2

Approximation Theory

LECTURE 12: *Introduction to Approximation Theory*

INTERPOLATION IS AN INVALUABLE TOOL in numerical analysis: it provides an easy way to replace a complicated function by a polynomial (or piecewise polynomial), and, *at least as importantly*, it provides a mechanism for developing numerical algorithms for more sophisticated problems. Interpolation is not the only way to approximate a function, though: and indeed, we have seen that the quality of the approximation can depend perilously on the choice of interpolation points.

We saw one example in Section 1.7: finite difference formulas for approximating derivatives and solving differential equation boundary value problems. Several other applications will follow later in the semester.

If approximation is our goal, interpolation is only one means to that end. In this chapter we investigate alternative approaches that directly optimize the quality of the approximation. How do we measure this quality? That depends on the application. Perhaps the most natural means is to *minimize the maximum error* of the approximation.

> Given $f \in C[a,b]$, find $p_* \in \mathcal{P}_n$ such that
>
> $$\max_{x\in[a,b]} |f(x) - p_*(x)| = \min_{p\in\mathcal{P}_n} \max_{x\in[a,b]} |f(x) - p(x)|.$$

This is called the *minimax approximation problem*.

Norms clarify the notation. For any $g \in C[a,b]$, define

$$\|g\|_\infty := \max_{x\in[a,b]} |g(x)|,$$

the 'infinity norm of g'. One can show that $\|\cdot\|_\infty$ satisfies the basic norm axioms on the vector space $C[a,b]$ of continuous functions. Thus the minimax approximation problem seeks $p_* \in \mathcal{P}_n$ such that

$\|g\|_\infty \geq 0$ for all $g \in C[a,b]$
$\|g\|_\infty = 0 \iff g(x) = 0$ for all $x \in [a,b]$.
$\|\alpha g\|_\infty = |\alpha|\|g\|_\infty$ for all $\alpha \in \mathbb{C}$, $g \in C[a,b]$.
$\|g+h\|_\infty \leq \|g\|_\infty + \|h\|_\infty$, for all $g,h \in C[a,b]$.

$$\|f - p_*\|_\infty = \min_{p\in\mathcal{P}_n} \|f - p\|_\infty.$$

Notice that, for better or worse, this approximation will be heavily influenced by extreme values of $f(x)$, even if they occur over only a small range of $x \in [a, b]$.

Some applications call instead for an approximation that balances the size of the errors against the range of x values over which they are attained. In such cases it is most common to minimize the integral of the square of the error, the *least squares approximation problem*.

> Given $f \in C[a, b]$, find $p_* \in \mathcal{P}_n$ such that
>
> $$\Big(\int_a^b (f(x) - p_*(x))^2 \, \mathrm{d}x\Big)^{1/2} = \min_{p \in \mathcal{P}_n} \Big(\int_a^b (f(x) - p(x))^2 \, \mathrm{d}x\Big)^{1/2}.$$

This problem is often associated with *energy minimization* in mechanics, giving one motivation for its widespread appeal. As before, we express this more compactly by introducing the *two-norm* of $g \in [a, b]$:

$$\|g\|_2 = \Big(\int_a^b |g(x)|^2 \, \mathrm{d}x\Big)^{1/2},$$

so the least squares problem becomes

$$\|f - p_*\|_2 = \min_{p \in \mathcal{P}_n} \|f - p\|_2.$$

This chapter focuses on these two problems. Before attacking them we mention one other possibility, minimizing the absolute value of the integral of the error: the *least absolute deviations* problem.

This problem has become quite important in recent years. In particular, the analogous problem resulting when f is replaced by its vector discretization $\mathbf{f} \in \mathbb{C}^n$ plays a pivotal role in *compressive sensing*.

> Given $f \in C[a, b]$, find $p_* \in \mathcal{P}_n$ such that
>
> $$\int_a^b |f(x) - p_*(x)| \, \mathrm{d}x = \min_{p \in \mathcal{P}_n} \int_a^b |f(x) - p(x)| \, \mathrm{d}x.$$

With this problem we associate the *one-norm* of $g \in C[a, b]$,

$$\|g\|_1 = \int_a^b |g(x)| \, \mathrm{d}x,$$

giving the least absolute deviations problem as

$$\|f - p_*\|_1 = \min_{p \in \mathcal{P}_n} \|f - p\|_1.$$

2.1 Minimax Approximation: General Theory

The goal of minimizing the maximum error of a polynomial p from the function $f \in C[a,b]$ is called *minimax* (or *uniform*, or L^∞) approximation: Find $p_* \in \mathcal{P}_n$ such that

$$\|f - p_*\|_\infty = \min_{p \in \mathcal{P}_n} \|f - p\|_\infty.$$

Let us begin by connecting this problem to polynomial interpolation. On Problem Set 2 you were asked to prove that

$$\|f - \mathbf{\Pi}_n f\|_\infty \leq \big(1 + \|\mathbf{\Pi}_n\|_\infty\big)\|f - p_*\|_\infty, \tag{2.1}$$

where $\mathbf{\Pi}_n$ is the linear interpolation operator for

That is, $p = \mathbf{\Pi}_n f \in \mathcal{P}_n$ is the polynomial that interpolates f at $x_0, \ldots, x_n$.

$$x_0 < x_1 < \cdots < x_n$$

with $x_0, \ldots, x_n \in [a,b]$. Here $\|\mathbf{\Pi}_n\|_\infty$ is the *operator norm* of $\mathbf{\Pi}_n$:

$$\|\mathbf{\Pi}_n\|_\infty = \max_{f \in C[a,b]} \frac{\|\mathbf{\Pi}_n f\|_\infty}{\|f\|_\infty}$$

You further show that

$$\|\mathbf{\Pi}_n\| = \max_{x \in [a,b]} \sum_{j=0}^{n} |\ell_j(x)|,$$

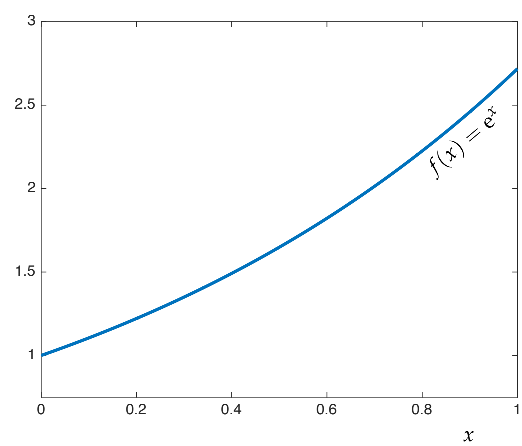
where ℓ_j denotes the jth Lagrange interpolation basis function

$$\ell_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k}.$$

Now appreciate the utility of bound (2.1): the linear interpolant $\mathbf{\Pi}_n f$ (*which is easy to compute*) is within a factor of $1 + \|\mathbf{\Pi}_n\|_\infty$ of the optimal approximation p_*. Note that $\|\mathbf{\Pi}_n\|_\infty \geq 1$: how much larger than one depends on the distribution of the interpolation points.

In the following sections we shall characterize and compute p_* (indeed more difficult than computing the interpolant), then use the theory of minimax approximation to find an excellent set of *almost fail-safe* interpolation points.

We begin by working out a simple example by hand.

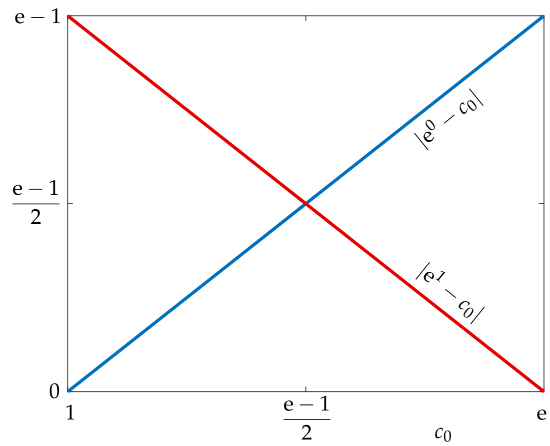


Example 2.1. Suppose we seek the constant that best approximates $f(x) = e^x$ over the interval $[0,1]$, shown in the margin. Before going on, sketch out a constant function (degree-0 polynomial) that approximates f in a manner that *minimizes the maximum error*.

Since $f(x)$ increases monotonically for $x \in [0,1]$, the optimal constant approximation $p_* = c_0$ must fall between $f(0) = 1$ and

$f(1) = e$, i.e., $1 \le c_0 \le e$. Moreover, since f is monotonic and p_* is a constant, the function $f - p_*$ is also monotonic, so the maximum error $\max_{x\in[a,b]} |f(x) - p_*(x)|$ must be attained at one of the end points, $x = 0$ or $x = 1$. Thus,

$$\|f - p_*\|_\infty = \max\{|e^0 - c_0|, |e^1 - c_0|\}.$$



The picture to the right shows $|e^0 - c_0|$ (blue) and $|e^1 - c_0|$ (red) for $c_0 \in [1, e]$. The optimal value for c_0 will be the point at which *the larger of these two lines is minimal*. The figure clearly reveals that this happens when the errors are equal, at $c_0 = (1 + e)/2$. We conclude that the optimal minimax constant polynomial approximation to e^x on $x \in [0, 1]$ is $p_*(x) = c_0 = (1 + e)/2$.

The plots in Figure 2.1 compare f to the optimal polynomial p_* (top), and show the error $f - p_*$ (bottom). We picked c_0 so that the error $f - p_*$ was equal in magnitude at the end points $x = 0$ and $x = 1$; in fact, it is equal in magnitude, but opposite in sign,

$$e^0 - c_0 = -(e^1 - c_0).$$

This property—maximal error being attained with alternating sign—is a key feature of minimax approximation.

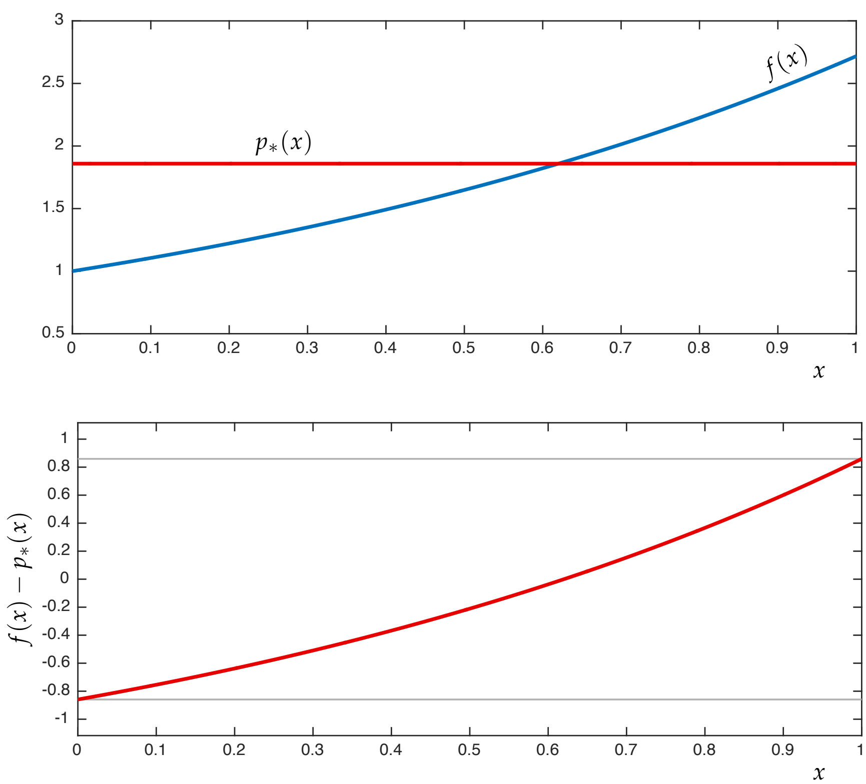


Figure 2.1: Minimax approximation of degree $k = 0$ to $f(x) = e^x$ on $x \in [0, 1]$. The top plot compares f to p_*; the bottom plot shows the error $f - p_*$, whose extreme magnitude is *attained, with opposite sign, at two values of $x \in [0, 1]$*.