Stability and Transient Dynamics for Linearized Reduced Order Models

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Nonlinear Model Reduction for Control
Blacksburg, Virginia

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Thanks to: US NSF DMS-0449973, DMS-1720257.
On Monday, Boris Kramer mentioned the simple model cf. [Kawano & Scherpen, 2017]

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} = \begin{bmatrix}
-1 & 1 \\
0 & -1
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + \begin{bmatrix}
-x_2(t)^2 \\
0
\end{bmatrix} + \begin{bmatrix}
1 \\
1
\end{bmatrix} u(t).
\]
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0
\end{bmatrix} + \begin{bmatrix}
1 \\
1
\end{bmatrix} u(t).
\]

Move the nonlinearity to the second component, and adjust the off-diagonal, and drop the input:

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} = \begin{bmatrix}
-1 & \gamma \\
0 & -1
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
x_1(t)^2
\end{bmatrix}.
\]

Note that \( x = \begin{bmatrix}
0 \\
0
\end{bmatrix} \) is a fixed point.

Is it stable?
Prelude

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x_1(t)^2
\end{bmatrix}.
\]

Note that \( x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) is a fixed point.

Is it stable?

Linear stability analysis: linearize about \( x = 0 \) to get \( \dot{\xi} = A\xi \),

\[
\begin{bmatrix}
\dot{\xi}_1(t) \\
\dot{\xi}_2(t)
\end{bmatrix} = \begin{bmatrix}
-1 & \gamma \\
0 & -1
\end{bmatrix} \begin{bmatrix}
\xi_1(t) \\
\xi_2(t)
\end{bmatrix},
\]

and note that \( A \) has negative eigenvalues: therefore, \( x = 0 \) is stable.

What is the basin of attraction? Does it depend on \( \gamma \)?

Let's use numerical simulations to assess the stability.
\begin{align*}
\gamma &= 0 \\
\end{align*}

blue: basin of attraction of stable fixed point \( x = 0 \)
blue: basin of attraction of stable fixed point $x = 0$
Prelude

\[ \gamma = 2 \]

\[ \blue{\text{blue: basin of attraction of stable fixed point } x = 0} \]
blue: basin of attraction of stable fixed point $x = 0$
Prelude

\[ \gamma = 8 \]

\[
\begin{array}{c}
\text{unstable} \\
\text{blue: basin of attraction of stable fixed point } x = 0
\end{array}
\]
blue: basin of attraction of stable fixed point $x = 0$
Prelude

\[ \gamma = 32 \]

\[ x_1(0) \]

\[ x_2(0) \]

\text{blue: basin of attraction of stable fixed point } x = 0
blue: basin of attraction of stable fixed point $x = 0$
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This transient linear + nonlinear coupling has been proposed as model for transition to turbulence in fluid mechanics.

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This transient linear + nonlinear coupling has been proposed as model for transition to turbulence in fluid mechanics.

The Mechanism Behind Transient Growth

Consider the (diagonalizable) example

\[ A = \begin{bmatrix} -1 & 0 \\ 100 & -2 \end{bmatrix} \]

with eigenvalues and (nearly aligned) eigenvectors

\[ \lambda_1 = -1, \quad v_1 = \begin{bmatrix} 1/100 \\ 1 \end{bmatrix}, \quad \lambda_2 = -2, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

Expand the initial condition in this basis (much cancellation):

\[ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 100 \begin{bmatrix} 1/100 \\ 1 \end{bmatrix} - 99 \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

Now evolve the system in time:

\[ x(t) = e^{tA}x(0) = 100e^{-t} \begin{bmatrix} 1/100 \\ 1 \end{bmatrix} - 99e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

The exponentials decay at different rates, breaking the cancellation.
The Mechanism Behind Transient Growth

Seven snapshots of the state vector

\[ x(t) = 100e^{-t} \begin{bmatrix} 1/100 \\ 1 \end{bmatrix} - 99e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

Any reduction of this system to an \( r = 1 \) dimensional system will miss this transient growth.
The Mechanism Behind Transient Growth

For $A \in \mathbb{C}^{n \times n}$, the \textit{numerical range} is the set

$$W(A) = \left\{ \frac{x^* Ax}{x^* x} : x \in \mathbb{C}^n \right\}.$$ 

- $W(A)$ is a closed, bounded, convex subset of $\mathbb{C}$ that contains the origin.
- If $A$ is normal, $W(A)$ is the \textit{convex hull of the spectrum}.
- If $A$ is Hermitian, $W(A) = [\lambda_{\text{min}}, \lambda_{\text{max}}] \subset \mathbb{R}.$

The \textit{numerical abscissa} is the rightmost point in $W(A)$:

$$\omega(A) = \max_{z \in W(A)} \text{Re}(z) = \lambda_{\text{max}} \left( \frac{A + A^*}{2} \right).$$
The Mechanism Behind Transient Growth

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Classical results from semigroup theory...
The Mechanism Behind Transient Growth

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Classical results from semigroup theory…

\textbf{Theorem (see, e.g., Trefethen & E. 2005, Part IV)}

$$\frac{d}{dt} \| e^{tA} \| \bigg|_{t=0} = \omega(A), \quad \| e^{tA} \| \leq e^{t\omega(A)}$$

- Solutions $e^{tA}x(0)$ to $\dot{x}(t) = Ax(t)$ can transiently grow only if $\omega(A) > 0$.
- Potentially $\omega(A) > 0$ even if all eigenvalues of $A$ are in the left-half plane.
Let $V \in \mathbb{C}^{n \times r}$ have orthonormal columns, $V^*V = I$.

To compute eigenvalues and to reduce models, we can restrict

$$A \in \mathbb{C}^{n \times n} \quad \text{down to} \quad V^*AV \in \mathbb{C}^{r \times r}.$$ 

For the bulk of this talk we focus on *Galerkin* projection of a SISO system

$$\dot{x}(t) = Ax(t) + bu(t)$$

$$y(t) = c^*x(t),$$

e.g., as generated by the *Arnoldi process* applied to $(A, b)$, or POD:

$$\dot{x}_r(t) = (V^*AV)x_r(t) + (V^*b)u(t)$$

$$y_r(t) = (c^*V)x_r(t).$$

- The eigenvalues of $V^*AV$ are in the numerical range $W(A)$:

  $$(V^*AV)\xi = \theta \xi \quad \implies \quad \frac{(V\xi)^*A(V\xi)}{(V\xi)^*(V\xi)} = \theta.$$ 

- When $A = A^*$, the *Cauchy Interlacing Theorem* describes precisely how the eigenvalues of $V^*AV$ distribute amongst the eigenvalues of $A$.

- For nonnormal $A$, very little is understood about the eigenvalues of $V^*AV$. 
Does there exist some notion of “interlacing” for non-Hermitian matrices?

Consider an extreme example:

\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \]

Repeat the following experiment many times:

- Generate random two dimensional subspaces, \( \mathcal{V} = \text{Ran} \, V \), where \( V^* V = I \).
- Form \( V^* A V \in \mathbb{C}^{2 \times 2} \) and compute its eigenvalues: \( \theta_1, \theta_2 \).
- Sort by real part: \( \text{Re} \, \theta_1 \geq \text{Re} \, \theta_2 \).
- Since \( A \) has eigenvalues \( \lambda_1 = \lambda_2 = 0 \), “interlacing” is meaningless here.
Eigenvalues of $V^*AV$

Eigenvalues of $V^*AV$ for random (complex) two dimensional subspaces

Black circle shows boundary of $W(A) = \{z \in \mathbb{C} : |z| \leq \sqrt{2}/2\}$
Denote the eigenvalues of the Hermitian part $\frac{1}{2}(A + A^*)$, labeled

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n.$$ 

**Theorem (Carden)**

Let $\theta_1, \ldots, \theta_r$ denote the eigenvalues of $V^*AV \in \mathbb{C}^{r \times r}$ for an $r < n$ dimensional subspace $\text{Range}(V)$, labeled by decreasing real part: $\text{Re} \theta_1 \geq \cdots \geq \text{Re} \theta_r$.

Then for $k = 1, \ldots, r$,

$$\frac{\mu_{n-r+k} + \cdots + \mu_n}{r - k + 1} \leq \text{Re} \theta_k \leq \frac{\mu_1 + \cdots + \mu_k}{k}.$$

- Ky Fan similarly bounded the real parts of the eigenvalues of $A$ [Fan 1950].
- The fact that $\theta_j \in W(A)$ gives the well-known bound

  $\mu_n \leq \text{Re} \theta_j \leq \mu_1, \quad j = 1, \ldots, r.$

The theorem provides sharper bounds for interior eigenvalues of $V^*AV$. 
Denote the eigenvalues of the Hermitian part \( \frac{1}{2}(A + A^*) \), labeled
\[ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n. \]

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Then for \( k = 1, \ldots, r, \)
\[
\frac{\mu_{n-r+k} + \cdots + \mu_n}{r - k + 1} \leq \text{Re} \theta_k \leq \frac{\mu_1 + \cdots + \mu_k}{k}.
\]

**Corollary (for Galerkin Model Reduction)**

If for some \( 1 \leq k \leq r, \)
\[ \mu_1 + \cdots + \mu_k < 0, \]
then \( V^*AV \) has no more than \( k - 1 \) eigenvalues in the right-half plane.
Bounds on the Number of Unstable Modes: Example

\[ A = \frac{1}{8} \begin{bmatrix} 
-10 & 32\varrho \\
1 & -10 & 32\varrho^2 \\
& & \ddots & \ddots \\
& & & -10 & 32\varrho^{n-1} \\
& & & & 1 & -10 
\end{bmatrix}, \quad \varrho = \frac{3}{4} \]

\[ n = 128 \]

\( A \) is stable, but \( W(A) \) extends into the RHP.

How many unstable modes can \( V^*AV \) have?
The containment regions for $\theta_k$ for $r = 8$ guarantee that $V_r^*AV_r$ has at most two unstable modes.
Two Matrices with Identical $W(A)$

Compute $r = 4$ eigenvalues of $V^*AV$ for these $8 \times 8$ matrices $A$:

\[
\begin{bmatrix}
0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\quad \gamma
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \varrho^1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \varrho^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \varrho^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \varrho^4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \varrho^5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \varrho^6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

(Choose $\gamma$ to give the same $W(A)$ for both examples; $\varrho = 1/8$.)

Smallest magnitude eigenvalue of $V^*AV$, 10,000 random complex subspaces.
Now sort the eigenvalues of $V^*AV$ by magnitude: $|\theta_1| \geq |\theta_2| \geq \cdots \geq |\theta_r|$.

For any $A \in \mathbb{C}^{n \times n}$, the product of eigenvalues is log-majorized by the product of singular values; see, e.g., [Marshall, Olkin, Arnold 2011]. Sort the eigenvalues and singular values of $A$ by magnitude, $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Then

$$\prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} \sigma_j.$$

**Theorem (Carden)**

Let $\theta_1, \dotsc, \theta_r$ denote the eigenvalues of $V^*AV \in \mathbb{C}^{r \times r}$ for an $r < n$ dimensional subspace $\text{Range}(V)$, labeled by decreasing magnitude: $|\theta_1| \geq \cdots \geq |\theta_r|$. Then for $k = 1, \dotsc, r$,

$$|\theta_k| \leq (\sigma_1 \cdots \sigma_k)^{1/k},$$

where $\sigma_1 \geq \cdots \geq \sigma_n$ are the singular values of $A$. 

Now sort the eigenvalues of $V^*AV$ by magnitude: $|\theta_1| \geq |\theta_2| \geq \cdots \geq |\theta_r|$.

### Theorem (Carden)

Let $\theta_1, \ldots, \theta_r$ denote the eigenvalues of $V^*AV \in \mathbb{C}^{r \times r}$ for an $r < n$ dimensional subspace $\text{Range}(V)$, labeled by decreasing magnitude: $|\theta_1| \geq \cdots \geq |\theta_r|$. Then for $k = 1, \ldots, r$,

$$|\theta_k| \leq (\sigma_1 \cdots \sigma_k)^{1/k},$$

where $\sigma_1 \geq \cdots \geq \sigma_n$ are the singular values of $A$.

### Corollary (for Galerkin Model Reduction)

If for some $1 \leq k \leq r$,

$$\sigma_1 \cdots \sigma_k < 1,$$

then $V^*AV$ has no more than $k - 1$ eigenvalues outside the unit disk.
Lid driven cavity fluid stability problem from IFISS [Elman, Ramage Silvester]. Q2-Q1 elements, $32 \times 32$ mesh, viscosity $\nu = 0.01$, dimension $n = 2178$.

We seek the rightmost eigenvalue of a generalized eigenvalue problem. Compute eigenvalues via shift-invert Arnoldi: $A_\gamma := (A - \gamma B)^{-1}B$.

We now seek the largest magnitude eigenvalue of $A_\gamma$.

finite eigenvalues of $A - \lambda B$

By the theorem, at least $r - 1$ eigenvalues of $V^*AV$ are located in the blue disk.
How many unstable modes can $V^*AV$ have when $A$ is stable?

**Theorem (Duintjer Tebbens & Meurant 2012)**

*Specify the following complex scalars:*

- $\lambda_1, \ldots, \lambda_n$;
- $\theta_1^{(1)}$;
- $\theta_1^{(2)}, \theta_2^{(2)}$;
- $\ldots$
- $\theta_1^{(n-1)}, \theta_2^{(n-1)}, \ldots, \theta_{n-1}^{(n-1)}$.

*There exists $A \in \mathbb{C}^{n \times n}$ and $b \in \mathbb{C}^n$ such that*

- $A$ has the specified eigenvalues: $\lambda_1, \ldots, \lambda_n$;
- $V_r^*AV_r$ has the specified eigenvalues: for $r = 1, \ldots, n-1$,

$\text{eigenvalues of } V_r^*AV_r = \{\theta_1^{(r)}, \ldots, \theta_r^{(r)}\}$

*when the columns of $V_r$ are an orthonormal basis for the Krylov subspace*

$\mathcal{K}_r(A, b) = \text{span}\{b, Ab, \ldots, A^{r-1}b\}$.

**IMPORTANT NOTE:**

This construction allows you to specify the eigenvalues of $A$, but you cannot specify $W(A)$.
Adversarial Construction for Galerkin Reduction

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -362880 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & -1451520 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 & -1693440 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 & -846720 \\ 1 & 5 & 0 & 0 & 0 & 0 & 0 & -211680 \\ 1 & 6 & 0 & 0 & 0 & 0 & 0 & -28224 \\ 1 & 7 & 0 & 0 & 0 & 0 & 0 & -2016 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 64 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

All modes of $$V_r^*AV_r$$ are unstable for $$1 \leq r < n.$$
Petrov–Galerkin Projection

\[ \dot{x}(t) = Ax(t) + bu(t) \]
\[ y(t) = c^*x(t). \]

Thus far we have focused on *Galerkin* projection, \( V^*_rAV_r \) with \( V^*_rV_r = I \),
e.g., as generated by the *Arnoldi process* applied to \((A, b)\).
The resulting model will match \( r \) moments of the transfer function at \( z = \infty \):

\[ \dot{x}_r(t) = (V^*_rAV_r)x_r(t) + (V^*_rb)u(t) \]
\[ y_r(t) = (c^*V_r)x_r(t). \]

We briefly consider *Petrov–Galerkin* projection, \( W^*_rAV_r \) with \( W^*_rV_r = I \),
e.g., as generated by the *bi-Lanczos process* applied to \((A, b, c)\).
The resulting model will match \( 2r \) moments of the transfer function at \( z = \infty \):

\[ \dot{x}_r(t) = (W^*_rAV_r)x_r(t) + (W^*_rb)u(t) \]
\[ y_r(t) = (c^*V_r)x_r(t). \]

What are the stability properties of this Petrov–Galerkin reduced order model?
Can $W^*AV$ have unstable modes when $A$ is stable?

**Theorem (Greenbaum 1998)**

Let $A \in \mathbb{C}^{n \times n}$, and suppose $1 \leq r \leq n/2$. Specify:

- $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$ and $\beta_1, \ldots, \beta_{r-1} \in \mathbb{C}$;
- nonzero starting vector, $b \in \mathbb{C}^n$ with $v_1 := b/\|b\|$;
- vectors $v_2, \ldots, v_{r+1}$ and scalars $\gamma_1, \ldots, \gamma_{r-1} \in \mathbb{C}$ generated by:

\[
\begin{align*}
\hat{v}_{j+1} & := Av_j - \alpha_j v_j - \beta_{j-1} v_{j-1} \\
\gamma_j & := \|\hat{v}_{j+1}\| \\
v_{j+1} & := \hat{v}_{j+1}/\gamma_j
\end{align*}
\]

- vector $c \perp \text{span}\{v_2, \ldots, v_{r+1}, Av_{r+1}, \ldots, A^{r-1}v_{r+1}\}$.

Then $r$ steps of the bi-Lanczos process applied to $(A, b, c)$ either breaks down, or generates

\[
W^rAV_r = \begin{bmatrix}
\alpha_1 & \beta_1 \\
\gamma_1 & \alpha_2 & \ddots \\
& \ddots & \ddots & \beta_{r-1} \\
& & \gamma_{r-1} & \alpha_r
\end{bmatrix}.
\]
Consider the \textit{Hermitian matrix} $A$ and the Greenbaum construction:

$$
A = \begin{bmatrix}
-2 & 1 & & & \\
1 & -2 & 1 & & \\
& 1 & -2 & 1 & \\
& & 1 & -2 & \ddots \\
& & & \ddots & 1 \\
& & & & 1 & -2 & 1 \\
& & & & & 1 & -2 \\
\end{bmatrix} \in \mathbb{C}^{16 \times 16}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \in \mathbb{C}^{16}.
$$

- $A = A^*$ has eigenvalues $\lambda_j = -2 + \cos(j\pi/17)$ in the left-half plane.
- $W(A) = [\lambda_n, \lambda_1]$ is also contained in the left-half plane.
- \textit{Any Galerkin projection of} $A$ \textit{will produce a stable reduced order model.}

Use Greenbaum's Theorem to construct

$$
W^*_r AV_r = \begin{bmatrix} 
+2 & & & 1 \\
& \gamma_1 & +2 & & \\
& & \ddots & \ddots & \\
& & & \ddots & 1 \\
& & & & \gamma_{r-1} & +2 \\
\end{bmatrix} \in \mathbb{C}^{r \times r}.
$$
Adversarial Construction for Petrov–Galerkin Reduction

\[ A = \begin{bmatrix} -2 & 1 & \ddots & \ddots & 1 \\ 1 & -2 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 1 & \ddots & \ddots & \ddots & -2 \\ 1 & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \in \mathbb{C}^{16 \times 16} \]

\[ W_r^* A V_r = \begin{bmatrix} +2 & 1 & \ddots & \ddots & \ddots \\ \gamma_1 & +2 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \gamma_{r-1} & \ddots & \ddots & \ddots & +2 \\ \end{bmatrix} \in \mathbb{C}^{r \times r} \]

All modes of \( W_r^* A V_r \) are unstable for \( 1 \leq r \leq n/2 \) here, despite the fact that \( A \) is a stable Hermitian matrix.
Unstable ROMs for stable systems are distasteful. One might go to lengths to *suppress the instability*; see, e.g., [Grimme, Sorensen, van Dooren 1995].

*However, an unstable ROM might better capture transient dynamics than a stabilized version.*

Boeing 767 example: stable linear system, $n = 55$; reduce to dimension $r = 20$ [Anderson, Ly, Liu 1990; Burke, Lewis, Overton 2003]
On the domain $x \in (0, \ell)$, $t > 0$, consider the nonlinear heat equation

$$u_t(x, t) = u_{xx}(x, t) + u_x(x, t) + \frac{1}{8} u(x, t) + u(x, t)^3,$$

with Dirichlet boundary conditions: $u(0, t) = u(\ell, t) = 0$.

[Sandsted & Scheel, 2005], [Galkowski, 2012] consider stability of this equation with small initial data, as a function of $\ell$.

We take $\ell = 30$ and $u_0(x) = 10^{-5} x(x - \ell)(x - \ell/2)$ and reduce to $r = 40$. 
Concluding Thoughts

▶ The interplay of linear transient growth and nonlinearity requires care.

▶ Reduction methods that preserve structure, nonlinearity, energy provide a major step in the right direction.

▶ Use a physically relevant inner product / norm.

*Eigenvalues (and the transfer function) are independent of the state-space representation, but \( W(\mathbf{A}) \) depends highly only the choice of coordinates. It is possible that \( W(\mathbf{A}) \) extends into the right-half plane in the Euclidean (vector) inner product, but not in the “energy inner product” motivated by the application.*

▶ We still have much to learn about the eigenvalues of \( \mathbf{V}^* \mathbf{A} \mathbf{V} \).

*Insight about these eigenvalues informs both model reduction and algorithms for solving large-scale eigenvalue problems.*


