Lecture 24: Handling Inhomogeneous Boundary Conditions

Consider the heat equation

$$u_t = u_{xx} + f, \quad u(x,0) = u_0(x)$$

with <u>inhomogeneous</u> boundary conditions

$$u(0,t) = g(t), \quad u(1,t) = h(t)$$

for functions $g(t)$ and $h(t)$. (Thus the boundary conditions can <u>vary in time</u>.)

How can we incorporate these boundary conditions?

As usual, L is defined on a domain with <u>homogeneous</u> boundary conditions of the same kind:

$$Lv = -v_{xx} \quad \text{for } v \in C_D^2[0,1].$$

We seek to build a solution

$$u(x,t) = v(x,t) + w(x,t)$$

with $w(x,t)$ engineered to satisfy the inhomogeneous boundary conditions, and $v(x,t)$ the solution of a related differential equation with homogeneous boundary conditions.

FOLLOWING THE LEAD OF THE APPROACH WE TOOK FOR $-u_{xx}=0$, LET

$$w(x,t) = f(t) + x\,(g(t)-f(t))$$

SO THAT $w(0,t) = f(t)$ AND $w(1,t) = g(t)$.

NOW CONSIDER $u(x,t) = v(x,t) + w(x,t)$. PLUG THIS INTO THE PDE TO SEE WHAT v MUST SATISFY:

$$u_t(x,t) = u_{xx}(x,t) + f(x,t)$$

$$\Rightarrow \quad v_t(x,t) + w_t(x,t) = v_{xx}(x,t) + w_{xx}(x,t) + f(x,t)$$

NOW $w_t(x,t) = f'(t) + x(g'(t) - f'(t))$ } NONZERO (EASY TO OVERLOOK!)

AND $w_{xx}(x,t) = 0$.

THUS $\boxed{v_t(x,t) = v_{xx}(x,t) + f(x,t) - w_t(x,t)}$

DEFINE $\tilde{f}(x,t) = f(x,t) - w_t(x,t)$.

THEN FIND $v(x,t)$ TO SOLVE

$$\boxed{\begin{gathered} v_t(x,t) = v_{xx}(x,t) + \tilde{f}(x,t) \\ \text{i.e.,} \quad v_t(x,t) = -L\,v(x,t) + \tilde{f}(x,t). \end{gathered}}$$

DON'T FORGET ABOUT INITIAL CONDITIONS!

$$u_0(x) = u(x,0) = v(x,0) + w(x,0) = v(x,0) + f(0) + x(g(0)-f(0))$$

$$\Rightarrow \quad \boxed{v_0(x) = v(x,0) = u_0(x) - \big(f(0) + x\,(g(0)-f(0))\big).}$$

SOLVE FOR v, BUILD $u(x,t) = v(x,t) + w(x,t)$.