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This story begins with an absorbing recent book that contained an ancient puzzle … whose answer startled the first author who noticed something combinatorial about the problem … and who then conjectured a theorem to the second author … who went off and proved the theorem. The two of them studied the literature and concluded that this piece of combinatorics, contained in one of the legendary mathematics books from ancient China, had gone unnoticed until now.

The well problem

Tim Chartier's *Math Bytes* [**2**] contains many topics of great interest to those intrigued by the interplay between mathematics and computing. In particular, it contains the so-called well problem from the two-thousand-year-old *Nine Chapters of the Mathematical Art* [**5**, ch. 8, prob. 13]:

> Given are five families who share a well. The deficit of two of A's well-ropes is the same as one of B's well-ropes [that is, two of A's well-ropes plus one of B's well-ropes equals the depth of the well]. The deficit of three of B's well-ropes

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is the same as one of C's well-ropes. The deficit of four of C's well-ropes is the same as one of D's well-ropes. The deficit of five of D's well-ropes is the same as one of E's well-ropes. The deficit of six of E's well-ropes is the same as one of A's well-ropes. Each then obtains the well-rope that makes up the deficit, and all reach the water. Problem: How deep is the well, and how long is each of the well-ropes?

To solve this, let $A, \dots, E$ represent the lengths of the given family's well-ropes and let w be the depth of the well. It is straightforward to obtain the following system of five linear equations in six unknowns.

$$\begin{aligned} 2A + B &= w \\ 3B + C &= w \\ 4C + D &= w \\ 5D + E &= w \\ 6E + A &= w \end{aligned}$$

Two thousand years ago, the writer (or writers) of the *Nine Chapters* anticipated our modern methods of solving systems of linear equations, so we follow their lead and recast the well problem in terms of matrix algebra. It is convenient to view this set of equations as a nonhomogeneous system of five linear equations in the five unknowns A, B, C, D, E and the parameter w, written as the augmented matrix of a 5×6 linear system.

$$\left[\begin{array}{ccccc|c} 2 & 1 & 0 & 0 & 0 & w \\ 0 & 3 & 1 & 0 & 0 & w \\ 0 & 0 & 4 & 1 & 0 & w \\ 0 & 0 & 0 & 5 & 1 & w \\ 1 & 0 & 0 & 0 & 6 & w \end{array}\right]$$

Elementary row operations reduce the above matrix to the following matrix in upper-triangular form.

$$\left[\begin{array}{ccccr|r} 1 & 0 & 0 & 0 & 6 & -w \\ 0 & 1 & 0 & 0 & -12 & w \\ 0 & 0 & 1 & 0 & 36 & -4w \\ 0 & 0 & 0 & 1 & -144 & 15w \\ 0 & 0 & 0 & 0 & 721 & -76w \end{array}\right]$$

With more unknowns than equations, this is an underdetermined system with infinitely many real solutions. However, the solution of the original problem gives positive integers for all the unknowns. With this in mind, we see that $w = 721$ and $E = 76$ are solutions in positive integers. Back-solving this system gives $D = 129$, $C = 148$, $B = 191$, and $A = 265$. Moreover, because $w = 721$ is relatively prime to each of A, B, C, D, E, the stated values are the least positive integer solutions of the system.

For a careful choice of units, this is a reasonable problem: Measured in inches, the well is a bit more than sixty feet deep, the longest rope is about 22 feet long, and the shortest rope is a little more than six feet in length.

Now, if you are a combinatorialist or if you have taught combinatorics, and you see that value for A, then you will likely think,

"Wait a minute ... what's 265 doing in a problem from ancient China?"
Otherwise, you may be thinking
"Wait a minute ... what's so special about 265?"
An excellent question and, in isolation, the well problem is silent regarding the presence of 265 in its solution. But what if there were an arbitrary number of families? Generalizing the problem from five families to n families is a mathematician's way of thinking, so here we go.

The generalized well problem

Here is the restatement of the well problem for n families.

> Given are n families who share a well. The deficit of two of the first family's well-ropes is the same as one of the second family's well-ropes. The deficit of three of the second family's well-ropes is the same as one of the third family's well-ropes ... Finally, the deficit of $n + 1$ of the nth family's well-ropes is the same as one of the first family's well-ropes. Each family then obtains the well-rope that makes up the deficit, and all reach the water. Problem: How deep is the well, and how long are the families' well-ropes?

As before, this is an underdetermined system, so we look for the smallest positive integers that solve the problem. Call the first family family A. The case $n = 1$ is special: It says that the deficit of 2 of A's ropes is the same as 1 of A's ropes. That is, $2A + A = w$ giving minimal positive integer solutions $w = 3$ and $A = 1$. For $n = 2$, we find $A = 2$, $B = 1$, and $w = 5$. For n families, write A_n for the longest rope length, s_n for the shortest rope length (belonging to the nth family), and w_n for the well depth. Values up to $n = 8$ are shown in Table 1.

Eventually we see some patterns: The well depths w_n seem close to certain factorials and the longest rope lengths A_n seem to depend of lengths for fewer families, suggesting recurrence relations. In more detail,

- note that $w_3 = 25 = 4! + 1$ and $w_4 = 119 = 5! - 1$, suggesting
$$w_n = (n + 1)! + (-1)^{n+1};$$
- relating A_n to A_{n-1}, we see $A_4 = 44 = 5 \cdot 9 - 1$ and $A_5 = 265 = 6 \cdot 44 + 1$, so perhaps
$$A_n = (n + 1)A_{n-1} + (-1)^{n-1};$$
- relating A_n to A_{n-1} and A_{n-2}, notice that $A_3 = 9 = 3 \cdot 2 + 3 \cdot 1$ and also $A_4 = 44 = 4 \cdot 9 + 4 \cdot 2$, so one can hope that
$$A_n = (n - 1)A_{n-1} + (n - 1)A_{n-2}.$$

Table 1. Longest and shortest rope lengths and well depths for up to eight families.

n	1	2	3	4	5	6	7	8
A_n	1	2	9	44	265	1854	14,833	133,496
s_n	1	1	4	15	76	455	3186	25,487
w_n	3	5	25	119	721	5039	40,321	362,879

Derangements

Derangements are permutations with no fixed points; we give more details below. The problem of counting derangements first dates from Pierre Rémond de Montmort in 1708 [4]. He and Nicholas Bernoulli independently solved the problem around 1713 ([4] includes their correspondence). If you are familiar with these special permutations, you may recognize 265 as the number of derangements on six letters. If not, putting the A_n values from Table 1 into the *Online Encyclopedia of Integer Sequences* gives [6, A000166] with derangements as the title interpretation. Either way, you can now join the first author in the startled question about a derangement number being connected to an ancient Chinese problem about well-rope lengths.

More formally, let an n-permutation be an ordered arrangement of $\{1, \dots, n\}$. An n-derangement is an n-permutation with no number appearing in its original position. For example, the permutation 24315 is not a derangement because it fixes 3 and 5, while 41523 is a derangement because all five numbers have been moved from where they started.

Let D_n be the number of n-derangements. There are no 1-derangements, there is only one 2-derangement, namely 21, and there are two 3-derangements, 231 and 312, so $D_1 = 0$, $D_2 = 1$, and $D_3 = 2$. Here are some general results about D_n.

Theorem 1. *Let D_n be the number of derangements of an n-element set. Then*

(a) $D_1 = 0$, $D_2 = 1$, *and* $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$ *for* $n > 2$,

(b) $D_n = nD_{n-1} + (-1)^n$ *for* $n \geq 2$, *and*

(c) $D_n = n! \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k!}$ *for* $n \geq 1$.

These are proved in many combinatorics books, such as [1, pp. 128–129].

Connections

We can now establish the connection between combinatorics and the generalized well problem.

Theorem 2. *Given a positive integer n, consider the well problem for n families, so that A_n is the length of the longest rope and w_n is the depth of the well.*

(a) $\dfrac{A_n}{w_n} = \dfrac{D_{n+1}}{(n+1)! + (-1)^{n+1}}$.

(b) *If* $\gcd(D_{n+1}, (n+1)! + (-1)^{n+1}) = 1$, *then the smallest positive integer values for A_n and w_n are* $A_n = D_{n+1}$ *and* $w_n = (n+1)! + (-1)^{n+1}$, *respectively.*

Proof. Table 1 establishes (a) for small values of n. We proceed by induction. Define the $n \times n$ matrices

$$M_n = \begin{pmatrix} 2 & 1 & & & \\ & 3 & 1 & & \\ & & \ddots & \ddots & \\ & & & n & 1 \\ 1 & & & & n+1 \end{pmatrix}, \quad M'_n = \begin{pmatrix} 1 & 1 & & & \\ 1 & 3 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 1 & & & n & 1 \\ 1 & & & & n+1 \end{pmatrix}$$

where only nonzero values are indicated. In words, M_n has diagonal entries 2 through $n+1$ and $m_{i,i+1} = 1$ for $1 \le i \le n-1$ with $m_{n,1} = 1$ also, while M_n' is the same as M_n except that the first column is replaced by all 1s. As we saw above for $n = 5$, the first column of M_n contains the coefficients of A_n for each of the n equations. We obtain a formula for $|M_n|$, the determinant of M_n, then a recurrence satisfied by $|M_n'|$, one which is also satisfied by the derangement numbers.

First, we prove that $|M_n| = (n+1)! + (-1)^{n+1}$. Using cofactor expansion along the bottom row,

$$|M_n| = (-1)^{n+1} \cdot 1 \cdot \begin{vmatrix} 1 & & & \\ 3 & 1 & & \\ & \ddots & \ddots & \\ & & n & 1 \end{vmatrix} + (-1)^{2n-2} \cdot (n+1) \cdot \begin{vmatrix} 2 & 1 & & \\ & 3 & \ddots & \\ & & \ddots & 1 \\ & & & n \end{vmatrix}.$$

Both of these $(n-1) \times (n-1)$ matrices are triangular, so their determinants equal the product of their respective diagonal elements, 1 and $n!$, respectively. Therefore,

$$|M_n| = (-1)^{n+1} + (n+1)n! = (n+1)! + (-1)^{n+1}.$$

Now, we find a recurrence for $|M_n'|$, also by means of cofactor expansion along the bottom row:

$$\begin{aligned} |M_n'| &= (-1)^{n+1} \cdot 1 \cdot \begin{vmatrix} 1 & & & \\ 3 & 1 & & \\ & \ddots & \ddots & \\ & & n & 1 \end{vmatrix} + (-1)^{2n-2} \cdot (n+1) \cdot \begin{vmatrix} 1 & 1 & & \\ 1 & 3 & \ddots & \\ \vdots & & \ddots & 1 \\ 1 & & & n \end{vmatrix} \\ &= (-1)^{n+1} + (n+1)|M_{n-1}'|. \end{aligned}$$

Since $|M_2'| = 2 = D_3$ and $|M_3'| = 4|M_2'| + 1 = 9 = D_4$, the sequence $|M_n'|$ satisfies the recurrence relation (b) of Theorem 1 for D_{n+1} (notice that the indices are offset by one because of the initial values). Therefore, $|M_n'| = D_{n+1}$.

Now, define matrices L_n to be the same M_n' except that the first column is multiplied by w_n. Using Cramer's rule and the previous results,

$$A_n = \frac{|L_n|}{|M_n|} = \frac{w_n|M_n'|}{|M_n|} = \frac{w_n D_{n+1}}{(n+1)! + (-1)^{n+1}}$$

which establishes (a).

As a consequence of (a), we see that $A_n((n+1)! + (-1)^{n+1}) = w_n D_{n+1}$ and, supposing $\gcd(D_{n+1}, (n+1)! + (-1)^{n+1}) = 1$, no common factor greater than one can be cancelled from both sides of this equation. Hence, A_n divides D_{n+1} and vice versa, so that $A_n = D_{n+1}$ and $w_n = (n+1)! + (-1)^{n+1}$, as claimed in (b). ■

It turns out that $\gcd(D_{n+1}, (n+1)! + (-1)^{n+1}) = 1$ for all $n < 10^5$ with one exception, $n = 8$. In fact, $D_9 = 133{,}496 = 11 \cdot 12{,}136$ and $9! - 1 = 11 \cdot 32{,}989$. So, for eight families around a well, the smallest integer value for the longest rope length is 12,136, shorter than 14,833, the smallest integer value for the longest rope length for seven families!

Is $n = 8$ the only value of n for which the relevant greatest common divisor is greater than 1? No one knows.

Table 2. The longest rope lengths and two ratios.

n	$D_{n+1} = A_n$	$(n+1)!/D_{n+1}$	$D_{n+1}/(n+1)!$
1	1	2.00000...	0.500000...
2	2	3.00000...	0.333333...
3	9	2.66666...	0.275000...
4	44	2.72727...	0.366666...
5	265	2.71698...	0.368056...
6	1854	2.71845...	0.367857...
7	14,833	2.71826...	0.367882...
8	133,496	2.71828...	0.367879...

The longest rope lengths contain some more surprises. Table 2 shows $D_{n+1} = A_n$ along with the ratio $(n+1)!/D_{n+1}$ and its reciprocal $D_{n+1}/(n+1)!$. The third column of Table 2 tells us that the ratios of the depths of the well to the longest rope lengths converge rapidly to e (the $(-1)^{n+1}$ term in w_n quickly becomes negligible, so w_n is essentially $(n+1)!$). The fourth column tells us that for large n, the probability that a given permutation on n elements is a derangement is approximately $1/e$. This is the basis for a famous problem from recreational mathematics dating back to Montmort:

> A hat checker at a restaurant has checked n hats and gets the hat-check tickets totally mixed-up. What is the probability that none of the hats ends up with the right customer?

The number of such occurrences is the derangement number D_n and the relevant probability is $D_n/n!$, a four-decimal digit approximation to $1/e$ if there are at least seven customers wearing hats (corresponding to the $n = 6$ row of Table 2).

We have not explored the s_n row of Table 1, which corresponds to [**6**, A002467]. By the description of the generalized well problem, $s_n = (w_n - A_n)/(n+1)$.

Several sequences we have considered share the same recurrence relation. Specifically, let $x_1, x_2, \ldots$ be a sequence satisfying the recurrence $x_{n+1} = n(x_n + x_{n-1})$ and vary the initial conditions:

- $x_1 = 1$ and $x_2 = 2$ give the factorials, $x_n = n!$.
- $x_1 = 0$ and $x_2 = 1$ give the derangement numbers / the lengths of the longest well ropes, $x_n = D_n = A_{n-1}$.
- $x_1 = x_2 = 1$ give the sequence of shortest well ropes, $x_n = s_n$.

It follows that $s_n = n! - D_n$. That is, the length of the shortest well-ropes s_n is the number of n-permutations with at least one fixed point. Isn't that interesting?

Conclusions

The most recent version of *Nine Chapters of the Mathematical Art* [**5**], originally compiled in the first millennium BCE, predates Montmort's work by 16 centuries. And, apparently, no one noticed the connection to derangements until now.

But there is more to the story, because after being lost for many centuries, parts of the *Nine Chapters* were finally brought to light in the nineteenth century. Florian Cajori made partial translations into English in 1893. After additional work by David Smith (1925) and Dirk Struik (1948), the first complete English translation [**5**] dates from 1999.

More recently, in his 2011 *The Chinese Roots of Linear Algebra* [**3**], Roger Hart devotes a 40-page chapter plus part of an appendix to the well problem alone. But he does not connect the 265 in the solution of problem 9 of chapter 8 of *The Nine Chapters on the Mathematical Arts* to the number of derangements of six items and provides no combinatorial perspective on the well problem.

Why, then, did we notice this result? The reason is that for many years, the first author has taught recurrence relations in elementary combinatorics courses (and the derangement numbers are always present), saw the well problem, and was able to hook the second author on it.

Looking at a problem with combinatorial eyes is all it took.

Summary. *The Nine Chapter on the Mathematical Arts* from ancient China includes a problem concerning five families who share a well. We generalize the problem and offer a combinatorial perspective that appears to be new.

References

[1] Andreescu, T., Feng, Z. (2004). *A Pathway to Combinatorics for Undergraduates: Counting Strategies*. Boston, MA: Birkhäuser.

[2] Chartier, T. (2014). *Math Bytes: Google Bombs, Chocolate-Covered Pi, and Other Cool Bits in Computing*. Princeton, NJ: Princeton Univ. Press.

[3] Hart, R. (2011). *The Chinese Roots of Linear Algebra*. Baltimore, MD: Johns Hopkins Univ. Press.

[4] Montmort, P. R. de (1713). *Essay d'analyse sur les jeux de hazard*, 2nd ed. Paris: Quillau.

[5] Shen, K., Crossley, J. N., Lun, A. W.-C., eds. (1999). *The Nine Chapters on the Mathematical Art: Companion and Commentary*. Beijing: Oxford Univ. Press.

[6] Sloane, N. J. A., ed. (2018). Online encyclopedia of integer sequences. oeis.org.

[7] Straffin, P. D. (1998). Liu Hui and the first golden age of Chinese mathematics. *Math. Mag.* 71: 163–181. doi.org/10.2307/2691200.

