Lecture 32: Resonance in Waves

Let L denote some linear operator with eigenvalues λ_j and corresponding eigenfunctions ψ_j (orthonormal, form a basis):

$$L\psi_j = \lambda_j \psi_j \qquad j = 1, 2, \ldots$$

In this lecture we focus on the inhomogeneous equation

$$u_{tt}(x,t) = -Lu + \sin(\omega t)$$

with trivial initial conditions

$$u(x,0) = u_t(x,0) = 0.$$

Of course, this is a more general setting for the equation

$$u_{tt}(x,t) = u_{xx}(x,t) + \sin(\omega t)$$

$$u(0,t) = u(1,t) = 0.$$

The particular forcing function $\sin(\omega t)$ (which applies equally at all points in space) has important physical implications — we want to emphasize that these apply in general settings — not just $Lu = -u_{xx}$.

We seek a solution of the form

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\psi_j(x).$$

Substitute this into the PDE to get

$$\frac{d^2}{dt^2}\sum_{j=1}^{\infty} a_j(t)\psi_j(x) = -L\sum_{j=1}^{\infty} a_j(t)\psi_j(x) + \sin(\omega t)$$

$$\Rightarrow \quad \sum_{j=1}^{\infty} a_j''(t)\psi_j(x) = -\sum_{j=1}^{\infty} a_j(t)\, L\psi_j(x) + \sin(\omega t)$$

$$= -\sum_{j=1}^{\infty} a_j(t)\lambda_j\psi_j(x) + \sin(\omega t)$$

Take the inner product with φ_k to get

$$\left(\sum_{j=1}^{\infty} a_j''(t)\, \varphi_j, \varphi_k \right) = \left(-\sum_{j=1}^{\infty} a_j(t)\, \lambda_j\, \varphi_j + \sin(\omega t), \varphi_k \right)$$

Use linearity of the inner product:

$$\sum_{j=1}^{\infty} a_j''(t) \underbrace{(\varphi_j, \varphi_k)}_{=0 \text{ if } j \neq k} = -\sum_{j=1}^{\infty} a_j(t)\, \lambda_j \underbrace{(\varphi_j, \varphi_k)}_{=0 \text{ if } j \neq k} + \sin(\omega t) \underbrace{(1, \varphi_k)}_{\text{call this } C_k}$$

(constant – no t dependence)

$$\Rightarrow \quad a_k''(t) = -\lambda_k a_k(t) + \sin(\omega t)\, C_k$$

The trivial initial conditions imply

$$a_k(0) = a_k'(0) = 0$$

For these initial conditions and inhomogeneity, the exact solution for $a_k(t)$ can be written out explicitly. If $\omega \neq \sqrt{\lambda_k}$, then

$$a_k(t) = C_k \left(\frac{\sqrt{\lambda_k}\, \sin(\omega t) - \omega \sin(\sqrt{\lambda_k}\, t)}{\sqrt{\lambda_k}\, (\lambda_k - \omega^2)} \right)$$

while if $\omega = \sqrt{\lambda_k}$,

$$a_k(t) = C_k \left(\frac{\sin(\sqrt{\lambda_k}\, t) - t \sqrt{\lambda_k} \cos(\sqrt{\lambda_k}\, t)}{2 \lambda_k} \right)$$

Note the "t" dependence in this last formula.

Now suppose $\omega = \sqrt{\lambda_k}$ for some eigenvalue λ_k.

Then

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\,\psi_j(x)$$

$$= \sum_{j\neq k} c_j \left(\frac{\sqrt{\lambda_j}\sin(\sqrt{\lambda_k}t) - \sqrt{\lambda_k}\sin(\sqrt{\lambda_j}t)}{\sqrt{\lambda_j}(\lambda_j - \lambda_k)} \right) \psi_j(x)$$

$$+ \; c_k \left(\frac{\sin(\sqrt{\lambda_k}t) - t\sqrt{\lambda_k}\cos(\sqrt{\lambda_k}t)}{2\lambda_k} \right) \psi_k(x)$$

These terms oscillate about in a generally nondescript way.

This term grows in amplitude with t, oscillating due to the $\cos(\sqrt{\lambda_k}t)$ factor.

Thus, the solution starts at $u(x,0) = 0$ but will grow in the shape of $\psi_k(x)$ without bound.

This phenomenon is called "resonance": If you drive a physical system in a sinusoidal fashion at a frequency ω that is tuned to an eigenvalue, $\omega = \sqrt{\lambda_k}$, The system will grow and grow - until failure/collapse!!

If ω is near $\sqrt{\lambda_k}$, you will similarly see excitation in the $\psi_k(x)$ shape.

A particularly beautiful demonstration of this effect occurs when one drives a 2d plate, covered in sand, at $\sin(\omega t)$. The sand collects on nodal lines - beautiful figures named for their discoverer Ernst Chladni.