LECTURE 26: SOLVING THE HEAT EQUATION WITH FINITE ELEMENTS

9: HEAT EQUATION: APPROXIMATE SOLUTION VIA THE FINITE ELEMENT METHOD

THE SPECTRAL METHOD GIVES EXACT SERIES SOLUTIONS TO "NICE" DIFFERENTIAL EQUATIONS — PROBLEMS FOR WHICH WE CAN EASILY FIND THE ASSOCIATED EIGENVALUES AND EIGENFUNCTIONS. WHEN THOSE ARE NOT EASY TO COMPUTE, WE TURN TO APPROXIMATE METHODS (JUST AS IN THE STATIC CASE STUDIED EARLIER).

① DERIVE THE WEAK FORM OF THE PDE.

FOR
$$u_t(x,t) = u_{xx}(x,t) + f(x,t) \qquad (*)$$

WITH DIRICHLET BOUNDARY CONDITIONS
$$u(0,t) = u(1,t) = 0$$

WE USE THE SPACE OF TEST FUNCTIONS
$$V = C_0^2[0,1].$$

MULTIPLY $(*)$ BY $v \in V$ TO GET

$$u_t(x,t)\,v(x) = u_{xx}(x,t)v(x) + f(x,t)\,v(x).$$

INTEGRATE OVER THE DOMAIN TO GET

$$\int_0^1 U_t(x,t)\, v(x)\, dx = \int_0^1 U_{xx}(x,t)\, v(x)\, dx + \int_0^1 f(x,t) v(x)\, dx$$

INTEGRATE THE FIRST TERM ON THE RIGHT BY PARTS!

$$\int_0^1 U_t(x,t) v(x)\, dx = \left[U_x(x,t)\, v(x) \right]_0^1 - \int_0^1 U_x(x,t) V_x(x,t)\, dx$$
$$+ \int_0^1 f(x,t) v(x)\, dx.$$

SINCE $v \in V = C_0^2[0,1]$, $V(0) = V(1) = 0$ (AN "ESSENTIAL" BOUNDARY CONDITION) THE BOUNDARY TERM IS ZERO, GIVING:

$$\int_0^1 U_t(x,t) v(x)\, dx = \underbrace{- \int_0^1 U_x(x,t) V_x(x,t)\, dx}_{= \, a(u,v)} + \int_0^1 f(x,t) v(x)\, dx$$

$= a(u,v) \Leftarrow$ "energy inner product"

WRITING IN INNER PRODUCT FORM,

WEAK FORM

$$\boxed{ (U_t, v) = -a(u,v) + (f,v) \quad \forall v \in V. }$$

② IMPOSE WEAK FORM ON A FINITE DIMENSIONAL SPACE: GALERKIN APPROXIMATION

Let $V_N = \text{span}\{\phi_1, \dots, \phi_N\}$

BE AN N-DIMENSIONAL SUBSPACE OF V.

NOW SEEK $\qquad U_N(x,t) = \sum_{j=1}^{N} a_j(t)\, \phi_j(x)$

THAT SATISFIES THE WEAK FORM FOR ALL $v \in V_N$:

GALERKIN
PROBLEM
$$\left(\tfrac{\partial}{\partial t} U_N,\, v\right) = -a(U_N, v) + (f, v) \quad \forall v \in V_N$$

③ EXTRACT A LINEAR ALGEBRA PROBLEM FROM THE
GALERKIN FORMULATION

THE GALERKIN PROBLEM IS SOLVED IF AND ONLY IF

$$\left(\tfrac{d}{dt} U_N,\, \phi_k\right) = -a(U_N, \phi_k) + (f, \phi_k) \qquad (**)$$

FOR ALL $k = 1, \ldots, N$ (SINCE $\{\phi_k\}_{k=1}^{N}$ IS A BASIS FOR V_N). WRITE $(**)$ OUT AS

$$\left(\tfrac{\partial}{\partial t} \sum_{j=1}^{N} a_j(t)\phi_j(x),\, \phi_k\right) = -a\left(\sum_{j=1}^{N} a_j(t)\phi_j(x),\, \phi_k\right) + (f, \phi_k)$$

USE LINEARITY OF THE INNER PRODUCTS TO GET

$$\sum_{j=1}^{N} a_j'(t)\,(\phi_j, \phi_k) = -\sum_{j=1}^{N} a_j(t)\, a(\phi_j, \phi_k) + (f, \phi_k).$$

FOR EXAMPLE, WHEN $k=1$, THIS MEANS

$$\begin{bmatrix} (\phi_1, \phi_1) & \cdots & (\phi_N, \phi_1) \end{bmatrix} \begin{bmatrix} a_1'(t) \\ \vdots \\ a_N'(t) \end{bmatrix} = -\begin{bmatrix} a(\phi_1, \phi_1) & \cdots & a(\phi_N, \phi_1) \end{bmatrix} \begin{bmatrix} a_1(t) \\ \vdots \\ a_N(t) \end{bmatrix} + \begin{bmatrix} (f, \phi_1) \end{bmatrix}$$

STACKING THESE EQUATIONS FOR $k=1,\dots,N$:

$$
\underbrace{\begin{bmatrix} (\phi_1,\phi_1) & \cdots & (\phi_N,\phi_1) \\ \vdots & & \vdots \\ (\phi_1,\phi_N) & \cdots & (\phi_N,\phi_N) \end{bmatrix}}_{\substack{\text{"MASS MATRIX"} \\ M}} \underbrace{\begin{bmatrix} a_1'(t) \\ \vdots \\ a_N'(t) \end{bmatrix}}_{a'(t)} = - \underbrace{\begin{bmatrix} a(\phi_1,\phi_1) & \cdots & a(\phi_N,\phi_1) \\ \vdots & & \vdots \\ a(\phi_1,\phi_N) & \cdots & a(\phi_N,\phi_N) \end{bmatrix}}_{\substack{\text{"STIFFNESS MATRIX"} \\ K}} \underbrace{\begin{bmatrix} a_1(t) \\ \vdots \\ a_N(t) \end{bmatrix}}_{a(t)} + \underbrace{\begin{bmatrix} (f,\phi_1) \\ \vdots \\ (f,\phi_N) \end{bmatrix}}_{\substack{\text{"LOAD VECTOR"} \\ f(t)}}
$$

"MASS MATRIX" M — JUST THE GRAM MATRIX FOR $\{\phi_k\}_{k=1}^N$

"STIFFNESS MATRIX" K — JUST AS BEFORE

So WE COMPACTLY WRITE THIS AS A FIRST ORDER SYSTEM OF N COUPLED LINEAR ODEs:

$$
\boxed{M\,a'(t) = -K\,a(t) + f(t)} \qquad (***)
$$

HERE, FOR OUR USUAL HAT FUNCTIONS,

$$
M = h \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & & O \\ \frac{1}{6} & & \ddots & \\ & \ddots & & \frac{1}{6} \\ O & & \frac{1}{6} & \frac{2}{3} \end{bmatrix}, \quad K = \frac{1}{h}\begin{bmatrix} -2 & 1 & & O \\ 1 & & \ddots & \\ & \ddots & & 1 \\ O & & 1 & -2 \end{bmatrix}
$$

To MAKE $(***)$ WELL DEFINED, WE NEED <u>INITIAL CONDITIONS</u> FOR $a(t)$.

WE CAN EXTRACT INITIAL CONDITIONS
IN TWO WAYS:

(a) MAKE $U_N(x,0)$ EQUAL THE BEST APPROXIMATION
OF $U_0(x)$ FROM SPAN $\{\phi_1, \ldots, \phi_N\}$.

USING TECHNIQUES FROM EARLIER THIS
SEMESTER, THIS WILL REQUIRE

$$\begin{bmatrix} (\phi_1, \phi_1) & \cdots & (\phi_N, \phi_1) \\ \vdots & & \\ (\phi_1, \phi_N) & \cdots & (\phi_N, \phi_N) \end{bmatrix} \begin{bmatrix} a_1(0) \\ \vdots \\ a_N(0) \end{bmatrix} = \underbrace{\begin{bmatrix} (U_0, \phi_1) \\ \vdots \\ (U_0, \phi_N) \end{bmatrix}}_{g}$$

$$\Rightarrow \boxed{M\, a(0) = g} \quad \text{(Solve for } a(0))$$

(b) THIS IS THE EASY/CONVENTIONAL APPROACH
FOR HAT FUNCTIONS: IF WE WANT
$U_N(x,0)$ TO <u>INTERPOLATE</u> $U_0(x)$ AT
$X_1, X_2, \ldots, X_N$, USE THE FACT THAT

$$U_N(X_k, 0) = \sum_{j=1}^{N} a_j(0)\, \underbrace{\phi_j(X_k)}_{= \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}} = a_k(0)$$

TO SIMPLY SET

$$\boxed{a_k(0) = U_0(X_k)} \qquad k = 1, \ldots, N$$

$$\Rightarrow \quad a(0) = \begin{bmatrix} U_0(X_1) \\ \vdots \\ U_0(X_N) \end{bmatrix}.$$

Now, solve $a'(t) = -M^{-1}(K\, a(t) + f(t))$ for $a(t)$ . . . . .