Lecture 34: Finite Elements for the Wave Equation

11: Approximate Solution of the Wave Equation

Given our experience solving the heat equation with finite elements, the wave equation will be easy to handle. We follow the usual steps.

① Derive a weak form for the equation.

Begin with
$$u_{tt}(x,t) = u_{xx}(x,t) + f(x,t).$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x).$$

Multiply the PDE by a test function $v \in C_D^2[0,1]$ and integrate over the spatial domain, i.e., take the inner product of each side of the PDE with v:

$$(u_{tt}, v) = (u_{xx}, v) + (f, v)$$

$$\int_0^1 u_{tt}(x,t)\,v(x)\,dx = \int_0^1 u_{xx}(x,t)\,v(x)\,dx + \int_0^1 f(x)\,v(x)\,dx$$

$$= \underbrace{\Big[u_x(x,t)\,v(x)\Big]_0^1}_{\left(=0 \text{ since } v(0) = v(1) = 0\right)} - \int_0^1 u_x(x,t)\,v_x(x)\,dx + \int_0^1 f(x)\,v(x)\,dx$$

WEAK FORM OF THE PDE:

$$\int_0^1 u_{tt}(x,t)\,v(x) = -\int_0^1 u_x(x,t)\,v_x(x)\,dx + \int_0^1 f(x)\,v(x)\,dx$$

$$(u_{tt}, v) = -a(u,v) + (f,v)$$

where $a(\cdot,\cdot)$ is the energy inner product,

$$a(u,v) = \int_0^1 u_x(x,t)\,v_x(x)\,dx.$$

② GALERKIN APPROXIMATION

Since we will struggle to find u by testing the PDE against all $v \in C_0^2[0,1]$, we instead seek

$$u_N(x,t) \in \underbrace{V_N}_{\text{spatial component}} \times \underbrace{C[0,\infty)}_{\text{time component}}$$

for an N-dimensional subspace $V_N = \operatorname{span}\{\phi_1, \dots, \phi_N\}$, that satisfies the weak form for test functions in V_N:

$$(*) \quad \left(\frac{\partial^2}{\partial t^2} u_N, v\right) = -a(u_N, v) + (f, v) \quad \text{for all } v \in V_N.$$

③ LINEAR ALGEBRA PROBLEM

THE GALERKIN CONDITION (*) IS SATISFIED IF IT HOLDS FOR ALL VECTORS $\phi_1, \dots, \phi_N$ IN A BASIS FOR V_N.

WE WRITE $u_N \in V_N \times C[0,\infty)$ IN THIS BASIS:

$$u_N(x,t) = \sum_{j=1}^{N} a_j(t)\, \phi_j(x)$$

and then require

$$\left(\frac{\partial^2}{\partial t^2} u_N, \phi_k\right) = -a(u_N, \phi_k) + (f, \phi_k) \quad k = 1, \dots, N.$$

EXPANDING:

$$\left(\frac{\partial^2}{\partial t^2} \sum_{j=1}^{N} a_j(t)\phi_j(x), \phi_k(x)\right) = -a\left(\sum_{j=1}^{N} a_j(t)\phi_j(x), \phi_k(x)\right) + (f, \phi_k)$$

$$\Rightarrow \boxed{\sum_{j=1}^{N} a_j''(t)\,(\phi_j, \phi_k) = -\sum_{j=1}^{N} a_j(t)\, a(\phi_j, \phi_k) + (f, \phi_k) \quad \text{for } k = 1, \dots, N.}$$

$$k=1: \quad \sum_{j=1}^{N} a_j''(t)(\phi_j, \phi_1) = -\sum_{j=1}^{N} a_j(t)\, a(\phi_j, \phi_1) + (f, \phi_1)$$

$$\vdots$$

$$k=N \quad \sum_{j=1}^{N} a_j''(t)(\phi_j, \phi_N) = -\sum_{j=1}^{N} a_j(t)\, a(\phi_j, \phi_N) + (f, \phi_N)$$

ARRANGING IN MATRIX FORM:

$$\underbrace{\begin{bmatrix} (\phi_1,\phi_1) & \cdots & (\phi_N,\phi_1) \\ \vdots & & \vdots \\ (\phi_1,\phi_N) & \cdots & (\phi_N,\phi_N) \end{bmatrix}}_{\text{MASS MATRIX}} \begin{bmatrix} a_1''(t) \\ \vdots \\ a_N''(t) \end{bmatrix} = -\underbrace{\begin{bmatrix} a(\phi_1,\phi_1) & \cdots & a(\phi_N,\phi_1) \\ \vdots & & \\ a(\phi_1,\phi_N) & \cdots & a(\phi_N,\phi_N) \end{bmatrix}}_{\text{STIFFNESS MATRIX}} \begin{bmatrix} a_1(t) \\ \vdots \\ a_N(t) \end{bmatrix} + \underbrace{\begin{bmatrix} (f,\phi_1) \\ \vdots \\ (f,\phi_N) \end{bmatrix}}_{\text{LOAD VECTOR}}$$

WE WRITE THIS AS:

$$M a''(t) = -K a(t) + f(t)$$

i.e.,

$$\boxed{a''(t) = -M^{-1} K a(t) + M^{-1} f(t)}$$

As in the previous lecture, we shall write this second order equation as a first order system:

$$\begin{bmatrix} a'(t) \\ a''(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}}_{} \begin{bmatrix} a'(t) \\ a''(t) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1} f(t) \end{bmatrix}$$

THIS IS A $2N \times 2N$ MATRIX:
A 2×2 "BLOCK MATRIX" WITH
$N \times N$ BLOCKS.
NOTE HOW NICELY THIS PARALLELS THE OPERATOR VERSION IN THE LAST LECTURE!!

④ Solve the Linear algebra differential equation. 34.4

First, consider the exact solution in time:

$$\begin{bmatrix} a(t) \\ a'(t) \end{bmatrix} = e^{t\begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}} \begin{bmatrix} a(0) \\ a'(0) \end{bmatrix}$$

$$+ \int_0^t e^{(t-s)\begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}} f(s)\, ds$$

What should we use for the initial conditions?

Initial position $\quad U_0(x) = a(x,0)$
Initial velocity $\quad V_0(x) = a_t(x,0)$

If our finite element basis is made up of hat functions with

$$\phi_j(x_k) = \begin{cases} 1, & j=k; \\ 0, & j \neq k; \end{cases}$$

then we can set $a(0)$ and $a'(0)$ so the solution

$U_N(x,0) = U_0(x)$ when x is a grid point:

$$U_0(x_k) = U_N(x_k,0) = \sum_{j=1}^{N} a_j(0) \underbrace{\phi_j(x_k)}_{=0 \text{ or } 1}$$

$$= a_k(0)$$

and similarly for velocity:

$$\frac{\partial}{\partial t} U_N(x,0) = V_0(x) = \sum_{j=1}^{N} a_j'(0) \underbrace{\phi_j(x_k)}_{=0 \text{ or } 1} .$$

$$= a_k'(0)$$

So we set

$$a(0) = \begin{bmatrix} a_1(0) \\ \vdots \\ a_N(0) \end{bmatrix} \qquad a'(0) = \begin{bmatrix} a_1'(0) \\ \vdots \\ a_N'(0) \end{bmatrix}$$

Next, consider the approximate solution of the linear algebra differential equations in time: 34.

To simplify the notation, let

$$y(t) = \begin{bmatrix} \partial(t) \\ \partial'(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}, \quad g(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

Then

$$y_k \simeq y(k\Delta t), \quad g_k = g(k\Delta t)$$

for a time step Δt. We can advance y_k via:

$$y_{k+1} = y_k + \Delta t \left(A y_k + g_k\right) \qquad \text{FORWARD EULER}$$

or

$$y_{k+1} = y_k + \Delta t \left(A y_{k+1} + g_{k+1}\right) \qquad \text{BACKWARD EULER}$$

THE BACKWARD EULER METHOD IS AN IMPLICIT METHOD (just as for the heat equation), so we find y_{k+1} via

$$(I - \Delta t A)\, y_{k+1} = y_k + \Delta t\, g_{k+1}$$

$$\Longrightarrow \quad y_{k+1} = (I - \Delta t A)^{-1} \left(y_k + \Delta t\, g_{k+1}\right) \qquad \text{BACKWARD EULER.}$$

OF COURSE, THESE ARE JUST TWO OF MANY OPTIONS.
HOW WILL THEY BEHAVE AS $k \to \infty$?
That will depend on properties of the method, and the eigenvalues of $A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}$ .....