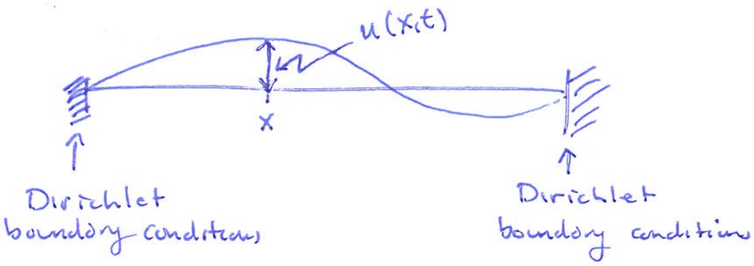
10: WAVE EQUATION: EXACT SOLUTION VIA THE SPECTRAL METHOD

We seek to model the transverse vibrations of a taut string:



$u(x,t)$ describes the distance of the string from its equilibrium position ($u(x,t)=0$) at the point $x \in [0,1]$ at time $t > 0$. We shall not derive the wave equation from first principles (see the book "Nonlinear Problems in Elasticity" by Stuart Antman for a masterful derivation), but merely state the simplest version

$$(*) \qquad u_{tt}(x,t) = u_{xx}(x,t) + f(x,t)$$

external forcing

$$u(0,t) = u(1,t) = 0$$

Since the equation is second-order in time, we need initial conditions for both <u>position</u> and <u>velocity</u>:

$$u(x,0) = u_0(x) = \text{initial displacement of string}$$
$$u_t(x,0) = v_0(x) = \text{initial velocity of string.}$$

More generally, we will replace $(*)$ by

$$(**) \qquad u_{tt} = -Lu + f,$$

where L is a symmetric linear operator whose eigenfunctions $\varphi_1, \varphi_2, \ldots$ allow us to write

(***) $$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\,\varphi_j(x) \qquad (\{\varphi_j\} \text{ orthogonal})$$

for any fixed time. Similarly, we write

$$f(x,t) = \sum_{j=1}^{\infty} c_j(t)\,\varphi_j(x),$$

but in this lecture focus on the case of $f(x,t)=0$.

We follow the same strategy we used for the heat eqn:

Use the PDE (**) to derive ordinary differential equations that govern the coefficients $a_j(t)$.

Substitute (***) into (**) to obtain

$$\frac{\partial^2}{\partial t^2} \sum_{j=1}^{\infty} a_j(t)\,\varphi_j(x) = -L \sum_{j=1}^{\infty} a_j(t)\,\varphi_j(x)$$

$$\Rightarrow \quad \sum_{j=1}^{\infty} a_j''(t)\,\varphi_j(x) = \sum_{j=1}^{\infty} a_j(t)\,(-L\varphi_j(x))$$

Use $L\varphi_j = \lambda_j \varphi_j$ to obtain:

$$\sum_{j=1}^{\infty} a_j''(t)\,\varphi_j(x) = \sum_{j=1}^{\infty} a_j(t)\,(-\lambda_j \varphi_j(x))$$

Take the inner product of both sides with φ_k to get

$$\Big(\sum_{j=1}^{\infty} a_j''(t)\,\varphi_j, \varphi_k\Big) = \Big(\sum_{j=1}^{\infty} a_j(t)(-\lambda_j)\,\varphi_j, \varphi_k\Big).$$

$$\Rightarrow \sum_{j=1}^{\infty} a_j''(x)\,(\varphi_j, \varphi_k) = \sum_{j=1}^{\infty} a_j(t)(-\lambda_j)\,(\varphi_j, \varphi_k).$$

Use orthogonality of the eigenfunctions

$$(\varphi_j, \varphi_k) = \begin{cases} 0 & \text{if } j \neq k \\ \neq 0 & \text{if } j = k \end{cases}$$

to obtain

$$a_k''(t) = -\lambda_k a_k(t)$$

(Compare this to the equation $a_k'(t) = -\lambda_k a_k(t)$ obtained for the heat equation.)

This ODE for $a_k(t)$ has the same form as the ODE we have been using all semester to find eigenfunctions (in space, x). It has the general solution

$$a_k(t) = A\sin(\sqrt{\lambda_k}\, t) + B\cos(\sqrt{\lambda_k}\, t)$$

but now we use the initial conditions to determine A and B. At $t=0$, use

$$a_k(0) = A\sin(0) + B\cos(0) = B$$

$$a_k'(t) = \sqrt{\lambda_k}\left(A\cos(\sqrt{\lambda_k}\, t) - B\sin(\sqrt{\lambda_k}\, t)\right)$$

$$\Rightarrow a_k'(0) = \sqrt{\lambda_k}\,(A\cos(0) - B\sin(0)) = A\sqrt{\lambda_k}.$$

If we expand the initial condition as

$$u(x,0) = \sum_{j=1}^{\infty} a_j(0)\,\varphi_j(x) = u_0(x)$$

take an inner product with φ_k to find

$$a_k(0) = \frac{(u_0, \varphi_k)}{(\varphi_k, \varphi_k)} \qquad k = 1, 2, 3, \ldots$$

Similarly, expand the initial velocity as

$$u_t(x,0) = \sum_{j=1}^{\infty} a_j'(0)\,\psi_j(x) = v_0(x)$$

and take the inner product with ψ_k to obtain

$$a_k'(0) = \frac{(v_0, \psi_k)}{(\psi_k, \psi_k)}.$$

These formulas for $a_k(0)$ and $a_k'(0)$ are, of course just the best approximation coefficients for u_0 and v_0.

Now we can identify the A and B in the formula for $a_k(t)$:

$$a_k(0) = B \implies B = \frac{(u_0, \psi_k)}{(\psi_k, \psi_k)}$$

$$a_k'(0) = \sqrt{\lambda_k}\,A \implies A = \frac{1}{\sqrt{\lambda_k}}\,\frac{(v_0, \psi_k)}{(\psi_k, \psi_k)} \qquad (\text{provided } \lambda_k \neq 0.)$$

Thus, presuming L has no zero eigenvalue, we can express

$$a_k(t) = \frac{(u_0, \psi_k)}{(\psi_k, \psi_k)}\cos(\sqrt{\lambda_k}\,t) + \frac{1}{\sqrt{\lambda_k}}\,\frac{(v_0, \psi_k)}{(\psi_k, \psi_k)}\sin(\sqrt{\lambda_k}\,t)$$

with

$$u(x,t) = \sum_{k=1}^{\infty} \underbrace{\left(\frac{(u_0, \psi_k)}{(\psi_k, \psi_k)}\cos(\sqrt{\lambda_k}\,t) + \frac{1}{\sqrt{\lambda_k}}\,\frac{(v_0, \psi_k)}{(\psi_k, \psi_k)}\sin(\sqrt{\lambda_k}\,t)\right)}_{\text{Note that this term oscillates in } t\text{: fundamentally different behavior than for the heat equation.}}\psi_k(x)$$

In particular, note that these oscillations occur as a result of the $\frac{\partial^2}{\partial t^2}$ term, independent of the eigenfunctions $\varphi_k(x)$.

For a vibrating string, this suggests that an initial pluck will cause the string to vibrate forever!

To get around this apparently unrealistic behavior, we can add <u>damping</u> to the model:

$$u_{tt}(x,t) = u_{xx}(x,t) \underbrace{- \gamma u_t(x,t)}_{\text{"viscous damping" is proportional to the strings velocity.}}$$

Instead of pursuing this direction here, we shall instead consider in the next lecture what happens when we drive the string with a forcing function.