism

$$\sigma : (S_1, \cdots, S_n) \to (T_1, \cdots, T_n)$$

for each mono $\sigma : m \to p$ in Σ for which $\sigma(S_i) = T_i$ for each i. As in (9.8), we must present a small homotopy model for $B_{\mathcal{C}(n)}$.

For each $(S_1, \cdots, S_n)$, there are unique monos $\delta_{S_i} : k_i \to m$ in Δ_+ whose image is S_i. Thus for each object $S = (S_1, \cdots, S_n)$, we get the induced morphism of the coproduct,

$$\delta_S : k_1 \oplus \cdots \oplus k_n \to m,$$

which is a mono in Σ. Thus as in (9.9) we have a functor

$$\phi : Iso\, \Sigma \times \cdots \times Iso\, \Sigma \to \mathcal{C}(n),$$

and precisely as in (9.8) one can show that

$$\phi_* : B_{Iso\, \Sigma \times \cdots \times Iso\, \Sigma} \to B_{\mathcal{C}(n)}$$

is a homotopy equivalence. □

Elementary Properties of Γ-Spaces

It is unfair to make entry into Γ-spaces only with the above complicated Γ-space, basic though it may turn out to be. Here we back off and point out that Γ arises for elementary purposes as well.

First of all, each abelian topological monoid G gives a Γ-space. Think of G as written additively and as having as its natural base point its neutral element 0, and think of G^m as the space of all functions

$$g : \{0, 1, \cdots, m\} \to G$$

which preserve base point, i.e. have $g(0) = 0$, and agree to write an element either as a function or alternatively simply by writing $g = (g_1, \cdots, g_m)$ where g_i is the value of the function at i. Then we get the Γ-space $\mathcal{N}G = \coprod_{m \geq 0} G^m$, where a morphism $\gamma : m \to n$ induces the action map $\gamma_* : G^m \to G^n$ given by

$$\gamma_*(g_1, \cdots, g_m) = (g'_1, \cdots, g'_n)$$

where

$$g'_j = \begin{cases} \sum_{\sigma(i)=j} g_i, & \text{for } \sigma^{-1}(j) \neq \emptyset \\ 0, & \text{for } \sigma^{-1}(j) = \emptyset. \end{cases}$$

Thus we have a natural functor $\mathcal{N} : \text{AB TOP MON} \to \text{TOP}^{\Gamma}$, which is perhaps the most elementary reason for paying attention to Γ-spaces. This functor relates well to the nerve functor $N : \text{TOP MON} \to \text{TOP}^{\Delta^o}$, as we will soon see.

In order to do so, we recall (7.2), which asserted that $(\Delta_+)^o$ is naturally isomorphic to the subcategory Λ of Δ consisting of all order preserving functions

$$\lambda : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}$$

for which $\lambda(0) = 0$ and $\lambda(m) = n$. By dropping the element 0 of Δ_+, and renaming objects, we get the following.

(9.10) *The category Δ^o can be taken to be the category whose objects are the non-negative integers and whose morphisms $\mu : m \to n$ are all order preserving functions*

$$\mu : \{0, 1, \cdots, m+1\} \to \{0, 1, \cdots, n+1\}$$

for which $\mu(0) = 0$ *and* $\mu(m+1) = n+1$. *The* Δ*-space* $\nabla = \coprod \nabla(n)$ *in this model has* $\nabla(n)$ *the space of all order preserving functions*

$$t : \{0, 1, \cdots, n+1\} \to I$$

for which $t(0) = 0$ *and* $t(n+1) = 1$, *i.e.* $\nabla(n)$ *consists of all*

$$0 = t(0) \leq t(1) \leq \cdots \leq t(n) \leq t(n+1) = 1.$$

We can now define the natural functor

$$\theta : \Delta^o \to \Gamma$$

by identifying the two base points 0 and $n+1$ of an object $\{0, 1, \cdots, n+1\}$ of Δ^o to obtain the single base point 0 of the object $\{0, 1, \cdots, n\}$ of Γ. Clearly each morphism $\mu : m \to n$ in Δ^o then gives a well defined morphism $\theta(\mu) : m \to n$ of Γ.

There is also a natural functor

$$\mathrm{TOP}_* \to \mathrm{TOP}^{\Gamma^o},$$

where given a compactly generated space A with base point a_0, we can interpret A^n as all base point preserving functions

$$a : \{0, 1, \cdots, n\} \to A,$$

and thus obtain the Γ^o-space $\coprod A^n$ with its natural right action of Γ. Of central importance here is the Γ^o-space arising from the choice $A = I/\partial I$, which can be written as the Γ^o-space

$$\mathcal{S}^1 = \coprod_{n \geq 0} (I/\partial I)^n.$$

(9.11) *The* Γ^o*-space* $\mathcal{S}^1 = \coprod (I/\partial I)^n$ *given above is related to the* Δ^o*-space* ∇ *by*

$$\mathcal{S}^1 \simeq \theta_\# \nabla = (\coprod \nabla(m) \times \Gamma(m, n))/\sim,$$

where $\sim$ *is the least equivalence relation such that if*

$$t \in \nabla(m), \quad m \xleftarrow{\mu} m', \quad m' \xleftarrow{\gamma} m,$$

where μ *is in* Δ^o *and* γ *is in* Γ, *then*

$$(t\mu, \gamma) \sim (t, \theta(\mu)\gamma).$$

Proof. Denote the elements of the Γ^o-space $\theta_\# \nabla$ by $t \times_{\Delta^o} \gamma$. There is a well defined map

$$(\theta_\# \nabla)(n) \to (I/\partial I)^n$$

assigning to $(t_1, \cdots, t_m) \times_{\Delta^o} \gamma$ the point represented by $(t_1, \cdots, t_m)\gamma$ in $(I/\partial I)^n$. We need a Γ^o-map

$$\mathcal{S}^1 \to \nabla \times_{\Delta^o} \Gamma.$$

In order to get it, we must have a relationship between the $\nabla(m)$'s and the $(I/\partial I)^n$'s. Given a point u of $I^n/\partial I^n$, there is a unique pair consisting of a point

$$0 < t_1 < \cdots < t_m < 1$$

of $\nabla(m) - \partial\nabla(m)$ and an epi $m \xleftarrow{\gamma} n$ in Γ so that $(t_1, \cdots, t_m)\gamma$ represents u in $(I/\partial I)^n$. Given this uniqueness and given $u \in (I/\partial I)^n$, one can map u into $(t_1, \cdots, t_m) \times_{\Delta^o} \gamma$. The remark follows. □

The Realization Functor $|\diamond|_{\Gamma^o} : \mathbf{TOP}^\Gamma \to \mathbf{TOP}$

The category Γ will not by itself lead to interesting homotopy limit or homotopy colimit problems. This is because 0 is both an initial and a terminal object of Γ, so that if Y is a Γ-space then $Y(0)$ is a non-standard homotopy colimit and a non-standard homotopy limit. To associate interesting homotopy colimit problems with TOP^Γ one has to have such a functor as $\theta : \Delta^o \to \Gamma$, so that one can use the composition

$$\mathrm{TOP}^\Gamma \xrightarrow{\theta^\#} \mathrm{TOP}^{\Delta^o} \xrightarrow{hocolim} \mathrm{TOP}$$

as an interesting alternative. Alternatively, one has the composition

$$\mathrm{TOP}^\Gamma \xrightarrow{\theta^\#} \mathrm{TOP}^{\Delta^o} \xrightarrow{|\diamond|} \mathrm{TOP}$$

to exploit. In the beginning stages, as with simplicial topology, the latter choice is the more compelling geometrically, and we examine it first. It is best to express this composition internally as a functor $|\diamond|_{\Gamma^o} : \mathrm{TOP}^\Gamma \to \mathrm{TOP}$.

Let Y be a Γ-space. We then have from the pairing

$$\times_\Gamma : \mathrm{TOP}^{\Gamma^o} \times \mathrm{TOP}^\Gamma \to \mathrm{Top},$$

and the choice $\mathcal{S}^1 \in \mathrm{TOP}^{\Gamma^o}$, the functor

$$|\diamond|_\Gamma : \mathrm{TOP}^\Gamma \to \mathrm{Top}, \quad Y \mapsto \mathcal{S}^1 \times_\Gamma Y.$$

Having chosen now a direct model for Δ^o, the associated bifunctor for it is now written as

$$\mathrm{TOP}^{(\Delta^o)^o} \times_{\Delta^o} \mathrm{TOP}^{\Delta^o} \to \mathrm{Top},$$

and the realization $|\diamond| : \mathrm{TOP}^{\Delta^o} \to \mathrm{TOP}$ is now written as

$$|Z| = \nabla \times_{\Delta^o} Z.$$

We now have the following theorem.

Theorem 9.12 *The realization functor $|\diamond|_\Gamma$ is a functor*

$$|\diamond|_\Gamma : \mathit{TOP}^\Gamma \to \mathit{TOP},$$

and is related to the simplicial realization by

$$|Y|_\Gamma = \mathcal{S}^1 \times_\Gamma Y \simeq \theta_\# \nabla \times_\Gamma Y \simeq \nabla \times_{\Delta^o} \theta^\# Y.$$

Thus $|Y|_\Gamma \simeq |\theta^\# Y|$.

If G is an abelian topological monoid with cofibered base point, we can now relate the permutative nerve $\mathcal{N}G$ to the simplicial nerve NG. Clearly $\theta^{\#}\mathcal{N}G \simeq NG$, so that from (9.12) we have

$$|\mathcal{N}G|_\Gamma \simeq |NG| = B_G.$$

Thus for such a G we can now write the points of B_G as points of $|\mathcal{N}G|_\Gamma$, i.e. in the form

$$(t_1, \cdots, t_n) \times_\Gamma (g_1, \cdots, g_n)$$

where permutations as well as other identifications are allowed. It is this interpretation which makes (9.3) more obvious at a glance.

Note that there are numerous properties of $| \diamond |_\Gamma$ which follow from Chapter 2 and (9.12). One example is a product theorem.

(9.13) *If Y and Y' are in TOP^Γ, let $Y \times_{Z_+} Y'$ denote the product Γ-space $\coprod Y(n) \times Y'(n)$ with its diagonal action. The natural map*

$$|Y \times_{Z_+} Y'|_\Gamma \to |Y|_\Gamma \times |Y'|_\Gamma$$

is then a homeomorphism.

The Cofibration Condition for Γ-Spaces

There are various useful choices of unique factorization pairs for the category Γ. One of these assigns to each morphism $\gamma : m \to p$ in Γ the unique factorization $\gamma = \delta\gamma_1$ where $\gamma_1 : m \to p$ is an epi in Γ and where

$$\delta : \{0, 1, \cdots, n\} \to \{0, 1, \cdots, p\}$$

is an order preserving mono with $\delta(0) = 0$.

Define the *cofibration condition* for a Γ-space Y to be the condition that each order preserving mono

$$\delta : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}$$

for which $\delta(0) = 0$ induces an action map $\delta_* : Y(m) \to Y(n)$ which is an inclusion map onto a cofibered closed subset $\delta_* Y(m) \subset Y(n)$. It is automatic that δ_* is an inclusion map onto a closed subset. Every mono $\gamma : m \to n$ in Γ can be written as an isomorphism followed by a δ, hence it then follows that if Y satisfies the cofibration condition, then for every mono $\gamma : m \to n$ we have $\gamma_* Y(m)$ cofibered in $Y(n)$. The subset $Y^{deg}(n)$ of $Y(n)$ can be defined to be either the union of the images $\delta_* Y(m)$ for all monos δ satisfying the above condition and for which $m < n$, or as the union of all $\gamma_* Y(m)$ for all monos $\gamma : m \to n$ in Γ for which $m < n$. If Y satisfies the cofibration condition, then $Y^{deg}(n)$ is cofibered in $Y(n)$.

We will assume the cofibration conditions on our Γ-space Y whenever it seems to make the work easier. It is no big deal to do so. Thus, for example, it can be checked that every principal Γ-space satisfies the cofibration condition because of the above unique factorization. Thus every Γ-space Y is isomorphic in TOP^Γ $[\mathrm{WHE}^{-1}]$ to a Γ-space satisfying the cofibration condition.

For a Γ-space Y satisfying the cofibration condition, the composition

$$\mathrm{TOP}^\Gamma \xrightarrow{\theta^\#} \mathrm{TOP}^{\Delta^o} \xrightarrow{|\diamond|} \mathrm{TOP}$$

gives from (6.9) a non-standard homotopy colimit for the Δ^o-space $\theta^\# Y$. Thus there is no real restriction in confining ourselves to Γ-spaces Y satisfying the cofibration condition and to using $|Y|_\Gamma$ as the closest thing we have to a homotopy colimit for Y.

The Infinite Symmetric Product $SP^\infty(A;Y)$ with Coefficients

One can now take take a second step in the evolution of the infinite symmetric product.

First one needs that the functor

$$\mathrm{TOP}_* \times \mathrm{TOP}^\Gamma \to \mathrm{Top}, \quad (A,Y) \mapsto (\coprod A^n) \times_\Gamma Y$$

actually maps into TOP. We leave full details to the reader. The bottom line is that $B = (\coprod A^n) \times_\Gamma Y$ is naturally filtered as $B = \bigcup B_n$. Given $n > 0$, consider the pair $(A^n, A^{n,deg})$ where $A^{n,deg}$ consists of all $(a_1, \cdots, a_n)$ where either for some i we have $a_i = a_0$ or else for some $i \neq j$ we have $a_i = a_j$. Consider also the pair $(Y(n), Y^{deg}(n))$ where $Y^{deg}(n)$ is the union of all the images of all $\delta_* : Y(m) \to Y(n)$ as δ ranges over all monos

$$\delta : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}$$

which preserve order, have $\delta(0) = 0$, and have $m < n$. Then there is a relative homeomorphism

$$(A^n, A^{n,deg}) \times (Y(n), Y^{deg}(n)) \to (B_n, B_{n-1})$$

in the style of (2.8). The root fact is the unique factorization of morphisms of Γ into epimorphisms of Γ followed by those monos

$$\{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}$$

which preserve order and map 0 into 0.

We thus leave to the reader the full details of the following.

(9.14) *For each compactly generated space A and each Γ-space Y, define the infinite symmetric product $SP^\infty(A;Y)$ by*

$$SP^\infty(A;Y) = (\coprod A^n) \times_\Gamma Y.$$

There results the functor

$$SP^\infty : TOP_* \times TOP^\Gamma \to TOP.$$

The space $SP^\infty(A;Y)$ is naturally filtered as

$$SP^\infty(A;Y) = \bigcup SP^p(A;Y).$$

If G is an abelian topological monoid, then we obtain $SP^\infty(A; \mathcal{N}G)$, which we denote simply by $SP^\infty(A; G)$. Then $SP^\infty(A; G)$ is an abelian topological monoid and we thus obtain the special case

$$SP^\infty : TOP_* \times AB\ TOP\ MON \to AB\ TOP\ MON.$$

PROOF. We have left to observe only that given a compactly generated space A with base point a_0, and given an abelian topological monoid G, then $SP^\infty(A; G)$ is an abelian topological monoid. The points of $SP^\infty(A; G)$ are of the form

$$s = (a_1, \cdots, a_m) \times_\Gamma (g_1, \cdots, g_m).$$

These are subject to the following relations.

(i) If $a_i = a_0$, then a_i and g_i can be deleted.
(ii) If $g_i = 0$, then a_i and g_i can be deleted.
(iii) Both sides can be permuted by the same permutation of n letters.
(iv) If $a_i = a_j$ for some $i < j$, then a_j and g_j can be deleted with g_i replaced by $g_i + g_j$.

These relations generate all the relations. That being the case, if

$$t = (a'_1, \cdots, a'_n) \times_\Gamma (g'_1, \cdots, g'_n)$$

is another element, we can define the sum to be

$$s + t = (a_1, \cdots, a_m, a'_1, \cdots, a'_n) \times_\Gamma (g_1, \cdots, g_m, g'_1, \cdots, g'_n),$$

and $SP^\infty(A; G)$ is clearly an abelian topological monoid. Of course, this can be presented in more abstract form, but we wait until it is essential to do so. □

For Y a Γ-space, the spaces $SP^\infty(S^0; Y)$ and $SP^\infty(S^1; Y)$ deserve special note. In fact, (9.12) has already pointed out that $SP^\infty(S^1; Y)$ is just the chosen realization $|Y|_\Gamma$, or for that matter the simplicial realization $|\theta^\# Y|$. We leave it to the reader to show that $SP^\infty(S^0; Y) \simeq Y(1)$.

McCord's Theorem $SP^\infty(A; SP^\infty(B; G)) \simeq SP^\infty(A \wedge B; G)$

We confine ourselves for the moment to the bifunctor

$$SP^\infty : \mathrm{TOP}_* \times \mathrm{AB\ TOP\ MON} \to \mathrm{AB\ TOP\ MON},$$

and are now able to put (9.4) in a more satisfactory form.

Fix the abelian topological monoid G, and fix the compactly generated space A with base point a_0. We must understand better the abelian topological monoid $SP^\infty(A; G)$, and especially if H is another abelian topological monoid we must understand how many morphisms $SP^\infty(A; G) \to H$ there are of abelian topological monoids.

Consider the filtration $SP^\infty(A; G) = \bigcup SP^p(A; G)$, where $SP^p(A; G)$ consists of all

$$s = (a_1, \cdots, a_p) \times_\Gamma (g_1, \cdots, g_p).$$

Then $SP^0(A;G)$ is the base point, and $SP^1(A;G)$ is the smashed product $A \wedge G$. The basic fact is then that for each base point preserving map

$$\phi : A \wedge G \to H, \quad a \wedge g \mapsto \phi(a \wedge g)$$

such that $\phi(a \wedge g) + \phi(a \wedge g') = \phi(a \wedge (g + g'))$, we get a unique morphism

$$\Phi : SP^\infty(A;G) \to H$$

which is given by ϕ on the 1-skeleton. For one can simply define

$$\Phi((a_1, \cdots, a_n) \times_\Gamma (g_1, \cdots, g_n)) = \phi(a_1 \wedge g_1) + \cdots + \phi(a_n \wedge g_n)$$

and check that the generating relations are all preserved.

We seek to use this by defining natural maps

$$\phi : A \wedge B \wedge G \to SP^\infty(A; SP^\infty(B;G)).$$

The right hand side has 1-skeleton $A \wedge (SP^\infty(B,G)) \supset A \wedge B \wedge G$ so that we have only to take the natural map from $A \wedge B \wedge G$ to $SP^\infty(A; SP^\infty(B,G))$, to check the above condition of ϕ, and thus get the morphism

$$\Phi : SP^\infty(A \wedge B;G) \to SP^\infty(A, SP^\infty(B;G)).$$

We also need natural maps $\Theta : SP^\infty(A; SP^\infty(B;G)) \to SP^\infty(A \wedge B;G)$. Here we need for starters a map

$$\theta : A \wedge (SP^\infty(B;G)) \to SP^\infty(A \wedge B;G).$$

One checks that

$$\theta(a \wedge ((b_1, \cdots b_n) \times_\Gamma (g_1, \cdots, g_n)) = (a \wedge b_1, \cdots a \wedge b_n) \times_\Gamma (g_1, \cdots, g_n)$$

suffices. Thus the reader can readily finish the check of the following theorem.

Theorem 9.15 *For compactly generated spaces A and B with base point and for G an abelian topological monoid, there is the natural isomorphism*

$$SP^\infty(A \wedge B;G) \simeq SP^\infty(A; SP^\infty(B;G))$$

of abelian topological monoids.

The following corollary then gives the extended form of (9.4).

Corollary 9.16 *Let G be an abelian topological monoid with cofibered base point, and let B^n_G denote the sequence of abelian topological monoids with cofibered base point which has $B^0_G = G$ and which has B^{n+1}_G the classifying space of B^n_G. Then*

$$B^n_G \simeq SP^\infty(S^n;G)$$

for all $n \geq 0$.

PROOF. We have already expressed B_G as $SP^\infty(S^1;G)$. Hence its classifying space B^2_G is

$$SP^\infty(S^1; SP^\infty(S^1;G)) \simeq SP^\infty(S^2;G).$$

One continues this use of (9.15) for all $n > 0$. The case $n = 0$ can be shown directly. $\square$

We can now put (9.6) in an extended form due to McCord [1.3].

(9.17) *Let G be an abelian topological monoid which is of the homotopy type of a CW-complex and has cofibered base point. Then for every $n > 0$ we get that the natural inclusion*

$$SP^{\infty}(S^n; G) \subset \Omega SP^{\infty}(S^{n+1}; G)$$

is a homotopy equivalence. If $\pi_0(G)$ is a group, then also

$$G \subset \Omega SP^{\infty}(S^1; G)$$

is a homotopy equivalence. Thus if G is a discrete abelian group, then $SP^{\infty}(S^n; G)$ is the $K(G,n)$-spectrum, and is an Ω-spectrum.

Outline of a Functor $B : \mathbf{TOP}^{\Gamma} \to \mathbf{TOP}^{\Gamma}$

We have now completed McCord's topological presentation of the spectrum $\{K(\pi, n)\}$ for each abelian group π. The bottom line is a functor

$$\text{AB TOP MON} \to \text{AB TOP MON}, \quad G \mapsto SP^{\infty}(S^1; G),$$

or its equivalent formulation in terms of classifying spaces, so that starting with $G = \pi$ one can produce the spectrum iteratively.

As one can look at the above in two ways, one can look at Segal's generalizations in two ways. We first present his generalization of the above classifying space approach. Since classifying spaces are special cases of Milnor realizations, this generalization is based on the above remarks about realizations.

Here we need Segal's functor

$$B : \text{TOP}^{\Gamma} \to \text{TOP}^{\Gamma},$$

for which commutativity holds in

$$\begin{array}{ccc} \text{AB TOP MON} & \xrightarrow{B_{\diamond}} & \text{AB TOP MON} \\ \downarrow & & \downarrow \\ \text{TOP}^{\Gamma} & \xrightarrow{B} & \text{TOP}^{\Gamma}. \end{array}$$

We now outline his construction in its most minimal form, leaving some details to be checked later when they are done in a more general form anyway.

We can readily define realizations $|\diamond|_{\Gamma^n} : \text{TOP}^{\Gamma^n} \to \text{TOP}$. One simply uses the product $(\Gamma^n)^o$-space $\mathcal{S}^1 \times \cdots \times \mathcal{S}^1 = (\mathcal{S}^1)^n$ and the functor

$$\text{TOP}^{(\Gamma^n)^o} \times \text{TOP}^{\Gamma^n} \to \text{Top}$$

to produce

$$|Z|_{\Gamma^n} = (\mathcal{S}^1)^n \times_{\Gamma^n} Z,$$

for Z in TOP^{Γ^n}. Then

$$|Z|_{\Gamma^n} = (\mathcal{S}^1)^n \times_{\Gamma^n} Z = (\theta_\# \nabla)^n \times_{\Gamma^n} Z$$
$$= \theta_\#(\nabla^n) \times_{\Gamma^n} Z = \nabla^n \times_{\Delta^n} \theta^\# Z = \nabla \times_\Delta i^\# \theta^\# Z,$$

where i is the diagonal inclusion $\Delta \hookrightarrow \Delta^n$. It follows that $|Z|_{\Gamma^n}$ is compactly generated, thus we have the functor

$$|\diamond|_{\Gamma^n} : \mathrm{TOP}^{\Gamma^n} \to \mathrm{TOP}$$

and the various ways of presenting $|Z|_{\Gamma^n}$. Thus $|Z|_{\Gamma^n}$ is the Γ-realization of the Γ-space $j^\# Z$, where j is the diagonal inclusion $\Gamma \hookrightarrow \Gamma^n$, or is the Milnor realization $|i^\# \theta^\# Z|$. Writing it out in more detail,

$$|Z|_{\Gamma^n} = \coprod((S^1)^{p_1} \times \cdots \times (S^1)^{p_n} \times Z(p_1, \cdots, p_n))/\sim$$
$$= (\coprod(S^1)^p \times Z(p, \cdots, p))/\sim \quad = (\coprod \nabla(p) \times Z(p, \cdots, p))/\sim .$$

These observations are due to Segal.

Suppose now that we fix a Γ-space Y. There is the functor

$$\oplus_n : \Gamma^n \to \Gamma, \quad (p_1, \cdots, p_n) \mapsto p_1 + \cdots + p_n, \quad (\gamma_1, \cdots, \gamma_n) \mapsto \gamma_1 \oplus \cdots \oplus \gamma_n.$$

This induces the functor

$$\oplus_n^\# : \mathrm{TOP}^\Gamma \to \mathrm{TOP}^{\Gamma^n},$$

and we get a sequence $\{Y_n\}$ where Y_n is the Γ^n-space $Y_n = \oplus_n^\# Y$ given by

$$Y_n(p_1, \cdots, p_n) = Y(p_1 + \cdots + p_n).$$

(9.18) *Let Y be a Γ-space, and let Y_n be the Γ^n-space given by*

$$Y_n(p_1, \cdots, p_n) = Y(p_1 + \cdots + p_n).$$

Then the sequence $\{|Y_n|_{\Gamma^n}\}$ is a Γ-space, so that we get a functor

$$B : TOP^\Gamma \to TOP^\Gamma, \quad Y \mapsto \coprod |Y_n|_{\Gamma^n}.$$

This functor can also be described as follows. For each $n \geq 0$, let Y'_n denote the Γ-space given by

$$Y'_n(p) = Y(np) = Y(p + \cdots + p)$$

where Γ acts diagonally on $Y(p + \cdots + p)$. Then we get the Γ- space

$$\coprod |Y'_n|_\Gamma = \coprod SP^\infty(S^1; Y'_n)$$

which is naturally homeomorphic to BY in TOP^Γ.

The equivalence of the two presentations is clear, up to presenting the action of Γ. The details of the action is buried in the more general arithmetic outline which we now develop. This general arithmetic is in the fashion of May [7.4], but is done in a special case.

Small Categories with Based Choice Functions for Finite Coproducts

In order to prepare for the more extensive production of the Γ-spaces Y which play the role of coefficients in $SP^\infty(A;Y)$, we will use some categorical machinery concerning permutations of the type that May has provided. It seems clearest if done at an intermediate level of generality, and we choose to put it in terms of small categories G with finite coproducts. In fact, we put the following conditions on G.

(i) We require that G have associated with each ordered pair p, q of objects a standard coproduct

$$p \xrightarrow{\alpha_{p,q}} p \oplus q \xleftarrow{\beta_{p,q}} q,$$

with the associativity condition that for each ordered triple p, q, r of objects we have that $(p \oplus q) \oplus r = p \oplus (q \oplus r)$ as well as that the diagram

$$\begin{array}{ccc} q & \xrightarrow{\beta_{p,q}} & p \oplus q \\ {\scriptstyle \alpha_{q,r}}\downarrow & & \downarrow{\scriptstyle \alpha_{p\oplus q,r}} \\ q \oplus r & \xrightarrow{\beta_{p,q\oplus r}} & p \oplus q \oplus r \end{array}$$

commutes. We then have a standard choice for a coproduct $p_1 \oplus \cdots \oplus p_n$ for any ordered n-tuple $(p_1, \cdots, p_n)$ of objects. For each n-tuple $(p_1, \cdots, p_n)$ of objects and each permutation σ of n letters we get an isomorphism

$$p_1 \oplus \cdots \oplus p_n \to p_{\sigma^{-1}(1)} \oplus \cdots \oplus p_{\sigma^{-1}(n)}$$

coming from the uniqueness of the coproduct up to natural isomorphism.

(ii) We require that $p \oplus q = q \oplus p$ for every ordered pair of objects.

(iii) We require that G have a given object 0 which is both an initial object and a final object. We also require that $p \oplus 0 = p = 0 \oplus p$ for all objects p, and that

$$\alpha_{p,0}; p \to p \oplus 0 = p, \quad \beta_{0,p} : p \to 0 \oplus p = p$$

are both identity morphisms.

If these are satisfied, we say that G is a small category with a *based choice function for finite coproducts* . Then Γ is an example of a small category with a based choice function for finite coproducts, as the reader should check. In fact, Γ is the only small category with a based choice function for finite coproducts that we use, and the generality is only to provide clarity in the proofs.

If G has a based choice function for finite coproducts, then G is naturally a strict monoidal category. We also use

$$\oplus : G \times G \to G$$

for the defining bifunctor for this structure. Here assign to each ordered pair p, q of objects the object $p \oplus q$, and given $g : p \to p'$ and $h : q \to q'$, let $g \oplus h : p \oplus q \to p' \oplus q'$ be the unique morphism such that

$$\begin{tikzcd}
p \arrow[r, "\alpha_{p,q}"] \arrow[d, "g"'] & p \oplus q \arrow[d, "g\oplus h"'] & q \arrow[l, "\beta_{p,q}"'] \arrow[d, "h"'] \\
p' \arrow[r, "\alpha_{p',q'}"] & p' \oplus q' & q' \arrow[l, "\beta_{p',q'}"']
\end{tikzcd}$$

commutes.

If G is a small category with a based choice function for finite coproducts, then for each ordered pair p, q of objects we have the diagram

$$p \xleftarrow{\rho_{p,q}} p \oplus q \xrightarrow{\mu_{p,q}} q$$

where $\rho_{p,q} = 1_p \oplus \epsilon_q$ with ϵ_q the unique morphism $q \to 0$, and similarly for $\mu_{p,q}$. These are functorial in p and q in a way that the reader can supply. As a consequence, for each $(p_1, \cdots, p_i, \cdots, p_n)$ we get a natural morphism

$$p_1 \oplus \cdots \oplus p_i \oplus \cdots \oplus p_n \to p_1 \oplus \cdots \oplus p_{i-1} \oplus p_{i+1} \oplus \cdots \oplus p_n.$$

Of course, using the unique morphisms $\nu_p : 0 \to p_i$ one also gets a natural morphism

$$p_1 \oplus \cdots \oplus p_{i-1} \oplus p_{i+1} \oplus \cdots \oplus p_n \to p_1 \oplus \cdots \oplus p_i \oplus \cdots \oplus p_n.$$

It is important to observe that all these natural morphisms come in pairs, with source and target reversed. Similarly there are natural morphisms corresponding to omitting any subcollection of the objects, or to inserting a collection of new objects.

The problem is now to put the natural isomorphisms and the natural morphisms above into a context that fits what we want to do. Perhaps it is best to simply plunge into what we want to do, which is to define for any G, having a based choice function for finite coproducts, a bifunctor

$$\mathrm{TOP}^{G^o} \times \mathrm{TOP}^G \to \mathrm{Top}^\Gamma.$$

The Bifunctor $\mathbf{TOP}^{G^o} \times \mathbf{TOP}^G \to \mathbf{Top}^\Gamma$

Fix a G^o-space X and a G-space Y. For each $n \geq 0$, let

$$\oplus_n : G^n \to G$$

denote the multifunctor given by

$$\oplus_n(p_1, \cdots, p_n) = \bigoplus_{1 \leq i \leq n} p_i, \quad \oplus_n(g_1, \cdots g_n) = \bigoplus_{1 \leq i \leq n} g_i.$$

There are the functors

$$\oplus_n^\# : \mathrm{TOP}^G \to \mathrm{TOP}^{G^n}, \quad \oplus_n^\# : \mathrm{TOP}^{G^o} \to \mathrm{TOP}^{(G^n)^o},$$

and we denote $\oplus_n^\# X$ by X_n and $\oplus_n^\# Y$ by Y_n. These spaces then have

$$X_n(p_1, \cdots, p_n) = X(p_1 \oplus \cdots \oplus p_n), \quad Y_n(p_1, \cdots, p_n) = Y(p_1 \oplus \cdots \oplus p_n).$$

One can then form the space $X_n \times_{G^n} Y_n$.

We have next to provide a map $\gamma_* : X_m \times_{G^m} Y_m \to X_n \times_{G^n} Y_n$ for each morphism $\gamma : m \to n$ in Γ. First we need a functor $F_\gamma : G^m \to G^n$. We can partition $\{1, \cdots, m\}$ into the subsets

$$S_j = \gamma^{-1}(j) \cap \{1, \cdots, m\}, \quad j \in \{1, \cdots, n\}.$$

Denote those S_j which are non-empty by

$$S_j = \{j_1 < \cdots < j_{k_j}\}.$$

Define the functor $F_\gamma : G^m \to G^n$ by

$$F_\gamma(g_1, \cdots, g_m) = (g_1', \cdots, g_n'),$$

where

$$g_j' = \begin{cases} g_{j_1} \oplus \cdots \oplus g_{j_{k_j}}, & \text{for } \gamma^{-1}(j) \neq \emptyset \\ 1_0, & \text{for } \gamma^{-1}(j) = \emptyset. \end{cases}$$

The formula for $F_\gamma(p_1, \cdots, p_m)$ is entirely similar. Let $p_\omega = (p_1, \cdots, p_m)$ be an object of G^m and let

$$p_{\omega'} = (p_1', \cdots, p_n')$$

be its image under the functor F_γ. Denote by $\oplus(p_\omega)$ the object

$$\oplus(p_\omega) = p_1 \oplus \cdots \oplus p_m$$

of G. We then need maps

$$X_m(p_\omega) \to X_n(p_{\omega'}), \quad Y_m(p_\omega) \to Y_n(p_{\omega'}),$$

equivalently maps

$$X(\oplus(p_\omega)) \to X(\oplus(p_{\omega'})), \quad Y(\oplus(p_\omega)) \to Y(\oplus(p_{\omega'}))$$

which are equivariant with respect to the functor. Since X is a right G-space, a morphism $g' : \oplus(p_{\omega'}) \to \oplus(p_\omega)$ in G serves to give a map

$$X(\oplus(p_\omega)) \to X(\oplus(p_{\omega'})), \quad x \mapsto xg'.$$

Since Y is a left G-space, a morphism $g : \oplus(p_\omega) \to \oplus(p_{\omega'})$ serves to give a map

$$Y(\oplus(p_\omega) \to Y(\oplus(p_{\omega'})), \quad y \to gy.$$

Hence our problem is to associate with $\gamma : m \to n$ in Γ two natural morphisms

$$g_\gamma : p_\omega \to p_{\omega'}, \quad g_\gamma' : p_{\omega'} \to p_\omega$$

for each $p_\omega = (p_1, \cdots, p_m)$.

We have

$$p_{\omega'} = (p_{1_1} \oplus \cdots \oplus p_{1_{k_1}}, \cdots, p_{n_1} \oplus \cdots \oplus p_{n_{k_n}}),$$

where zeroes are to be inserted wherever $\gamma^{-1}(j) = \emptyset$. There is then a natural morphism $g_\gamma : \oplus(p_\omega) \to \oplus(p_{\omega'})$ as follows from the above discussion. Those p_i

for which $\gamma(i) = 0$ can be deleted and there results a natural morphism, zeros can be inserted for those j for which $\gamma^{-1}(j) = \emptyset$ and the identity morphism can be used, and finally the natural isomorphism connected with a permutation can be used.

Also, there is a natural morphism $g'_\gamma : \oplus(p_{\omega'}) \to \oplus(p_\omega)$. Here one uses the permutation inverse to that used in g_γ, and follows with a natural inclusion.

The natural morphisms give the action maps

$$(g'_\gamma)^* : X(\oplus(p_\omega)) \to X(\oplus(p_{\omega'})), \quad (g_\gamma)_* : Y(\oplus(p_\omega)) \to Y(\oplus(p_{\omega'}))$$

that we need. These can also be displayed as well defined maps

$$X_m(p_\omega) \to X_m(p_{\omega'}), \quad x \mapsto xg'_\gamma$$

and similarly for Y.

We define the map

$$\gamma_* : X_m \times_{G^m} Y_m \to X_n \times_{G^n} Y_n$$

by

$$\gamma_*(x \times_{G^m} y) = xg'_\gamma \times_{G^n} g_\gamma y.$$

The morphisms g_γ can be regarded as a natural transformation $\oplus_m \to \oplus_n\gamma$, similarly g'_γ as a natural transformation $\oplus_n\gamma \to \oplus_m$, hence γ_* is well defined.

Theorem 9.19 *Let G be a small category with a based choice function for finite coproducts. . Then there is the functor*

$$TOP^{G^o} \times TOP^G \to Top^\Gamma$$

defined as above. Given a G^o-space X and a G-space Y, form the $(G^n)^o$-spaces X_n and the G^n-spaces Y_n for each $n \geq 0$. Then there is the natural action of Γ on $\coprod_{n\geq 0} X_n \times_{G^n} Y_n$, thus the functor takes (X,Y) into the Γ-space $\coprod X_n \times_{G^n} Y_n$.

Proof. The question is whether the array of maps γ_* constitutes an action of Γ. Consider a diagram

$$p \xleftarrow{\gamma'} n \xleftarrow{\gamma} m$$

in Γ, and the resulting functors

$$F(\gamma'\gamma) : G^m \to G^p, \quad F(\gamma')F(\gamma) : G^m \to G^p.$$

When written out in coordinate form, then in each jth-coordinate the functors differ only by a permutation. Hence the two functors are related by a natural isomorphism

$$T_{\gamma',\gamma} : F(\gamma'\gamma) \to F(\gamma')F(\gamma).$$

One can next consider the compositions

$$G \xleftarrow{\oplus_p} G^p \leftarrow G^m$$

and get the natural isomorphism

$$\oplus_p T_{\gamma',\gamma} : \oplus_p F_{\gamma'\gamma} \to \oplus_p F_{\gamma'} F_\gamma.$$

For each $(p_1, \cdots, p_m)$, we can denote the resulting isomorphism of G by

$$h_{\gamma',\gamma} : \oplus_p F_{\gamma'\gamma} \to \oplus_p F_{\gamma'} F_\gamma$$

and get

$$h_{\gamma,\gamma'} g_{\gamma'\gamma} = g_{\gamma'} g_\gamma,$$
$$g'_\gamma g'_{\gamma'} h_{\gamma',\gamma} = g'_{\gamma\gamma'}.$$

We can now compute

$$\gamma'_* \gamma_*(x \times_{G^m} y) = x g'_\gamma g'_{\gamma'} \times_{G^p} g_{\gamma'} g_\gamma(y)$$

as

$$x g'_{\gamma'\gamma} h^{-1}_{\gamma',\gamma} \times_{G^p} h_{\gamma',\gamma} g_{\gamma'\gamma}(y),$$

which is equal to

$$(\gamma'\gamma)_*(x \times_{G^m} y). \quad \square$$

The Extended Functor $\mathcal{SP}^\infty : \mathbf{TOP}_* \times \mathbf{TOP}^\Gamma \to \mathbf{TOP}^\Gamma$

We can now finish generalizing the infinite symmetric product by presenting a bifunctor

$$\mathcal{SP}^\infty : \mathrm{TOP}_* \times \mathrm{TOP}^\Gamma \to \mathrm{TOP}^\Gamma,$$

which is a form of a result of Segal [4.4]. If we fix A to be S^1, then the Γ-space $\mathcal{SP}^\infty(S^1; Y)$ will be precisely the Γ-space $\coprod |Y_n|_{\Gamma^n}$ exhibited in (9.18). The general case will have completed the proof of (9.18).

Fix a compactly generated space A with base point a_0, and fix a Γ-space Y. We then have the Γ^o-space $\coprod A^p$, and first need to review its properties. We have $A^0 = pt$, and the natural homeomorphism $A^{p+q} \simeq A^p \times A^q$, so that $\coprod A^p$ is a strictly multiplicative Γ^o-space with the natural homeomorphisms

$$A^{p_1 + \cdots + p_m} \simeq A^{p_1} \times \cdots \times A^{p_m}.$$

If we denote $\coprod A^p$ by $\mathcal{A}$, then the spaces $\mathcal{A}_n$ of the preceding paragraphs are the n-fold products

$$\mathcal{A}_n = \mathcal{A} \times \cdots \times \mathcal{A}.$$

Applying the bifunctor $\mathrm{TOP}^{\Gamma^o} \times \mathrm{TOP}^\Gamma \to \mathrm{Top}^\Gamma$ to the Γ^o-space $\mathcal{A} = \coprod A^p$ and the Γ-space Y, we define

$$\mathcal{SP}^\infty(A; Y) = \coprod \mathcal{A}_n \times_{\Gamma^n} Y_n$$

to be the resulting Γ-space in Top. Note that each $\mathcal{A}_m \times_{\Gamma^m} Y_m$ can be presented as the image of a quotient map

$$\pi_n : \coprod A^{p_1 + \cdots p_m} \times Y(p_1 + \cdots + p_m) \to \mathcal{A}_m \times_{\Gamma^m} Y_m,$$

by taking the equivalence relation $\sim$ such that if

$$a \in A^{q_1 + \cdots + q_m}, \quad q_i \xleftarrow{\gamma_i} p_i, \quad y \in Y(p_1 + \cdots p_m)$$

then

$$(a(\gamma_1 \oplus \cdots \oplus \gamma_m), y) \sim (a, (\gamma_1 \oplus \cdots \oplus \gamma_m)y).$$

We denote the image of $A^{p_1+\cdots+p_m} \times Y(p_1 + \cdots + p_m)$ in $\mathcal{A}_m \times_{\Gamma^m} Y_m$ by $\mathcal{SP}^{p_1,\cdots,p_m}(A;Y)$, thus obtaining a generalized filtration.

(9.20) *The bifunctor $\mathcal{SP}^\infty$ is a bifunctor*

$$\mathcal{SP}^\infty : TOP_* \times TOP^\Gamma \to TOP^\Gamma.$$

Each of the spaces $\mathcal{A}_m \times_{\Gamma^m} Y_m$ is filtered as

$$\bigcup \mathcal{SP}^{p_1,\cdots,p_m}(A;Y)$$

and there is a relative homeomorphism

$$(A^{p_1}, A^{p_1,deg}) \times \cdots \times (A^{p_m}, A^{p_m,deg}) \times_{\Sigma(p_1)\times\cdots\times\Sigma(p_m)} (Y(p), Y^{deg}(p))$$

$$\to (\mathcal{SP}^{p_1,\cdots,p_m}(A;Y), \bigcup \mathcal{SP}^{q_1,\cdots,q_m}(A;Y)),$$

where $p = p_1 + \cdots + p_m$ and the union is over all $(q_1, \cdots, q_m)$ such that $q_i \leq p_i$ for all i and $q_i < p_i$ for some i.

One should examine the elementary case $\mathcal{SP}^\infty(S^0;Y)$ and show that $Y \simeq \mathcal{SP}^\infty(S^0;Y)$. Note also the relationship between $\mathcal{SP}^\infty(A;Y)$ and $SP^\infty(A;Y)$. Namely, as a Γ-space $\mathcal{SP}^\infty(A;Y)$ has a space assigned for each $m \geq 0$, and $SP^\infty(A;Y)$ is the space assigned when $m = 1$.

If Y is the permutative nerve $\mathcal{N}G$ of an abelian topological monoid, it is the case that $\mathcal{SP}^\infty(A;\mathcal{N}G)$ is precisely the permutative nerve of the abelian topological monoid $SP^\infty(A;G)$. Thus the generalization coincides in this case with the more classical construct.

The Associative Law $SP^\infty(A; \mathcal{SP}^\infty(B;Y)) \simeq SP^\infty(A \wedge B;Y)$

Following Segal, we can now put (9.15) in the above extended form. We will need first of all a map

$$\phi : SP^\infty(A; \mathcal{SP}^\infty(B;Y)) \to SP^\infty(A \wedge B, Y).$$

This is obtained from the family of maps

$$\phi_{p_1,\cdots,p_m} : A^m \times (B^{p_1+\cdots+p_m} \times Y(p_1 + \cdots + p_m)) \to SP^\infty(A \wedge B, Y)$$

given by

$$((a_1, \cdots, a_m), ((b_1, \cdots, b_{p_1+\cdots+p_m}), y) \mapsto$$

$$(a_1 \wedge b_1, \cdots, a_1 \wedge b_{p_1}, \cdots, a_m \wedge b_{p_1+\cdots+p_{m-1}+1}, \cdots, a_m \wedge b_{p_1+\cdots+p_m}) \times_\Gamma y.$$

We write this map as $\phi_\omega : A^m \times (B^\omega \times Y(\omega)) \to SP^\infty(A \wedge B, Y)$. The maps ϕ_ω give a well defined map if for each morphism $\gamma : m \to n$ in Γ, the two compositions

$$A^n \times (B^\omega \times Y(\omega)) \xrightarrow{\gamma^* \times 1} A^m \times (B^\omega \times Y(\omega)) \to SP^\infty(A \wedge B, Y)$$

and

$$A^n \times (B^\omega \times Y(\omega)) \xrightarrow{1\times\gamma_*} A^n \times (B^{\omega'} \times Y(\omega')) \to SP^\infty(A \wedge B, Y)$$

coincide. We leave it to the reader to make this check. There results a well defined map

$$\phi : SP^\infty(A, \mathcal{SP}^\infty(B;Y)) \to SP^\infty(A \wedge B, Y).$$

We need also a map

$$\theta : SP^\infty(A \wedge B, Y) \to SP^\infty(A; \mathcal{SP}^\infty(B;Y)).$$

This map will be given by a family of maps

$$\theta_m : (A \wedge B)^m \times Y(m) \to SP^\infty(A; \mathcal{SP}^\infty(B;Y)),$$

$$((a_1 \wedge b_1, \cdots, a_m \wedge b_m), y) \mapsto (a_1 \cdots, a_m) \times_\Gamma ((b_1, \cdots, b_m) \times_{\Gamma^m} y).$$

We also leave it to the reader to make the tedious check that this gives a well defined map

$$\theta : SP^\infty(A \wedge B, Y) \to SP^\infty(A, \mathcal{SP}^\infty(B;Y))$$

and that both compositions are the identity.

Theorem 9.21 *Given compactly generated spaces A and B with base points, and given a Γ-space Y, we have the natural homeomorphism*

$$\phi : SP^\infty(A, \mathcal{SP}^\infty(B;Y)) \to SP^\infty(A \wedge B; Y).$$

The Spectra Generated by Γ-Spaces

The most general fact about producing spectra from Γ-spaces is as follows.

Theorem 9.22 *For each Γ-space Y with $Y(0)$ contractible, one gets a spectrum as follows. In TOP^Γ $[WHE^{-1}]$, one may as well suppose that $Y(0) = pt$. Then form the spectrum whose spaces are $SP^\infty(S^n;Y)$ and whose maps are the natural maps*

$$SP^\infty(S^n;Y) \to \Omega SP^\infty(S^{n+1};Y) = \Omega SP^\infty(S^1; \mathcal{SP}^\infty(S^n;Y).$$

PROOF. We have noted previously that we can replace Y by a Γ space satisfying the cofibration condition, for example by replacing Y by a principal Γ-space over it. Suppose this has been done. The unique morphisms $n \to 0$ and $0 \to n$ give a natural embedding of $Y(0)$ in each $Y(n)$ as a cofibered closed subspace. One can then form $Y(n)/Y(0)$ for each n, thus the Γ-space $\coprod Y(n)/Y(0)$. There is the Γ-map

$$\coprod Y(n) \to \coprod Y(n)/Y(0),$$

a weak homotopy equivalence in TOP^Γ. Since now $Y(0) = pt$, we can then form the spectrum as stated. □

Note that every Γ-space Y is comultiplicative, with the maps

$$\rho_{p,q*} \times \mu_{p,q*} : Y(p+q) \to Y(p) \times Y(q).$$

These maps are to be considered as maps

$$Y(p+q) \to Y(p) \times_{Y(0)} Y(q).$$

Since we assume $Y(0) = pt$, we can drop the subscript. We then say that Y is a *special* Γ*-space*, or is *strictly comultiplicative up to homotopy*, if each map $Y(p+q) \to Y(p) \times Y(q)$ is a homotopy equivalence in TOP.

We can now obtain Segal's generalizations of (9.17).

Theorem 9.23 *Let Y be a Γ-space which has $Y(0) = pt$, which satisfies the cofibration condition, which has each $Y(m)$ of the homotopy type of a CW-complex, and which is strictly comultiplicative up to homotopy. Then the inclusion maps*

$$f_n : SP^\infty(S^n; Y) \to \Omega SP^\infty(S^{n+1}; Y)$$

are homotopy equivalences for $n > 0$. If in addition $\pi_0(Y(1))$ with its natural abelian monoid structure is a group, then the natural inclusion

$$Y(1) = SP^\infty(S^0; Y) \to SP^\infty(S^1; Y)$$

is also a homotopy equivalence.

Proof. Here we have only to mesh with earlier results. Consider the Δ^o-space $X = \theta^\# Y$ of (9.12). It has $X(0) = pt$, it satisfies the cofibration condition, it is a special Δ^o-space, and each $X(n)$ is of the homotopy type of a CW-complex. We also have $X(1) = Y(1)$ and $|\theta^\# Y| = SP^\infty(S^1; Y)$. Whenever we know that $\pi_0(Y(1))$, with its natural monoid structure, is a group it follows from (7.16) that

$$Y(1) \to \Omega SP^\infty(S^1; Y)$$

is a homotopy equivalence. All that remains is to prove that $\mathcal{SP}^\infty(S^1; Y)$ inherits the above conditions from Y, noting also that since $SP^\infty(S^1; Y)$ is automatically path connected then $\pi_0(SP^\infty(S^1; Y))$ is trivial. The spectrum $\mathcal{SP}^\infty(S^1; Y)$ has been displayed in (9.18) and the reader can check the conditions. □

Corollary 9.24 *Let Y be a Γ-space such that $Y(0)$ is contractible, such that Y is strictly comultiplicative up to homotopy and such that each $Y(n)$ is of the homotopy type of a CW-complex. Then one can replace it in $TOP^\Gamma[WHE^{-1}]$ by a Γ-space satisfying all the hypotheses of (9.23) and thus obtain a spectrum satisfying the conclusions of (9.23).*

References

(9.1) J. F. Adams, Infinite Loop Spaces, Ann. of Math. Study 90, Princeton University Press, Princeton, 1978.

(9.2) A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische produkte*, Ann. Math. **67** (1958), 239–281.

(9.3) J.P. May and R. Thomason, *The uniqueness of infinite loop space machines*, Topology **17** (1978), 205–224.

CHAPTER X

Homotopy Colimit Problems Associated with the Symmetric Groups

This chapter is devoted to two problems. In Chapter 9, we interpreted the infinite symmetric product $A \mapsto SP^\infty A$ as a composition of functors

$$\mathrm{TOP}_* \xrightarrow{\diamond^\infty} \mathrm{TOP}^{Mono\ \Sigma} \xrightarrow{colim} \mathrm{Top},$$

where $A^\infty = \coprod A^p$ with its natural left action of $Mono\ \Sigma$. The first purpose of this chapter is to review the corresponding homotopy colimit problem; i.e. the analysis of the composition

$$\mathrm{TOP}_* \xrightarrow{\diamond^\infty} \mathrm{TOP}^{Mono\ \Sigma} \xrightarrow{hocolim} \mathrm{TOP}.$$

The goal here is to interpret work of J.P. May [2.8] as implying that there exists a natural homotopy class of maps

$$hocolim\ A^\infty = E_{(Mono\ \Sigma)^o} \times_{Mono\ \Sigma} A^\infty \to \Omega^\infty S^\infty A,$$

these maps being homotopy equivalences whenever A is path connected, is of the homotopy type of a CW-complex, and has cofibered base point. That is, for such A we use results of May to show that $\Omega^\infty S^\infty A$ is a non-standard homotopy colimit of the $Mono\ \Sigma$-space A^∞. A quick outline of the reduction of this problem to results of May is as follows.

In the style of Chapter 9, the functor $\diamond^\infty : \mathrm{TOP}_* \to \mathrm{TOP}^{Mono\ \Sigma}$ and the reduced product bifunctor for $Mono\ \Sigma$ give a functor

$$\mathrm{TOP}^{(Mono\ \Sigma)^o} \times \mathrm{TOP}_* \to \mathrm{TOP}, \quad (V, A) \mapsto V \times_{Mono\ \Sigma} A^\infty.$$

Choosing $V = E_{(Mono\ \Sigma)^o}$ gives the the standard homotopy colimit of A^∞; we then follow the lead of May by investigating the freedom of choice in the selection of V such that $V \times_{Mono\ \Sigma} A^\infty$ is a (possibly non-standard) homotopy colimit of A^∞. Given any $(Mono\ \Sigma)^o$-space V, one gets by restriction a right action of $\Sigma(p)$ on each $V(p)$, and thus an element $V(p)$ of $\mathrm{TOP}^{(\Sigma(p))^o}$ for each p. We show that a homotopy colimit of A^∞ results for any choice of V for which each $V(p)$ is homotopy equivalent in $\mathrm{TOP}^{(\Sigma(p))^o}$ to a universal $(\Sigma(p))^o$-space.

One can then follow Boardman-Vogt [4.1] by producing appropriate models for V. The first candidate V has $V(p)$ the space of distinct p-tuples $(x_1, \cdots, x_p)$ in R^∞, where if $\sigma : q \to p$ is a mono of Σ, then

$$(x_1, \cdots, x_p)\sigma = (x_{\sigma(1)}, \cdots, x_{\sigma(q)}).$$

We then give the classic result that each $V(p)$ is a universal $(\Sigma(p))^o$-space, thus V is admissible. Note that each $V(p)$ is naturally filtered as $\bigcup V_n(p)$, where $V_n(p)$ is the space of distinct p-tuples in R^n.

The final Boardman-Vogt model uses the space $V_n'(p)$ of p-tuples $(J_1, \cdots, J_p)$ of little n-cubes in $I^n = [-1,1]^n$, where the interiors of the little cubes are required to be disjoint and where each little n-cube is a product of closed subintervals of $[-1,1]$. This gives a $(Mono\ \Sigma)^o$-space V_n' for which there is a natural map

$$V_n' \times_{Mono\ \Sigma} A^\infty \to \Omega^n S^n A.$$

Passing to the colimit of the natural inclusions

$$V_1' \to \cdots \to V_n' \to \cdots$$

induced by sending an n-cube J into the $(n+1)$-cube $J \times [-1,1]$, one receives a $(Mono\ \Sigma)^o$-space V' for which $V'(p)$ consists of p-tuples $(J_1, \cdots, J_p)$ of little ∞-cubes with disjoint interiors, with an action as above. One has the natural $(Mono\ \Sigma)^o$-map $V' \to V$ induced by the maps $V_n'(p) \to V_n(p)$ which send each cube into its centroid. We refer to May for the straight-forward check that each $V'(p) \to V(p)$ is a homotopy equivalence in $\mathrm{TOP}^{(\Sigma(p))^o}$; thus V' is an admissible model.

For A in TOP_*, there is the compactly generated space $\Omega^\infty S^\infty A$, obtained as the colimit of the inclusions

$$A \to \Omega S A \to \cdots \Omega^n S^n A \to .$$

Passing to the colimit from the maps

$$V_n' \times_{Mono\ \Sigma} A^\infty \to \Omega^n S^n A,$$

one gets a natural map

$$V' \times_{Mono\ \Sigma} A^\infty \to \Omega^\infty S^\infty A.$$

We then simply quote the classic theorem of May, in our form that this map is a homotopy equivalence whenever A is path connected, has cofibered base point, and is of the homotopy type of a CW-complex. He proves it by proving that each $V_n' \times_{Mono\ \Sigma} A^\infty$ is homotopy equivalent to $\Omega^n S^n A$; the interested reader will consult May [2.8,Th. 6.1].

In the course of this review, we are plunged a little way into the body of work which presents universal and classifying spaces for the symmetric groups. The second purpose of this chapter is to plunge a little further by presenting some of the work of Japanese topologists on this topic. In particular, we interpret work of Nakamura [10.2], and the related work of Fox-Neuwirth [10.1], as presenting definitive models for the universal and classifying spaces of the symmetric groups.

The Bifunctor $\mathbf{TOP}^{(Mono\ \Sigma)^o} \times \mathbf{TOP}_* \to \mathbf{TOP}$

Recall that Σ is the category whose objects are the non-negative integers and whose morphisms $\sigma : m \to n$ are the functions

$$\sigma : \{1, \cdots, m\} \to \{1, \cdots, n\}.$$

Thus $Mono\, \Sigma$ is the subcategory of all such one-to-one functions.

The subcategory $Mono\, \Delta_+$ denotes all $\sigma : \underline{p} \to \underline{q}$ which are not only one-to-one but also order preserving. $Iso\, \Sigma$ denotes the subcategory of all isomorphisms, i.e. all $\sigma : \underline{p} \to \underline{p}$ which are one-to-one and onto. Thus $Iso\, \Sigma = \coprod_{p \geq 0} \Sigma(p)$ where $\Sigma(p)$ denotes the symmetric group.

The subcategories $Mono\ \Delta_+$, $Iso\ \Sigma$ provide a unique factorization pair for $Mono\ \Sigma$. That is, every morphism σ of Σ can be written uniquely as $\sigma = \delta\rho$ where δ is in $Mono\, \Delta_+$ and ρ is in $Iso\, \Sigma$.

Let X be a $(Mono\ \Sigma)^o$-space. For each compactly generated space A with base point a_0, we then get the space

$$X \times_{Mono\, \Sigma} A^\infty = (\coprod X(p) \times A^p)/\sim.$$

(10.1) *The equivalence relation $\sim$ on $\coprod X(p) \times A^p$ has the following properties. Call an element $(a_1, \cdots, a_p) \in A^p$* nondegenerate *if no a_i is the base point. Every equivalence class in $\coprod X(p) \times A^p$ admits a representative of the form $(x, a) \in X(q) \times A^q$ where a is nondegenerate. If $(x', a') \in X(r) \times A^r$ is another representative with a' nondegenerate, then $r = q$ and for some $\rho \in \Sigma(q)$ we have $x' = x\rho$ and $a = \rho a'$. For any other representative $(x'', a'') \in X(s) \times A^s$ of the equivalence class we have $s > q$.*

Proof. We assume that given $a \in A^p$ there exists a unique $a' \in A^q$ and morphism $\delta : q \to p$ in $Mono\, \Delta_+$ with $a = \delta a'$. We can define an explicit map

$$\Phi : \coprod X(p) \times A^p \to \coprod X(p) \times A^p$$

sending $(x, a) \in X(p) \times A^p$ into a well-defined $(x', a') \in X(q) \times A^q$ with a' nondegenerate. Namely, take the unique nondegenerate $a' \in A^q$ and $\delta : q \to p$ in $Mono\, \Delta_+$ for which $a = \delta a'$ and define

$$\Phi(x, a) = (x\delta, a').$$

Note that if a is nondegenerate, then $\Phi(x, a) = (x, a)$. Note also that $(x, a) \sim \Phi(x, a)$.

Among the things we must see is that q is constant on each equivalence class. Next we must interpret $X(q) \times A^q$ as a space upon which $\Sigma(q)$ acts; this is evident. Having done so, we must check that as the (x, a) vary over an equivalence class of $\sim$, then the $\Phi(x, a)$ vary over a single orbit of the action of $\Sigma(q)$ on $X(q) \times A^q$.

In order to show these, one notes first that $\Phi(x\delta', a) = \Phi(x, \delta' a)$ for any

$$x \in X(p'), \quad \delta' : p \to p' \text{ in } Mono\, \Delta_+, \quad a \in A^p.$$

This is trivial. Only slightly less trivial is showing that if

$$x \in X(p), \quad \rho \in \Sigma(p), \quad a \in A^p,$$

then $\Phi(x\rho, a)$ and $\Phi(x, \rho a)$ have the same q-value and lie in the same orbit of $\Sigma(q)$ on $X(q) \times A^q$. The remark follows readily. $\square$

It follows from the above that $X \times_{Mono\ \Sigma} A^\infty$ is nicely filtered, say as

$$X \times_{Mono\ \Sigma} A^\infty = \bigcup (X \times_{Mono\ \Sigma} A^\infty)_p,$$

where $(X \times_{Mono\ \Sigma} A^\infty)_p$ is all

$$x \times_{Mono\ \Sigma} (a_1, \cdots, a_p)$$

for which $x \in X(p)$. Let $A^{p,deg} \subset A^p$ be all $(a_1, \cdots, a_p) \in A^p$ such that some a_i is the base point. One then gets readily the following remark.

(10.2) *For each $(Mono\ \Sigma)^o$-space X and each A in TOP_*, the space $X \times_{Mono\ \Sigma} A^\infty$ is filtered as*

$$X \times_{Mono\ \Sigma} A^\infty = \bigcup (X \times_{Mono\ \Sigma} A^\infty)_p,$$

where $(X \times_{Mono\ \Sigma} A^\infty)_p$ is the pushout of a diagram

$$X(p) \times_{\Sigma(p)} A^p \longleftarrow X(p) \times_{\Sigma(p)} A^{p,deg} \longrightarrow (X \times_{Mono\ \Sigma} A^\infty)_{p-1}.$$

Since $\Sigma(p)$ is a finite group, the first two spaces of this pushout diagram can be checked to be weakly Hausdorff, hence by induction so is $X \times_{Mono\ \Sigma} A^\infty$. Thus we have the functor

$$TOP^{(Mono\ \Sigma)^o} \times TOP_* \to TOP.$$

If A has cofibered base point, it follows from an argument of the type of (9.2) that the filtration is cofibered.

We can now return to the question of when $X \times_{Mono\ \Sigma} A^\infty$ yields a homotopy colimit of A^∞.

Theorem 10.3 *Consider the class C of all $(Mono\ \Sigma)^o$- spaces X such that each $(\Sigma(p))^o$-space $X(p)$, obtained by restricting the action of $(Mono\ \Sigma)^o$, is homotopy equivalent in $TOP^{(\Sigma(p))^o}$ to $E_{(\Sigma(p))^o}$. Then*

(i) $E_{(Mono\ \Sigma)^o}$ is in the class C,
(ii) if X is in the class C, then there exists a natural homotopy class of $(Mono\ \Sigma)^o$- maps $f : E_{(Mono\ \Sigma)^o} \to X$, and
(iii) if $f : X \to X'$ is a $(Mono\ \Sigma)^o$-map, if X and X' are in the class C, and if A has cofibered base point, then

$$f \times_{Mono\ \Sigma} 1 : X \times_{Mono\ \Sigma} A^\infty \to X' \times_{Mono\ \Sigma} A^\infty$$

is a homotopy equivalence of spaces.

Thus if X is in the class C and A has cofibered base point, then $X \times_{Mono\ \Sigma} A^\infty$ is a homotopy colimit for A^∞.

PROOF. To show (i), we must consider the space $E(p) = E_{(Mono\ \Sigma)^o}(p)$ together with its action of $\Sigma(p)$. It will suffice to see that $E(p)$ is a principal $(\Sigma(p))^o$-space, for being contractible it will then be a (non-standard) universal $(\Sigma(p))^o$-space. This in turn will be true for all p if E is a principal $(Iso\ \Sigma)^o$-space. But this follows from the fact that if $B = \coprod_{p\geq 0} B(p)$ is a Z_+-space, then

$$B \times_{Z_+} (Mono\ \Sigma) = (B \times_{Z_+} (Mono\ \Delta_+)) \times_{Z_+} (Iso\ \Sigma),$$

for this implies that the restriction of a principal $(Mono\ \Sigma)^o$- space to $(Iso\ \Sigma)^o$ is principal.

Property (ii) follows from generalities. Given X in the class C, consider the $(Mono\ \Sigma)^o$-space

$$EX = E_{(Mono\ \Sigma)^o} X.$$

Each $X(p)$ is contractible, thus each $(EX)(p)$ is contractible. Hence EX is a universal $(Mono\ \Sigma)^0$-space. Hence there is a unique homotopy class of $(Mono\ \Sigma)^o$-maps $E_{(Mono\ \Sigma)^o} \to EX$, and composing with the map $EX \to X$, we get a well defined homotopy class of maps $E_{(Mono\ \Sigma)^o} \to X$.

Of course the major proposition is (iii). Let X and X' be in C, and let $f : X \to X'$ be a $(Mono\ \Sigma)^o$-map. For each p, we get the commutative diagram

$$\begin{tikzcd}
X(p) \times_{\Sigma(p)} A^p \arrow[d, "f'"'] & X(p) \times_{\Sigma(p)} A^{p,deg} \arrow[l] \arrow[r] \arrow[d, "f''"'] & X \times_{Mono\ \Sigma} A^\infty \arrow[d, "f'''"'] \\
X'(p) \times_{\Sigma(p)} A^p & X'(p) \times_{\Sigma(p)} A^{p,deg} \arrow[l] \arrow[r] & X' \times_{Mono\ \Sigma} A^\infty.
\end{tikzcd}$$

The maps $f_p : X(p) \to X'(p)$, being homotopy equivalences in $\mathrm{TOP}^{(\Sigma(p))^o}$, induce homotopy equivalences f' and f''. It follows inductively that the f''' are homotopy equivalences for all p. Hence the conclusion follows. □

The Space $V(p)$ of Distinct p-Tuples in R^∞

At the heart of model-making involving the symmetric group $\Sigma(p)$ is the right action of $\Sigma(p)$ on $(R^\infty)^p$, and on $(R^n)^p$, by

$$(x_1, \cdots, x_p)\rho = (x_{\rho(1)}, \cdots, x_{\rho(p)}).$$

We start in a very elementary way with this important action.

Allow p to be any non-negative integer, with $(R^\infty)^0$ interpreted as a singleton, the empty set.

(10.4) *We can consider $(R^\infty)^p$ as a simplicial complex in such a way that the action of $\Sigma(p)$ is simplicial. If ∇ is a simplex of $(R^\infty)^p$, and if $\rho \in \Sigma(p)$ has $\rho^*(\nabla) \subset \nabla$ or $\rho^*(\nabla) \supset \nabla$, then ρ is the identity on ∇.*

PROOF. First make each $(R^\infty)^p$ into a regular cell complex. Choose a representation of the real numbers as a regular cell complex in which 0 is a vertex. To be explicit, let the 1-cells be the closed intervals $[i, i+1]$. Then R^n is a product of regular cell complexes, and as such is a regular cell complex. The inclusion $R^n \to R^{n+1}$ obtained by setting the last coordinate equal to 0 establishes R^n as a regular cell subcomplex of R^{n+1}. Hence R^∞ is a regular cell complex. Then each $(R^\infty)^p$ is also naturally a regular cell complex. Moreover, the cells of all these regular cell complexes are bounded Euclidean cells. Hence each cell has a well defined barycenter.

We can then make each $(R^\infty)^p$ into a simplicial complex by taking its barycentric subdivision. This is the simplicial complex which is associated with the poset whose objects C are the open cells of the regular cell complex $(R^\infty)^p$, and which has $C \leq D$ iff $C \subset \overline{D}$.

Thus one sees that the vertices of $(R^\infty)^p$ are all $(v_1, \cdots, v_p)$ where v_i is a vertex of the simplicial complex R^∞. Moreover,

$$(v_1, \cdots, v_p) < (w_1, \cdots, w_p)$$

iff $v_i \leq w_i$ for $1 \leq i \leq p$ and $v_i < w_i$ for some i. One can then proceed to check the assertion. $\square$

(10.5) *Let X denote the subspace of $(R^\infty)^p$ consisting of all $(x_1, \cdots, x_p)$ such that $x_i = x_j$ for some $i \neq j$. Then X is the total space of the simplicial subcomplex of $(R^\infty)^p$ which consists of all simplices ∇ of $(R^\infty)^p$ which are pointwise fixed by some $\rho \in \Sigma(p)$ other than the identity.*

This is left as an exercise. For the finite dimensional case $(R^n)^p$, we denote the corresponding subset by X_n.

We now consider the open subset $V(p) = (R^\infty)^p - X$ of $(R^\infty)^p$, equivalently, we consider the space of one-to-one functions

$$x : \{1, \cdots, p\} \to R^\infty.$$

If K denotes the simplicial complex $(R^\infty)^p$ and if L denotes the subcomplex of (10.5), then

$$V(p) = |K| - |L|,$$

where $|K|$ denotes the union of the simplices of K. We outline a classical construction which then exhibits $V(p) = |K| - |L|$ as a simplicial complex.

(10.6) *Let K be a simplicial complex, and let L be a subcomplex such that if ∇ is a simplex of K with $\nabla \cap |L| \neq \emptyset$, then $\nabla \cap |L|$ is a face of ∇. Then $|K| - |L|$ is naturally a simplicial complex. Moreover, if (K, L) is any simplicial pair then $(Sd\ K, Sd\ L)$ satisfies the above condition, thus $|K| - |L|$ is naturally a simplicial complex.*

Proof. One utilizes a subdivision $Sd_*\ K$ of the simplicial complex K, one which subdivides simplices of K which intersect $|L|$ and which leaves all the rest undivided.

The simplices of K are of the form

(i) a simplex ∇' of $|K| - |L|$, or

(ii) a simplex ∇'' of $|L|$, or
(iii) a join $\nabla' * \nabla''$ where ∇' is a simplex of $|K|-|L|$ and ∇'' is a simplex of $|L|$.

To obtain $Sd_*\ K$, one subdivides each simplex ∇ of K, proceeding inductively on the dimension of the simplex. Suppose the subdivision has been defined for all dimensions $< k$, and that we now have a k-simplex ∇. If $\nabla = \nabla'$ as above, let $Sd_*\ \nabla = \nabla$. If either $\nabla = \nabla''$ or $\nabla = \nabla' * \nabla''$ as above, then let $Sd_*\ \nabla$ be the simplicial cone from the barycenter of ∇ over $Sd_*\ (\partial\nabla)$. Thus $Sd_*\ K$ is defined.

Denote by $K-L$ the subcomplex of K whose simplices ∇ have $\nabla \cap |L| = \emptyset$. Denote by $Sd\ L$ the standard barycentric subdivision of L. Then $K-L$ and $Sd\ L$ are simplicial subcomplexes of $Sd_*\ K$. Since we now have the pair $(Sd_*\ K, Sd\ L)$, we can iterate the construction on this pair to obtain a subdivision $Sd_*^2\ K$ of $Sd_*\ K$, which has $Sd_*\ K - Sd\ L$ and $Sd^2\ L$ as subcomplexes. Continuing inductively, we get $(Sd_*^n\ K, Sd^n\ L)$ and we get a diagram of simplicial inclusions

$$K - L \to Sd_*\ K - Sd\ L \to Sd_*^2\ K - Sd^2 L \to .$$

We thus get a simplicial complex

$$M = \bigcup (Sd_*^n\ K - Sd^n\ L).$$

We leave it to the reader to check that $|M| = |K| - |L|$. $\square$

Theorem 10.7 *The space $V(p) = (R^\infty)^p - X$ with its right action of $\Sigma(p)$ is a (non-standard) universal right $\Sigma(p)$-space. The corresponding model for a non-standard classifying space $B(p)$ for $\Sigma(p)$ is then the colimit $B(p) = V(p)/\Sigma(p)$, which can be regarded as the suitably topologized set of all finite subsets of R^∞ which have precisely p elements. The space $B(p)$ is filtered as $B(p) = \bigcup B_n(p)$, where $B_n(p) = ((R^n)^p - X_n)/\Sigma(p)$ is called the configuration space of all subsets of R^n with exactly p elements.*

Proof. We must first show $V(p)$ to be principal. We may assume that $V(p)$ is a simplicial complex, that the action is simplicial, and that if ∇ is a simplex and ρ is a non-identity element of $\Sigma(p)$ then the interiors of ∇ and $\rho^*(\nabla)$ are disjoint.

Let $V^k(p)$ denote the k-skeleton of $V(p)$. We can then pick one k-dimensional simplex ∇_i^k from each orbit class of k- dimensional simplices, and thus obtain a relative homeomorphism

$$\coprod (\nabla_i^k, \partial\nabla_i^k) \times \Sigma(p) \to (V^k(p), V^{k-1}(p)).$$

Hence $V(p) = \bigcup V^k(p)$ is a principal right $\Sigma(p)$-space.

We must show that $V(p)$ is contractible. Let $V_n(p) \subset V(p)$ denote all one-to-one functions

$$x : \{1, \cdots, p\} \to R^n.$$

Then $V(p) = \bigcup V_n(p)$, and each $V_n(p)$ is a subcomplex of $V(p)$.

One proves that $V(p)$ is ∞-connected by proving that each $V_n(p)$ is $(n-2)$-connected. Since $V(p)$ is a CW-complex, it then follows that $V(p)$ is contractible. We leave it as an exercise to the reader to prove the classical assertion that $V_n(p)$ is $(n-2)$-connected. As a hint, the most elegant proof applies to $V_n(p) = (R^n)^p - X_n$, where X_n is contained in the $(np-n)$-skeleton of $(R^n)^p$, and then uses the fact that X_n is disjoint from the $(n-1)$-skeleton of the dual cellular subdivision of the simplicial manifold $(R^n)^p$. $\square$

We now combine (10.7) and (10.3) to exhibit a basic model for the homotopy colimit of A^∞ when A has cofibered base point.

Corollary 10.8 *Consider the $(Mono\ \Sigma)^o$-space $\coprod_{p\geq 0} V(p)$, where if a point $x = (x_1, \cdots, x_q) \in V(q)$ and $\sigma : \underline{p} \to \underline{q}$ is one-to-one, then $x\sigma \in V(p)$ is the composition $x\sigma$ of one-to-one functions. Then $V \times_{Mono\ \Sigma} A^\infty$ is a homotopy colimit for A^∞.*

Boardman-Vogt Spaces of Little Cubes and Their Use by May

We need now a variant of the above space $V(p)$ of all $x = (x_1, \cdots, x_p)$ in R^∞ with $x_i \neq x_j$ for $i \neq j$.

There is a slight convenience here in considering the basic closed interval of real numbers as $I = [-1, 1]$. Thus in this section, the suspension SA of a space with base point is $(I/\partial I) \wedge A$ and the loop space ΩA is the space of based maps $I/\partial I \to A$, all with $I = [-1, 1]$.

A *little interval* J is a nondegenerate subinterval $[a, b]$ of the interval I. There is a space $\mathcal{D}(1)$ of little intervals, where the topology is that of the space

$$\mathcal{D}(1) = \{(a, b) \in R^2 : -1 \leq a < b \leq 1\}.$$

Regard the space $\mathcal{D}(1)$ as having the base point $I = [-1, 1]$.

A *little n-cube* J is a product

$$[a_1, b_1] \times \cdots \times [a_n, b_n]$$

of little intervals, and is a subset of the n-cube I^n. There is then the space $\mathcal{D}(n)$ of little n-cubes, topologized as a product. By the interior of J we will mean the product

$$(a_1, b_1) \times \cdots \times (a_n, b_n)$$

of the open intervals.

There is the natural map $\mathcal{D}(n) \to R^n$ which assigns to a little n-cube its centroid.

There is a closed inclusion $\mathcal{D}(n) \to \mathcal{D}(n+1)$ given by

$$[a_1, b_1] \times \cdots \times [a_n, b_n] \mapsto [a_1, b_1] \times \cdots \times [a_n, b_n] \times [-1, 1].$$

From the closed inclusions

$$\mathcal{D}(1) \to \cdots \to \mathcal{D}(n) \to \cdots$$

we can form the colimit, the space $\mathcal{D}(\infty)$ of *little ∞-cubes* J. We can regard $\mathcal{D}(\infty) = \bigcup \mathcal{D}(n)$ as a filtered space. There is the natural map $\mathcal{D}(\infty) \to R^\infty$

induced by the maps $\mathcal{D}(n) \to R^n$. In fact, we have chosen $I = [-1, 1]$ so that its centroid is 0, so that this map is well defined.

The Boardman-Vogt space $V'_n(p)$ of disjoint little n-cubes is then the space of all p-tuples $(J_1, \cdots, J_p)$ of little n-cubes such that the interiors are disjoint. Considering the space $V_n(p)$ as all distinct p-tuples in R^n, we get a natural map $V'_n(p) \to V_n(p)$ which assigns to each little n-cube its centroid.

Two little n-cubes have disjoint interiors iff the corresponding $(n+1)$-cubes have disjoint interiors. Thus it is meaningful to speak of two little ∞-cubes as having disjoint interiors.

The Boardman-Vogt space $V'(p)$ is then the space of p-tuples of little ∞-cubes whose interiors are disjoint. This space is filtered as $V'(p) = \bigcup V'_n(p)$. The elements of this space are the functions $\underline{p} \to \mathcal{D}(\infty)$ whose images have disjoint interiors.

Theorem 10.9 *Consider the* $(Mono\ \Sigma)^o$*-space* $V' = \coprod V'(p)$ *of* p*-tuples of little* ∞*-cubes with disjoint interiors. Then* V' *is in the class* C *of (10.3). Hence if* A *has cofibered base point then* $V' \times_{Mono\ \Sigma} A^\infty$ *is a homotopy colimit of* A^∞*, and the map*

$$V' \times_{Mono\ \Sigma} A^\infty \to V \times_{Mono\ \Sigma} A^\infty$$

is a homotopy equivalence of spaces.

That V' is in the class C is written out in May [2.8,p. 34-36]. The proof is elementary but detailed, and we simply refer the reader to it. The rest follows.

The Basic Theorem of May

We now come to the point of the little cubes, namely their relationship to iterated loop spaces. Suppose we have a p-tuple $(J_1, \cdots, J_p)$ of little n-cubes with disjoint interiors, and also a p-tuple $(f_1, \cdots, f_p)$ in the iterated loop space $\Omega^n A$. First consider each f_i as a map of pairs $(I^n, \partial I^n) \to (A, a_0)$. By change of scale, next consider each $f_i : (J_i, \partial J_i) \to (A, a_0)$. Then map the complement

$$I^n - (J_1 \cup \cdots \cup J_p)$$

into the base point. There results from the given data a single map $f : (I^n, \partial I^n) \to (A, a_0)$, thus a well defined point of $\Omega^n A$. In the following elementary proposition, this is formalized into a map

$$\mu_n : V'_n \times_{Mono\ \Sigma} (\Omega^n A)^\infty \to \Omega^n A$$

which sends

$$(J_1, \cdots, J_p) \times_{Mono\ \Sigma} (f_1, \cdots, f_p)$$

into f.

Theorem 10.10 *Let* A *be a compactly generated space with base point. Consider the* $(Mono\ \Sigma)^o$*-space* $V'_n = \coprod V'_n(p)$*. The nth-loop space* $\Omega^n A$ *is a compactly generated space with base point, and there is a natural map*

$$\mu_n : V'_n \times_{Mono\ \Sigma} (\Omega^n A)^\infty \to \Omega^n A.$$

Replacing A by the nth-suspension $S^n A$, we get a natural map

$$V_n' \times_{Mono\ \Sigma} (\Omega^n S^n A)^\infty \to \Omega^n S^n A.$$

Using the natural map $A \to \Omega^n S^n A$, we get a natural map

$$\tau_n : V_n' \times_{Mono\ \Sigma} A^\infty \to \Omega^n S^n A.$$

Letting n tend to infinity, we get a natural map

$$\tau : V' \times_{Mono\ \Sigma} A^\infty \to \Omega^\infty S^\infty A.$$

We leave it to the reader either to invent the proofs, or to look them up in May.

We come now to the basic theorem of May [2.8,p.52]. Here at the very heart of the matter we have no new proof to offer, and simply refer the reader to May's Theorem 6.1 [2.8,p. 50].

Theorem 10.11 *Let A be a path connected space, of the homotopy type of a CW-complex, and with cofibered base point. Then the natural maps*

$$V_n' \times_{Mono\ \Sigma} A^\infty \to \Omega^n S^n A, \quad V' \times_{Mono\ \Sigma} A^\infty \to \Omega^\infty S^\infty A$$

are homotopy equivalences of spaces. Hence $\Omega^\infty S^\infty A$ is a homotopy colimit of A^∞.

$B_{Mono\ \Sigma}(A^\infty)$ Interpreted in Terms of Γ-Spaces

Recall that we have interpreted Segal's basic Γ-space Y as follows. For $p \geq 0$, let $\mathcal{C}(p)$ denote the category whose objects are all the disjoint p-tuples of subsets

$$S = (S_1, \cdots, S_p) \subset \{1, \cdots, m\}.$$

Given another object

$$T = (T_1, \cdots, T_p) \subset \{1, \cdots, n\}$$

then we get a morphism $\sigma : S \to T$ in $\mathcal{C}(p)$ for each morphism $\sigma : m \to n$ in $Mono\ \Sigma$ with $\sigma(S_i) = T_i$ for all i. For each morphism $\gamma : p \to q$ in Γ one writes down a functor $\gamma : \mathcal{C}(p) \to \mathcal{C}(q)$ and checks that there results a functor $F : \Gamma \to \mathrm{CAT}$. Using the classifying space functor, one gets the composition

$$\Gamma \xrightarrow{F} \mathrm{CAT} \to \mathrm{TOP},$$

and thus the special Γ-space Y. If $F' : \Gamma \to \mathrm{CAT}$ denotes the functor related to F by $F'(n) = F(n)^o$ then we can also regard Segal's basic Γ-space as the composition

$$\Gamma \xrightarrow{F'} \mathrm{CAT} \to \mathrm{TOP}.$$

Theorem 10.12 *The homotopy colimit $B_{Mono\ \Sigma}(A^\infty)$ is naturally homeomorphic to the infinite symmetric product $SP^\infty(A; Y)$ of Chapter 9, where Y is*

Segal's special Γ- *space.*

PROOF. There is a natural inclusion functor $i : (Mono\ \Sigma)^o \to \Gamma$. If $\sigma : p \to q$ is a morphism of $Mono\ \Sigma$, then one assigns to it the morphism $\gamma : q \to p$ in Γ where $\gamma(j) = \sigma^{-1}(j)$ if $\sigma^{-1}(j) \neq \emptyset$ and $\gamma(j) = 0$ if $\sigma^{-1}(j) = \emptyset$. The category $(Mono\ \Sigma)^o$ is then identified with the subcategory Π of Γ whose morphisms $\alpha : q \to p$ are the functions

$$\{0, 1, \cdots, q\} \to \{0, 1, \cdots, p\}$$

which send 0 into 0 and which map $\alpha^{-1}\{1, \cdots, p\}$ one-to-one onto $\{1, \cdots, p\}$.

The induced functor $i_\# : \mathrm{Top}^\Pi \to \mathrm{Top}^\Gamma$ has $i_\#(E_\Pi) \simeq Y$. For let $\mathcal{D}(p)$ denote the category whose objects are all morphisms $\alpha : \diamond \to p$ in Π, and whose morphisms are all the commutative diagrams

$$\begin{array}{ccc} q' & \xleftarrow{\alpha''} & q \\ {\scriptstyle\alpha'}\big\downarrow & & {\scriptstyle\alpha}\big\downarrow \\ p & = & p \end{array}$$

in Π, so that $E_\Pi(p) = B_{\mathcal{D}(p)}$. From (5.9), we can present $(i_\# E_\Pi)(r)$ as $B_{\mathcal{C}'(r)}$ where $\mathcal{C}'(r)$ has objects all $\gamma : \diamond \to r$ in Γ and morphisms all commutative diagrams

$$\begin{array}{ccc} q' & \xleftarrow{\alpha} & q \\ {\scriptstyle\gamma'}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow \\ r & = & r. \end{array}$$

It is then checked that $\mathcal{C}'(r) \simeq (\mathcal{C}(r))^o$. The theorem follows. $\square$

We can now put May's results in terms of the infinite symmetric product $SP^\infty(A; Y)$, where Y is Segal's special Γ-space.

Theorem 10.13 *Consider the categories $\mathcal{C}(p)$ for which $B_{\mathcal{C}(p)} = Y(p)$. We have natural bifunctors*

$$\mathcal{C}(p) \times \mathcal{C}(q) \to \mathcal{C}(p + q)$$

which make the Γ-space Y a multiplicative Γ-space. It then follows from (7.9) that $SP^\infty(A; Y)$ is a topological monoid. If A is path connected, has the homotopy type of a CW-complex, and has cofibered base point, then $SP^\infty(A; Y)$ is naturally homotopy equivalent to $\Omega^\infty S^\infty A$.

An Equivariant Partitioning of $(R^m)^p$ into Convex Sets

We now review partitions of $(R^m)^p$ of the type of Nakamura [10.2] and Fox-Neuwirth [10.1].

Fix the positive number p and consider simultaneously all the spaces $(R^m)^p$. A point x of $(R^m)^p$ can be considered as a function

$$x : \underline{p} = \{1, \cdots, p\} \to R^m.$$

Here we regard x as a function from the linearly ordered set $\underline{p}$ to the linearly ordered set R^m, where R^m has the lexicographic order. There is a unique mono-epi factorization of such maps x; i.e. given x there is a unique positive integer p_m, a unique epi $\sigma : \underline{p} \to \underline{p_m}$, and a unique order preserving mono $x_m : \underline{p_m} \to R^m$ with $x = x_m \sigma$. Moreover, if x is order preserving, then σ is order preserving.

Let $\pi_{m-1} : R^m \to R^{m-1}$ denote the order preserving map

$$\pi_{m-1}(t_1, \cdots, t_m) = (t_1, \cdots, t_{m-1}).$$

We then have the order preserving map

$$\pi_{m-1} x_m : \underline{p_m} \to R^{m-1}.$$

Applying the unique mono-epi factorization, we get a positive integer p_{m-1}, an order preserving epi $\delta_{m-1} : \underline{p_m} \to \underline{p_{m-1}}$, and an order preserving mono $x_{m-1} : \underline{p_{m-1}} \to R^{m-1}$ with $\pi_{m-1} x_m = x_{m-1} \delta_{m-1}$. We can continue the process to obtain from x a unique commutative diagram

$$\begin{tikzcd}
\underline{p_1} \arrow[d, "x_1"'] & \cdots \arrow[l, "\delta_1"'] & \underline{p_{m-1}} \arrow[l] \arrow[d, "x_{m-1}"'] & \underline{p_m} \arrow[l, "\delta_{m-1}"'] \arrow[d, "x_m"'] & \underline{p} \arrow[l, "\sigma"'] \arrow[d, equal] \\
R^1 & \cdots \arrow[l, "\pi_1"'] & R^{m-1} \arrow[l] & R^m \arrow[l, "\pi_{m-1}"'] & \underline{p}, \arrow[l, "x"']
\end{tikzcd}$$

where each δ_i is an order preserving epi, where σ is an epi, and where each x_i is an order preserving mono. The top line of this diagram will be denoted by ω, and we let $C(\omega) \subset (R^m)^p$ denote the set of all $x \in (R^m)^p$ which yield the above diagram with top line ω.

The subset $C(\omega)$ can be checked to be a finite intersection of closed halfspaces and open halfspaces in the Euclidean space $(R^m)^p$. As ω ranges over the set $\mathcal{C}_m$ of all

$$\underline{p_1} \xleftarrow{\delta_1} \cdots \xleftarrow{\delta_{m-1}} \underline{p_m} \xleftarrow{\sigma} \underline{p}$$

with each δ_i an order preserving epi and σ an epi, we thus have the partition of $(R^m)^p$ into non-empty disjoint subsets $C(\omega)$, with each $C(\omega)$ a finite intersection of open and closed halfspaces. For $x \in (R^m)^p$, p_1 counts the number of distinct first coordinates of the various functional values $x(j)$, p_2 the maximum number of distinct second coordinates of the various $x \in C(\omega)$, etc. Thus $C(\omega)$ is homeomorphic to an open disk of dimension $p_1 + \cdots + p_m$. The action of $\Sigma(p)$ permutes the $C(\omega)$ by

$$C(\delta_1, \cdots, \delta_{m-1}, \sigma)\tau = C(\delta_1, \cdots, \delta_{m-1}, \sigma\tau).$$

(10.14) *Fix an element ω of $\mathcal{C}_m$ as the diagram*

$$\underline{p_1} \xleftarrow{\delta_1} \cdots \leftarrow \underline{p_{m-1}} \xleftarrow{\delta_{m-1}} \underline{p_m} \xleftarrow{\sigma} \underline{p},$$

where each δ_i is an order preserving epi and where σ is an epi. Then the closure of $C(\omega)$ is the disjoint union of all $C(\omega')$, where ω' is

$$\underline{q_1} \xleftarrow{\delta'_1} \cdots \leftarrow \underline{q_{m-1}} \xleftarrow{\delta'_{m-1}} \underline{q_m} \xleftarrow{\sigma'} \underline{p},$$

for which there exists a commutative diagram

$$\begin{tikzcd}
\underline{p_1} \arrow[d, "\tau_1"'] & \cdots \arrow[l, "\delta_1"'] & \underline{p_{m-1}} \arrow[l] \arrow[d, "\tau_{m-1}"'] & \underline{p_m} \arrow[l, "\delta'_{m-1}"'] \arrow[d, "\tau_m"'] & \underline{p} \arrow[l, "\sigma"'] \arrow[d, equal] \\
\underline{q_1} & \cdots \arrow[l, "\delta'_1"'] & \underline{q_{m-1}} \arrow[l] & \underline{q_m} \arrow[l, "\delta'_{m-1}"'] & \underline{p}, \arrow[l, "\sigma'"']
\end{tikzcd}$$

where τ_1 is an order preserving epi and where τ_i for $i > 1$ is an epi which is order preserving on each $\delta_{i-1}^{-1}(pt)$. For each ω', there is at most one such commutative diagram and the diagram is completely determined by τ_m.

If we regard $\mathcal{C}_m$ as a category whose objects are the ω and whose morphisms are the $\tau : \omega \to \omega'$ in the above proposition, then $\mathcal{C}_m$ is then a poset with a unique morphism $\tau : C(\omega) \to C(\omega')$ whenever $\overline{C(\omega)} \supset C(\omega')$.

Proof. In this outline, we leave the following to the reader. The closure $\overline{C(\omega)}$ consists of all $x : \underline{p} \to R^m$ for which there exists a commutative diagram

$$\begin{tikzcd}
\underline{p_1} \arrow[d, "x_1"'] & \cdots \arrow[l, "\delta_1"'] & \underline{p_{m-1}} \arrow[l] \arrow[d, "x_{m-1}"'] & \underline{p_m} \arrow[l, "\delta_{m-1}"'] \arrow[d, "x_m"'] & \underline{p} \arrow[l, "\sigma"'] \arrow[d, equal] \\
R^1 & \cdots \arrow[l, "\pi_1"'] & R^{m-1} \arrow[l] & R^m \arrow[l, "\pi_{m-1}"'] & \underline{p}, \arrow[l, "x"']
\end{tikzcd}$$

where x_1 is order preserving and for $i > 0$ each x_i is order preserving on any $\delta_{i-1}^{-1}(pt)$. Assuming this, we outline the theorem. Fix x in the closure, and consider the above commutative diagram. Consider the diagram

$$x_1(\underline{p_1}) \xleftarrow{\pi_1} \cdots \leftarrow x_{m-1}(\underline{p_{m-1}}) \xleftarrow{\pi_{m-1}} x_m(\underline{p_m}) \xleftarrow{x} \underline{p}.$$

Note that the functions are epis, and use commutativity to show that all except the last of the functions are order preserving. Each $x_i(\underline{p_i})$ is a finite set which is also linearly ordered by lexicographic order. Hence there is a unique q_i and order preserving isomorphism of $x_i(\underline{p_1})$ with $\underline{q_i}$. Thus we have a commutative diagram

$$\begin{tikzcd}
\underline{p_1} \arrow[d, "\tau_1"'] & \cdots \arrow[l, "\delta_1"'] & \underline{p_{m-1}} \arrow[l] \arrow[d, "\tau_{m-1}"'] & \underline{p_m} \arrow[l, "\delta_{m-1}"'] \arrow[d, "\tau_m"'] & \underline{p} \arrow[l, "\sigma"'] \arrow[d, equal] \\
\underline{q_1} \arrow[d, "i_1"'] & \cdots \arrow[l, "\delta'_1"'] & \underline{q_{m-1}} \arrow[l] \arrow[d, "i_{m-1}"'] & \underline{q_m} \arrow[l, "\delta'_{m-1}"'] \arrow[d, "i_m"'] & \underline{p} \arrow[l, "\sigma'"'] \arrow[d, equal] \\
R^1 & \cdots \arrow[l, "\pi_1"'] & R^{m-1} \arrow[l] & R^m \arrow[l, "\pi_{m-1}"'] & \underline{p}, \arrow[l, "x"']
\end{tikzcd}$$

where τ_j is an epi, where i_j is an order preserving epi, and where $i_j\tau_j = x_j$. The theorem follows. $\square$

Cellular Categories Related to the Symmetric Groups

The following piecewise linear theorem is a variant of that found in Stallings [5.6], but we will assume that it follows similarly.

(10.15) *Let U be an open subset of R^n for which we have a finite, disjoint partition $\mathcal{C}$ into nonempty sets $C \in \mathcal{C}$, where*

(i) each $C \in \mathcal{C}$ is a finite intersection of open and closed halfspaces, and
(ii) the closure of each $C \in \mathcal{C}$ in U is a union of $C' \in \mathcal{C}$.

Consider $\mathcal{C}$ as a category which has a morphism $C \to C'$ whenever the closure of C contains C'. Choose a point $x(C) \in C$ for each C, and consider the union X of all the simplices

$$\langle x(C_0), \cdots, x(C_k)\rangle$$

for all sequences such that C_i is contained in the boundary of C_{i+1}. Then X is a deformation retract of U. Note that X is naturally homeomorphic to $B_{\mathcal{C}}$. For each $C \in \mathcal{C}$, fix $C_0 = C$ and take the union of all the

$$\langle x(C), x(C_1), \cdots, x(C_k)\rangle$$

as above. Then we get a combinatorial $(n-k)$-cell where k is the dimension of C. Thus $\mathcal{C}$ is then a cellular category in the sense of Chapter 5.

We can now apply the above in several slightly different ways.

Example 1. Consider $U = (R^m)^p$ with the partition $\{C(\omega)\}$ of (10.14). The associated category we have denoted by $\mathcal{C}_m$ in (10.14). It has objects

$$\underline{p_1} \xleftarrow{\delta_1} \cdots \leftarrow \underline{p_{m-1}} \xleftarrow{\delta_{m-1}} \underline{p_m} \xleftarrow{\sigma} \underline{p},$$

where each δ_i is an order preserving epi and where σ is an epi. Moreover, it has morphisms all $\tau : \omega \to \omega'$ corresponding to commutative diagrams

$$\begin{tikzcd}
\underline{p_1} \arrow[d, "\tau_1"'] & \cdots \arrow[l, "\delta_1"'] & \underline{p_{m-1}} \arrow[l] \arrow[d, "\tau_{m-1}"'] & \underline{p_m} \arrow[l, "\delta'_{m-1}"'] \arrow[d, "\tau_m"'] & \underline{p} \arrow[l, "\sigma"'] \arrow[d, equal] \\
\underline{q_1} & \cdots \arrow[l, "\delta'_1"'] & \underline{q_{m-1}} \arrow[l] & \underline{q_m} \arrow[l, "\delta'_{m-1}"'] & \underline{p}, \arrow[l, "\sigma'"']
\end{tikzcd}$$

where τ_1 is an order preserving epi and where τ_i for $i > 1$ is an epi which is order preserving on each $\delta_{i-1}^{-1}(pt)$. Pick the points $x(\omega) \in C(\omega)$ for all ω for which σ is order preserving, and then pick the others equivariantly with respect to the $\Sigma(p)$-action. There is a natural right action of $\Sigma(p)$ on both X and $B_{\mathcal{C}_m}$ and they are equivariantly homeomorphic. Moreover, X is an equivariant deformation retract of $(R^m)^p$. No doubt one can make the choices so that X lies in the $m(p-1)$-dimensional subspace of $(R^m)^p$ which is orthogonal to the diagonal, and take X to be a closed equivariant neighborhood of the origin in the subspace.

Example 2. Let $U = (R^m)^p - X_m$, where X_m denotes all $x : \underline{p} \to R^m$ which are not one-to-one. Consider the category $\mathcal{D}_m$ whose objects are all

$$\omega : \underline{p_1} \xleftarrow{\delta_1} \cdots \leftarrow \underline{p_{m-1}} \xleftarrow{\delta_{m-1}} \underline{p} \xleftarrow{\sigma} \underline{p}$$

for which each δ_i is an order preserving epi and for which $\sigma \in \Sigma(p)$, with a morphism $\tau : \omega \to \omega'$ for each commutative diagram

$$\begin{array}{ccccccccc}
\underline{p_1} & \xleftarrow{\delta_1} & \cdots & \longleftarrow & \underline{p_{m-1}} & \xleftarrow{\delta_{m-1}} & \underline{p} & \xleftarrow{\sigma} & \underline{p} \\
\tau_1 \downarrow & & & & \tau_{m-1} \downarrow & & \tau_m \downarrow & & \| \\
\underline{q_1} & \xleftarrow{\delta'_1} & \cdots & \longleftarrow & \underline{q_{m-1}} & \xleftarrow{\delta'_{m-1}} & \underline{p} & \xleftarrow{\sigma'} & \underline{p},
\end{array}$$

where τ_1 is an order preserving epi and where τ_i for $i > 0$ is an epi and also is order preserving on each $\delta_{i-1}^{-1}(pt)$. Then $B_{\mathcal{D}_n}$ can be considered as an equivariant deformation retract of the space $(R^m)^p - X_m$ and thus its orbit space $B_{\mathcal{D}_m}/\Sigma(p)$ is a homotopy model for the configuration space of all subsets of R^m with exactly p elements.

Example 3. If one wishes to go directly to orbit spaces, then consider the category $\mathcal{E}_m$ which has an object for each

$$\omega : \underline{p_1} \xleftarrow{\delta_1} \cdots \leftarrow \underline{p_{m-1}} \xleftarrow{\delta_{m-1}} \underline{p}$$

with each δ_i an order preserving epi, and a morphism $\tau : \omega \to \omega'$ for each commutative diagram

$$\begin{array}{ccccccc}
\underline{p_1} & \xleftarrow{\delta_1} & \cdots & \longleftarrow & \underline{p_{m-1}} & \xleftarrow{\delta_{m-1}} & \underline{p} \\
\tau_1 \downarrow & & & & \tau_{m-1} \downarrow & & \tau_m \downarrow \\
\underline{q_1} & \xleftarrow{\delta'_1} & \cdots & \longleftarrow & \underline{q_{m-1}} & \xleftarrow{\delta'_{m-1}} & \underline{p}
\end{array}$$

for which the τ_i have the above properties. Note that there is a functor

$$\theta : \mathcal{E}_m \to \Sigma(p)$$

which sends a morphism τ into $\tau_m \in \Sigma(p)$. It can be seen that $\mathcal{E}_m$ is a cellular category, since $\mathcal{D}_m$ is. Moreover, $B_{\mathcal{E}_m}$ is homotopy equivalent to the configuration space of subsets of R^m with exactly p elements.

Example 4. One has natural inclusions of $\mathcal{E}_m$ into $\mathcal{E}_{m+1}$ as full subcategories. Thus one can take the colimit of

$$\mathcal{E}_1 \longrightarrow \cdots \longrightarrow \mathcal{E}_m \longrightarrow,$$

giving a category $\mathcal{E}$ which has objects all diagrams

$$\underline{p_1} \xleftarrow{\delta_1} \cdots \leftarrow \underline{p_k} \xleftarrow{\delta_k} \cdots$$

for which each δ_k is an order preserving epi and for which for all k large each $p_k = p$. The morphisms $\tau : \omega \to \omega'$ are all commutative diagrams

$$\begin{array}{ccccccc}
\underline{p_1} & \xleftarrow{\delta_1} & \cdots & \xleftarrow{\delta_{k-1}} & \underline{p_k} & \xleftarrow{\delta_k} & \cdots \\
\tau_1 \downarrow & & & & \tau_k \downarrow & & \\
\underline{q_1} & \xleftarrow{\delta'_1} & \cdots & \xleftarrow{\delta'_{k-1}} & \underline{q_k} & \xleftarrow{\delta'_k} & \cdots
\end{array}$$

which have τ_1 an order preserving epi and for $i > 1$ have τ_i an epi and also order preserving on each $\delta_{i-1}^{-1}(pt)$. Moreover, $\mathcal{E}$ is cellular since each $\mathcal{E}_m$ is. There is the functor $\theta : \mathcal{E} \to \Sigma(p)$ which takes a morphism $\tau : \omega \to \omega'$ into the common value of the τ_k for k large. This is a topological resolution of $\Sigma(p)$ in the sense of Chapter 5. Hence $B_{\mathcal{E}} \sim B_{\Sigma(p)}$, and we have a CW-model for $B_{\Sigma(p)}$ with a cell of dimension $\sum(p - p_k)$ for each object ω.

References

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