MATH 5524 · MATRIX THEORY

Project Specification

Posted Monday 27 February 2017. Due at 5:25pm on **Saturday 6 May 2017**.

1. *Declare the topic of your project by meeting with the instructor by Thursday March 23.*
2. You are welcome to select a topic from the list below, or to pursue a different topic related to matrix theory that interests you. Topics connected to your own research are particularly encouraged. Please discuss the subject matter and scope with the instructor before embarking on such a project.
3. Final write-ups are due (by email) by 5:25pm on Saturday 6 May 2017 at the latest.
   No late submissions will be accepted on this assignment.
4. Type your report using LaTeX (or your favorite word processor).
5. The report should be *at least* five pages, potentially including graphics.
6. Each student should complete an independent project, but you are welcome to discuss the topic with your fellow students, the instructor, your advisor, etc.
7. The project will count for 15% of your final grade in the course.
8. 80% of the project grade will be based on mathematical/scientific content.
   20% will be based on the quality of the exposition.
   See the attached rubric for details.
9. Projects will generally be based on a careful reading of an article or section of a book.
10. *Basic expectations.* All projects should: (a) tackle a topic beyond the scope of the lectures; (b) refer in detail to at least one source in the literature (papers/books), and show an appreciation for the broader literature on the topic; (c) clearly describe the theoretical setting and assumptions; (d) provide illustrations (e.g., small examples, MATLAB plots, etc.) of the concept in action, ideally beyond those contained in the article/book itself.

A few project ideas follow. Some of these articles are lengthy works with great theoretical depth; digesting a solid piece will make for a fine project. Other papers are approachable survey articles; a complete project will dig deeper into the literature to follow points raised in these surveys. You will discuss the scope of your project when you meet with the instructor to declare your project intention.

- Tamara G. Kolda and Brett W. Bader. "Tensor Decompositions and Applications," *SIAM Review* 51 (2009) 455–500.

  Tensors are extensions of matrices for higher dimensional data. The most common tensors (beyond vectors and matrices) have three *modes*, i.e., $T \in \mathbb{C}^{m\times n\times p}$, with entries $t_{j,k,\ell} \in \mathbb{C}$. For example, each "slice" of T, say $T_{:,:,\ell} \in \mathbb{C}^{m\times n}$, could represent an image, and the collection of slices could represent frames of a video, or different planes of a 3d medical scan. This survey article provides a great introduction to tensors and some of their interesting complexity, e.g., how low-rank approximation of tensors differs from low-rank approximation of matrices.

- N. Halko, P. G. Martinsson, and J. A. Tropp. "Finding Structure with Randomness: Probabilistic Algorithms for Constructing Approximate Matrix Decompositions," *SIAM Review* 53 (2011) 217–288.

  Recent years have seen tremendous interest in "RandNLA," the use of randomization in traditional numerical linear algebra algorithms to tackle very large scale problems. Often this field takes classical algorithms (such as subspace iteration for eigenvalue computation) and uses recent results from random matrix theory to derive convergence bounds that hold *with high probability*. This article includes deep and important results whose proofs are nontrivial. Working through a section of the article would make a fine project.

- Françoise Tisseur and Karl Meerbergen. "The Quadratic Eigenvalue Problem," *SIAM Review* 43 (2001) 235–286.

  Quadratic eigenvalue problems arose in Problem Sets 1 and 2; they are naturally associated with second-order differential equations, and have many applications in damped mechanical systems. This important survey article will introduce you to the variety of spectral behavior possible for such problems, along with many applications where they arise.

- Julio Moro, James V. Burke, Michael L. Overton. "On the Lidskii–Vishik–Lyusternik Perturbation Theory for Eigenvalues of matrices with Arbitrary Jordan Structure," *SIAM J. Matrix Anal. Appl.* 18 (1997) 793–817.

  How do eigenvalues change when a matrix is subjected to very small perturbations? We will answer this question in class for diagonalizable matrices. The analogous theory for Jordan blocks is substantially more intricate, but also rather beautiful (if you like that kind of thing). This article provides the best description of these results. Understand this article, and you will prove your mastery of the Jordan form in all its subtlety.

- Chandler Davis and W. M. Kahan. "The Rotation of Eigenvectors by a Perturbation. III," *SIAM J. Numer. Anal.* 7 (1970) 1–46.

  How do the eigenvectors of a Hermitian matrix change when the matrix is perturbed? This paper contains the landmark Davis–Kahan "sin Θ" theorems describing how invariant subspaces are perturbed. If you seek a deep theoretical investigation of angles between subspaces, this masterpiece is for you.

- Extended investigation of the numerical range (field of values).

  The numerical range has many intriguing properties beyond the basic facts we can cover in our lectures. There are many projects lurking here. Start by reading Chapter 1 of R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991 (linked from the class website, with free access from VT). A few potential projects: (1) work through the proof of the convexity of the numerical range and investigate the geometric properties of its boundary; (2) investigate "inverse numerical range" problems: given a point $z \in W(\mathbf{A})$, find a unit vector $\mathbf{x} \in \mathbb{C}^n$ such that $\mathbf{x}^*\mathbf{A}\mathbf{x} = z$; (3) investigate a few of the many generalizations of the numerical range.

- Toeplitz matrix theory.

  Toeplitz matrices have constant values on each diagonal; this structure arises in a vast array of applications ranging from differential equations to signal processing. The spectral properties of such matrices are incredibly rich; the topic supports a wide variety of projects. For example, one could study the asymptotic distribution of eigenvalues of Hermitian Toeplitz matrices (the theorem of Grenander and Szegő), or non-Hermitian Toeplitz matrices (the theorem of Schmidt and Spitzer). The non-Hermitian case is particularly intriguing, as the eigenvalues of finite Toeplitz matrices tend to a limit that is generally different from the spectrum of the analogous infinite dimensional Toeplitz operator. Pseudospectra unravel this puzzle; see the paper by Reichel and Trefethen (1992) or the book by Böttcher and Grudsky (2005). The latter authors also study condition numbers of Toeplitz matrices.

- Random matrix theory.

  Random matrices have a rich history, with intriguing spectral properties. (For an introduction, simply type `plot(eig(randn(1600)/40),'k.'); axis equal` and enjoy.) For an overview of recent results and some applications:

  Terence Tao and Van Vu. "From the Littlewood–Offord problem to the circular law: universality of the spectral distribution of random matrices," *Bull. AMS* 46 (2009) 377–396.

  For a discussion of *condition numbers* of random matrices, see:

  Alan Edelman. "Eigenvalues and condition numbers of random matrices," *SIAM J. Matrix Anal. Appl.* 9 (1988) 543–560.

  Triangular random matrices are much more ill-conditioned than dense ones:

  D. Viswanath and L. N. Trefethen. "Condition numbers of random triangular matrices," *SIAM J. Matrix Anal. Appl.* 19 (1998) 564–581.

- Inverse eigenvalue problems.

  Given a set of eigenvalues, can one construct a matrix *of a specified form* that has those eigenvalues? This *inverse eigenvalue problem* has attracted considerable attention over the years, and it is relevant, for example, in the design of structures that do not vibrate when stimulated at certain frequencies (e.g., by earthquakes). The case of Hermitian tridiagonal matrices is particularly clean, and is classically motivated by the problem of locating beads on a vibrating string. For a discussion involving real experimental data, see:

  Steven J. Cox, Mark Embree, and Jeffrey M. Hokanson. "One can hear the composition of a string: experiments with an inverse eigenvalue problem," *SIAM Review* 54 (2012) 157–178.

- The Lanczos method, tridiagonal matrices, quadrature, model reduction, continued fractions.

  Jörg Liesen and Zdenek Strakoš. *Krylov Subspace Methods.* Oxford University Press, 2013.

  The Lanczos process reduces a Hermitian matrix to tridiagonal form. As this algorithm proceeds, one learns a wealth of information about the matrix: eigenvalue estimates (obeying Cauchy interlacing), quadrature rules based on approximate spectral measures, reduced-order models that match moments; one even finds connections to continued fractions and orthogonal polynomials. Chapter 3 of the book by Liesen and Strakoš provides a masterful description of the connections between all these objects: one of the most complete and satisfying corners of matrix theory.

- Karl Meerbergen and Alastair Spence. "Inverse Iteration for Purely Imaginary Eigenvalues with Application to the Detection of Hopf Bifurcations in Large-Scale Problems," *SIAM J. Matrix Anal. Appl.* 31 (2010) 1982–1999.

  A key problem in the study of dynamical systems concerns the identification of parameter values for which a stable system goes unstable. Consider systems of the form $\mathbf{M}\mathbf{x}'(t) = \mathbf{A}(p)\mathbf{x}(t)$. What is the smallest value of $|p|$ for which the matrix pencil $(\mathbf{A}(p), \mathbf{M})$ has a purely imaginary eigenvalue? This paper shows how this question can be answered elegantly in terms of *Lyapunov equations*, leading to an algorithm that requires the solution of such an equation at each iteration.

- Daniel A. Spielman. "Spectral Graph Theory and Its Applications," in *48th Annual IEEE Symposium on the Foundations of Computer Science*, 2007. DOI 10.1109/FOCS.2007.56.

  Fan R. K. Chung, *Spectral Graph Theory*, American Mathematical Society, 1997.

  Spectral graph theory is a vital subject concerned with eigenvalues and eigenvectors of graph adjacency matrices. From such matrices one can learn a wealth of information about the graph, including, for example, how best to partition a graph into subgraphs. Spielman's survey article will give you an introduction to this rich and fascinating field; pick an appealing thread and do a deeper investigation of the topic. Chung's book is the classic reference with many more theoretical details.

- Ernesto Estrada and Desmond J. Higham. “Network Properties Revealed through Matrix Functions,” *SIAM Review* 52 (2010) 696–714.

  Matrix theory can be used to characterize the proximity of two nodes in a graph (or “network”). This article describes one way (using the exponential of a matrix) that we shall discuss in class, as well as a variant using the resolvent. Use this paper as a jumping-off point to study “centrality measures” for network analysis.

- James P. Keener. “The Perron–Frobenius Theorem and the Ranking of Football Teams,” *SIAM Review* 35 (1993) 80–93.

  In 2014, Virginia Tech famously beat Ohio State, who went on to be National Champions. If we beat the champions, don’t we have a case for being champions ourselves? (Certainly not that year...!) No doubt you have worked similar logic: If Team A beat Team B, and Team B beat Team C, does that not suggest that Team A is better than Team C? Are sports victories transitive? A simple nonnegative matrix model can be used to rank teams in a manner that incorporates head-to-head results. Study this model and its more sophisticated variants.

- Michael W. Berry, Susan T. Dumais, and Gavin W. O’Brien. “Using Linear Algebra for Intelligent Information Retrieval,” *SIAM Review* 37 (1995) 573-595.

  In class we will briefly discuss *latent semantic indexing*, an application of the singular value decomposition for text analysis. How can you detect the proximity of documents? How do you adjust an existing matrix factorization when new documents are entered into your database? This survey paper gives an introduction to these ideas.

MATH 5524: Matrix Theory Project

Grading Rubric

Name: ______________________________

An *excellent* evaluation earns full points; a *satisfactory* evaluation earns partial points; an *unsatisfactory* evaluation earns few or no points.

total points		excellent	satisfactory	unsatisfactory	points
10	Introduction/summary	☐	☐	☐	______
10	Literature review	☐	☐	☐	______
30	Theoretical exposition	☐	☐	☐	______
30	Examples / experiments	☐	☐	☐	______
15	Quality of presentation	☐	☐	☐	______
5	Spelling and grammar	☐	☐	☐	______
				total score	______

Comments: ______________________________

GUIDELINES FOR EVALUATION

Intro/Summary	Excellent:	Fluent description of the problem and summary of project
	Satisfactory:	Some relevant details missing from problem statement, summary
	Unsatisfactory:	Poor motivation; scope or content of project unclear
Literature review	Excellent:	Cites main paper(s) and puts in the context of broader literature
	Satisfactory:	Cites main paper(s), but limited investigation beyond
	Unsatisfactory:	Contribution of main paper not clearly explained
Theoretical exposition	Excellent:	Clear discussion of key theoretical findings, perhaps with a proof or two
	Satisfactory:	Describes the key ideas of the subject but weak on a few details
	Unsatisfactory:	The report only shows a superficial understanding of the subject
Examples / experiments	Excellent:	Includes carefully thought-out experiments (by hand or computation)
	Satisfactory:	A few examples/experiments are included, repeated from the paper
	Unsatisfactory:	Examples/experiments are missing, incorrect, or directly copied
Quality of presentation	Excellent:	Carefully written and revised document, good typesetting, illustrations
	Satisfactory:	Document is fluent and clear, but room for improvement in text or plots
	Unsatisfactory:	Sloppy text, incoherent presentation, poor graphics, typesetting
Spelling/grammar	Excellent:	Few/no spelling or grammatical errors
	Satisfactory:	A handful of minor spelling or grammatical errors
	Unsatisfactory:	Errors make the text difficult to read