Lecture 23: Heat equation with inhomogeneous forcing.

In the last lecture we solved the homogeneous heat equation

$$u_t = u_{xx} \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = u_0(x)$$

This lecture adds an inhomogeneous forcing term

$$u_t = u_{xx} + f \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = u_0(x).$$

Begin with the general operator setting

$$u_t = -Lu + f$$

where $L : C_0^2[0,1] \to C[0,1]$, $Lu = -u_{xx}$.

with eigenvalues and eigenfunctions $\lambda_j = j^2\pi^2$, $\psi_j(x) = \sqrt{2}\sin(j\pi x)$.

At every fixed t, we can write the solution as

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\,\psi_j(x), \qquad (*)$$

where $a_1(t), a_2(t), \ldots$ give the best approximation/projection coefficient for $u(\cdot,t)$ onto the subspace $\operatorname{span}\{\psi_j\}$.

Plug the expansion $(*)$ into $u_t = -Lu + f$ to obtain

$$\frac{\partial}{\partial t}\sum_{j=1}^{\infty} a_j(t)\,\psi_j(x) = -L\sum_{j=1}^{\infty} a_j(t)\,\psi_j(x) + f(x,t)$$

$$\Rightarrow \quad \sum_{j=1}^{\infty} a_j'(t)\,\psi_j(x) = -\sum_{j=1}^{\infty} a_j(t)\,L\psi_j(x) + f(x,t)$$

$$= -\sum_{j=1}^{\infty} a_j(t)\,\lambda_j\,\psi_j(x) + f(x,t)$$

since $L\psi_j = \lambda_j\psi_j$.

Take inner products with ψ_k,

$$\left(\sum_{j=1}^{\infty} a_j'(t)\,\psi_j,\ \psi_k\right) = -\left(\sum_{j=1}^{\infty} a_j(t)\lambda_j\psi_j, \psi_k\right) + (f, \psi_k)$$

$$\Rightarrow \quad \sum_{j=1}^{\infty} a_j'(t)\,(\psi_j, \psi_k) = -\sum_{j=1}^{\infty} \lambda_j\, a_j(t)\,(\psi_j, \psi_k) + (f, \psi_k)$$

Use orthogonality of the eigenfunctions to simplify:

$$a_k'(t)(\psi_k, \psi_k) = -\lambda_k\, a_k(t)\,(\psi_k, \psi_k) + (f, \psi_k)$$

giving the ODE for $a_k(t)$:

$$a_k'(t) = -\lambda_k\, a_k(t) + \frac{(f, \psi_k)}{(\psi_k, \psi_k)}$$

Note that

$$\frac{(f, \psi_k)}{(\psi_k, \psi_k)} = \frac{\int_0^1 f(x,t)\,\psi_k(x)\,dx}{\int_0^1 \psi_k(x)\,\psi_k(x)\,dx} = c_k(t)$$

gives the best-approximation coefficients for $f(\cdot,t)$, the function f with t frozen:

$$f(\cdot,t) = \sum_{j=1}^{\infty} \frac{(f(\cdot,t), \psi_j)}{(\psi_j, \psi_j)}\,\psi_j(x) = \sum_{j=1}^{\infty} c_j(t)\,\psi_j(x).$$

So we must now solve

$$\boxed{a_k'(t) = -\lambda_k\, a_k(t) + c_k(t)}$$

This first-order inhomogeneous linear equation is a staple of first courses in the solution of ODEs. We shall quickly recapitulate the solution.

Multiply both sides by $e^{\lambda_k t}$; and rearrange:

$$e^{\lambda_k t} a_k'(t) + \lambda_k e^{\lambda_k t} a_k(t) = e^{\lambda_k t} c_k(t)$$

Integrate both sides:

$$\int_0^t \left(e^{\lambda_k s} a_k'(s) + \lambda_k e^{\lambda_k s} a_k(s)\right) ds = \int_0^t e^{\lambda_k s} c_k(s)\, ds$$

The integrand on the left is a derivative (product rule):

$$\int_0^t \frac{d}{ds}\left(e^{\lambda_k s} a_k(s)\right) ds = \int_0^t e^{\lambda_k s} c_k(s)\, ds$$

So by the Fundamental Theorem of Calculus:

$$\left[e^{\lambda_k s} a_k(s)\right]_{s=0}^{s=t} = \int_0^t e^{\lambda_k s} c_k(s)\, ds$$

$$\Longrightarrow \quad e^{\lambda_k t} a_k(t) - a_k(0) = \int_0^t e^{\lambda_k s} c_k(s)\, ds$$

$$\Longrightarrow \quad \boxed{a_k(t) = a_k(0)\, e^{-\lambda_k t} + \int_0^t e^{\lambda_k (s-t)} c_k(s)\, ds}$$

Hence the solution to the inhomogeneous PDE is

$$\boxed{\begin{aligned} u(x,t) &= \sum_{j=1}^{\infty} a_j(t)\, \varphi_j(x) \\ &= \sum_{j=1}^{\infty} \left(a_j(0)\, e^{-\lambda_j t} + \int_0^t e^{\lambda_j (s-t)} c_j(s)\, ds \right) \varphi_j(x) \end{aligned}}$$

Recall that the coefficients $a_k(0)$ come from the best approximation expansion of the initial condition:

$$u(x,0) = u_0(x) = \sum_{j=1}^{\infty} a_j(0)\,\varphi_j(x)$$

given by $\quad a_j(0) = \dfrac{(u_0, \varphi_j)}{(\varphi_j, \varphi_j)}.$

Special case: Time-independent f

If $f(x,t) = f(x)$ has no dependence on t, then the coefficients $c_j(t)$ are constant, $c_j(t) = c_j$.

Then we can significantly simplify the formula for $a_j(t)$.

$$a_j(t) = a_j(0)\,e^{-\lambda_j t} + \int_0^t e^{\lambda_j(s-t)}\,c_j\,ds$$

$$= a_j(0)\,e^{-\lambda_j t} + c_j e^{-\lambda_j t}\int_0^t e^{\lambda_j s}\,ds$$

$$= a_j(0)\,e^{-\lambda_j t} + c_j e^{-\lambda_j t}\underbrace{\left[\frac{e^{\lambda_j s}}{\lambda_j}\right]_0^t}_{=\frac{e^{\lambda_j t}-1}{\lambda_j}}$$

(Assuming $\lambda_j \neq 0$) Need to handle $\lambda_j = 0$ separately, if the problem has a zero eigenvalue.

$$= a_j(0)\,e^{-\lambda_j t} + c_j\left(\frac{1-e^{-\lambda_j t}}{\lambda_j}\right)$$

For the operator $Lu = -u''$ on $C_0^2[0,1] \to C[0,1]$, $\lambda_j = j^2\pi^2$; $j = 1, 2, 3, \ldots$, so

$$a_j(t)\,e^{-\lambda_j t} \longrightarrow 0, \qquad \frac{1-e^{-\lambda_j t}}{\lambda_j} \longrightarrow \frac{1}{\lambda_j}.$$

Hence

$$a_j(t) \to \frac{c_j}{\lambda_j} \quad \text{as} \quad t \to \infty$$

and the solution

$$u(x,t) = \sum_{j=1}^{\infty} \left(a_j(0) e^{-\lambda_j t} + c_j \left(\frac{1 - e^{-\lambda_j t}}{\lambda_j} \right) \right) \varphi_j(x)$$

tends to

$$\sum_{j=1}^{\infty} \frac{c_j}{\lambda_j} \varphi_j = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \frac{(f, \varphi_j)}{(\varphi_j, \varphi_j)} \varphi_j$$

Note: this is exactly the solution to $Lu = f$ we obtained from the spectral method!
We have just confirmed that the heat equation with Dirichlet boundary conditions and time-independent f tends to a steady state, justifying our earlier study of $Lu = f$.

Question: How general was this analysis?
Will this apply to other boundary conditions?
What do you need to require of the eigenvalues?