MATH 5524 · MATRIX THEORY

Problem Set 1

Posted Wednesday 25 January 2017. Due 5pm, Wednesday, 1 February 2017.

Complete any four problems, 25 points each.

You are welcome to complete more problems if you like. The latter problems are generally more challenging than the early problems. If you are already familiar with this material, please tackle the latter problems. If you submit more than four solutions, specify those four you would like to be graded.

1. Compute the resolvent $\mathbf{R}(z) = (z\mathbf{I} - \mathbf{A})^{-1}$ for the following three matrices.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   (These are simple enough to compute by hand.)

2. The structure of the resolvents in Problem 1 reveal a considerable amount about the sensitivity of the eigenvalues to small perturbations. We will develop the full theory later in the semester; for now, build your intuition by conducting the following computational experiment.

   For each matrix in Problem 1, compute the eigenvalues of $\mathbf{A} + \mathbf{E}$ for random matrices $\mathbf{E}$ having norm $10^{-15}, 10^{-12}, 10^{-9}, 10^{-6}, 10^{-3}, 10^{0}$. (For example, use `E = randn(3); E = 1e-15*E/norm(E)`.) Produce a `loglog` plot in MATLAB comparing the size of the perturbation, $\|\mathbf{E}\|$ (horizontal axis) versus the maximum amount the eigenvalues of $\mathbf{A}+\mathbf{E}$ differ from those of $\mathbf{A}$ (vertical axis). Your plot should have three lines, one for each of the three $\mathbf{A}$ matrices.

   Can you spot a relationship between the entries in your resolvents and the slopes the lines on your log-log plot?

3. For each of the following two matrices, compute (by hand) *two different* Schur factorizations (i.e., different values of $\mathbf{T}$).

$$\mathbf{A} = \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix}; \qquad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

   (You should have a total of four Schur factorizations for this problem, two for each matrix.)

4. (a) Suppose $\mathbf{A} \in \mathbb{C}^{n\times n}$ is a *normal* matrix (meaning $\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^*$) and has the Schur factorization $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^*$. Show that the upper triangular matrix $\mathbf{T}$ is actually *diagonal*, i.e., $t_{j,k} = 0$ if $j \neq k$.

   (b) What does this imply about the eigenvectors of $\mathbf{A}$?

   (c) Suppose $\mathbf{A}$ is a normal matrix and all of its eigenvalues are real. Show that $\mathbf{A}$ must be Hermitian, i.e., $\mathbf{A}^* = \mathbf{A}$.

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5. In class we will show that every square matrix has at least one eigenvalue. If you have some experience with linear algebra, this is not very surprising. However, you might find it more surprising that a slight change to the eigenvalue problem can lead to rather more exotic behavior.

   The differential equation $\mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ (which arises in a variety of engineering finite element problems, for example) gives rise to the *generalized eigenvalue problem* $\mathbf{A}\mathbf{u} = \lambda\mathbf{B}\mathbf{u}$.

   (a) Show that $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{u}$ is a solution to the differential equation $\mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ if and only if $\mathbf{A}\mathbf{u} = \lambda\mathbf{B}\mathbf{u}$.

   (b) For each of the following four problems, list all values of λ for which $\mathbf{A}\mathbf{u} = \lambda\mathbf{B}\mathbf{u}$ for some nonzero $\mathbf{u} \in \mathbb{C}^n$. The spectrum of this generalized eigenvalue problem is denoted $\sigma(\mathbf{A}, \mathbf{B})$.

   $$\text{(i) } \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \text{(ii) } \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

   $$\text{(iii) } \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{(iv) } \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

   (c) To appreciate the difference between cases (i) and (ii) in part (b), compute $\sigma(\mathbf{A}, \mathbf{B})$ for the pair

   $$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}$$

   with $\varepsilon \neq 0$. How does $\sigma(\mathbf{A}, \mathbf{B})$ evolve as $\varepsilon \to 0$?

6. More sophisticated dynamical systems give rise to still more subtle eigenvalue problems than those encountered in the last problem.

   Damped vibrating systems give second order differential equations of the form

   $$\mathbf{C}\mathbf{x}''(t) + \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t).$$

   (a) Show that $\mathrm{e}^{\lambda t}\mathbf{u}$ is a solution to $\mathbf{C}\mathbf{x}''(t) + \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ if and only if $\lambda^2\mathbf{C}\mathbf{u} + \lambda\mathbf{B}\mathbf{u} = \mathbf{A}\mathbf{u}$. (This is called a *quadratic eigenvalue problem*.)

   (b) Find *all* solutions $\lambda \in \mathbb{C}$ for which $\lambda^2\mathbf{C}\mathbf{u} + \lambda\mathbf{B}\mathbf{u} = \mathbf{A}\mathbf{u}$ for some nonzero $\mathbf{u} \in \mathbb{C}^n$, where

   $$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Are the eigenvectors $\mathbf{u}$ associated with distinct eigenvalues always linearly independent?

   In recent years there has been increasing interest in *delay differential equations*, the simplest example of which is $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t - s)$ for some $s > 0$.

   (c) Show that $\mathrm{e}^{\lambda t}\mathbf{u}$ is a solution to $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t - s)$ if and only if $\mathbf{A}\mathbf{u} = \lambda\mathrm{e}^{s\lambda}\mathbf{u}$.

   (d) This *nonlinear eigenvalue problem* is even interesting in the scalar case, $n = 1$. Consider

   $$\mathbf{A} = [\,1\,], \quad s = 1,$$

   for which $\mathbf{A}\mathbf{u} = \lambda\mathrm{e}^{s\lambda}\mathbf{u}$ reduces to the transcendental equation $1 = \lambda\mathrm{e}^{\lambda}$. This problem explores those values of λ that satisfy this equation.

   In MATLAB, type `help lambertw` to learn about Lambert's W function. Verify (computationally) that $\lambda_k = \texttt{lambertw}(k, 1)$ gives a solution of $\lambda\mathrm{e}^{\lambda} = 1$, for integer values of $k = -10, \ldots, 10$. Make a plot that shows λ_k for $k = -20, \ldots, 20$ (in the complex plane). How many distinct eigenvalues does this nonlinear eigenvalue problem have, even though $n = 1$?

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7. (a) Show that if $\mathbf{S}$ is skew-Hermitian (i.e., $\mathbf{S}^* = -\mathbf{S}$), then

$$\mathbf{Q} = (\mathbf{I} - \mathbf{S})(\mathbf{I} + \mathbf{S})^{-1}$$

is a unitary matrix. (You may use that $(\mathbf{A}^{-1})^* = (\mathbf{A}^*)^{-1}$ for any invertible matrix $\mathbf{A}$.)

(b) Show that all unitary matrices $\mathbf{Q} \in \mathbb{C}^{n \times n}$ are normal. What can be said of the eigenvalues of unitary matrices?

(c) Show that, if $\mathbf{Q}$ is a unitary matrix and -1 is *not* an eigenvalue of $\mathbf{Q}$, then there exists a skew-Hermitian matrix $\mathbf{S}$ such that

$$\mathbf{Q} = (\mathbf{I} - \mathbf{S})(\mathbf{I} + \mathbf{S})^{-1}.$$

[Stewart and Sun; Loewy (1898)]

8. A matrix $\mathbf{P} \in \mathbb{C}^{n \times n}$ is a *projector* provided $\mathbf{P}^2 = \mathbf{P}$. We say that such a $\mathbf{P}$ is an *orthogonal projector* provided that $\mathbf{P}$ is also Hermitian, $\mathbf{P}^* = \mathbf{P}$.

Prove that a nonzero projector $\mathbf{P}$ is an orthogonal projector if and only if $\|\mathbf{P}\| = 1$.
(The norm is the matrix 2-norm induced by the vector 2-norm.)

[Trefethen and Bau]