CMDA 4604 · INTERMEDIATE TOPICS IN MATHEMATICAL MODELING

End-of-Term Project

1. Declare the topic of your project by meeting with the instructor by Friday November 13.
2. Final write-ups are due (by email) by Wednesday, December 9 at the latest.
   No late submissions will be accepted on this assignment.
3. Type your report (using LaTeX or your favorite word processor).
4. The report should be *at least* four pages, potentially including graphics.
5. You may work alone, or with one other student.
   (Two students are expected to produce a more extensive project than single students.)
6. The project will count double the weight of a normal 100-point problem set.
7. 80% of the project grade will be based on mathematical/scientific content.
   20% will be based on the quality of the exposition.
8. *Experimental projects* should carefully describe (a) the mathematical questions being investigated; (b) the experimental apparatus; (c) the data collected; (d) how the data fits mathematical expectations; (e) sources of error; (f) how more accurate results might be obtained.
9. *Theoretical/computational projects* should (a) tackle a topic beyond the scope of the lectures; (b) refer to at least one source in the literature (papers/books); (c) clearly describe the theoretical setting and assumptions; (d) provide illustrations (e.g., MATLAB plots, movies, etc.) of the concept in action.

Students are welcome to select one of the project ideas described below, or propose a different topic of their own design having similar scope. The instructor expects to interact with each student to provide additional background information and tips for interesting avenues to explore that align with their interest.

Experimental projects

E1. *Quantitative experiments with Chladni plates*
Experiment further with the instructor's Chladni plate hardware. Conduct *careful* measurements of the resonant frequencies (square roots of eigenvalues) of a given plate. Measure how these eigenvalues change with the thickness of the plate, size of the plate, and flaws in the plate. (A variety of plates are available for your study.)

Martin J. Gander and Felix Kwok, "Chladni Figures and the Tacoma Bridge: Motivating PDE Eigenvalue Problems via Vibrating Plates," *SIAM Review* 54 (2012) 573–596.
`http://dx.doi.org/10.1137/10081931X`

E2. *Implement a discrete heat equation in a resistor-capacitor circuit.*
The equations describing resistor-capacitor circuits (of the proper arrangement) lead to equations of the form $\mathbf{a}'(t) = \mathbf{K}\mathbf{a}(t)$, here $\mathbf{K}$ is the stiffness matrix from the finite element method. Thus, circuits can be used to simulate the heat equation. (Indeed such circuits were used in the oil industry in the 1950s as efficient computational devices for simulating the heat equation!) Implement such a circuit and conduct conduct careful measurements. How does the output compare to the expected solution of the heat equation?

For basics of resistor circuit modeling, see Chapter 2 of the CMDA 3606 notes.
`http://www.math.vt.edu/people/embree/cmda3606/chapter2.pdf`

A. F. Robertson and Daniel Gross, "An Electrical-Analog Method for Transient Heat Flow Analysis," *J. Res. Nat. Bureau Standards* 61 (1958) 105–115.
`http://nvlpubs.nist.gov/nistpubs/jres/61/jresv61n2p105_A1b.pdf`

E3. *Implement a damped wave equation in a resistor-inductor-capacitor circuit.*
A simple RLC (resistor-inductor-capacitor) circuit gives a second order linear differential equation that obeys the equation of damped simple harmonic motion ($x''(t) = -\kappa x(t) - 2dx'(t)$). Implement this circuit and conduct basic experiments with the damping constant d to illustrate weakly damped, critically damped, and over-damped systems. Then attempt to aggregate several such circuits to simulate the damped wave equation.

E4. *Careful laboratory studies of steady-state heat distribution.*
We would like to develop some experimental data from the heat equation to drive our uncertainty quantification studies. Attempt to reproduce the results of Smith's Example 3.5 measuring the temperature $u(x)$ at various points x along the length of bars made of aluminum and copper (or other metals). In this experiment, the temperature at one end of the bar is fixed, while the heat flux is set at the other end of the bar.

Ralph C. Smith, *Uncertainty Quantification: Theory, Implementation, and Algorithms*, SIAM, Philadelphia, 2014. `http://www4.ncsu.edu/~rsmith/UQ_TIA/`

E5. *Uncertainty quantification for a mass-spring-damper system.*
Construct a simple mass-spring-damper system. For a given spring, estimate the spring constant using Hooke's law based on loading the spring with different masses, then collect displacement-versus-time data for a vibrating system. Use least squares to recover values of the spring constant and damping parameter. Does the spring constant agree with what you measured using Hooke's Law?

Ralph C. Smith, *Uncertainty Quantification: Theory, Implementation, and Algorithms*, SIAM, Philadelphia, 2014. `http://www4.ncsu.edu/~rsmith/UQ_TIA/`

Theoretical/numerical projects

T1. *Vibrations of piano wires strings.*
Stiff strings, like those made out of wire used in pianos, obey a fourth order version of the wave equation having the form $u_{tt} = -\alpha u_{xxxx} + \beta u_{xx} + \gamma u_{xxt} - \delta u_t$, which we write as $u_{tt} = Lu$, with hinged boundary conditions. Compute the spectrum of the L when the parameters α, β, γ, δ are constant. Conduct simulations of these vibrations in Chebfun. How do they compare to solutions of the conventional wave equation? Do they *sound* different? Address the finite element modeling for this fourth-order problem.

Julien Bensa, Stefan Bilbao, Richard Kronland-Martinet, and Julius O. Smith III, "The simulation of piano string vibration: from physical models to finite difference schemes and digital waveguides," *J. Acoustical Soc. America* 114 (2003) 1095–1107.

T2. *Vibrations of spider webs.*
Spider webs can be modeled as strings that obey the wave equation, but have a special boundary condition where strings meet to enforce continuity of the network (the vibrating web does not become disconnected at the junctions between strings) and a force balance. Study the solution to the wave equation for simple webs and/or implement the finite element method to conduct mathematical models of more complicated vibrating structures.

E. J. P. Georg Schmidt, "On the modelling and exact controllability of networks of vibrating strings," *SIAM J. Control Optimization* 30 (229–245) 1992.
`http://dx.doi.org/10.1137/0330015`

For background, see the work of Fritz Vollraths's "Oxford Silk Group" in the Department of Zoology at Oxford University: `http://users.ox.ac.uk/~abrg/spider_site`

(If you are interested in this project, I can provide you with a more gentle introduction to simple problems than you will find in Schmidt's more general analysis.)

T3. *Optimal damping for vibrating strings.*
We have studied some elementary questions about the optimal constant δ to damp solutions to the wave equation $u_{tt} = u_{xx} - 2\delta u_t$. If you allow δ to vary, can you beat the optimal constant, $\delta = \pi$?

This question was open for some time before better damping parameters were found. Explore this question and conduct numerical experiments with spatially-varying damping. (Except for a few very special values of δ, you cannot compute the eigenvalues and eigenfunctions exactly.) How do you handle spatially-varying δ in the finite element method?

Steven J. Cox and Michael L. Overton, "Perturbing the critically damped wave equation," *SIAM J. Appl. Math.* 56 (1996) 1353–1362.
http://dx.doi.org/10.1137/S0036139994277403

Pedro Freitas, "Optimizing the rate of decay of solutions of the wave equation using genetic algorithms: a counterexample to the constant damping conjecture," *SIAM J. Control Optim.* 37 (1998) 376–387.
http://dx.doi.org/10.1137/S0363012997329445

Carlos Castro and Steven J. Cox, "Achieving arbitrarily large decay in the damped wave equation," *SIAM J. Control Optim.* 39 (2001) 1748–1755.
http://dx.doi.org/10.1137/S0363012900370971

T4. *Finite element methods for the advection–diffusion equation.*
The advection diffusion equation $-u'' + cu' = f$ has appeared on several problem sets. The underlying operator L defined by $Lu = -u'' + cu'$ is non-symmetric, which complicates the use of the spectral method; it also raises some issues for the finite element method. Study how to solve this equation using the standard Galerkin finite element method. How does the quality of the approximation relate to the mesh parameter N and the constant c?

B. Fischer, A. Ramage, D. J. Silvester, and A. J. Wathen, "On parameter choice and iterative convergence for stabilised discretizations of advection-diffusion problems," *Comp. Methods Appl. Mech. Eng.* 179 (1999) 179–195.
http://dx.doi.org/10.1016/S0045-7825(99)00037-7

T5. *Spectral stability of the advection–diffusion operator.*
A hint of the challenge raised by the advection–diffusion operator ($Lu = -u'' + cu$) comes from the sensitivity of the eigenvalues to small perturbations in the operator. One way to investigate this phenomenon comes from the *pseudospectra* of L, which consist of sets in the complex plane containing the eigenvalues of L and all "nearby" operators. Learn about pseudospectra of matrices and operators, describe the finite element discretization of the advection–diffusion eigenvalue problem, $-u'' + cu = \lambda u$, and compute some pseudospectra of these discretizations.

Lloyd N. Trefethen and Mark Embree, *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*, Princeton University Press, Princeton, 2005. (See especially Chapters 2, 5, 12.)
http://pup.princeton.edu/titles/8113.html

EigTool (MATLAB software for computing pseudospectra): https://github.com/eigtool/eigtool

T6. *Galerkin eigenvalue approximation.*
This semester we have focused on symmetric linear operators with constant coefficients, e.g., $L_0 u = -u''$ or, more generally, the shifted operator $L_c u = -u'' + cu$. How can one approximate the eigenvalues of an operator with variable coefficients, like $Lu = -u'' + w(x)u$? The theory of *Galerkin eigenvalue approximation* addresses this issue. Apply the Galerkin method to the eigenvalue equation $Lu = \lambda u$; your stiffness matrix will now have entries that depend on the variable coefficient $w(x)$; its entries can still be computed easily using Chebfun. You can show that the eigenvalues of the finite-dimensional Galerkin problem provide *upper bounds* on the exact eigenvalues. Compare the results you get from using hat functions in the Galerkin method, to using the exact eigenfunctions $\psi_n(x) = \sqrt{2}\sin(n\pi x)$ for the constant coefficient problem. How do your results change as the number of basis functions N in the Galerkin approximation increases?

Hans F. Weinberger. *Variational Methods for Eigenvalue Approximation*, SIAM, Philadelphia, 1974.
http://dx.doi.org/10.1137/1.9781611970531