MATH 5524 · MATRIX THEORY

Problem Set 2

Posted Monday 6 February 2016. Due at 5pm on **Tuesday 14 February 2016**.
[Late assignments due at 5pm on Wednesday 15 February 2016.]

Complete any four problems, 25 points each.
(Note that Problem 6 includes a bonus of up to 5 points.)

You are welcome to complete more problems if you like. The latter problems are generally more challenging than the early problems. If you are already familiar with this material, please tackle the latter problems. If you submit more than four solutions, specify those four you would like to be graded.

All norms refer to the Euclidean vector 2-norm and the matrix norm it induces.

1. Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ are such that $\mathbf{v}^*\mathbf{u} = 1$, so that $\mathbf{P} = \mathbf{u}\mathbf{v}^*$ is a projector. Prove that

$$\|\mathbf{P}\| = \frac{1}{\cos \angle(\mathbf{u}, \mathbf{v})}.$$

   Interpret this result in light of Problem 8 on Problem Set 1. (Suppose $\mathbf{P}$ is a projector onto a one-dimensional subspace, $\mathcal{R}(\mathbf{u})$. Then $\|\mathbf{P}\| = 1$ if and only $\mathbf{P} = \mathbf{P}^*$.)

2. Compute the Jordan Canonical Form of the following matrices:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & -2 \\ -1 & 0 & 0 \end{bmatrix}.$$

   In each case, write down the spectral projectors $\mathbf{P}_j$ associated with each distinct eigenvalue λ_j.

3. Use the analytical approach (contour integrals of the resolvent) to compute all the terms in the spectral representation

$$\mathbf{A} = \sum_{j=1}^{m} (\lambda_j \mathbf{P}_j + \mathbf{D}_j)$$

   for each of the three matrices in Problem 2.

4. In class we deal with Jordan blocks of the form

$$\mathbf{J} = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

   Prove that we can replace the ones on the superdiagonal with any nonzero value ε; that is, show that there exists invertible $\mathbf{S}$ such that

$$\mathbf{S}\mathbf{J}\mathbf{S}^{-1} = \begin{bmatrix} \lambda & \varepsilon & & \\ & \lambda & \ddots & \\ & & \ddots & \varepsilon \\ & & & \lambda \end{bmatrix}.$$

What happens to $\|\mathbf{S}\|\|\mathbf{S}^{-1}\|$ as $\varepsilon \to 0$?

Explain how to generalize this argument to the case where $\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}$ has two Jordan blocks, i.e.,

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{bmatrix}.$$

(That is, change the superdiagonal in both $\mathbf{J}_1$ and $\mathbf{J}_2$ from 1 to ε.)

(Once you have seen how to do this for two Jordan blocks, you will notice that the general case is obvious. You do not need to discuss it.)

5. Given matrices $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{B} \in \mathbb{C}^{m\times m}$, consider the *Sylvester operator* $\mathcal{S} : \mathbb{C}^{n\times m} \to \mathbb{C}^{n\times m}$, defined for any $\mathbf{X} \in \mathbb{C}^{n\times m}$ by

$$\mathcal{S}\mathbf{X} = \mathbf{A}\mathbf{X} - \mathbf{X}\mathbf{B}.$$

   (a) Prove that $\mathcal{S}$ is a *linear operator*, i.e.,
   (i) $\mathcal{S}(\mathbf{X} + \mathbf{Y}) = \mathcal{S}\mathbf{X} + \mathcal{S}\mathbf{Y}$ for all $\mathbf{X}, \mathbf{Y} \in \mathbb{C}^{n\times m}$;
   (ii) $\mathcal{S}(c\mathbf{X}) = c(\mathcal{S}\mathbf{X})$ for all $c \in \mathbb{C}$ and $\mathbf{X} \in \mathbb{C}^{n\times m}$.

   (b) Since $\mathcal{S}$ is a linear operator that maps a space ($\mathbb{C}^{n\times m}$) into itself, it makes sense to ask: What are the eigenvalues and eigenvectors (eigenmatrices?) of $\mathcal{S}$? That is, we seek $\lambda \in \mathbb{C}$ and $\mathbf{V} \in \mathbb{C}^{n\times m}$ such that

$$\mathcal{S}\mathbf{V} = \lambda\mathbf{V}.$$

   Determine all the eigenvalues and “eigenvectors” (in $\mathbb{C}^{n\times m}$) of $\mathcal{S}$, in terms of the eigenvalues $\{\alpha_j\}$ and $\{\beta_j\}$, right eigenvectors $\{\mathbf{x}_j\}$ and $\{\mathbf{y}_j\}$, and left eigenvectors $\{\widehat{\mathbf{x}}_j\}$ and $\{\widehat{\mathbf{y}}_j\}$ of $\mathbf{A}$ and $\mathbf{B}$. That is,

$$\mathbf{A}\mathbf{x}_j = \alpha_j\mathbf{x}_j, \qquad \widehat{\mathbf{x}}_j^*\mathbf{A} = \alpha_j\widehat{\mathbf{x}}_j^*, \qquad \mathbf{B}\mathbf{y}_j = \beta_j\mathbf{y}_j, \qquad \widehat{\mathbf{y}}_j^*\mathbf{B} = \beta_j\widehat{\mathbf{y}}_j^*.$$

   Hint: Generalizing what we know about eigenvalues of matrices, the equation $\mathcal{S}\mathbf{X} = \mathbf{C}$ *has a solution if and only if* $\mathcal{S}$ *has no zero eigenvalues. You may use this fact for part (c).*

   (c) A matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is said to be *stable* if all of its eigenvalues have negative real parts: $\operatorname{Re}\alpha < 0$ for all $\alpha \in \sigma(\mathbf{A})$.

   Use part (b) to conclude that if $\mathbf{A}$ is stable, then the Lyapunov equation $\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^* = \mathbf{C}$ has a unique solution for $\mathbf{C} \in \mathbb{C}^{n\times n}$.

   Lyapunov equations play an essential role in control theory; for example, the controllability *and* observability Gramians *are computed as solutions to Lyapunov equations.*

6. Bernoulli’s description of the compound pendulum with three equal masses (see Section 1.2 of the class notes) models an ideal situation: there is no energy loss in the system. When we add a viscous damping term, the displacement $x_j(t)$ of the jth mass is governed by the differential equation

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} - 2a \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix}$$

for damping constant $a \geq 0$. We write this equation in matrix form,

$$\mathbf{x}''(t) = -\mathbf{A}\mathbf{x}(t) - 2a\mathbf{x}'(t).$$

As with the damped harmonic oscillator (see Section 1.7 of the notes), we introduce $\mathbf{y}(t) := \mathbf{x}'(t)$ and write the second-order system in first-order form:

$$\begin{bmatrix} \mathbf{x}'(t) \\ \mathbf{y}'(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{A} & -2a\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix}.$$

Denote the eigenvalues of $\mathbf{A}$ as $\gamma_1 < \gamma_2 < \gamma_3$ with corresponding eigenvectors $\mathbf{u}_1$ $\mathbf{u}_2$, and $\mathbf{u}_3$.

(a) What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{S}(a) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{A} & -2a\mathbf{I} \end{bmatrix}$$

in terms of the constant $a \geq 0$ and the eigenvalues and eigenvectors of $\mathbf{A}$? (Give symbolic values in terms of γ_1, γ_2, and γ_3.)

(b) For what values of $a \geq 0$ does the matrix $\mathbf{S}(a)$ have a double eigenvalue? What can you say about the eigenvectors associated with this double eigenvalue? (Give symbolic values in terms of γ_1, γ_2, and γ_3.)

(c) Produce a plot in MATLAB (or the program of your choice) superimposing the eigenvalues of $\mathbf{S}(a)$ for $a \in [0, 3]$. (To make this figure, plot the eigenvalues for many distinct values of $a \in [0, 3]$. Compare to Figure 1.4 in the notes.)

(d) What value of a minimizes the maximum real part of the eigenvalues? That is, find the $a \geq 0$ that minimizes the *spectral abscissa*

$$\alpha(\mathbf{S}(a)) := \max_{\lambda \in \sigma(\mathbf{S}(a))} \operatorname{Re} \lambda.$$

(e) [up to 5 bonus points]
Now suppose that you damp each mass differently, so that the damping component $-2a\mathbf{I}$ in $\mathbf{S}(a)$ is replaced by

$$\begin{bmatrix} -2a_1 & 0 & 0 \\ 0 & -2a_2 & 0 \\ 0 & 0 & -2a_3 \end{bmatrix}.$$

Determine values of a_1, a_2, and a_3 that give a smaller spectral abscissa than that given by the best constant in part (d).

(You do not need to compute exact eigenvalues: it is fine to use numerical calculations using `eig` in MATLAB. Credit will be awarded based on the degree by which you beat the best constant.)

7. Let $\mathbf{U}$ be a unitary matrix and let

$$\mathbf{P}_n := \frac{1}{n}(\mathbf{I} + \mathbf{U} + \mathbf{U}^2 + \cdots \mathbf{U}^{n-1}).$$

Prove the Ergodic Theorem:

$$\lim_{n \to \infty} \mathbf{P}_n = \mathbf{P},$$

where $\mathbf{P}$ denotes the orthogonal projector onto the space

$$\mathcal{V} = \{\mathbf{x} \in \mathbb{C}^n : \mathbf{U}\mathbf{x} = \mathbf{x}\}.$$

(If $\mathbf{U}$ has an eigenvalue $\lambda = 1$, then $\mathcal{V}$ is the associated invariant subspace.) [Halmos]

8. Let $\mathcal{U}$ and $\mathcal{V}$ be two nontrivial subspaces of $\mathbb{C}^n$. The *containment gap* between $\mathcal{U}$ and $\mathcal{V}$ is the sine of the largest angle between a vector in $\mathcal{U}$ and a vector in $\mathcal{V}$:

$$\delta(\mathcal{U},\mathcal{V}) := \max_{\mathbf{u}\in\mathcal{U}} \min_{\mathbf{v}\in\mathcal{V}} \frac{\|\mathbf{u}-\mathbf{v}\|}{\|\mathbf{u}\|}.$$

Prove that $\delta(\mathcal{U},\mathcal{V}) = \|(\mathbf{I}-\mathbf{P}_{\mathcal{V}})\mathbf{P}_{\mathcal{U}}\|$, where $\mathbf{P}_{\mathcal{U}}$ and $\mathbf{P}_{\mathcal{V}}$ denote the orthogonal projectors onto $\mathcal{U}$ and $\mathcal{V}$, respectively.

9. Given any $\mathbf{U},\mathbf{V} \in \mathbb{C}^{n\times k}$ such that $\mathbf{V}^*\mathbf{U} = \mathbf{I}$, the matrix

$$\mathbf{P} := \mathbf{U}\mathbf{V}^*$$

defines a projector onto $\mathcal{U} := \mathcal{R}(\mathbf{U})$ and along $\mathcal{V}^\perp = \mathcal{N}(\mathbf{V}^*)$, where $\mathcal{V} := \mathcal{R}(\mathbf{V})$. (That is, $\mathcal{R}(\mathbf{P}) = \mathcal{U}$ and $\mathcal{N}(\mathbf{P}) = \mathcal{V}^\perp$.) In Problem 8 we see that the containment gap between $\mathcal{U}$ and $\mathcal{V}$ is given by

$$\delta(\mathcal{U},\mathcal{V}) := \max_{\mathbf{u}\in\mathcal{U}} \min_{\mathbf{v}\in\mathcal{V}} \frac{\|\mathbf{u}-\mathbf{v}\|}{\|\mathbf{u}\|}.$$

Prove that, provided $\mathcal{U} \not\subseteq \mathcal{V}$,

$$\|\mathbf{P}\| = \frac{1}{\sqrt{1-\delta(\mathcal{U},\mathcal{V})^2}},$$

which is a generalization of Problem 1 to subspaces.