Lecture 40: Bayesian Approach to Parameter Estimation

Having seen the frequentist approach to parameter estimation, we now consider basic Bayesian techniques; see Smith, Section 4.8 and Chapter 8.

Bayes Theorem

Let "A" and "B" denote two events

$P(A)$ = Probability event A occurs $\in [0,1]$

$P(B)$ = Probability event B occurs $\in [0,1]$

$P(A|B)$ = Probability A occurs, given that B occurs

$P(B|A)$ = Probability B occurs, given that A occurs

Probability of A and B is given by

$$P(A) \times P(B|A)$$

or, equivalently,

$$P(B) \times P(A|B).$$

Hence, we can equate these expressions:

$$P(A)\, P(B|A) = P(B)\, P(A|B)$$

From which follows Bayes Theorem:

$$\boxed{P(A|B) = \frac{P(B|A)\, P(A)}{P(B)}}.$$

SIMILARLY, WE BAYES THEOREM FOR PROBABILITY DENSITIES:

WE HAVE A GENERAL MODEL

$$y = f(q)$$

AND WE WISH TO ESTIMATE q FROM OBSERVATIONS y.

LET $\pi(y)$ = PROBABILITY DENSITY FOR y

$\pi_0(q)$ = PROBABILITY DENSITY FOR q ("PRIOR")

$\pi(y|q)$ = PROBABILITY DENSITY OF $\cancel{y}$, GIVEN q

$\pi(q|y)$ = PROBABILITY DENSITY OF q, GIVEN y (THE "POSTERIOR")] OUR GOAL

BAYES THEOREM GIVES

$$\pi(q|y) = \frac{\pi(y|q)\,\pi_0(q)}{\pi(y)}$$

TYPICALLY WE CANNOT EASILY COMPUTE $\pi(y)$, BUT WE CAN INSTEAD NORMALIZE (SO $\pi(q|y)$ IS A PROBABILITY DENSITY: $\int \pi(q|y)\,dq = 1$

$$\pi(q|y) = \frac{\pi(y|q)\,\pi_0(q)}{\int_{q\in Q} \pi(y|q)\,\pi_0(q)\,dq}$$

WHERE Q IS THE SET OF VALUES q CAN TAKE.

(IN PRACTICE THIS COULD BE A HIGH-DIMENSIONAL INTEGRAL — NUMERICALLY INTEGRATE VIA "SPARSE GRIDS" OR MONTE CARLO TECHNIQUES

NOW, IT AMOUNTS TO CHARACTERIZE $\pi(y|q)$ AND $\pi_0(q)$. AN EXAMPLE (FROM SMITH, SECTION 4.8) HELPS EXPLAIN THE DETAILS.

SUPPOSE WE OBSERVE SOME SET OF COIN TOSSES, E.G.

$$\underbrace{HTTTHTHHTT}_{N \text{ TOSSES}}$$

FROM THESE OBSERVATIONS, I SEEK TO ESTIMATE THE PROBABILITY q OF GETTING HEADS ON A SINGLE COIN TOSS.

NOW <u>GIVEN</u> A VALUE OF q and

N_0 = # OF OBSERVED TAILS IN OUR N TOSSES

N_1 = # OF OBSERVED HEADS IN OUR N TOSSES

WE CAN COMPUTE THE PROBABILITY OF HAVING MADE THESE OBSERVATIONS!

$$\pi(y|q) = q^{N_1}(1-q)^{N_0}$$

NOTE: THIS IS A <u>FUNCTION OF q</u>.

NOW FOR $\pi_0(q)$ WE MIGHT USE THE "UNIFORMED PRIOR"

$$\pi_0(q) = 1$$

MEANING THAT WE HAVE NO SPECIAL PREVIOUS INSIGHT ABOUT q.

IF WE HAD SOME PRIOR KNOWLEDGE (E.G., BASED ON PAST STUDIES OF COINS MADE BY THE SAME MANUFACTURER) WE CAN USE SOMETHING MORE SOPHISTICATED, LIKE THE NORMAL DISTRIBUTION

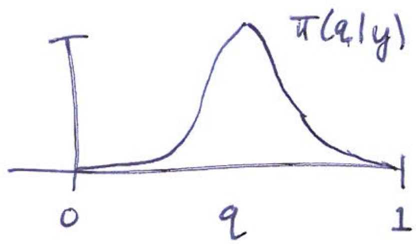
$$\pi_0(q) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(q-\mu)^2}{2\sigma^2}}$$

OF MEAN $\mu \in [0,1]$ AND VARIANCE σ^2. (NOTE THAT WE TRUNCATE THIS TO POSSIBLE VALUES $q \in [0,1]$.)

THEN BAYES THEOREM GIVES

$$\pi(q|y) = \frac{\pi(y|q)\,\pi_0(q)}{\int_0^1 \pi(y|q)\,\pi_0(q)\,dq}$$

WHICH IS A FUNCTION OF q.



the desired distribution.

SEE COINTOSS1.m (UNIFORMED PRIOR)
COINTOSS2.m (POOR CHOICE OF PRIOR)

ON THE CLASS WEBSITE.