8. HEAT EQUATION: EXACT SOLUTION VIA THE SPECTRAL METHOD

WE NOW BEGIN OUR DISCUSSION OF TIME-DEPENDENT PROBLEMS, such AS THE HEAT EQUATION

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + f(x,t)$$

FOR A HOMOGENEOUS BAR, WITH DIRICHLET BOUNDARY CONDITIONS $u(0,t) = u(1,t) = 0$

AND NOW AN INITIAL CONDITION

$$u(x,0) = u_0(x).$$

MORE GENERALLY, WE THINK OF PROBLEMS OF THE FORM

$$(*) \qquad \frac{\partial}{\partial t} u(x,t) = -L\, u(x,t) + f(x,t)$$

WHERE L IS A SYMMETRIC LINEAR OPERATOR WITH EIGENVALUES AND EIGENFUNCTIONS λ_j AND φ_j:

$$L\, \varphi_j = \lambda_j\, \varphi_j$$

for which we can expand the solution $u(x,t)$ as

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\, \varphi_j(x).$$

SUBSTITUTE THIS FORM INTO $(*)$ TO OBTAIN

$$\frac{\partial}{\partial t} \sum_{j=1}^{\infty} a_j(t)\, \varphi_j(x) = -L \sum_{j=1}^{\infty} a_j(t)\, \varphi_j(x) + f(x,t)$$

TAKE THE TIME DERIVATIVE AND LINEAR OPERATOR L UNDER THE SUMS TO GET

$$\sum_{j=1}^{\infty} a_j'(t)\,\varphi_j(x) = -\sum_{j=1}^{\infty} a_j(t)\,\underbrace{L\varphi_j(x)}_{=\lambda_j \varphi_j(x)} + f(x,t)$$

$$\Rightarrow \quad \sum_{j=1}^{\infty} a_j'(t)\,\varphi_j(x) = \sum_{j=1}^{\infty} -\lambda_j\, a_j(t)\,\varphi_j(x) + f(x,t)$$

TAKE THE INNER PRODUCT WITH φ_k TO OBTAIN

$$\left(\sum_{j=1}^{\infty} a_j'(t)\,\varphi_j(x),\ \varphi_k\right) = \left(\sum_{j=1}^{\infty} -\lambda_j\, a_j(t)\,\varphi_j(x),\ \varphi_k\right) + (f,\varphi_k)$$

USE LINEARITY OF THE INNER PRODUCT:

$$\sum_{j=1}^{\infty} a_j'(t)\,(\varphi_j,\varphi_k) = \sum_{j=1}^{\infty} -\lambda_j\, a_j\,(\varphi_j,\varphi_k) + (f,\varphi_k).$$

USE ORTHOGONALITY OF THE EIGENFUNCTIONS:

$$(\varphi_j,\varphi_k) = \begin{cases} 0, & j \neq k \\ (\varphi_j,\varphi_j), & j = k \end{cases}$$

TO SIMPLIFY TO AN EQUATIONS FOR $a_k'(t)$:

$$a_k'(t) = -\lambda_k\, a_k(t) + \frac{(f,\varphi_k)}{(\varphi_k,\varphi_k)}$$

NOTE THAT

$$(f,\varphi_k) = \int_0^1 f(x,\underset{\uparrow}{t})\,\varphi_k(x)\,dx \quad \underline{\text{for fixed } t},$$

SO

$$\frac{(f,\varphi_k)}{(\varphi_k,\varphi_k)} = C_k(t) \quad \text{IS A FUNCTION OF } t.$$

WE THUS HAVE

$$a_k'(t) = -\lambda_k a_k(t) + c_k(t).$$

IN THIS LECTURE WE SET $f(x,t) = 0$, GIVING SIMPLY

$$a_k'(t) = -\lambda_k a_k(t)$$

WHOSE SOLUTION IS SIMPLY

$$a_k(t) = e^{-\lambda_k t} a_k(0).$$

HOW DO WE FIND $a_k(0)$? RECALL THAT $u(x,t)$ SATISFIES THE INITIAL CONDITION

$$u(x,t) = u_0(x).$$

AT $T=0$, WE THUS WANT

$$u(x,t) = \sum_{j=1}^{\infty} a_j(0)\, \varphi_j(x) = u_0(x).$$

TAKE THE INNER PRODUCT WITH φ_k TO GET

$$\sum_{j=1}^{\infty} a_j(0)\,(\varphi_j, \varphi_k) = (u_0, \varphi_k)$$

AND USE ORTHOGONALITY OF THE EIGENFUNCTIONS TO GET

$$a_k(0) = \frac{(u_0, \varphi_k)}{(\varphi_k, \varphi_k)}$$

Thus

$$a_k(t) = e^{-\lambda_k t} a_k(0) = e^{-\lambda_k t} \frac{(u_0, \varphi_k)}{(\varphi_k, \varphi_k)}.$$

PUTTING THE PIECES TOGETHER WE HAVE

$$u(x,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \frac{(u_0, \varphi_j)}{(\varphi_j, \varphi_j)} \varphi_j(x).$$

FOR THE MOTIVATING PROBLEM

$$u_t = u_{xx}, \quad x \in [0,1], \quad u(0,t) = u(1,t) = 0$$

with $\quad Lu = -u_{xx}, \quad u \in C_D^2[0,1]$

so $\quad \lambda_j = j^2\pi^2, \quad \varphi_j(x) = \sqrt{2}\sin(j\pi x),$

WE HAVE

$$u(x,t) = \sum_{j=1}^{\infty} e^{-j^2\pi^2 t} \frac{(u_0, \varphi_j)}{(\varphi_j, \varphi_j)} \left(\sqrt{2}\sin(j\pi x)\right).$$

TAKE NOTE!

As $t \to \infty$, we see $e^{-j^2\pi^2 t} \to 0$, so

- $u(x,t) \to 0$ as $t \to 0$
- As t INCREASES, $e^{-\pi^2 t}$ $(j=1)$ DECAYS TO 0 QUITE A BIT MORE SLOWLY THAN $e^{-4\pi^2 t}$ $(j=2)$, $e^{-9\pi^2}$ $(j=3)$, etc.,

  So AS $u(x,t) \to 0$, IT WILL ASSUME THE SHAPE OF $\varphi_1(x)$ (ASSUMING THAT $(u_0\ \varphi_1) \neq 0$).