Lecture 33: Wave Equation as a First-Order System

In this lecture we approach the wave equation

$$u_{tt}(x,t) = u_{xx}(x,t)$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = u_0(x)$$
$$u_t(x,0) = v_0(x)$$

from a different perspective. Introduce the variable

$$v(x,t) = u_t(x,t).$$

Now we will write

$$u_t = v$$

$$u_{tt} = v_t = u_{xx}$$

as the first order system

$$\begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} v \\ u_{xx} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

A matrix whose entries are themselves linear operators.

In summary

$$\begin{bmatrix} u \\ v \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix}}_{A} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$A : C_D^2[0,1] \times C_D^2[0,1] \to C[0,1] \times C[0,1]$$

This means that $u \in C_D^2[0,1]$ and $v \in C_D^2[0,1]$.

Does it make sense to require $v \in C_0^2[0,1]$, i.e.,

$$v(0,t) = v(1,t) = 0 \ ?$$

Yes: $u(0,t) = u(1,t) = 0$ for all t, so

$$v(0,t) = \frac{\partial u}{\partial t}(0,t) = 0, \quad v(1,t) = \frac{\partial u}{\partial t}(1,t) = 0.$$

Note that we can also write the PDE as

$$\begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} 0 & I \\ -L & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

where $L : C_0^2[0,1] \to C[0,1]$ is the usual Laplace operator

$$Lu = -u_{xx}$$

with eigenvalues $\lambda_j = j^2\pi^2$

and eigenfunctions $\psi_j(x) = \sin(j\pi x)$

(we drop the usual $\sqrt{2}$ normalization here for simplicity.)

Energy Inner Product

One of the reasons to study this approach to the wave equation: it shows the importance of using an inner product that is informed by the physics of the problem.

Consider this inner product on $C_0^2[0,1] \times C_0^2[0,1]$:

$$\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix} \right) = \int_0^1 u'(x) \overline{f'(x)} \, dx + \int_0^1 v(x) \overline{g(x)} \, dx,$$

note derivatives here (with respect to x.)

We need complex conjugates here because we will see complex components arise shortly....

With this inner product we associate the <u>energy norm</u>

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\| = \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right)^{1/2}$$

$$= \left(\int_0^1 u'(x)\overline{u'(x)} + v(x)\overline{v(x)}\, dx \right)^{1/2}$$

$$= \left(\int_0^1 |u'(x)|^2 + |v(x)|^2\, dx \right)^{1/2}$$

So that

$$\left\| \begin{bmatrix} u \\ u_t \end{bmatrix} \right\|^2 = \int_0^1 \underbrace{|u_x(x,t)|^2}_{\text{potential energy}} + \underbrace{|u_t(x,t)|^2}_{\text{kinetic energy}}\, dx$$

represents the energy in the vibrating string at time t. (Note that the terms "energy inner product" and "energy norm" mean something different here than they did in the finite element context — but the underlying motivation for the name is similar.)

With this inner product, it is natural to ask:

Is $A = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix}$ symmetric? This would imply orthogonal eigenfunctions and real eigenvalues...

If A is symmetric, we would have

$$\left(\begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix} \right) = \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \right)$$

for all $\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix} \in C_0^2[0,1] \times C_0^2[0,1]$. We compute

$$\left(\begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix} \right) = \left(\begin{bmatrix} v \\ u'' \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \right)$$

$$(*) \qquad = \int_0^1 v'(x) \overline{f'(x)}\, dx + \int_0^1 u''(x) \overline{g(x)}\, dx$$

$$\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \right) = \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} g \\ f'' \end{bmatrix} \right)$$

$$(**) \qquad = \int_0^1 u'(x) \overline{g'(x)}\, dx + \int_0^1 v(x) \overline{f''(x)}\, dx$$

To prove symmetry, we would need to show that $(*) = (**)$.

Integrate each term in $(*)$ by parts:

$$(*) = \underbrace{\left[v(x) \overline{f'(x)} \right]_{x=0}^1}_{\substack{=0 \text{ since } v \in C_0^2[0,1], \\ v(0)=v(1)=0}} - \int_0^1 v(x) \overline{f''(x)}\, dx + \underbrace{\left[u'(x) \overline{g(x)} \right]_{x=0}^1}_{\substack{=0 \text{ since } g \in C_0^2[0,1] \\ g(0)=g(1)=0.}} - \int_0^1 u'(x) \overline{g'(x)}\, dx$$

$$= -\int_0^1 v(x) \overline{f''(x)}\, dx - \int_0^1 u'(x) \overline{g'(x)}\, dx = - (**)$$

note minus sign!

$$\Rightarrow \left(\begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix} \right) = \left(\begin{bmatrix} u \\ v \end{bmatrix}, - \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \right)$$

So $\left(A\begin{bmatrix}u\\v\end{bmatrix}, \begin{bmatrix}f\\g\end{bmatrix}\right) = \left(\begin{bmatrix}u\\v\end{bmatrix}, -A\begin{bmatrix}f\\g\end{bmatrix}\right)$

We say that A is <u>skew-symmetric</u>.

Eigenvalues of <u>skew-symmetric</u> operators must be <u>purely imaginary</u>:

If A is skew-symmetric and $A\phi = \lambda\phi$ with $(\phi,\phi) = 1$, then

$$\lambda = \lambda\underbrace{(\phi,\phi)}_{=1} = (\lambda\phi, \phi) = (A\phi, \phi) = (\phi, -A\phi)$$
$$= (\phi, -\lambda\phi) = \overline{(-\lambda\phi, \phi)} = -\overline{\lambda}\underbrace{(\phi,\phi)}_{=1} = -\overline{\lambda}.$$

If $\lambda = \alpha + i\beta$, then $\lambda = -\overline{\lambda}$ means:

$$\alpha + i\beta = -\overline{(\alpha+i\beta)} = -(\alpha - i\beta) = -\alpha + i\beta$$

So $\alpha = -\alpha \Rightarrow \alpha = 0$

Thus $\lambda = i\beta$, purely imaginary.

(This is a simple modification of the earlier proof that eigenvalues of symmetric operators are real.)

One can similarly verify the eigenfunctions of a skew-symmetric operator associated with distinct eigenvalues must be orthogonal. (We will prove it in a special case momentarily.)

What are the eigenvalues and eigenfunctions of $A = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix}$?

$$A \begin{bmatrix} u \\ v \end{bmatrix} = \gamma \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \gamma \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\Rightarrow \quad v = \gamma u$$

$$u'' = \gamma v = \gamma^2 u \Rightarrow \boxed{-u'' = -\gamma^2 u}$$

Recall $Lu = -u''$ for $u \in C_0^2[0,1]$, with L having eigenvalues and eigenfunctions $\lambda_j = j^2\pi^2$, $\psi_j(x) = \sin(j\pi x)$.

So $-u'' = \underbrace{-\gamma^2}_{\text{eigenvalue of } L} u$ if and only if $-\gamma^2 = \lambda_j$, $u = \sin(j\pi x)$ for some $j = 1, 2, \ldots$

$$-\gamma^2 = j^2\pi^2 \Rightarrow \gamma^2 = -j^2\pi^2 \Rightarrow \boxed{\gamma = \pm i j \pi}$$

We label

$$\boxed{\gamma_{\pm j} = \pm i j \pi,} \quad j = 1, 2, 3, \ldots$$

The corresponding eigenfunction has

$$u = \sin(j\pi x)$$
$$v = \gamma u = \gamma \sin(j\pi x)$$

i.e.

$$\boxed{\Psi_{\pm j} = \begin{bmatrix} \sin(j\pi x) \\ \pm i j\pi \sin(j\pi x) \end{bmatrix},} \quad j = 1, 2, 3, \ldots$$

These eigenfunctions are orthogonal:

Let $j,k \in \{\pm 1, \pm 2, \dots\}$ be distinct numbers:

$$(\Psi_j, \Psi_k) = \left(\begin{bmatrix} \sin(|j|\pi x) \\ ij\pi \sin(|j|\pi x) \end{bmatrix}, \begin{bmatrix} \sin(|k|\pi x) \\ ik\pi \sin(|k|\pi x) \end{bmatrix} \right)$$

$$= \int_0^1 |j||k|\pi^2 \cos(|j|\pi x) \overline{\cos(|k|\pi x)}\, dx$$

$$+ \int ij\pi \sin(|j|\pi x)\, \overline{ik\pi \sin(|k|\pi x)}\, dx$$

$\uparrow \; \bar{i} = -i$

$$= |j||k|\pi^2 \underbrace{\int_0^1 \cos(|j|\pi x)\cos(|k|\pi x)\, dx}_{\substack{=0 \text{ if } |j| \neq |k| \\ = \frac{1}{\sqrt{2}} \text{ if } |j| = |k|}}$$

$$\underbrace{-i^2}_{=1} jk\pi^2 \underbrace{\int_0^1 \sin(|j|\pi x)\sin(|k|\pi x)\, dx}_{\substack{=0 \text{ if } |j| \neq |k| \\ = \frac{1}{\sqrt{2}} \text{ if } |j| = |k|}}$$

$$= \begin{cases} 0 & \text{if } j \neq k \\ \frac{j^2\pi^2}{2} & \text{if } j = k \end{cases} \qquad \left(\begin{array}{l} \text{if } j = -k, \text{ then} \\ jk = -j^2, \text{ so} \\ (\Psi_j, \Psi_{-j}) = 0. \end{array} \right)$$

Now we consider the solution of the wave equation

$$(***) \qquad \begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Using the exact same technique that we used for the heat equation.

We shall seek a solution of the form

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \sum_{j=\pm 1}^{\pm\infty} a_j(t)\, \Phi_j(x).$$

Substitute this into (***) to get

$$\frac{\partial}{\partial t} \sum_{j=\pm 1}^{\pm\infty} a_j(t)\, \Phi_j(x) = A \sum_{j=\pm 1}^{\pm\infty} a_j(t)\, \Phi_j(x)$$

$$\Rightarrow \sum_{j=\pm 1}^{\pm\infty} a_j'(t)\, \Phi_j(x) = \sum_{j=\pm 1}^{\pm\infty} a_j(t)\, A\Phi_j(x)$$

$$(A\Phi_j = \gamma_j \Phi_j)$$

$$= \sum_{j=\pm 1}^{\pm\infty} a_j(t)\, \gamma_j\, \Phi_j(x)$$

Take the inner product of each side with Φ_k:

$$\left(\sum_{j=\pm 1}^{\pm\infty} a_j'(t)\, \Phi_j, \Phi_k \right) = \left(\sum_{j=\pm 1}^{\pm\infty} \gamma_j a_j(t)\, \Phi_j, \Phi_k \right)$$

Use linearity of the inner product to obtain

$$\sum_{j=\pm 1}^{\pm\infty} a_j'(t)\, (\Phi_j, \Phi_k) = \sum_{j=\pm 1}^{\pm\infty} \gamma_j a_j(t) (\Phi_j, \Phi_k)$$

Use orthogonality to get

$$a_j'(t)\, (\Phi_k, \Phi_k) = \gamma_j\, a_j(t)\, (\Phi_k, \Phi_k)$$

$$\Rightarrow \boxed{a_j'(t) = \gamma_j\, a_j(t)}$$

Which has the solution

$$a_j(t) = e^{t\gamma_j} a_j(0).$$

Where does $a_j(0)$ come from? From the initial conditions, naturally!

At $t=0$, we want

$$\begin{bmatrix} u_0(x) \\ v_0(x) \end{bmatrix} = \begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \sum_{j=\pm 1}^{\pm\infty} a_j(0)\, \Phi_j(x)$$

So, using the usual best approximation method,

$$\begin{bmatrix} u_0(x) \\ v_0(x) \end{bmatrix} = \sum_{j=\pm 1}^{\pm\infty} \underbrace{\frac{\left(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \Phi_j\right)}{(\Phi_j, \Phi_j)}}_{a_j(0)} \Phi_j(x)$$

$$\Rightarrow \boxed{a_j(0) = \frac{\left(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \Phi_j\right)}{(\Phi_j, \Phi_j)}}$$

Thus, the solution to the <u>wave</u> equation is

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \sum_{j=\pm 1}^{\pm\infty} e^{\gamma_j t} a_j(0)\, \Phi_j(x)$$

Picking out the first component:

$$u(x,t) = \sum_{j=\pm 1}^{\pm\infty} \underbrace{e^{ij\pi t}} a_j(0) \sin(|j|\pi x)$$

↑ exponentials, like the heat eqn – but <u>imaginary</u>.

$$= \sum_{j=\pm 1}^{\pm\infty} \left(\cos(j\pi t) + i\sin(j\pi t)\right) a_j(0) \sin(|j|\pi x)$$

↑ oscillates – a wave.

But this looks like an answer that involves <u>complex numbers</u>. This is only an accident of how we've written this out.

If $u_0(x)$ and $v_0(x)$ are real-valued, then

$a_j(0)$ and $a_{-j}(0)$ will be closely related, and eliminate the complex parts.

- Useful exercise: Work out $a_j(0)$ to verify this, and to show that this formula for $u(x,t)$ agrees with the spectral method solution written out in Lecture 31.

- Another useful exercise: Incorporate an inhomogeneous term, in the same way we did with the heat equation:

$$u_{tt}(x,t) = u_{xx}(x,t) + f(x,t).$$