Lecture 27: Heat equation: Galerkin approximation in space, exact solution in time via the matrix exponential.

In the last lecture we derived the Galerkin approximation $u_N(x,t) = \sum_{j=1}^{N} a_j(t)\phi_j(x)$ to the heat equation, where the coefficients

$$a(t) = \begin{bmatrix} a_1(t) \\ \vdots \\ a_N(t) \end{bmatrix}$$

satisfy the differential equation

$$Ma'(t) = -Ka(t) + f(t)$$

with solution

$$a(t) = e^{-M^{-1}Kt}a(0) + \int_0^t e^{M^{-1}K(s-t)} f(s)\,ds.$$

In this lecture we address two points:

(a) How do we obtain the initial condition $a(0)$?

(b) How does $u_N(x,t)$ behave as $t \to \infty$ if $f = 0$?

(a) The initial condition $a(0)$ requires some reflection. In general, $u_0(x) \notin \operatorname{span}\{\phi_1, \ldots, \phi_N\} = V_N$, so it will be impossible to write

$$u_0(x) = \sum_{j=1}^{N} a_j(0)\phi_j(x)$$

as we would wish.

So, we must settle for

$$\sum_{j=1}^{N} a_j(0)\,\phi_j(x) \approx u_0(x).$$

Two options to consider:

(i) For general spaces $V_N = \text{span}\{\phi_1, \ldots, \phi_N\}$, we could simply let $\sum_{j=1}^{n} a_j(0)\,\phi_j(x)$ be the best approximation to $u_0(x)$.

Thus we must solve

$$\begin{bmatrix} (\phi_1, \phi_1) & \cdots & (\phi_1, \phi_N) \\ \vdots & \ddots & \vdots \\ (\phi_N, \phi_1) & \cdots & (\phi_N, \phi_N) \end{bmatrix} \begin{bmatrix} a_1(0) \\ \vdots \\ a_N(0) \end{bmatrix} = \begin{bmatrix} (u_0, \phi_1) \\ \vdots \\ (u_0, \phi_N) \end{bmatrix}$$

for $a_1(0), \ldots, a_N(0)$. (Note that this is the Mass matrix here!)

(ii) In the case of hat functions, $V_N = \text{span}\{\phi_1, \ldots, \phi_N\}$, a simpler approach may be more appealing.

At any grid point $x_k = \frac{k}{N+1} = kh$, we have

$$\phi_j(x_k) = \begin{cases} 1, & j = k; \\ 0 & j \neq k. \end{cases}$$

Thus we can ask that $\sum_{j=1}^{N} a_j(0)\,\phi_j(x_k) = u_0(x_k)$ for $k = 1, \ldots, N$:

$$u_0(x_k) = \sum_{j=1}^{N} a_j(0)\,\phi_j(x_k) = a_k(0)$$

Thus $$a(0) = \begin{bmatrix} a_1(0) \\ \vdots \\ \cdot \\ a_N(0) \end{bmatrix} = \begin{bmatrix} u_0(x_1) \\ \vdots \\ u_0(x_N) \end{bmatrix}$$

This choice for $a(0)$ has the virtues that

- It interpolates the exact initial data on the finite element grid.
- It is trivial to compute (no matrix equation to solve, as in option (i)).

Now we address the asymptotic ($t \to \infty$) behavior of the solution to the homogeneous case ($f=0$), for which we have

$$a(t) = e^{-M^{-1}Kt}\, a(0).$$

We need to better understand the matrix exponential: the series definition does not alone provide sufficient insight.

We shall consider the generic problem

$$y'(t) = A\,y(t) \qquad A = N \times N \text{ matrix}$$

with solution $y(t) = e^{tA} y(0)$.

Suppose A has eigenvalues $\lambda_1, \dots, \lambda_N$ with associated eigenvectors $v_1, \dots, v_N$, and suppose these eigenvectors are linearly independent. (We can find such eigenvectors

When $A = A^T$ or when all eigenvalues of A are distinct (those are sufficient conditions):

$$Av_1 = \lambda_1 v_1 \quad Av_2 = \lambda_2 v_2, \quad \cdots, \quad Av_N = \lambda_N v_N.$$

Stack these individual vector equalities as columns of a matrix equality:

$$\left[Av_1 \middle| Av_2 \middle| \cdots \middle| Av_N \right] = \left[\lambda_1 v_1 \middle| \lambda_2 v_2 \middle| \cdots \middle| \lambda_N v_N \right]$$

We can factor each side:

$$A \left[v_1 \middle| v_2 \middle| \cdots \middle| v_N \right] = \left[v_1 \middle| v_2 \middle| \cdots \middle| v_N \right] \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_N \end{bmatrix}$$

Which we write as

$$AV = V\Lambda$$

Note: post-multiply by a diagonal matrix to scale columns; pre-multiply by a diagonal matrix to scale rows.

Since the eigenvectors are linearly independent, V is invertible: so

$$A\underbrace{VV^{-1}}_{=I} = V\Lambda V^{-1} \implies \boxed{A = V\Lambda V^{-1}}$$

This is called a diagonalization of A.

Substitute this diagonal form of A into the definition of e^{tA} (from Lecture 26):

$$e^{tA} = I + tA + \tfrac{1}{2}t^2A^2 + \tfrac{1}{3!}t^3A^3 + \cdots$$

$$= \underbrace{I}_{=VV^{-1}} + tV\Lambda V^{-1} + \tfrac{1}{2}t^2 V\Lambda^2 V^{-1} + \tfrac{1}{3!}t^3 V\Lambda^3 V^{-1} + \cdots$$

Where we have used the fact that

$$A^2 = (V\Lambda V^{-1})(V\Lambda V^{-1}) = V\Lambda \underbrace{V^{-1}V}_{=I}\Lambda V^{-1} = V\Lambda^2 V^{-1}$$

and, in general, $A^k = V\Lambda^k V^{-1}$.

Factoring out the common V and V^{-1} from each side,

$$e^{tA} = V\underbrace{\left(I + t\Lambda + \tfrac{1}{2}t^2\Lambda^2 + \tfrac{1}{3!}t^3\Lambda^3 + \cdots\right)}_{\begin{bmatrix} 1+t\lambda_1+\frac{1}{2}t^2\lambda_1^2+\frac{1}{3!}t^3\lambda_1^3+\cdots & & & 0 \\ & 1+t\lambda_2+\frac{1}{2}t^2\lambda_2^2+\frac{1}{3!}t^3\lambda_2^3+\cdots & & \\ & & \ddots & \\ 0 & & & 1+t\lambda_N+\frac{1}{2}t^2\lambda_N^2+\frac{1}{3!}t^3\lambda_N^3+\cdots \end{bmatrix}} V^{-1}$$

(Since Λ is a diagonal matrix)

here entries are Taylor series for $e^{t\lambda_j}$

$$\Longrightarrow \quad e^{tA} = V\begin{bmatrix} e^{t\lambda_1} & & & \\ & e^{t\lambda_2} & & \\ & & \ddots & \\ & & & e^{t\lambda_N}\end{bmatrix}V^{-1}.$$

Now suppose $\lambda = \alpha + i\beta$
(recall that λ can be complex even if
A is real (and nonsymmetric))

$$e^{t\lambda} = e^{t(\alpha+i\beta)} = e^{t\alpha} e^{it\beta}$$
$$= e^{t\alpha}(\cos(t\beta) + i\sin(t\beta))$$

$$\Rightarrow |e^{t\lambda}|^2 = (\text{Real}(e^{t\lambda}))^2 + (\text{Imag}(e^{t\lambda}))^2$$
$$= (e^{t\alpha}\cos(t\beta))^2 + (e^{t\alpha}\sin(t\beta))^2$$
$$= e^{2t\alpha}\underbrace{(\cos^2(t\beta) + \sin^2(t\beta))}_{=1}$$

$$\Rightarrow |e^{t\lambda}| = e^{t\alpha}$$

Hence $e^{t\lambda} \to 0$ if and only if
$$\alpha = \text{Real}(\lambda) < 0$$

If $\alpha = \text{Real}(\lambda) = 0$, $|e^{t\lambda}| = 1$

If $\alpha = \text{Real}(\lambda) > 0$, $|e^{t\lambda}| \to \infty$ as $t \to \infty$.

What about the heat equation?
Recall that $a(t) = e^{-M^{-1}Kt} a(0)$.

Because of the negative sign in $-M^{-1}K$, we want all eigenvalues of $M^{-1}K$ to be <u>positive</u>, so that if

$$M^{-1}K = V \Lambda V^{-1}$$

we will have $e^{-M^{-1}Kt} = V \begin{bmatrix} e^{-\lambda_1 t} & & \\ & \ddots & \\ & & e^{-\lambda_N t} \end{bmatrix} V^{-1}$

and $e^{-\lambda_j t} \to 0$ if $\text{Re}(\lambda_j) > 0$.

[In fact, we shall see that this is the case when we study the eigenvalues of $M^{-1}K$ in detail.

See "show-eig.m" on the class website.

Thus $a(t) = e^{-M^{-1}Kt} a(0) \to 0$,

which implies $U_N(x,t) = \sum_{j=1}^{N} a_j(t)\, \phi_j(x)$

$\longrightarrow 0$

as $t \to \infty$, consistent with the long-time asymptotic behaviour of the exact solution $u(x,t)$.