Lecture 25: Heat Equation with Periodic Boundary Conditions and "The" Fourier series.

Consider the heat equation posed on a bar, where the ends are bent around so that they join together and form a ring. At the points where the ends meet, we will require that $u(x,t)$ and $u_x(x,t)$ be continuous. For convenience, we shall use the physical domain $x \in [-1,1]$.

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) \\ u(x,0) &= u_0(x) \\ \left.\begin{aligned} u(-1,t) &= u(1,t) \\ u_x(-1,t) &= u_x(1,t) \end{aligned}\right\} &\text{ Periodic boundary conditions.} \end{aligned}$$

As usual, we pose this problem as a linear operator equation $u_t = -Lu$, $u(x,0) = u_0(x)$, where

$$L : C_p^2[-1,1] \longrightarrow C[0,1]$$

$$Lu = -u_{xx}$$

for $C_p^2[-1,1] = \{ u \in C^2[-1,1] : u(-1) = u(1),\ u_x(-1) = u_x(1) \}$

One can show, using the usual techniques, that L is symmetric. What are the eigenvalues and eigenfunctions of L?

Eigenvalues and Eigenfunctions of L.

We see $\psi \neq 0$ such that

- $\psi \in C_p^2[-1,1]$
- $L\psi = \lambda \psi$

The second requirement implies $-\psi'' = \lambda \psi$, giving the general solution

$$\psi(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x).$$

What A, B, λ give nonzero ψ in $C_p^2[-1,1]$?

$$\psi(-1) = \psi(1), \qquad \psi'(-1) = \psi'(1)$$

$$\begin{aligned}\psi(-1) &= A\sin(-\sqrt{\lambda}) + B\cos(-\sqrt{\lambda}) \\ &= -A\sin(\sqrt{\lambda}) + B\cos(\sqrt{\lambda}) \\ \psi(1) &= A\sin(\sqrt{\lambda}) + B\cos(\sqrt{\lambda})\end{aligned} \qquad \left\{\begin{aligned}\sin(-\theta) &= -\sin\theta \\ \cos(-\theta) &= \cos\theta\end{aligned}\right.$$

Equating $\psi(-1)$ and $\psi(1)$ gives

$$-A\sin(\sqrt{\lambda}) + \cancel{B\cos(\sqrt{\lambda})} = A\sin(\sqrt{\lambda}) + \cancel{B\cos(\sqrt{\lambda})}$$

$$\Rightarrow \quad 2A\sin(\sqrt{\lambda}) = 0$$

$$\Rightarrow \quad \begin{cases} A = 0 \Rightarrow \psi(x) = B\cos(\sqrt{\lambda}x) \\ \text{or} \\ \sin(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda} = 0, \pi, 2\pi, \ldots \end{cases}$$

Now compute, in general,

$$\psi'(x) = \sqrt{\lambda}\,(A\cos(\sqrt{\lambda}x) - B\sin(\sqrt{\lambda}x)).$$

Hence

$$\psi'(-1) = \sqrt{\lambda}\left(A\cos(-\sqrt{\lambda}) - B\sin(-\sqrt{\lambda})\right)$$
$$= \sqrt{\lambda}\left(A\cos(\sqrt{\lambda}) + B\sin(\sqrt{\lambda})\right)$$
$$\psi'(1) = \sqrt{\lambda}\left(A\cos(\sqrt{\lambda}) - B\sin(\sqrt{\lambda})\right)$$

Equating $\psi'(-1)$ and $\psi'(1)$ gives

$$\sqrt{\lambda}\left(A\cancel{\cos(\sqrt{\lambda})} + B\sin(\sqrt{\lambda})\right) = \sqrt{\lambda}\left(A\cancel{\cos(\sqrt{\lambda})} - B\sin(\sqrt{\lambda})\right)$$

$$\Rightarrow \quad 2B\sqrt{\lambda}\sin(\sqrt{\lambda}) = 0$$

$$\Rightarrow \quad \begin{cases} B = 0 \Rightarrow \psi(x) = A\sin(\sqrt{\lambda}\,x) \\ \text{or} \\ \sqrt{\lambda}\sin(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda} = 0, \pi, 2\pi, \ldots \end{cases}$$

So $\psi(-1)=\psi(1)$ and $\psi'(-1)=\psi'(1)$ each give 2 cases — we must thus analyze $2\times 2 = 4$ possibilities:

①	②	③	④
$A=0$	$A=0$	$\sqrt{\lambda}=0,\pi,2\pi,\ldots$	$\sqrt{\lambda}=0,\pi,2\pi,\ldots$
$B=0$	$\sqrt{\lambda}=0,\pi,2\pi,\ldots$	$B=0$	$\sqrt{\lambda}=0,\pi,2\pi,\ldots$

① $A=0,\ B=0 \Rightarrow \psi(x)=0$: NOT AN EIGENFUNCTION!

② $A=0,\ \sqrt{\lambda}=0,\pi,2\pi,\ldots \Rightarrow \psi(x) = B\cos(\sqrt{\lambda}\,x),\quad \sqrt{\lambda}=0,\pi,2\pi,\ldots$

If $\sqrt{\lambda}=0$, pick $B=\frac{1}{2}$ so $\psi_0(x) = \frac{1}{2}\cos(0) = 1$, $\|\psi_0\| = 1$

If $\sqrt{\lambda}=\pi, 2\pi, \ldots$, pick $B=1$, so $\psi_n(x) = \cos(n\pi x)$, $\|\psi_n\|=1$, for $n = 1,2,3,\ldots$

(3) $\sqrt{\lambda} = 0, \pi, 2\pi, \ldots, \; B = 0 \Rightarrow \psi(x) = A\sin(\sqrt{\lambda}x)$

If $\sqrt{\lambda} = 0$, $\psi(x) = 0 \Rightarrow$ NOT AN EIGENFUNCTION!

If $\sqrt{\lambda} = \pi, 2\pi, \ldots$ Pick $A = 1$, so $\psi_{-n}(x) = \sin(n\pi x)$, $\|\psi_{-n}\| = 1$ for $n = 1, 2, 3, \ldots$

Note indexing here – the negative index is used to distinguish these eigenfunctions from the eigenfunctions $\psi_n(x) = \cos(n\pi x)$, found earlier. Note that both eigenfunctions have the same eigenvalue:

$$\lambda_{-n} = \lambda_n = n^2\pi^2$$

Yet (ψ_n, ψ_{-n})

$$= \int_{-1}^{1} \cos(n\pi x)\sin(n\pi x)\,dx = 0$$

So these eigenfunctions are orthogonal.

(4) $\sqrt{\lambda} = 0, \pi, 2\pi, \ldots$
$\sqrt{\lambda} = 0, \pi, 2\pi, \ldots$

$\Rightarrow \psi(x) = A\sin(n\pi x) + B\cos(n\pi x)$.

Note that this function is a linear combination of $\psi_n(x)$ and $\psi_{-n}(x)$:

$$\psi(x) = A\,\psi_{-n}(x) + B\,\psi_n(x)$$

So this solution does not lead to any new eigenfunctions.

In summary, the eigenvalues and eigenfunctions of L are:

$\lambda_0 = 0$, $\quad \psi_0(x) = 1/2$.

$\lambda_{-n} = n^2\pi^2$, $\quad \psi_{-n}(x) = \sin(n\pi x)$, $\quad n = 1, 2, 3, \ldots$

$\lambda_n = n^2\pi^2$, $\quad \psi_n(x) = \cos(n\pi x)$, $\quad n = 1, 2, 3, \ldots$

Using the techniques described earlier, we can easily write down a solution to the heat equation:

$$u(x,t) = e^{-\lambda_0 t} a_0(0)\,\varphi_0(x) + \sum_{n=1}^{\infty} e^{-\lambda_n t} a_n(0)\,\varphi_n(x) + \sum_{n=1}^{\infty} e^{-\lambda_{-n} t} a_{-n}(0)\,\varphi_{-n}(x)$$

$$= \frac{1}{2} a_0(0) + \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} a_n(0) \cos(n\pi x) + \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} a_{-n}(0) \sin(n\pi x)$$

where $a_k(0) = \dfrac{(u_0, \varphi_k)}{(\varphi_k, \varphi_k)}$ for $k = 0, \pm 1, \pm 2, \dots$

But there is another interesting point. We can expand a generic 2-periodic function $f \in C_p^2[-1,1]$ in this basis of eigenfunctions using the usual best approximation technique:

$$f = \sum_{n=-\infty}^{\infty} c_n \varphi_n(x), \qquad c_n = \frac{(f, \varphi_n)}{(\varphi_n, \varphi_n)}$$

This gives

$$\boxed{f(x) = \frac{1}{2}\int_{-1}^{1} f(x)\,dx + \sum_{n=1}^{\infty}\left(\int_{-1}^{1} f(x)\cos(n\pi x)\,dx\right)\cos(n\pi x) + \sum_{n=1}^{\infty}\left(\int_{-1}^{1} f(x)\sin(n\pi x)\,dx\right)\sin(n\pi x).}$$

This expansion is often simply called "<u>the Fourier Series</u>" for a function on $x \in [-1,1]$.

In the context of our class, it is simply an eigenfunction expansion associated with $Lu = -u_{xx}$ imposing periodicity in u and u_x.