Lecture 41: Recap for Final Exam

The exam will focus on material covered since the midterm, but the key concepts from the first half of the course are fair game:
e.g., best approximation, symmetry of linear operators, eigenvalues and eigenfunctions, etc.

Key ideas from second half of the course:

* Spectral Method for Time-Dependent Problems

e.g. $u_t = -Lu + f, \quad u_{tt} = -Lu + f$

① Write solution as an eigenfunction expansion:

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\,\psi_j(x).$$

② Substitute this form into the PDE

e.g., $$u_t = -Lu + f \Rightarrow \sum_{j=1}^{\infty} a_j'(t)\psi_j(x) = -\sum_{j=1}^{\infty} a_j(t)\,L\psi_j(x) + f(x,t)$$

$$\Rightarrow \sum_{j=1}^{\infty} a_j'(t)\psi_j(x) = \sum_{j=1}^{\infty}(-\lambda_j a_j(t))\psi_j(x) + f(x,t).$$

③ Take the inner product with ψ_k and use orthogonality of eigenfunctions:

$$\sum_{j=1}^{\infty} a_j'(t)\underbrace{(\psi_j,\psi_k)}_{=0 \text{ if } j\neq k} = \sum_{j=1}^{\infty}(-\lambda_j a_j(t))\underbrace{(\psi_j,\psi_k)}_{=0 \text{ if } j\neq k} + (f(\cdot,t),\psi_k)$$

$$\Rightarrow \quad a_k'(t) = -\lambda_k a_k(t) + \underbrace{\frac{(f(\cdot,t),\psi_k)}{(\psi_k,\psi_k)}}_{= c_k(t)}$$

$$\Rightarrow \quad \boxed{a_k'(t) = -\lambda_k a_k(t) + c_k(t)}$$ ODE for each coefficient $a_k(t)$. $k=1,2,\ldots$

INITIAL CONDITION COMES FROM THE INITIAL CONDITION FOR THE PDE:

$$a_k(0) = \frac{(u_0,\psi_k)}{(\psi_k,\psi_k)}.$$

(4) SOLVE THE SCALAR ODES:

e.g., $$\boxed{a_k(t) = e^{-\lambda_k t} a_k(0) + \int_0^t e^{\lambda_k(s-t)} c_k(s)\, ds}$$

(5) ASSEMBLE THE SOLUTION

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\,\psi_j(x).$$

THEN WE MIGHT ALSO ADD IN A PRELIMINARY STEP TO HANDLE INHOMOGENEOUS BOUNDARY CONDITIONS.

(0) HANDLE INHOMOGENEOUS BOUNDARY CONDITIONS.

WRITE $u(x,t) = v(x,t) + w(x,t)$

WHERE $w(x,t)$ HANDLES THE BOUNDARY CONDITIONS AND $v(x,t)$ SATISFIES A RELATED PDE WITH HOMOGENEOUS BOUNDARY CONDITIONS.

NOTE: ADJUST $u_0(x)$, $f(x,t)$ TO COMPENSATE FOR $w(x,t)$.

* FINITE ELEMENT METHODS FOR TIME DEPENDENT PROBLEMS

① DERIVE A WEAK FORM OF THE PDE

e.g. $u_t = u_{xx} + f \Rightarrow \frac{\partial}{\partial t}(u,v) = -a(u,v) + (f,v) \quad \forall v \in V$

$u_{tt} = u_{xx} + f \Rightarrow \frac{\partial^2}{\partial t^2}(u,v) = -a(u,v) + (f,v) \quad \forall v \in V$

WHERE V IS A SUITABLE SPACE OF TEST FUNCTIONS

(RECALL ESSENTIAL VS NATURAL BOUNDARY CONDITIONS)

② GALERKIN APPROXIMATION: RESTRICT V TO A SUBSPACE

$V_N = \text{span}\{\phi_1, \dots, \phi_N\}$: e.g., for $u_t = u_{xx} + f$, seek

$u_N = \sum_{j=1}^{N} a_j(t)\,\phi_j(x)$ SUCH THAT

$$\frac{\partial}{\partial t}(u_N, v) = -a(u_N, v) + (f,v) \quad \forall v \in V_N.$$

③ DERIVE A LINEAR ALGEBRA PROBLEM:

IMPOSE WEAK FORM ON $\phi_1, \dots, \phi_N$ TO GET, E.G.,

$$\begin{bmatrix} & & \\ & (\phi_j, \phi_k) & \\ & & \end{bmatrix} \begin{bmatrix} a_1'(t) \\ \vdots \\ a_N'(t) \end{bmatrix} = -\begin{bmatrix} & & \\ & a(\phi_j, \phi_k) & \\ & & \end{bmatrix} \begin{bmatrix} a_1(t) \\ \vdots \\ a_N(t) \end{bmatrix} + \begin{bmatrix} (f, \phi_1) \\ \vdots \\ (f, \phi_N) \end{bmatrix}$$

Initial data $a(0)$ comes from the initial condition, u_0.

$$M a'(t) = -K a(t) + f(t)$$

④a SOLVE THE LINEAR ALGEBRA PROBLEM <u>EXACTLY</u> (IN TIME)

$$a(t) = e^{-M^{-1}Kt} a(0) + \int_0^t e^{M^{-1}K(s-t)} f(s)\, ds$$

MATRIX EXPONENTIAL: KNOW BASIC PROPERTIES, E.G.,

$$e^{tA} = I + tA + \frac{t^2 A^2}{2} + \frac{t^3 A^3}{3!} + \cdots$$

Recall $e^{tA} \to 0$ as $t \to 0$ if and only if $Re(\lambda) < 0$ for all eigenvalues λ of A.

(4b) SOLVE THE LINEAR ALGEBRA PROBLEM APPROXIMATELY FOR THE GENERIC PROBLEM $y'(t) = Ay(t)$ (IN TIME)

FORWARD EULER: $y_{k+1} = y_k + \Delta t\, A y_k$

BACKWARD EULER: $y_{k+1} = y_k + \Delta t\, A y_{k+1}$

$$\Longrightarrow \quad y_{k+1} = (I - \Delta t A)^{-1} y_k$$

**UNDERSTAND STABILITY CONSIDERATIONS FOR THESE METHODS: EG. FOR THE HEAT EQUATION, BACKWARD EULER IS STABLE FOR ALL $\Delta t > 0$, BUT FORWARD EULER IS MORE RESTRICTIVE: DOUBLE N, QUARTER Δt. THIS IS THE "CFL CONDITION".

* UNCERTAINTY QUANTIFICATION

IF THIS APPEARS ON THE TEST, IT WILL BE AT THE CONCEPTUAL LEVEL — NO DETAILED CALCULATIONS

- FREQUENTIST PERSPECTIVE
  - LEAST SQUARES GIVES AN UNBIASED ESTIMATOR
  - WE CAN ALSO GET AN UNBIASED ESTIMATOR FOR THE VARIANCE OF THE NOISE
- BAYESIAN PERSPECTIVE
  - WE GET A DISTRIBUTION FOR THE QoIs INFORMED BY OBSERVATIONS.